# A NEW PROOF THAT $\Omega_3$ IS ZERO

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It is a fact that any closed orientable 3-manifold can be changed into $S^3$ by a finite number of elementary surgeries on embedded circles (which implies that $\Omega_3$, the 3-dimensional oriented cobordism group, is zero). Existing proofs of this fact either use a significant amount of algebraic topology (Thom [2]) or a lengthy calculation involving curves on a surface (Lickorish [1]). In this note, I shall give a short elementary proof which avoids both algebraic topology and calculation.

Suppose that $S$ is an orientable surface of genus $n$, then $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is said to be a *complete system of curves* on $S$ provided that each $x_i$ is a simple closed curve, the curves $\{x_i\}$ are pairwise disjoint and the union $\bigcup_i x_i$ does not separate $S$.

A *Heegaard diagram* $S(\mathbf{x}, \mathbf{y})$ is an orientable surface $S$ with two complete systems $\mathbf{x}, \mathbf{y}$. The diagram determines a closed orientable 3-manifold $M(\mathbf{x}, \mathbf{y})$ obtained by attaching thickened 2-discs to $S \times I$: along the $x_i$ on $S \times \{0\}$ and along the $y_i$ on $S \times \{1\}$, and then filling in the resulting $S^2$-boundaries with 3-balls. The resulting 3-manifold $M$ has a specific handle presentation with one 0-handle, one 3-handle, $n$ 1-handles and $n$ 2-handles; the curves $\mathbf{x}, \mathbf{y}$ are drawn on a level surface between the 1-handles and the 2-handles—the $\mathbf{x}$ being the $b$-spheres for the 1-handles and the $\mathbf{y}$ being the $a$-spheres for the 2-handles. Any handle presentation with one 0-handle and one 3-handle can be regarded as a Heegaard diagram in this way and it therefore follows from elementary results in handle theory that any orientable 3-manifold is given by some Heegaard diagram. Notice that if one of the $\mathbf{x}$ curves meets one of the $\mathbf{y}$ curves transversally in a single point then the corresponding handles are complementary and can be cancelled; therefore $M$ has a Heegaard diagram of lower genus. (In fact the reduced diagram can be obtained explicitly by cutting out a neighbourhood in $S$ of the two transverse curves, filling in the resulting circle boundary with a disc and completing any other curves, cut in the process, across the disc.)

I need the following observation.

LEMMA 1. *Suppose that $S(\mathbf{x}, \mathbf{y})$ is a Heegaard diagram and that $\mathbf{z}$ is a third complete system of curves on $S$. Let $\chi(M, \mathbf{z})$ denote the result of performing surgery on $M = M(\mathbf{x}, \mathbf{y})$ using the curves $\mathbf{z}$ (with framings given by parallel curves in the surface $S$). Then $\chi(M, \mathbf{z})$ is homeomorphic to the connected sum*

$$M(\mathbf{x}, \mathbf{z}) \# M(\mathbf{y}, \mathbf{z}).$$

*Proof.* Assume that the surgeries are performed at level $\{\frac{1}{2}\}$ in $S \times I$. Surgery on $z_i \times \{\frac{1}{2}\}$ is performed as follows. First, remove a neighbourhood of $z_i \times \{\frac{1}{2}\}$: this has the same effect as cutting $S \times I$ open at $S \times \{\frac{1}{2}\}$ near to $z_i$, and results in a new boundary torus $(\alpha_i \cup \beta_i) \times S^1$ when $\alpha_i \times S^1$ and $\beta_i \times S^1$ are annuli with common boundary in the two copies of $S \times \{\frac{1}{2}\}$, see Figure 1.

