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Geometric cone surfaces and $(2+1)$-gravity coupled to particles

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Abstract

We introduce the $(2+1)$-spacetimes with compact space of genus $g \geqslant 0$ and r gravitating particles which arise by three kinds of construction called: (a) the *Minkowskian suspension* of flat or hyperbolic cone surfaces; (b) the *distinguished deformation* of hyperbolic suspensions; (c) the *patchworking* of suspensions. Similarly to the matter-free case, these spacetimes have nice properties with respect to the canonical Cosmological Time Function. When the values of the masses are sufficiently large and the cone points are suitably spaced, the distinguished deformations of hyperbolic suspensions determine a relevant open subset of the full parameter space; this open subset is homeomorphic to $\mathcal{U} \times \mathbb{R}^{6g-6+2r}$, where $\mathcal{U}$ is a non empty open set of the Teichmüller space T_g^r. By patchworking of suspensions one can produce examples of spacetimes which are not distinguished deformations of any hyperbolic suspensions, although they have the same topology and same masses; in fact, we will guess that they belong to different connected components of the parameter space.

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1. Introduction

Globally hyperbolic matter-free $(2+1)$-spacetimes with compact space of genus $g \geqslant 1$ and cosmological constant $\Lambda = 0$ have been fairly well understood. These spacetimes can be arranged into classes with respect to the Teichmüller equivalence which is the equivalence up to isometry isotopic to the identity. The resulting parameter space of these universes can be identified with the cotangent bundle of the Teichmüller space T_g, which is homeomorphic to $B^{6g-6} \times \mathbb{R}^{6g-6}$, when $g \geqslant 2$, where B^m denotes the open m-dimensional

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ball. Two methods are particularly useful to study matter-free spacetimes: the *geometric-time-free* approach, which eventually identifies each spacetime by its *geometric holonomy* [9,19] and the *cosmological* approach which is based on fibrations by *constant mean curvature* space-like surfaces [1,12]. When $g \geqslant 2$, the correspondence between these two approaches is rather implicit. In [4] we have shown that the canonical *Cosmological Time Function* (CTF), that is the length of time that the events have been in existence, provides a very good cosmological resolution of the matter-free (2+1)-gravity. For instance, when $g \geqslant 2$, the asymptotic states associated with the CTF recover and decouple the linear Lorentz component and the translation part of the geometric holonomy; the orbit in T_g of the CTF is a real analytic curve connecting an interior point with a point of the Thurston's natural boundary of T_g; the initial singularity can be accurately described in terms of the degeneration of the geometry of the level surfaces of CTF (see Section 4 for more details).

It turns out that all matter-free spacetimes are obtained by means of two basic constructions:

(i) The *Minkowskian suspension* of flat or hyperbolic surfaces;

(ii) A distinguished kind of deformation of the hyperbolic suspensions.

The starting point of this paper is the remark that the constructions (i) and (ii) can be extended to the case of gravitating particles in $(2+1)$ dimensions, as we are going to briefly outline.

Let $S = \mathbb{H}^2/G$ be a compact hyperbolic surface; let the hyperbolic plane $\mathbb{H}^2$ be realized by the upper-hyperboloid embedded into the Minkowski space $\mathbb{M}^{2+1}$. So G is a torsion-free discrete subgroup of $SO^+(2,1)$ which is isomorphic to the fundamental group $\pi(S)$ of S. What we call the *Minkowskian suspension* $M(S)$ of S is sometimes also called the Lorentzian cone of S or the Löbell spacetime based on S. In fact, $M(S) = I^+(0)/G$, where $I^+(0)$ is the chronological future of $\{0\}$ in $\mathbb{M}^{2+1}$. $M(S)$ is a flat spacetime containing S as a Cauchy surface. Another equivalent way to define $M(S)$ is the following: S is endowed with a Riemannian metric ds^2 of constant curvature -1; then $M(S)$ is isomorphic to $S\times]0,\infty[$ with the metric $t^2\,ds^2 - dt^2$. Let S be now a hyperbolic surface with conical singularities (see Section 2 for the precise definition); though S is not in general a quotient of $\mathbb{H}^2$, the second description of $M(S)$ applies also to its regular part S' and leads to a spacetime $M(S)$ with gravitating particles corresponding to the cone points of S. If S is a flat torus there are actually two possible notions of Minkowskian suspension of S, leading to static or non-static flat spacetimes, respectively; again these notions can be suitably extended in the framework of flat surfaces with conical singularities (see Section 3 for more details about the suspensions).

Let us come to the deformations of hyperbolic suspensions. Let $S = \mathbb{H}^2/G$ be as above. A deformation of $M(S)$ is a flat spacetime of the form $V = U/G'$ where:

(1) G' is a subgroup of $ISO^+(2,1)$, isomorphic to $\pi(S)$, having G as linear Lorentz part;

(2) U is a maximal G'-invariant simply connected domain in $\mathbb{M}^{2+1}$ on which G' acts freely and properly discontinuously, V is diffeomorphic to $M(S)$ and contains a Cauchy surface diffeomorphic to S.

For a non trivial deformation $U \neq I^{+}(0)$, the initial singularity of V is non trivial an so on. It turns out (see [9] and also [4]) that any such a deformation $V = M(S, \mathcal{F})$ is completely determined by a suitable geometric object $\mathcal{F}$ embedded into S called a *measured geodesic lamination*. Measured geodesic laminations play a fundamental role in Thurston's works on the mapping class groups of surfaces and on the geometrization of 3-manifolds. It is a rather sophisticated mathematical theory. A good introduction to this theory can be found in [13]. Nevertheless the simplest measured geodesic laminations (called *multicurves*) are really very elementary objects as they consist of disjoint simple closed geodesics of S endowed with positive real weights. Multicurves are very significant because a generic geodesic lamination can be arbitrarily well approximated by multicurves. Moreover, the deformation $M(S, \mathcal{F})$ associated to a multicurve $\mathcal{F}$ can be described in a very elementary way. For the purposes of this paper it will be enough to understand this simple situation. If S has conical singularities we can use the geodesic laminations in its regular part S' and hence we have a corresponding notion of distinguished deformation of the Minkowskian suspension $M(S)$ (see Section 4 for more details about the deformations).

