Milan j. math. 72 (2004) 1–21

Milan Journal of Mathematics

About a Quantum Field Theory for 3D Gravity

Riccardo Benedetti

Abstract. We outline some aspects of a *dilogarithmic quantum field theory* (DQFT) for a 2+1 bordism category based on 3-manifolds equipped with principal flat $PSL(2,\mathbb{C})$ bundles. We give some geometric evidence that it could be pertinent to 3D gravity, and we describe the building blocks of DQFT: the *matrix dilogarithms*.

Keywords. Hyperbolic 3-manifolds, domains of dependence, Wick rotation, matrix dilogarithms, ideal triangulations.

1. Introduction

This paper arises from two talks given at the "Seminario Matematico e Fisico" (Milano) on November 2003, and at the Colloquium of the "Institut Mathematique de Jussieu" (Paris) on April 2004, respectively. The text keeps the rather informal talk style.

Firstly we are going to say a little bit more precise in which sense the terms that appear in the title are used. We stress that, in spite of the fact that we are going to use a somewhat allusive terminology, we do not claim any actual physical content; at most we are playing with a (non trivial) and hopefully suggestive toy model.

1.1. 3D Gravity

Thanks to the 3-dimensional peculiarity that

"Ricci implies Riemann"

we can roughly say that classical 3D (pure) gravity concerns the study of Riemannian or Lorentzian 3-manifolds of *constant curvature*. The sign of the curvature coincides with the sign of the cosmological constant. We stipulate that all manifolds are *oriented* and that the Lorentzian spacetimes are also *time-oriented*. We also include in the picture the presence of *world lines* of "particles"; along these lines we have concentrated singularities of the metric; a typical example is given by the *cone manifolds* of constant curvature with cone locus at some embedded link, where the cone angles reflect the "mass" of the particles. In the Lorentzian case we also require that the world lines are of causal type (see e.g. [7]). Another intriguing ingredient of the Lorentzian sector of the theory, is the *global causality*. A huge amount of literature concerns the cases which satisfy very strong global causality assumptions, in particular the *domains of dependence* $D(S)$ of compact Cauchy surfaces S. The strong global causality forces these domains to have the simple product topology $S \times \mathbb{R}$, and to admit global times that fibre the spacetime by space-like surfaces homeomorphic to S (accordingly to the naive image of a "space evolving in time", though *a priori* we do not dispose of any privileged time).

Another advantage of 3D gravity consists in the fact that we dispose of very explicit *local models* for the manifolds of constant curvature, and this allows us to adopt the very convenient technology of $(\mathbb{X}, \mathcal{G})$-manifolds, that is manifolds equipped with (maximal) *special atlas* (see e.g. Chapter B of [5] for more details). Here $\mathbb{X}$ denotes the local model, $\mathcal{G}$ the group of isometries of $\mathbb{X}$ which preserve the orientations. A special atlas has charts with values onto open sets of $\mathbb{X}$, and the chart changes are given by the restriction to each connected component of their domain of definition of elements $g \in \mathcal{G}$. For every $(\mathbb{X}, \mathcal{G})$-manifold M, a very general analytic continuation-like construction, gives us pairs (d, h), where

$$d : \widetilde{M} \to \mathbb{X}$$

is a *developing map* defined on the universal covering of M,

$$h : \pi(M) \to \mathcal{G}$$

is a *holonomy representation* of the fundamental group of M. The developing map is a local isometry, and it is unique up to post-composition with elements $g \in \mathcal{G}$:

$$d' = g \circ d \ .$$

The holonomy representation is unique up to conjugation

$$h' = ghg^{-1}\ .$$

Morever, for every $\gamma \in \pi(M)$ we have

$$d(\gamma(x)) = h(\gamma)(d(x))$$

where we consider on the left the natural action of the fundamental group on $\widetilde{M}$, on the right the action of $\mathcal{G}$ on $\mathbb{X}$. A $(\mathbb{X},\mathcal{G})$-structure on M lifts to a locally isometric structure on $\widetilde{M}$, and these share the same developing maps. In many situations it is more meaningful to consider this lifted structure, and keep track of the isometric action of the fundamental group of M on $\widetilde{M}$. This technology is in fact very flexible, and applies to the very general situation where $\mathcal{G}$ is any group of real analytic transformations of a model space $\mathbb{X}$ (not necessarily a group of isometries).

Normalizing the constant curvature κ to be equal to $0, \pm 1$, for the Euclidean signature of the metrics, we get the local models of the main 3-dimensional geometries (*flat*, *spherical*, *hyperbolic*). These (in particular the hyperbolic geometry) are the central objects of Thurston's geometrization program, which dominates the 3-dimensional geometry and topology on the last decades. For the Lorentzian signature, we get the well known *Minkowski space* ($\kappa = 0$), *de Sitter* or *anti- de Sitter* spaces ($\kappa = \pm 1$ resp.). Recall that the hyperbolic, de Sitter and anti- de Sitter spaces admit so called *projective* models for which the geodesics are represented by straight lines. In particular the quadric in $\mathbf{P}^3(\mathbb{R})$ $Q = \{f(x_1,\dots,x_4) = \sum_{i=1\dots3} x_i^2 = x_4^2\}$, separates the model of the hyperbolic space $\mathbb{H}^3 \cong \{f < 0\}$ (Q also makes its natural boundary) from the de Sitter space $\{f > 0\}$; in a sense hyperbolic and de Sitter spaces share a "common boundary".