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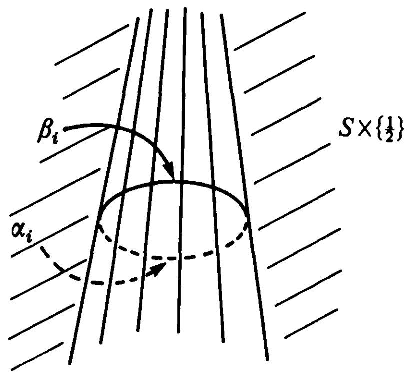


FIG. 1

Secondly, glue back a solid torus, namely $(\alpha_i \cup \beta_i) \times D^2$, where $\partial D^2$ is identified with $S^1$.

Now let $M_1, M_2$ be the results of cutting $M$ completely along $S \times \{\frac{1}{2}\}$ (where $M_1$ contains the lower half $S \times [0, \frac{1}{2}]$). Let

$$M_1^+ = M_1 \bigcup_i \alpha_i \times D^2, \quad M_2^+ = M_2 \bigcup_i \beta_i \times D^2.$$

Then $M_1^+, M_2^+$ are homeomorphic to $M(x, z)$, $M(y, z)$ respectively, with a 3-ball removed from each, and $\chi(M, z)$ is the union of $M_1^+, M_2^+$ along their common 2-sphere boundary.

I also need the following easy lemma.

LEMMA 2. *Suppose that $x, y$ are two non-separating curves on a surface $S$ which meet transversally in a finite number of points. Let $|x \cap y|$ denote the number of points of intersection.*

(a) *If $|x \cap y| = 0$ (that is, $x \cap y = \varnothing$) then there is a third non-separating curve $z$ which meets each of $x$ and $y$ transversally in a single point.*

(b) *If $|x \cap y| > 1$ there is a third non-separating curve $z$ such that $|x \cap z| < |x \cap y|$ and $|y \cap z| < |x \cap y|$.*

*Proof.* (a) Cut $S$ along $x$ and glue in discs $D_1, D_1'$ to get a new surface $S'$.

*Subcase* ($a_1$), *in which $y$ separates $S'$.* In this case $D_1, D_1'$ must lie on opposite sides of $y$ (or else $y$ would separate $S$). Join corresponding points of $D_1$ and $D_1'$ by a simple path $\alpha$ crossing $y$ once. Then $\alpha$ gives the required curve in $S$ (Figure 2).

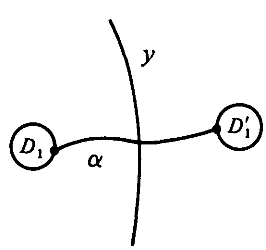


FIG. 2

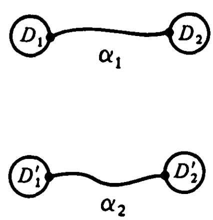


FIG. 3

*Subcase* (a$_2$), *in which $y$ does not separate $S'$.* Cut $S'$ along $y$ and glue in discs $D_2, D_2'$ then join $D_1$ to $D_2$ by a simple arc $\alpha_1$ and join the corresponding points of $D_1'$, $D_2'$ by another arc $\alpha_2$ not meeting $\alpha_1$. Then $\alpha_1 \cup \alpha_2$ gives the required curve in $S$ (Figure 3).

(b) By choosing two points of $x \cap y$ which are adjacent in $x$ we can find an arc $\alpha$ in $x$ which meets $y$ only at its end points $A, B$. Let $\beta, \gamma$ be the two arcs of $y$ joining $A$ to $B$. Since $y$ does not separate $S$, one of $\alpha \cup \beta$, $\alpha \cup \gamma$ does not separate $S$. Suppose, without loss of generality, that $\alpha \cup \beta$ does not separate $S$. Shift $\alpha$ off itself, starting by pushing in the $\beta$-direction at $A$. Complete by an arc close to $\beta$ to get a simple closed curve which meets $x$ in at least one fewer point and $y$ in at most one point (Figure 4(a) or (b)).