When gravity is coupled to particles, an explicit construction of all possible spacetimes has not been produced. We shall be concerned with compact spaces with a finite number of massive particles and vanishing cosmological constant. 't Hooft's approach [8] describes these spacetimes by means of the "linear" evolution of a special kind of Cauchy surfaces which are tiled by spatial planar polygons. The extrinsic curvature is null in the interior of each tile and it is singular along the edges; the evolution includes the changing of tiling combinatorics under codified transition rules. Each Cauchy surface of this type is intrinsically a flat surface with conical singularities. Some of these singularities correspond to the intersection with the particle world-lines; the spacetime has a concentrated curvature along these lines. The remaining singularities are 3-dimensionally *apparent singularities*, but the Gauss–Bonnet constraint implies that, in general, they cannot be avoided. Each globally hyperbolic spacetime contains such a kind of Cauchy surface with, at least locally, such a kind of evolution. However, it is not clear whether the evolution of a given surface necessarily fills all the spacetime and how the evolutions of different surfaces in the same spacetime are related each other. So, it seems hard to recover from this approach a clear identification of the parameter space.

Another experimented approach (see [2,3,11]) is the classical ADM formalism with the so called "instantaneous gauge", that requires fibration by spatial Cauchy surfaces with zero extrinsic curvature. This last requirement is technically very useful and permits to analytically find solutions by means of classical and very elegant mathematical tools. Unfortunately, it turns out that the only spacetimes with compact space covered by this approach are the static ones that is, by using the terminology of the present paper, the static Minkowskian suspensions of flat surfaces with conical singularities, that we shall describe below.

The aim of this note is to describe the spacetimes with compact space of genus $g \geqslant 0$ and r gravitating particles that one can obtain by means of three kinds of construction:

(a) The *Minkowskian suspensions* of flat or hyperbolic surfaces with conical singularities;

(b) The *distinguished deformations* of hyperbolic suspensions (in strict analogy with the matter-free case);

(c) The *patchworking* of Minkowskian suspensions (this is peculiar of gravity coupled with particles — see Section 5).

These spacetimes have very transparent structural properties and behave somewhat similarly to the matter-free universes with respect to the CTF, its asymptotic states, the initial singularity and so on. Moreover they form a rather wide class of spacetimes, so that we can derive from them some non trivial information about the actual parameter space. For example we will show that when the masses are big enough and the cone points are suitably spaced, the distinguished deformations of hyperbolic suspensions determine a relevant non empty open subset of the parameter space of the form $\mathcal{U} \times \mathbb{R}^{6g-6+2r}$, where $\mathcal{U}$ is an open set of the Teichmüller space $T_g^r \approx B^{6g-6+2r}$. On the other hand, by patchworking of suspensions, we will produce spacetimes with the same topology and the same masses of certain hyperbolic suspensions but which are not equivalent to any distinguished deformation of them. In fact we will guess that they belong to different connected components of the parameter space. So gravity coupled to particles seems to be much more flexible than pure gravity. In the last section we will state several related questions and we will develop a few speculations.

2. Geometric surfaces with conical singularities

Cone points. The local models of *flat* or *hyperbolic* surfaces with a conical singularity are respectively given, in complex coordinate, by the metrics on $\{|z| < 1\}$:

$$ds^2_{(E,\alpha)} = \alpha^2\, |z|^{2\alpha-2}\, |dz|^2 \,,$$

$$ds^2_{(H,\alpha)} = \alpha^2 \big[2\big/\big(1-|z|^{2\alpha}\big)\big]^2\, |z|^{2\alpha-2}\, |dz|^2,$$

where $\alpha > 0$. These metrics are obtained by pull-back of the standard Euclidean or Poincaré metrics on $\{|w| < 1\}$ via the map $w = z^\alpha$. In both cases the *concentrated curvature* at the conical point with coordinate $z = 0$ is $k = 2\pi(1-\alpha)$, the *cone angle* is $2\pi\alpha$. In order to have a genuine singularity, $\alpha \neq 1$.

Geometric cone surfaces. It is convenient to adopt the formalism of geometric (X, G)-manifolds (see, for instance, chapter B of [6]). Fix a *base* compact oriented surface F_g of genus $g \geqslant 0$ and fix $p_1, \ldots, p_r$ points on F_g. A *marked geometric (i.e., flat or hyperbolic) surface with conical singularities*, of cone angles $2\pi\alpha_i$, $i = 1, \ldots, r$, is a homeomorphism

$$\phi : \big(F_g, \{p_i\}\big) \to \big(S, \{q_i\}\big),$$

such that $S' = S \setminus \{q_i\}$ is a (X, G)-surface where $(X, G) = (\mathbb{R}^2, \mathit{Isom}^+(\mathbb{R}^2))$ or $(X, G) = (\mathbb{H}^2, \mathit{Isom}^+(\mathbb{H}^2))$, respectively and its metric completion has a conical singularity of cone angle $2\pi\alpha_i$ at q_i. $\mathit{Isom}^+(X)$ denotes the group of oriented isometries of X. We recall that a (X, G)-manifold is by definition endowed with an atlas $\{U_i, \psi_i\}$, where the homeomorphisms $\psi_i : U_i \to W_i \subset X$ are such that each transition map $\psi_i \circ \psi_j^{-1}$ coincides with an element of the specified group G of diffeomorphisms of X on each connected component of its domain of definition.

Gauss–Bonnet constraint. The classical Gauss–Bonnet formula leads to the following relations.