1.2. Quantum Field Theory

Having as model Atiyah's formalization of *topological quantum field theory* (TQFT), we use the term "3D-quantum field theory", roughly speaking, as synonimous of:

Representation of some $(2+1)$*-bordism category in the tensorial category of complex linear spaces.*

It is not necessary to recall here all the axioms. We simply say that in a series of papers in collaboration with S. Baseilhac [1, 2, 3, 4] we have developed a family $\mathcal{D}_N$, $N \geq 1$ being any odd integer, of so called *dilogarithmic quantum field theories* (DQFT), for a suitable $(2+1)$-bordism category based on oriented compact 3-manifolds Y, which include properly embedded 1-dimensional framed links L, and are equipped with flat connections (up to gauge equivalence) on principal $PSL(2,\mathbb{C})$-bundles on $Y \setminus L$ (equivalently, of conjugacy classes of $PSL(2,\mathbb{C})$-valued representations of the fundamental group of $Y \setminus L$), having *arbitrary* holonomy (non necessarily trivial) at the meridians of the link components.

Each theory $\mathcal{D}_N$ is finite dimensional and "exact" (in principle, everything can be explicitely computed). The name depends on the fact that the building blocks are so called *matrix dilogarithms of rank* N, that are determined automorphisms $\mathcal{R}_N$ of $\mathbb{C}^N \otimes \mathbb{C}^N$, associated to *hyperbolic ideal tetrahedra equipped with an elaborated extra-decoration*, and that satisfy certain fundamental *five terms identities.*

In a sense, one could look at the $\mathcal{D}_N$'s as a family of regularizations of one comprehensive field theory. The "classical member" of the family ($N = 1$) actually computes classical fundamental invariants (such as the *volume* and the *Chern-Simons invariant* of finite volume hyperbolic 3-manifolds). For links L in S^3 equipped with the *trivial* flat bundle, $\mathcal{D}_N$ computes the Kashaev's [27] invariant $< L >_N$, later identified by Murakami-Murakami [28] with $J_N(L)(\exp(2\pi i/N)$, where J_N denotes a suitably normalized colored Jones invariant. At least conjecturally, but with some consistent geometric motivations, $\mathcal{D}_1$ should be somewhat considered as $\mathcal{D}_\infty$, that is the "limit" of the "quantum" theories $\mathcal{D}_N$, $N > 1$, when $N \to \infty$ ("Volume Conjectures").

Our first aim here is to give some geometric motivations supporting the idea that these quantum field theories should be pertinent to 3D gravity. The actual construction of $\mathcal{D}_N$'s is quite complicated and also technically heavy. For this, we address the interested reader to the papers mentioned above. Here we will limit ourselves to say something about the building blocks, the matrix dilogarithms, and to indicate a first step towards global applications.

2. More 3D gravity

We want to motivate how a field theory, based on 3-manifolds equipped with flat connections on principal $PSL(2,\mathbb{C})$-bundles, should be, in principle, pertinent to 3D gravity. An immediate point of contact is given by hyperbolic 3-manifolds. In fact, it is well known that $PSL(2,\mathbb{C})$ is identified with the group of direct isometries of the hyperbolic space $\mathbb{H}^3$. The

Riemann sphere $S^2 = \mathbf{P}^1(\mathbb{C})$ forms the *natural boundary* of $\mathbb{H}^3$, and with the natural projective action of $PSL(2,\mathbb{C})$ on S^2, we get a global action on the compactification $\mathbb{H}^3 \cup S^2 \cong B^3$. Moreover, the action on the boundary enterely determines the action on the whole closed ball B^3.

Hence every hyperbolic manifold Y is naturally equipped with (the conjugacy class of) its holonomy representations, say ρ. If the manifold is complete, ρ is an injective representation onto a subgroup $\Gamma \cong \pi(Y)$ of $PSL(2,\mathbb{C})$ that acts freely and properly discontinuosly on $\mathbb{H}^3 \cong \widetilde{Y}$, so that Y is isometric to $\mathbb{H}^3/\Gamma$. More substantially, for finite volume complete hyperbolic 3-manifolds we have the following *volume rigidity* result. We summarize it in the compact case, but similar facts hold (with some technical complications) also for non compact *cusped* manifolds (see [15, 14]).