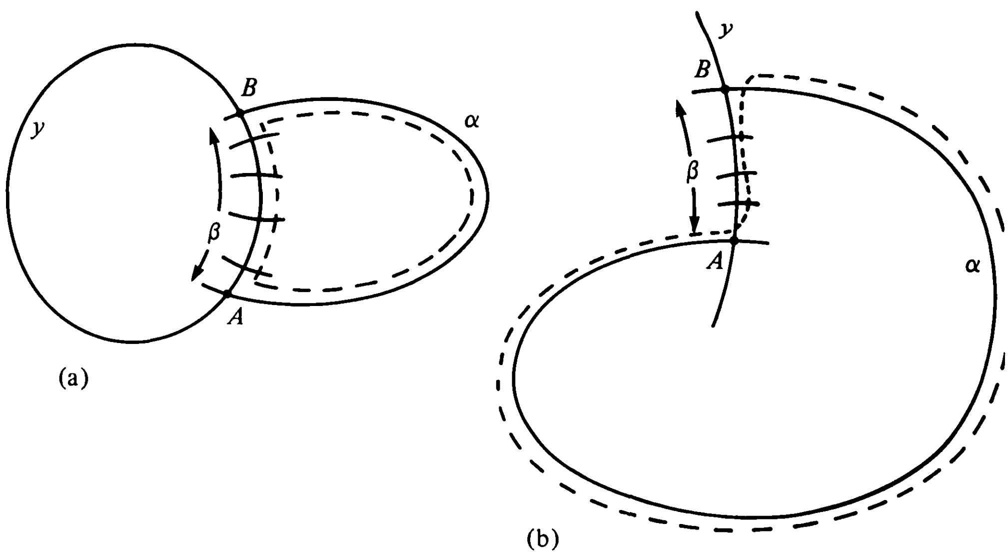


FIG. 4

THEOREM. *Any closed orientable 3-manifold can be reduced to $S^3$ by a finite number of surgeries on embedded curves.*

*Proof.* Let $M$ be a closed orientable 3-manifold and, without loss of generality, assume that $M = M(x, y)$. Associate to the diagram $S(x, y)$ two integers, namely

$$n = \text{genus}(S) \quad \text{and} \quad r = \min_{i,j} |x_i \cap y_j|.$$

The theorem is proved by double induction on $n$ and $r$. We assume, inductively, that the theorem is true for smaller $n$ or for the same $n$ and smaller $r$. The induction starts with $n = 0$, when $M$ is already $S^3$ and there is nothing to prove. The induction step is as follows.

*Case* 1, *in which $r > 1$.* Without loss of generality, we assume that $r = |x_1 \cap y_1|$. By Lemma 2(b), choose a curve $z_1$ such that $|x_1 \cap z_1| < r$ and $|y_1 \cap z_1| < r$. Extend $z_1$ to a complete system $z$ and apply Lemma 1:

$$\chi(M, z) \simeq M(x, z) \# M(y, z).$$

Each of the 3-manifolds on the right of the equation has diagram with the same $n$ but smaller $r$. By induction, each can be reduced to $S^3$ by surgery and thus, by performing all three sets of surgeries, $M$ can be reduced to $S^3$ by surgery.

*Case* 2, *in which* $r = 0$. In this case we use Lemma 2(a) to find $z_1$ meeting $x_1$, $y_1$ transversally in one point, and complete to $z$ as before. Then $M(x, z)$, $M(y, z)$ each has a diagram containing a transverse pair of curves meeting in one point, and therefore, as remarked earlier, each has another diagram of smaller genus. Hence, by induction, each can be reduced to $S^3$ by surgery and it follows, as in case 1, that $M$ can be reduced to $S^3$ by surgery.

*Case* 3, *in which* $r = 1$. In this case the diagram for $M$ contains a transverse pair of curves and therefore $M$ has a diagram of lower genus. Hence, by induction, $M$ can be reduced to $S^3$ by surgery.

## *References*

1. W. B. R. Lickorish, 'A representation of orientable combinatorial 3-manifolds', *Ann. of Math.* 76 (1962) 531–540.
2. R. Thom, 'Quelques propriétés globales des variétés différentiables', *Comm. Math. Helv.* 28 (1954) 17–86.

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