Flat case:

$$\sum_i k_i = 2\pi \sum_i (1-\alpha_i) = 2\pi(2-2g) \quad \text{(Gauss–Bonnet equality).}$$

Hyperbolic case:

$$\sum_i k_i = 2\pi \sum_i (1-\alpha_i) = 2\pi(2-2g) + \mathrm{Area}(S),$$

whence:

$$\sum_i (1-\alpha_i) > 2-2g \quad \text{(Gauss–Bonnet inequality).}$$

This implies, in any case, that when $g = 0$, necessarily $r \geqslant 3$, and we will make this assumption by default. We say that

$$\delta = (\mathbb{X}, g, [\alpha]_r) = \big(\mathbb{X}, g, (\alpha_1, \ldots, \alpha_r)\big),$$

(where $\mathbb{X} = \mathbb{R}^2$ or $\mathbb{H}^2$, $g \geqslant 0$ and the α_i' satisfy the appropriate Gauss–Bonnet equality or inequality), is a *virtual type* of geometric surfaces with conical singularities. We denote by T_δ the *Teichmüller space* of marked surfaces of type δ regarded up to *Teichmüller equivalence*. Two marked geometric surfaces of type δ, ϕ_1 and ϕ_2 are equivalent iff there exists an isometry $f : (S_1, \{q_i^1\}) \to (S_2, \{q_i^2\})$ such that $\phi_2^{-1} \circ f \circ \phi_1$ is isotopic to the identity of F_g relatively to $\{p_i\}$.

When $r > 0$, the fundamental group $\pi(F_g')$, where $F_g' = F_g \setminus \{p_i\}$, is a non-Abelian free group with $s = 2g + r + 1$ generators. For each $[\phi] \in T_\delta$, the associated *holonomy* representation

$$\rho_{[\phi]} : \pi(F_g') \to \mathit{Isom}^+(\mathbb{X})$$

is well defined up to conjugation.

The universal covering map $p : S^* \to S$ is, in a natural way, a local isometry so that S^* is homeomorphic to $\mathbb{R}^2$ and it is endowed with a geometric structure with conical singularities. $\pi(S)$ acts freely and properly discontinuously on S^* and $S = S^*/\pi(S)$. In general a geometric surface S with conical singularities cannot be realized in the form $S = \mathbb{X}/\Gamma$ where Γ is a discrete subgroup (not necessarily torsion-free) of the group of isometries of $\mathbb{X}$. When this is the case, the surface is called an orbifold.

Orbifolds. Geometric 2-dimensional compact orbifolds with only conical singularities make a special class of surfaces we are concerned with. Such an orbifold S is a quotient $\mathbb{X}/\Gamma$ where Γ is a group of isometries of $\mathbb{X}$ acting properly discontinuously and such that the set of points with non trivial stabilizer is made by isolated points. For a genuine orbifold this set is nonempty. Orbifolds might have an important role in the construction of a quantum version of $(2+1)$-gravity. They are classified as follows (see [15,16]).

Proposition 2.1. *A geometric cone surface is a genuine Euclidean orbifold iff it is of one of the types* $(\mathbb{R}^2, 0, (1/2, 1/3, 1/6))$, $(\mathbb{R}^2, 0, (1/2, 1/4, 1/4))$, $(\mathbb{R}^2, 0, (1/3, 1/3, 1/3))$, $(\mathbb{R}^2, 0, (1/2, 1/2, 1/2, 1/2))$. *A geometric cone surface is a genuine hyperbolic orbifold iff it is of a type* $(\mathbb{H}^2, g, [\alpha]_r)$ *satisfying the Gauss–Bonnet inequality and such that each* $\alpha_i \in [\alpha]_r$ *is of the form* $\alpha_i = 1/n_i$, $n_i \in \mathbb{N}^*$.

Conformal structures. Associated to each geometric structure with conical singularities there is a natural *conformal structure*. First, it is clear that any atlas of the geometric structure on the regular part S' is actually a conformal atlas (use the Poincaré disk model for $\mathbb{H}^2$, identify $\mathbb{R}^2$ with $\mathbb{C}$ and recall that any $f \in Isom^+(\mathbb{X})$ is also a biholomorphism). Then, in order to get a conformal atlas on the whole surface S, we only need to add the chart in complex coordinates around each conical singularity defined at the beginning of the present section.

Let T_g^r be the classical *Teichmüller space* of conformal structures on F_g, relatively to the marked points $\{p_i\}$; T_g^r is homeomorphic to an open ball $B^{6g-6+2r}$. For each virtual type $\delta = (\mathbb{X}, g, [\alpha]_r)$, if T_δ is non empty, there is a natural continuous map

$$\psi_\delta : T_\delta \to T_g^r,$$

which is obtained by associating to each geometric cone surface of type δ the conformal structure described above. In the case of flat surfaces we eliminate simple rescalings by normalizing the area to be equal to unity. Geometric surfaces with conical singularities are classified by the following proposition, which shows that the configurations space of geometric cone surfaces of a given type is isomorphic with the classical Teichmüller space T_g^r.

Proposition 2.2. *For any virtual type* δ, T_δ *is non empty and the natural map* ψ_δ *is a homeomorphism.*

Sketch of proof. The flat case is due to Troyanov (see [17]). The orbifold case is treated in [16]. Let us sketch the main steps of a proof in the general hyperbolic case.