Let W be a compact closed oriented 3-manifold. Let us denote by $\mathcal{R}(W)$ the set of conjugacy classes of representations of $\pi(W)$ with values in $PSL(2,\mathbb{C})$. Then it is well defined a volume function

$$\mathrm{Vol} : \mathcal{R}(W) \to \mathbb{R}$$

such that, if ρ is the holonomy of a hyperbolic structure h on W, then:

(1) $\mathrm{Vol}(\rho) = \mathrm{Vol}(W, h)$;

(2) *ρ is the unique maximum of the volume function (hence the hyperbolic structure on W is unique up to isometry).*

This geometric result is strictly related to the formulation of Euclidean 3D gravity with negative cosmological constant, in terms of the so called “new variables”, and a Chern-Simons action (see [9]). As fields one takes the connections on principal $PSL(2,\mathbb{C})$-bundles (instead of the metrics), the “constraint” equations imply that the “phase space” becomes the space of *flat* connections (up to gauge equivalence), and the action essentially consists of the above volume function. The classical solutions correspond to the extremal action. A moral from this is:

The partion functions of any pertinent field theory should recover at least this fundamental classical action, possibly in its complex version

$$\mathrm{CS}(h) + i\mathrm{Vol}(h)$$

where $\mathrm{CS}(h)$ *denotes the Chern-Simons invariant of the flat connection h.*

We already mentioned that the classical member $\mathcal{D}_1$ of the family $\{\mathcal{D}_N\}$ of DQFT actually computes this classical action. Any instance of confirmation of "$\mathcal{D}_N \to \mathcal{D}_1 = \mathcal{D}_\infty$", would support the pertinence of DFT to 3D gravity.

Now we want to outline how *geometrically finite* hyperbolic 3-manifolds of *infinite* volume, not only give us natural examples for our equipped $(2+1)$-bordism, but actually lead to concrete *interactions* between Lorentzian space-times of constant curvature. This also indicates that the separation of 3D gravity in different sectors, accordingly to the metric signature and (the sign of) the curvature, should be somewhat misleading; by the way *changes of space-times topology* are concretely realized in a purely classical set up. In this sense, these field theories we are discussing of, should be potentially pertinent to the whole 3D gravity, not only to its Euclidean sector with negative cosmological constant, that has the hyperbolic manifolds as classical solutions.

A few facts about geometrically finite hyperbolic 3-manifolds The following description of complete geometrically finite hyperbolic manifolds is essentially derived from Thurston's work [13]. First, every such a manifold is *topologically tame*, that is Y is diffeomorphic to the interior of a compact manifold $\widehat{Y}$. As we are assuming that $Y = \mathbb{H}^3/\Gamma$ has infinite volume, the boundary $\partial\widehat{Y}$ is non empty and contains at least one boundary component S of genus $g(S) > 1$. For simplicity, we assume also that all the components of $\partial\widehat{Y}$ are of genus at least 2, they are *incompressible* (i.e. their fundamental groups inject into the one of Y), Y is not diffeomorphic to a product $S \times \mathbb{R}$, and the group Γ doesn't contain any parabolic element (i.e. all elements different from the identity are of hyperbolic type). Geometrically finite manifolds are characterized by the fact that a copy of $\widehat{Y}$ can be (topologically at least) embedded into Y as its *convex core* $C(Y)$ (the maximal convex subset of Y). The connected components of $Y \setminus C(Y)$ are called the *ends* of Y; there is one end for each boundary component S of $C(Y)$, and this end is homeomorphic to $S \times \mathbb{R}$. The simplest case to figure out is when $C(Y)$ is a hyperbolic submanifold of Y with *totally geodesic boundary*. In this case the ends are said of *Fuchsian* type. The subgroup Γ_S of Γ which stabilizes an end is conjugate to a subgroup of $PSL(2,\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$, so that the totally geodesic surface S of $\partial C(Y)$ is identified with $S = \mathbb{H}^2/\Gamma_S$. In general we have *quasi-Fuchsian* deformation of the above situation. Let us summarize some facts that hold in general:

(1) Every component S of $\partial C(Y)$ inherits from the ambient hyperbolic structure of Y, an *intrinsic* structure of hyperbolic surface $S = \mathbb{H}^2/\Gamma^i_S$,

for a suitable subgroup Γ^i_S of $PSL(2,\mathbb{R})$ ($\Gamma^i_S = \Gamma_S$ only for a Fuchsian end). However, if S is not totally geodesic, the embedding of S into Y is not smooth; there is a *bending* locus which makes a *geodesic lamination* $\mathcal{L}$ of S. That is, $\mathcal{L}$ is a closed subset of S made by the disjoint union of complete (possibly closed) geodesic of S. The bending amount is encoded by a measure μ *transverse* to $\mathcal{L}$ (see e.g. [17, 16, 18]). Each component of $S \setminus \mathcal{L}$ is locally embedded in Y as a geodesically complete hyperplane of $\mathbb{H}^3$. The existence of the transverse measure implies that the closed leaves of $\mathcal{L}$, if any, are isolated.

(2) The end $E(S)$ corresponding to S is foliated by the level surfaces of the distance function from S. These surfaces are real analytically embedded into Y only in the Fuchsian case. Otherwise, they are C^1-embedded. By using the gradient of the distance function we can realize a retraction on $E(S)$ onto S. Each level surface contains a dense open set which is analytically embedded, and consists of a part that is of negative constant curvature (for the induced Riemannian metric), and of a flat part. The complement of this open set projects (via the retraction) onto the lamination. There are flat components iff there are closed leaves in $\mathcal{L}$, and each flat component projects onto a corresponding closed geodesic of $\mathcal{L}$.