(1) $\dim(T_\delta) = \dim(T_g^r)$.
Let us outline first a way to construct all hyperbolic cone surfaces. Fix $(F_g, \{p_1, \ldots, p_r\})$ as before. A *standard spine* of F_g' is a 1-complex P embedded in F_g', with only 3-valent vertices, such that F_g' is a regular neighbourhood of P (F_g' retracts onto P). Associated to such a P there is a dual (topological) *ideal triangulation* τ_P of F_g', that is a "relaxed" (i.e., multiple and self adjacencies between triangles are allowed) triangulation of F_g, having $\{p_1, \ldots, p_r\}$ as set of vertices. If $v(P) = |V(P)|$ denotes the number of vertices of P (i.e., the number of triangles of τ_P), $e(P) = |E(P)|$ the number of its edges (i.e., the number of the edges of the dual triangulation), one has $3v(P) = 2e(P)$ so that $e(P) =$

$6g-6+3r$. Clearly spines exist. Fix a spine P. For any *admissible* map $f: E(P) \to \mathbb{R}^+$ (i.e., a map such that at each vertex $v \in P$ the values of f on the three edges emanating from v satisfy the *triangular inequalities*), we can construct a hyperbolic surface with r conical singularities. This is obtained as a geometric realization of the dual triangulation τ_P, by using hyperbolic triangles with edge lengths prescribed by f. Recall that each hyperbolic triangle is determined by the edge lengths as well as by the interior angles, and there are classical explicit formulas relating lengths and angles. It is not too hard to see that varying the spine and the admissible function, one can realize all the hyperbolic virtual types. On the other hand, any cone hyperbolic surface arises in this way. In fact let $(F_g, F'_g) \sim (S, S')$ be such a surface. Consider the subset Q of S, such that for each $x \in Q$ there exist $i \neq j$ such that $d(x, p_i) = d(x, p_j)$. Generically Q is a standard spine of S'; the interior of an edge of Q consists of the points with exactly two equidistant marked points p_i, p_j, the same along the given edge. The "axis" of each edge, that is the geodesic arc connecting p_i and p_j and passing from the point of the edge of minimal distance from them, are the edges of a geometric realization on the dual triangulation τ_Q. In general Q is a spine, possibly with higher valency vertices; the same procedure produces a dual ideal cellularization of S' by convex hyperbolic polyhedra and we eventually obtain a geometric triangulation by subdividing without introducing new vertices. If a virtual type δ is realized by an admissible map f_0 on $E(P)$, the maps realizing the same type are obtained by imposing r independent conditions. So one can deduce, at least, that T_δ is a topological manifold of the right dimension $6g-6+2r$.

(2) *The map ψ_δ is injective.*
Consider $\mathbb{H}^2$ in the Poincaré disk model $D = \{|z| < 1\}$, and let $e^{2h}|dz|^2$ be the standard Poincaré distance. Realize a given element σ of T_g^r by a smooth hyperbolic surface (with marked points) $S = D/\Gamma$. Two hyperbolic surfaces with conical singularities of the same type, both representing σ, are given by two metrics $e^{2(h+h_i)}|dz|^2$, $i = 1, 2$, such that each h_i is a Γ-equivariant function on D, with the same kind of singularities over the marked points of S. It follows that $h_1 - h_2$ is a real analytic Γ-equivariant function on D satisfying the Liouville equation

$$\Delta(h_1 - h_2) = e^{2h}\left(e^{2h_1} - e^{2h_2}\right).$$

As S is compact $h_1 - h_2$ has maxima and minima. Either $\Delta(h_1 - h_2) > 0$ near a maximum, or $\Delta(h_1 - h_2) \leqslant 0$ near a minimum. By the maximum principle $h_1 - h_2$ is constant near the minimum or the maximum and hence it is constant (and necessarily equal to 0) everywhere.

(3) *Conclusion.*
By the *invariance of domain* theorem, ψ_δ is a homeomorphism onto a non empty open subset of T_g^r. To conclude it is enough to show that the image of ψ_δ is closed. This can be done by studying the convergence of the conformal factors (see the above step), or by arguing (via geometric considerations) that the image of a "diverging" sequence in T_δ is diverging in T_g^r. □

3. Minkowskian suspensions

Particle world-lines. Let us give, first, the local models of the line of universe of a massive particle. They are obtained by "suspension" of the local models for geometric cone surfaces. We can take indifferently, in coordinates (z, t),

$$d\sigma^2_{(E,\alpha)} = -dt^2 + ds^2_{(E,\alpha)},$$

or, assuming $t > 0$

$$d\sigma^2_{(H,\alpha)} = -dt^2 + t^2\, ds^2_{(H,\alpha)}.$$

They are equivalent *as local models*, in the sense that any point $(0, t_1)$ in the first model and any point $(0, t_2)$ in the second one have isometric neighbourhoods. They are not equivalent as global spacetimes; for instance if we take the time orientation in accordance with the t coordinate, the CTF of the first spacetime is degenerate, constant equal to ∞, while t is the CTF of the second one. We have a well defined cone angle $2\pi\alpha$ along such a universe line, which corresponds to a spacetime curvature concentrated along the line. In accordance with [7,8], if we normalize the gravitational constant to be $G = 1$, the *mass* of the particle is related to the cone angle by $m = (1/4)(1 - \alpha)$; in $(2 + 1)$-gravity there are not physical constraints on the sign of Gm, so that an arbitrarily big α is allowed.

Spacetimes with gravitating particles. A marked globally hyperbolic spacetime (coupled to massive particles) of type

$$\delta = \big(g, [\alpha]_r\big) = \big(g, (\alpha_1, \ldots, \alpha_r)\big)$$

is an homeomorphism

$$\phi : \big(F_g \times \mathbb{R}, \{p_i\} \times \mathbb{R}\big) \to (M, L_i),$$

such that $M' = M \setminus \{L_i\}$ is an oriented and time-oriented globally hyperbolic flat Lorentzian 3-manifold (i.e., a $\mathbb{M}^{2+1}$, $Isom^+(\mathbb{M}^{2+1})$-manifold, where $\mathbb{M}^{2+1}$ is the standard Minkowski space) and each point of L_i has a neighbourhood isometric to the above local models, with cone angle $2\pi\alpha_i$. It is convenient to restrict to *Geroch marking*, that is we stipulate that the surfaces $\phi(F_g \times \{t\})$ are Cauchy surfaces. As usual we work up to Teichmüller equivalence and we denote by T^{GR}_δ the corresponding Teichmüller space for a given type. We shall consider maximal spacetimes. Identifying F_g with $F_g \times \{0\}$, we have the holonomy representation

$$\rho_{[\phi]} : \pi(F'_g) \to Isom^+(\mathbb{M}^{2+1}).$$

We also make the usual assumption that the linear part of the holonomy takes values in $SO^+(2, 1)$, the group of time-orientation preserving Lorentz transformations.

Minkowskian suspensions of geometric cone surfaces. These are peculiar spacetimes such that M' is a $(Y, G(Y))$-manifold, for suitably chosen open subsets Y of $\mathbb{M}^{2+1}$, $G(Y)$ being the group of orientation preserving Minkowskian isometries keeping Y invariant. As Y we will take:

$$Y_E = \mathbb{M}^{2+1}$$

with metric $(dx^1)^2 + (dx^2)^2 - (dx^3)^2$, and thought fibred by the planes $\{x^3 = a\}$.

$$Y_H = \{x \in \mathbb{M}^{2+1}\colon (x^1)^2 + (x^2)^2 - (x^3)^2 < 0,\ x^3 > 0\}$$

thought fibred by the surfaces

$$\{x \in \mathbb{M}^{2+1}\colon (x^1)^2 + (x^2)^2 - (x^3)^2 = -a^2,\ x^3 > 0\}$$

finally

$$Y_T = \{x \in \mathbb{M}^{2+1}\colon (x^1)^2 - (x^3)^2 < 0,\ x^3 > 0\}$$

thought fibred by the surfaces

$$\{x \in \mathbb{M}^{2+1}\colon (x^1)^2 - (x^3)^2 = -a^2,\ x^3 > 0\}.$$

Note that Y_H is the chronological future of the point $\{0\}$ and that Y_T is the chronological future of the line $\{x_1 = x_3 = 0\}$. In both cases the square-root of a^2 is the CTF.

By the change of coordinates $x^1 = \tau sh(u),\ x^2 = y,\ x^3 = \tau ch(u)$, we see that Y_T is isometric to $P = \{(u, y, \tau) \in \mathbb{R}^{2+1}\colon \tau > 0\}$, with metric $\tau^2\, du^2 + dy^2 - d\tau^2$, and P is fibred by the level planes of the CTF τ.

Each Y_* is oriented and time-oriented in the usual way.

The group $G(Y_*)$ is $ISO(2,1)$ and $SO^+(2,1)$ for Y_E and Y_H, respectively; for Y_T it is more convenient to consider $G(P)$ which is generated by translations parallel to the level planes of τ and by the rotation of angle π around the τ-axis. Note that the foliations of these planes by vertical and horizontal lines are $G(P)$-invariant.

If S is a flat cone surface of type $(\mathbb{R}^2, g, [\alpha]_r)$, its Minkowskian suspension $M(S)$ is the obviously associated spacetime of type $(g, [\alpha]_r)$ such that $M'(S)$, that is the complement of the particle world-lines in $M(S)$, is a $(Y_E, G(Y_E))$-manifold with holonomy equal to the holonomy of S'. If S' has the flat metric ds^2, $M'(S)$ is isometric to $S' \times \mathbb{R}$ with metric $ds^2 - dt^2$. $M(S)$ is fibred by parallel copies of S. The CTF degenerates as it is constant equal to ∞. These are called *static Minkowskian suspensions*.

If S is a hyperbolic cone surface of type $(\mathbb{H}^2, g, [\alpha]_r)$, its Minkowskian suspension $M(S)$ is the obviously associated spacetime of type $(g, [\alpha]_r)$ such that $M'(S)$, that is the complement of the particle world-lines in $M(S)$, is a $(Y_H, G(Y_H))$-manifold with holonomy equal to the holonomy of S'. If S' has the metric ds^2 of constant curvature -1, $M'(S)$ is isometric to $S' \times]0, \infty[$ with metric $t^2 ds^2 - dt^2$. $M(S)$ is fibred by conformally rescaled copies of S; these surfaces are the level surfaces S_a of the CTF, in particular one has $S = S_1$; out of the particles, these surfaces S_a have constant mean curvature $1/a$ and constant intrinsic curvature equal to $-1/a^2$. The initial singularity consists of one point.

These suspensions are particularly nice when S is an orbifold (and the matter-free spacetimes are particular cases); if the orbifold $S = \mathbb{X}/\Gamma$, Γ acts isometrically also on the corresponding Y_*, and $M(S) = Y_*/\Gamma$.

The parameter space of Y_E- or Y_H-suspensions of a given type coincides, tautologically, with the parameter space of the suspended geometric cone surfaces (see the previous section).

The Y_T-Minkowskian suspensions involve the special flat cone surfaces S given by the *meromorphic quadratic differentials* with at most simple poles on Riemann surfaces.

The flat structures on the regular part S' of these cone surfaces have the peculiarity to be defined by atlas which have only translations or the rotation by angle π as transition maps. So the Y_T-Minkowskian suspension $M(S)$ is the natural spacetime such that $M'(S)$ is a $(P, G(P))$-manifold with holonomy equal to the holonomy of S'. In fact each such a suspension is determined by a couple (F, q), where F is a Riemann surface and q is a quadratic differential. That is, it is determined not only by the cone surface, but also by the horizontal and vertical measured foliations of the quadratic differential. We have already studied such spacetimes in [5] where we have shown how they "materialize" the classical *Teichmüller flow*. See also [4] for a description of the CTF. In fact in [5] we considered only holomorphic quadratic differentials, but everything runs verbatim if one allows also simple poles. Recall that in this way one can realize all the types with $2\pi\alpha_i = n_i\pi,\ n_i \geqslant 1$, satisfying the Gauss–Bonnet equality, with four exceptions (see [10]). Moreover, for any given realizable type, one knows the degrees of freedom (see [18]): if $\mu(a)$ denotes the number of cone points of cone angle a, then the degrees of freedom are

$$2g + \sum \mu(a) + (\epsilon - 3)/2,$$

where $\epsilon = -1$ iff there is at least one cone angle with odd n_i, and it is equal to 1 otherwise. For example, when the type contains only $n_i = 3$ (this corresponds to holomorphic quadratic differentials with simple zeros), the dimension of the corresponding space of Y_T-suspensions is $6g - 6$.