(3) The restriction h_S to S of the holonomy h of Y is in fact the holonomy of a *projective* structure on S, i.e. a $(S^2, PSL(2,\mathbb{C}))$- structure (see above). A developing map, for the projective structure on S arising in this way, is injective onto an open subset Ω_S of S^2 that is bounded by a *simple closed Jordan curve* (this curve is a round circle exactly in the Fuchsian case). The image $\Gamma_S \cong \pi(S)$ of h_S acts freely and properly discontinuously on Ω_S, so that the projective structure is given by Ω_S/Γ_S. Summarizing:

We have a compactification $\widehat{E}(S)$ of the end, that is homeomorphic to $S \times [0,1]$ and has a hyperbolic boundary component $\mathbb{H}^2/\Gamma^i_S$ and a projective component Ω_S/Γ_S. The level surfaces of the distance function from S interpolate these two boundary components.

The compactified end $\widehat{E}(S)$ is completely determined (up to isometry) by the pair $(\Gamma^i_S, (\mathcal{L},\mu))$. Moreover, the family of the $(\Gamma^i_S, (\mathcal{L},\mu))$'s, when $S \subset \partial C(Y)$ varies, completely determines the whole hyperbolic manifold Y.

The last claim is a weak version of Bers theorem (in fact, it is enough to consider the *complex structures* induced by the asymptotic projective structures); anyway, it already indicates a clear instance of "holographic" behaviour.

Wick rotation This is a very basic general procedure of interplaying Lorentzian and Riemannian manifolds.

Given a manifold M (of dimension $n+1$) equipped with a Riemannian metric g and a future oriented Lorentzian metric h, use g to identify h with a field of tangent space automorphisms; then there is a unique g-unitary and h-future directed vector fiels $v = v(g,h)$ which is made by eigenvectors of h with negative eigenvalues. Then the orthogonals w.r.t. g and h, to the line bundle $< v > \subset\ T(M)$ coincide ($T(M)$ is the tangent bundle of M).

Let v be now any nowhere vanishing vector field on M, $\alpha, \beta : M \to (0, +\infty)$ be positive functions, $\beta > 1$. For every $(0,2)$ symmetric tensor field K on M, such that $K(v(x), v(x)) \neq 0$ for every $x \in M$, the formula

$$v_{\alpha,\beta}{*}K(X,Y) := \alpha(x)K(X,Y) - \beta(x)K(v(x),X)K(v(x),Y)/K(v(x),v(x))$$

where $X, Y \in T(M)_x, x \in M$, define a new tensor obtained, by definition, via the *Wick rotation directed by v and with rescaling functions α, β*. Such a Wick rotation establishes a bijection $v_{\alpha,\beta}{*}$ between the set of Riemannian metrics and the set of future oriented Lorentzian metrics on M for which the field v is future directed of time type. In fact ${v_{\alpha,\beta}{*}}^{-1} = v_{1/\alpha,1/\beta}{*}$. If g and h are related by such a rotation, then the g-unitary field $v/||v||$ coincides with $v(g,h)$; the pair (g,v), and the rescaling functions, determine the global causal structure of the space-time (M,h). Note that, with the present definition of Wick rotation, generic g and h are not related by any rotation directed by $v(g,h)$, because their restrictions to $< v(g,h) >^{\perp}$ are in general not conformal. Recall that the particular case of Wick rotation relating the standard Euclidean and Minkowski metrics on $\mathbb{R}^{n+1}$, is sometimes indicated as "passing to the imaginary time".

Domains of dependence of constant curvature with compact Cauchy surfaces What we are going to summarize is largely due to Mess [10], with some complements (concerning the *canonical cosmological time*, and the Wick rotations) from [6, 8, 11]. We denote by $(\mathbb{X}_\kappa, G_\kappa)$ the local models for the Lorentzian spacetimes of constant curvature $\kappa = 0, \pm 1$. The key point is that any pair $(\Gamma^i_S, (\mathcal{L}, \mu))$ as above, that completely encodes an end $E(S)$ of some geometrically finite manifold Y, also encodes domains of dependences, with Cauchy surface homeomorphic to S, of arbitrary constant curvature. Moreover, the end and (some of) these spacetimes are related via canonical Wick rotations. More precisely:

For every $\kappa = 0, \pm 1$, and every pair $(\Gamma^i_S, (\mathcal{L}, \mu))$ as above, one can construct a canonical maximal domain of dependence $D_\kappa = D_\kappa(\Gamma^i_S, (\mathcal{L}, \mu))$ of constant curvature κ, with Cauchy surfaces homeomorphic to S such that:

(a) The universal covering $\widetilde{D}_\kappa$ can be realized as an open subset of $\mathbb{X}_\kappa$, and coincides with the image of an injective developing map. The image $\Gamma_\kappa \cong \pi(S)$ in G_κ of the holonomy acts freely and properly discontinuously on $\widetilde{D}_\kappa$, so that $D_\kappa = \widetilde{D}_\kappa / \Gamma_\kappa$.

(b) D_κ has a *canonical cosmological time*, say t (see [12] for some generality on this notion). For every event $x \in D_\kappa$, $t(x)$ is the proper time that x has been in existence and coincides with its *finite* Lorentz distance from the *initial singularity* Σ_κ of D_κ.