The only orbifolds which produce such a kind of suspension are the orbifolds of type $(\mathbb{R}^2, 0, (1/2, 1/2, 1/2, 1/2))$. They are obtained by the natural identification of the edges of two copies of a same "fundamental" Euclidean rectangle. The corresponding groups Γ are generated by two orthogonal translations and the rotation of angle π. Groups that determine the same Y_E-suspension (up to equivalence), do determine in general different Y_T-suspensions; in fact if we look at these groups acting on P, the horizontal and vertical foliations on each τ-level plane induce different foliations on the CTF level surfaces of the two suspensions.

4. Distinguished deformations of hyperbolic suspensions

As already mentioned in the introduction, all matter-free flat spacetimes with space of genus $g \geqslant 2$ can be obtained by deformation of Minkowskian suspensions of smooth hyperbolic surfaces S. Each deformation $M(S, \mathcal{F})$ is governed by a measured geodesic lamination $\mathcal{F}$ on S. We refer to [13] for an introduction to the general theory of geodesic laminations. For simplicity, we shall be concerned only with the case in which $\mathcal{F}$ is a multicurve, which is the simplest example of measured geodesic lamination. Multicurves are dense in the space of laminations, that is by making the multicurve "complicated" enough, we can fairly well approximate the shape of any spacetime; moreover the deformations associated to multicurves can be described in an elementary way.

Assume for a while that $S = \mathbb{H}^2/\Gamma$ is a smooth compact hyperbolic surface. A multicurve $\mathcal{F}$ consists of a finite union of disjoint simple geodesics endowed with positive weights. Assume, for simplicity, that there is one single geodesic σ, with weight s and length r. Consider the quotient $A'(s,r)$ of $B'(s,r) = \{(u, y, \tau) \in P;\ 0 \leqslant y \leqslant s\}$ by the group generated by the translation $(u, y, \tau) \to (u + r, y, \tau)$. Actually it is better to consider the isometric quotient $A(s,r)$ of $B(s,r) \subset Y_T$, obtained via the explicit change of coordinates given in Section 3. Then, in order to construct $M(S,\mathcal{F})$, cut-open $M(S)$ along the suspension of σ and insert $A(s,r)$ in the natural way. $M(S,\mathcal{F})$ is, by construction, fibred by C^1-embedded space-like surfaces made by the union of pieces of constant negative curvature and flat annuli; these surfaces are the level surfaces $S_a = \{\tau = a\}$ of the CTF τ.

$M(S)$ itself can be considered as a limit case of this procedure by taking $s = 0$. In such a case the level surfaces S_a are perfectly smoothly embedded and the initial singularity of $M(S)$ consists of one single point. In fact the lack of smoothness of the embedding of S_a is a "dual" large scale manifestation of the non trivial initial singularity of $M(S,\mathcal{F})$. We shall now elaborate on this point. Consider the universal covering $q : M(S,\mathcal{F})^* \to M(S,\mathcal{F})$; τ lifts to the CTF τ^* of $M(S,\mathcal{F})^*$ and each level surface S^*_a is the universal covering of the corresponding S_a. Consider also the universal covering map $p : \mathbb{H}^2 \to S$. Set $\Sigma = p^{-1}(\sigma)$; Σ consists of infinitely many disjoint complete geodesic lines of $\mathbb{H}^2$, called the leaves of Σ. One can define the so-called *dual metric tree* (T,d) of Σ. T embeds in $\mathbb{H}^2$ as follows: the vertices of T are obtained by choosing, in a Γ-invariant way, one point in each connected component of $\mathbb{H}^2 \setminus \Sigma$. Two vertices are connected by a geodesic edge iff the corresponding components are adjacent. The distance d is the length-space distance obtained by imposing that each edge of T, which crosses once one leaf of Σ, has length equal to s. (T,d) is endowed with a natural non trivial isometric action of $\pi(S)$. In a similar way (T,d) can be $\pi(S)$-equivariantly embedded in each level surface S^*_a; Σ is replaced by infinitely many disjoint flat bands of thickness equal to s. When $a \to \infty$, $(1/a)S^*_a$ converge to the hyperbolic plane $\mathbb{H}^2$ and the actions of $\pi(S)$ on $(1/a)S^*_a$ converge to the action of Γ on $\mathbb{H}^2$. When $a \to 0$ the actions of $\pi(S)$ on S^*_a "degenerate" to the action of $\pi(S)$ on T (more details can be found in [4]). Note that in this case T is a locally finite simplicial tree, but this is no longer true in the general case where a more complicated kind of dual *real trees* does occur. This complication is related to the fact that the typical intersection of a transverse interval with a generic geodesic lamination is a Cantor set.

If S is now a hyperbolic cone surface, the construction of the deformation $M(S,\mathcal{F})$ of $M(S)$ can be repeated if we take a multicurve or more generally a measured geodesic lamination $\mathcal{F}$ with compact support in the regular part S'; S^* plays the role of $\mathbb{H}^2$. The resulting spacetime $M(S,\mathcal{F})$ has the same type of $M(S)$ because the modifications occur far from the cone points.

Given a hyperbolic type $\delta = (g, [\alpha]_r)$, we denote by $D(\delta)$ the subset of T^{GR}_δ which is determined by the distinguished deformations of Minkowskian suspensions of hyperbolic cone surface of type δ. Of course, a suspension is meant as the trivial deformation of itself and there is a natural projection $p : D(\delta) \to T_\delta$, obtained by associating $M(S)$ and hence S to $M(S,\mathcal{F})$. The following proposition gives partial information on $D(\delta)$. We will use some notations introduced in Section 2. The set of hyperbolic "(g,r)-types" can be identified with an open set of $\mathbb{R}^r$.

Proposition 4.1.

(1) *For each hyperbolic type* δ *there is an open (possibly empty) maximal subset* $\mathcal{U}_\delta$ *of* T_δ *such that* $p^{-1}(\mathcal{U}_\delta) \subset D(\delta)$ *is homeomorphic to* $\mathcal{U}_\delta \times \mathbb{R}^{6g-6+2r}$ *(and* p *becomes the natural projection onto the first factor).*

(2) *For each* (g, r) *there is a maximal non empty open subset* $\mathcal{W}_{(g,r)}$ *of the space of* (g, r)*-types, such that for each* $\delta \in \mathcal{W}_{(g,r)}$, $\mathcal{U}_\delta$ *is non empty.*

(3) *For any* δ,

$$12g - 12 + 4r \geqslant \dim D(\delta) \geqslant 6g - 6 + 2r.$$

(4) *If* $\mathcal{U}_\delta$ *is non empty, then* $\mathcal{U}_\delta \times \mathbb{R}^{6g-6+2r}$ *is an open subset of* T_δ^{GR}.

Sketch of proof. By using the result of Section 2, the first statement is equivalent to show that the space of measured geodesic laminations with compact support on S', for a given hyperbolic cone surface S in $\mathcal{U}$ (for a suitable $\mathcal{U}$), is homeomorphic to $\mathbb{R}^{6g-6+2r}$. This fact is known in the "limit" case when each $\alpha_i = 0$, that is when S' is a complete finite area hyperbolic surface with r cusps (see [14]). Let us denote by HT_g^r (which is homeomorphic to T_g^r) the Teichmüller space of such hyperbolic surfaces with r cusps and fix one surface F. It is known that each geodesic lamination with compact support on F has support contained in F'' obtained by removing from F all the horocycles of length < 1 around all the cusp points (see [14, p. 72]). It turns out that any hyperbolic cone surface S which is "geometrically" close to F has, up to homeomorphism, the same space of measured geodesic laminations with compact support on S' as F. The crucial fact is that if S is close enough to a cusped F, each isotopy class of essential (i.e., non contractible nor contractible to one cone point) simple closed curves on S' is represented by a simple closed geodesic in S' of shortest length. "S geometrically close to F" means that, by removing suitable small "round" disks with centres at the cone points of S, we find S'' which is bi-Lipschitz homeomorphic to F'', by a homeomorphism close to an isometry. It follows that for any fixed compact subset K of HT_g^r there is an open subset U_K (possibly empty) of T_δ, which satisfies the first statement of the proposition.

To prove the second statement, it is enough to show that, for any fixed F as before, there are cone surfaces S close to F in the above sense. Fix a geodesic ideal triangulation $\mathcal{T}$ of F (i.e., a "relaxed" triangulation of F by ideal hyperbolic triangles). For each $0 < a < 1$ consider the horocycles of length a around the cusps of F. Associate to each edge of the triangulation the length of the subarc determined by the horocycles. Consider the cone surface S obtained accordingly with the construction after Proposition 2.2, by using the same $\mathcal{T}$ as topological ideal triangulation of F_g' and those lengths as edge-lengths. If a is small enough, S is close to F. S is not close enough to a cusped F when, at a qualitative level, the masses are not big enough or the particles are too close each other on a given level surface of the CTF of the corresponding Minkowskian suspension. In such a case the basic trouble consists in the fact that the shortest length representative of an essential isotopy class of simple closed curves on S' might be a broken geodesic passing through some cone points or even it might not exist as a simple curve.

The third statement is clear from the above discussion.

To achieve the last statement it is enough to show that T_δ^{GR} is of dimension $12g - 12 + 4r$; we are going to argue it without any assumption on the spacetime type $\delta = (g, [\alpha]_r)$.

The degrees of freedom of T_δ^{GR}. Fix a marked spacetime M of type δ and a relatively compact globally hyperbolic open neighbourhood U of the Cauchy surface image of $F_g \times \{0\}$. Let $\rho : \pi(F'_g) \to ISO^+(2,1)$ be its holonomy. As $\pi(F'_g)$ is a free group, a deformation of ρ is simply obtained by modifying ρ on a set of $2g - 2 + r + 1$ free generators. If a deformation ρ' is small enough, then, by the stability property of holonomies, ρ' is still the holonomy of a spacetime structure on the interior of U, with r gravitating particles. So, as the holonomy is defined only up to conjugation, the dimension of the set of all these spacetimes "close" to M is $12g - 12 + 6r$. In order to impose that the spacetimes have the specific cone angles prescribed by δ, we have to impose $2r$ (that is $(6 - d)r$, where d is the dimension of the conjugation orbit of a "rotation") more independent conditions, and we finally get the required number of degrees of freedom $12g - 12 + 4r$. □

5. Patchworking of Minkowskian suspensions

A simple variation of the construction of the distinguished modification of hyperbolic suspensions, based on multicurves, that we have described in the previous section, will produce interesting new examples of spacetimes.