The (lifted) initial singularity $\widetilde{\Sigma}_\kappa$ *of* $\widetilde{D}_\kappa$ *is a* real tree *which is embedded in space-like way on the frontier of* $\widetilde{D}_\kappa$ *in* $\mathbb{X}_\kappa$*. The fundamental group* $\pi(S)$ *acts on* $\widetilde{\Sigma}_\kappa$ *by isometry.*

The isometric actions of $\pi(S)$ *on the level surfaces of the canonical time* $\widetilde{t}$ *of* $\widetilde{D}_\kappa$*, converge in the sense of Gromov, when* $\widetilde{t} \to 0$*, to that action on the initial singularity. This real tree* $\widetilde{\Sigma}_\kappa$ *is the* dual *one to the (lifted) measured geodesic lamination* $(\widetilde{\mathcal{L}}, \widetilde{\mu})$*, and everything is* $\pi(S)$*-equivariant.*

For the notions of Gromov convergence, real tree and for Skora *duality theorem*, see e.g. [16]. Metric *simplicial* trees, possibly with vertices of infinite valence, are particular instance of real trees, that are dual to the simplest measured geodesic laminations made by a finite number of disjoint "weighted" simple closed geodesics. This special case is already very important because it makes a *dense* subset of the whole space of measured geodesic laminations.

(c) When $\kappa = -1$, t is a fibration over $(0, \pi)$. Consider the fibred subset

$$\mathcal{P} = \{t \leq \pi/2\}$$

of D_{-1}. The area of the level surfaces is strictly increasing with t on $\mathcal{P}$, and $S_{\pi/2}$ realizes the absolute maximum of the area of the level surfaces of t. All the level surfaces, with the exception of $S_{\pi/2}$, are C^1-embedded in $\mathcal{P}$, while the embedding of $S_{\pi/2}$ has a bending locus, so that:

The structure $(\Gamma^i_S, (\mathcal{L}, \mu))$ *eventually gets a further geometric realization in* $\mathbb{X}_{-1}$ *(instead of* $\mathbb{H}^3$*).*

When $\kappa = 0$, t is a fibration over $(0, +\infty)$. The area of the level surfaces is strictly increasing and tends to ∞ when $t \to +\infty$.

The rescaled level surfaces $(1/\widetilde{t})S_{\widetilde{t}}$ of $\widetilde{D}_0$, with the isometric action of $\pi(S)$, converge in the sense of Gromov, when $\widetilde{t} \to +\infty$, to $(\mathbb{H}^2, \Gamma^i_S)$.

(d) On $\mathcal{P}_{>\pi/4} = \mathcal{P} \cap \{\pi/2 > t > \pi/4\}$, when $\kappa = -1$, and on $(D_0)_{>1}$ when $\kappa = 0$ respectively,

There are canonical (and explicit) Wick rotations*, directed by the gradient of the canonical time, with rescaling functions that are constant on each t-level surfaces, and which convert $\mathcal{P}_{>\pi/4}$ and $(D_0)_{>1}$ respectively into the whole end $E(S)$, the t-level surfaces becoming the level surfaces of the distance function from the component S of the convex core $C(Y)$.*

These rotations extend continuously onto the compactification of the end $\widehat{E(S)}$, in such a way that the maximal area t-level surface $S_{\pi/2}$, with its bending lamination, maps isometrically onto the hyperbolic boundary component S, while $\widetilde{S}_{\pi/4}$ maps onto Ω_S, reproducing the developing map of the projective boundary component. A similar fact holds for the level surface $\widetilde{S}_1$, when $\kappa = 0$.

The above discussion applies to the space-times of constant negative curvature $(1/\kappa)D_{-1}(\Gamma^i_S, (\mathcal{L}, \kappa\mu))$, $\kappa \in [1, +\infty[$. It depends continuosly on the parameter κ, and interpolates D_0 and D_{-1}.

(e) One can define explicit rescaling functions on $\mathcal{P}_{<\pi/4}$, constant on the t-level surfaces, which convert $\mathcal{P}_{<\pi/4}$ into the dependence domain D_1 of positive constant curvature 1, in such a way that the t-level surfaces of $\mathcal{P}_{<\pi/4}$ become the t-level surfaces of D_1. A similar result holds for $(D_0)_{<1}$ contained in D_0. Moreover, all this "fits well" with the Wick rotations, at the "common boundary" of the hyperbolic and de Sitter space.

Some complementary remarks are in order.

Remarks 2.1.

(1) All the above discussion extends (in a simpler way) to the Fuchsian case. Here we assume that the lamination is empty (or that the transverse measure is equal to 0). For every κ, there is a natural inclusion of $PSL(2,\mathbb{R})$ into G_κ (for example, for $k = 0$ it embeds as the linear part of the Lorentz group; for $\kappa = -1$, we have the diagonal embedding into $G_{-1} \cong PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$). So there is a natural suspension of the hyperbolic surface $\mathbb{H}^2/\Gamma_S$ by the spacetime $\mathcal{D}_\kappa/\Gamma_S$; $\mathcal{D}_\kappa$ is a suitable simple $PSL(2,\mathbb{R})$-invariant domain of $\mathbb{X}_\kappa$ (for $\kappa = 0$ it is the future $I^+(0)$ of the origin in the Minkowski space $\mathbb{R}^{2+1}$; for $\kappa = -1$ it is the domain of dependence of a space-like hyperplane). Note that

These Fuchsian cases are characterized by the fact that the canonical time is smooth, and that the initial singularity reduces to one point.