Let $M(S, \mathcal{F})$ be as in the previous section. Assume that we have a finite union of simple closed geodesic on S' which are *disjoint from* $\mathcal{F}$. For simplicity, assume that there is a single geodesic σ of length a. Let (F, q) be a Riemann surface with a meromorphic quadratic differential q, with at most simple poles. Let $M(F, q)$ be the corresponding Y_T-Minkowskian suspension (see Section 3). Assume that the q-horizontal foliation on F contains a simple closed leaf c of length a. Then we can construct new spacetimes as follows: cut-open $M(S, \mathcal{F})$ along the suspension of σ and $M(F, q)$ along the suspension of c; then glue, pairwise, pieces of $M(S, \mathcal{F})$ with pieces of $M(F, q)$ along isometric boundary components in the natural way. Note that there is, in general, a finite number of possible combinations, and the resulting Lorentz manifolds may be not connected, so we can take each connected component as a new spacetime. Call $M([S, \mathcal{F}, \sigma], [F, q, c])$ any spacetime obtained in this way. By construction, it is fibred by space-like surfaces (made by rescaled pieces of S and by "stretched" pieces of F) which actually are the level surfaces of the CTF of $M([S, \mathcal{F}, \sigma], [F, q, c])$. Note also that the construction can be *iterated*, starting from suitable $M([S, \mathcal{F}, \sigma], [F, q, c])$; so one can produce a wide class of new examples. This *patchworking* is peculiar of spacetimes with gravitating particles; in fact if we formally apply it to matter-free spacetimes we get nothing else than distinguished deformations of hyperbolic suspensions.

In particular, let us use as (F, q) the orbifolds of type

$$\big(\mathbb{R}^2, 0, 4[1/2]\big) = \big(\mathbb{R}^2, 0, (1/2, 1/2, 1/2, 1/2)\big)$$

with the horizontal and vertical foliations of q (with 4 simple poles) parallel to the edges of the "fundamental" rectangle. It is not hard to construct by the patchworking procedure types of the form $\delta = (g, [\alpha]_r) = (g, [\alpha]_{r'} \cup 2h[1/2])$, $2h + r' = r$, which were also realizable by Minkowskian suspension of some hyperbolic cone surface. On the other hand, these new spacetimes do not belong to $D(\delta)$ because, for instance, the level surfaces of the respective CTF are not isometric as they have cone points of cone angle π with no isometric neighbourhoods. Other differences manifest themselves by studying the past asymptotic states of the respective CTF. By small perturbation of the holonomy of these examples one could produce examples out of $D(\delta')$ for any $[\alpha]'_r$ close to $[\alpha]_r$.

6. Final questions and considerations

We are going to conclude with some questions, problems and, sometimes, with a guess about them.

(1) *Is T^{GR}_δ connected*?

The answer could depend on the type. We guess that the above examples not belonging to $D(\delta)$, actually do not even belong to the same connected component of any element of $D(\delta)$.

(2) *Does any spacetime satisfy the Gauss–Bonnet constraint* $\sum(1-\alpha_i) \geqslant 2-2g$?

We guess that by suitable small perturbations of the holonomy of static Minkowskian suspensions (which satisfy the Gauss–Bonnet *equality*) one could obtain spacetimes with $\sum_i(1-\alpha_i) < 2-2g$.

We note that all the examples of spacetime that we have produced starting from non static Minkowskian suspensions have the following property:

Each particle line of universe has a neighbourhood isometric to the set of points of spatial distance $< bt$, *for some positive* b, *from the* t*-axis in the model* $\{(z,t),\ |z|<1,\ t>0\}$ *with metric* $d\sigma^2_{(H,\alpha)}$ (see Section 3).

(3) *Does the same property hold for any spacetime with tame — see [4] — CTF with values onto* $(0,\infty)$?

It would be interesting to find, if any, examples where the linear function bt must be replaced by some positive function $f(t)$ going faster to 0 when $t \to 0$.

(4) *Find an intrinsic characterization of hyperbolic cone surfaces belonging to* $\mathcal{U}_\delta$.

One expects that it could be expressed in terms of inequalities involving the cone angle, the genus and the distances between the cone points.

(5) *Describe* $\mathcal{W}_{(g,r)}$. *In particular, does* $m(g,r)$ *exist, with* $1 > m(g,r) > 0$, *such that for any* $\delta \in \mathcal{W}_{(g,r)}$ *and for any mass* m_i *associated to* δ, *one has* $m_i > m(g,r)$?

For example, beside the "rigid" case $(g,r) = (0,3)$, the very peculiar case $(g,r) = (0,4)$ has $\mathcal{W}_{(0,4)}$ which coincides with the whole space of $(0,4)$-types; moreover, for each type $\delta \in \mathcal{W}_{(0,4)}$, $\mathcal{U}_\delta$ coincides with T_δ, so that $D(\delta) = T_\delta \times \mathbb{R}^2$. On the other hand, we guess, for example, that for each $(0,r)$, $r > 4$, the last question has negative answer.

(6) *Is* $D(\delta)$ *always of dimension* $> 6g - 6 + 2r$?

Here one is asking whether there are always non trivial distinguished deformations. We guess that when $g \geqslant 2$ and the masses are all positive, then $D(\delta)$ contains at least $T_\delta \times \mathbb{R}^{6g-6}$; in other words one expects that there is at least the same "amount" of distinguished deformations of the matter-free case of the same genus.

(7) *Let C be any closed subset of $\mathcal{U}_\delta$. Is $p^{-1}(C) \subset \mathcal{U}_\delta \times \mathbb{R}^{6g-6+2r}$ closed in T_δ^{GR}?*

Finally, we note that in several instances of the present paper we have seen how very natural perturbations of a given spacetime *do not preserve the type* (see for instance the constructions of Section 2 or the argument at the end of Section 4). This would suggest that the study of $(2+1)$-gravity (coupled to particles) "type by type", or even "space-genus by space-genus", could be misleading. Spacetimes should be considered "all together" and it becomes quite demanding to figure out the structure of the corresponding (infinite dimensional) parameter space. We guess that Grothendieck theory of "Teichmüller Towers" could play an important role.

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