(2) The gradient of the cosmological time is C^0, however, the Wick rotations eventually determine real analytic identifications between the Lorentzian space-times and the hyperbolic ends.

(3) The measured geodesic laminations $(\mathcal{L}, \mu)$ occurring as bending laminations of ends of geometrically finite manifolds, are of special type (see [19]): in fact the convexity implies that *one bends always in the same direction*, and this imposes some constraints on the transverse measure. On the other hand, all the above discussion on the domains of dependence of constant curvature encoded by pairs $(S, (\mathcal{L}, \mu))$, where S is any hyperbolic surface, works anyway, for *arbitrary* measured geodesic laminations on S. We still have the Wick rotations that convert the Lorentzian space-times into hyperbolic 3-manifolds homeomorphic to $S \times \mathbb{R}$, of a more general type than the ends of geometrically finite manifolds. In particular, there are involved projective structures on S of a more general type, for instance having developing map which are surjective onto the whole S^2 (and not injective).

(4) Beware that we actually forget substantial geometric informations, by only keeping track of the holonomy of the projective boundary of the end: for examples ([20]), there are holonomies of a quasi-Fuchsian (even Fuchsian) structures (as before), that are also holonomies of projective structures with completely different developing map (i.e. surjective onto the whole S^2).

(5) It seems interesting to study the *maximal* analytic continuations in Y of the Wick rotations that we have defined on the ends, to develop a notion of "Wick cut locus" and so on. Moreover, it is interesting to study these Wick rotations on *higher dimensional* constant curvature (in particular flat - see [11]) space-times.

(6) As we have concretely interpreted any geometrically finite hyperbolic 3-manifold Y as an interaction between space-times of constant curvature, further classical invariants (such as the volume of the convex core $C(Y)$) give us a first measure of the "amplitude" of this interaction, and one would recover them in a pertinent field theory.

3. About the matrix dilogarithms

As promised we are going to say something about the building blocks of the dilogarithmic field theories $\mathcal{D}_N$.

On the geometric/combinatorial side, the basic building blocks of our constructions are certain decorated *$\mathcal{I}$-tetrahedra*.

An *$\mathcal{I}$-tetrahedron* (Δ, b, w) consists of

(1) An *oriented* tetrahedron Δ, (that we usually represent as positively embedded in $\mathbb{R}^3$, oriented by its standard basis).

(2) A *branching* b on Δ, that is a choice of edge orientation associated to a total ordering v_0, v_1, v_2, v_3 of the vertices by the rule: each edge is oriented by the arrow emanating from the smallest endpoint.
Denote by $E(\Delta)$ the set of b-oriented edges of Δ, and by e' the edge opposite to e. We put $e_0 = [v_0, v_1]$, $e_1 = [v_1, v_2]$ and $e_2 = [v_0, v_2] = -[v_2, v_0]$. These are the edges of the face opposite to the vertex v_3.

(3) A *modular triple*, $w = (w_0, w_1, w_2) = (w(e_0), w(e_1), w(e_2)) \in (\mathbb{C} \setminus \{0,\ 1\})^3$ such that (indices mod$(\mathbb{Z}/3\mathbb{Z})$):

$$w_{j+1} = 1/(1 - w_j) \quad . \tag{3.1}$$

Hence $w_0 w_1 w_2 = -1$, and this gives a *cross-ratio modulus* $w(e)$ to each edge e of Δ, by imposing that $w(e) = w(e')$.
We say that w is non degenerate if the imaginary parts of the w_j's are not equal to zero; in such a case they share the same sign $*_w = \pm 1$.
The ordered triple of edges

$$(e_0 = [v_0, v_1], e_2 = [v_0, v_2], e'_1 = [v_0, v_3]) \tag{3.2}$$

departing from v_0 defines a *b-orientation* of Δ. This orientation may or may not agree with the given orientation of Δ. In the first case we say that b is of index $*_b = 1$, and it is of index $*_b = -1$ otherwise.

The 2-faces of Δ can be named and ordered by their opposite vertices. Each 2-face f has two orientations: the boundary one, via the convenction "last the ingoing normal" and the b-orientation, i.e. the prevailing one among the three b-oriented edges which make the boundary of f.

Consider the half space model of the hyperbolic space $\mathbb{H}^3$. We orient it as an open set of $\mathbb{R}^3$. The natural boundary $\partial\bar{\mathbb{H}}^3 = \mathbb{CP}^1$ of $\mathbb{H}^3$ is oriented by its complex structure. Up to direct isometry, an $\mathcal{I}$-tetrahedron (Δ, b, w)

can be realized as an hyperbolic ideal tetrahedron with 4 distinct b-ordered vertices u_0, u_1, u_2, u_3 on $\partial\bar{\mathbb{H}}^3$, in such a way that

$$w_0 = (u_2 - u_1)(u_3 - u_0)/(u_2 - u_0)(u_3 - u_1) \quad .$$

These 4 points span a 'flat' (2-dimensional) tetrahedron exactly when the modular triple is degenerate (real). When it is non-degenerate, we get a positive embedding of Δ, with its own orientation, onto the corresponding hyperbolic ideal tetrahedron in $\mathbb{H}^3$ iff $*_b *_w = 1$.

Given any $\mathcal{I}$-tetrahedron (Δ, b, w), we consider an extra-decoration made by two $\mathbb{Z}$-valued functions defined on the edges of Δ, called *flattening* and *integral charge* respectively. These functions share the property that *opposite edges take the same value*, hence it is enough to specify their values on the edges e_0, e_1, e_2.

We denote by log the standard branch of the logarithm which has the arguments in $]-\pi, \pi]$.

For every $f = (f_0, f_1, f_2)$, $f_i = f(w_i) \in \mathbb{Z}$, set

$$\mathrm{l}_j = \mathrm{l}_j(b, w, f) = \log(w_j) + i\pi f_j$$

for $j = 1, 2, 3$. We say that (f_0, f_1, f_2) is a *flattening* of (Δ, b, w), if

$$\mathrm{l}_0 + \mathrm{l}_1 + \mathrm{l}_2 = 0 \quad .$$

We call l_j a *log-branch* of (Δ, b, w) for the edge e_j.

An *integral charge* is a function $c = (c_0, c_1, c_2)$, $c_i = c(w_i) \in \mathbb{Z}$, such that $c_0 + c_1 + c_2 = 1$. A $\mathcal{I}$-tetrahedron endowed with a flattening and an integral charge is said *flat/charged*.

For every $N > 0$, any function

$$A : \mathbb{C} \setminus \{0, 1\} \to \mathrm{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$$

can be interpreted as a function of $\mathcal{I}$-tetrahedra:

$$A(\Delta, b, w) := A(w_0)^{*_b}$$

as follows. $\mathbb{C}^N \otimes \mathbb{C}^N$ is endowed with the standard basis, so $A = A(x) \in \mathrm{Aut}(\mathbb{C}^N \otimes \mathbb{C}^N)$ is given by its matrix elements $A^{\delta,\gamma}_{\beta,\alpha}$, where $\alpha, \dots, \delta \in \{0, \dots, N-1\}$. We denote by $\bar{A} = \bar{A}(x)$ the inverse of $A(x)$, with entries $\bar{A}^{\beta,\alpha}_{\delta,\gamma}$. We have already used the branching b to select w_0 among the triple of cross-ratio moduli. We use again the branching to associate to each 2-face of Δ one index among $\gamma, \delta; \alpha, \beta$. The rule is shown in Fig. 3.1.

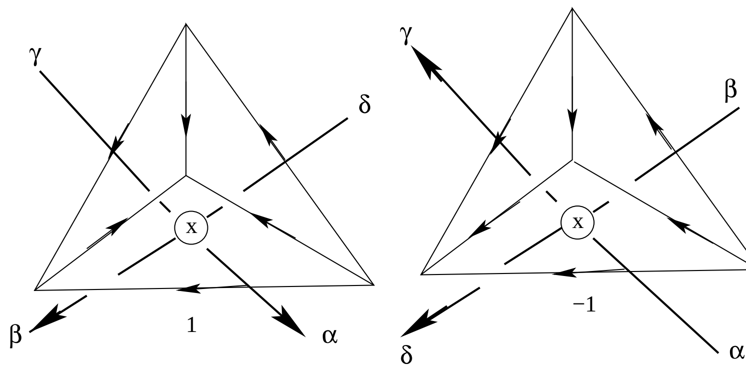

FIGURE 3.1. $A(\Delta, b, w) = A^{*_b}(w_0)$, $x = w_0$.

The matrix dilogarithm of rank N, $N \geq 1$ being any odd integer, is an explicitely given

$$\mathcal{R}_N(\Delta, b, w, f, c) = \mathcal{R}_N(w_0, f, c)^{*_b}$$

defined on flat/charged $\mathcal{I}$-tetrahedra, which satisfies fundamental *five terms identities*. The supports of these identities are suitable $\mathcal{I}$-flat/charged versions, called *transit configurations*, of the basic $2 \to 3$ bistellar (sometimes called Pachner or Matveev-Piergallini, see e.g. [21]) local move on 3D triangulations. The bare move is shown in Fig. 3.2.

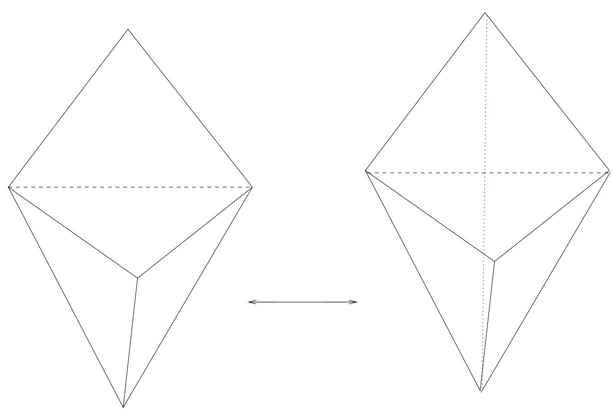

FIGURE 3.2. The bare $2 \to 3$ move.

We postulate that all the 5 tetrahedra involved in the move are oriented and that they induce opposite orientations on every common 2-face. Hence, we have two triangulations T and T' (by 2 and 3 tetrahedra resp.) of a same oriented polyhedron, and each tetrahedron inherits the induced orientation. A triangulation say H, like T or T', can be considered as a finite family of *abstract* tetrahedra, with a fixed identification rule of some pairs of abstract 2-faces. Denote by $E(H)$ the set of edges of H, by $E_\Delta(H)$ the whole set of edges of the associated abstract tetrahedra, and by $\epsilon_H : E_\Delta(H) \longrightarrow E(H)$ the natural identification map.

Assume now that each tetrahedron of T and T' is $\mathcal{I}$-flat/charged. We have to specify, for every ingredient of the decoration (branchings, modular triples, flattenings, integral charges), the constraints that they must satisfy, to have a transit configuration. First of all we require that T and T' are *globally* branched triangulations (T,b) and (T',b'), where b and b' are edge orientations which induce a branching on each abstract tetrahedron.

An $\mathcal{I}$-transit

$$(T,b,w) \to (T',b',w')$$

consists of a bare triangulation $2 \to 3$ move $T \to T'$ that extends to a branching move $(T,b) \to (T',b')$, i.e. the two branchings coincide on the 'common' edges of T and T'. Moreover the modular triples have the following behaviour. For each common edge $e \in \epsilon_T(E(T)) \cap \epsilon_{T'}(E(T'))$ we have

$$\prod_{a \in \epsilon_T^{-1}(e)} w(a)^* = \prod_{a' \in \epsilon_{T'}^{-1}(e)} w'(a')^* \tag{3.3}$$

where $* = \pm 1$ according to the b-orientation of the abstract tetrahedron containing a (resp. a').

Note that the above condition on the modular triples implies that the product of the $w'(a')^*$'s around the "new" edge of T' is equal to 1. So the inverse $3 \to 2$ transits are defined in the very same way, providing that this last condition is verified on T'.

Fig. 3.3 represents one specific instance of $\mathcal{I}$-transit. On each tetrahedron we have indicated the corresponding w_0. Note that in this peculiar case all $*_b$'s are equal to 1. Assume also that all the modular triples are non degenerate, and share the same sign $*_w = \pm 1$. In this case the transit conditions for the modular triples have a transparent geometric meaning. In fact, we are in presence of an oriented *convex* hyperbolic ideal polyhedron, endowed with two different geometric triangulations by two (resp. three) positevely embedded non degenerate ideal tetrahedra. The above transit conditions (including the exponents $*_b$'s) is the natural (algebraic) extension to situations including arbitrarily oriented ideal tetrahedra (where the convexity is possibly lost, possibly there are overlappings, and so on).

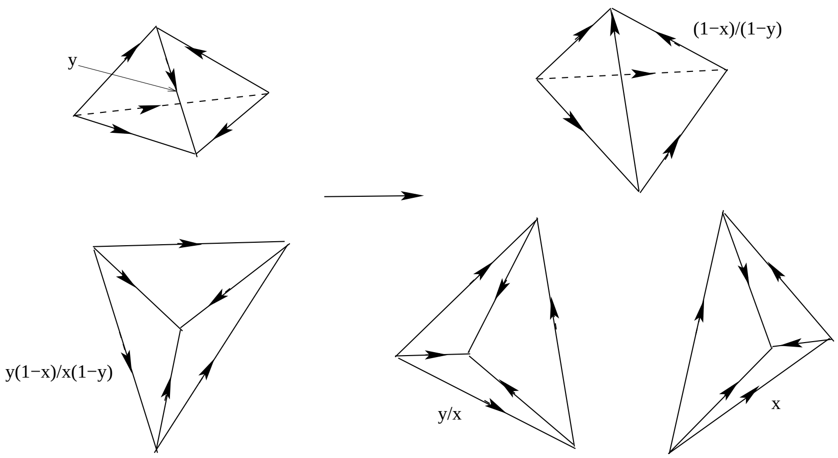

FIGURE 3.3. A peculiar instance of $\mathcal{I}$-transit.

We define now the notion of transit for flattened $\mathcal{I}$-tetrahedra. The simple idea is just to take formally the log of the $\mathcal{I}$-transits.
Consider a $2 \to 3$ $\mathcal{I}$-transit $(T, b, w) \to (T', b', w')$ as above. Give a flattening on each tetrahedron of the initial configuration, and denote by $\mathrm{l} : E_\Delta(T) \to \mathbb{C}$ the corresponding log-branch function on T. Recall the definition of the map ϵ_T given above. Then, a map $f' : E_\Delta(T') \to \mathbb{Z}$ defines a $2 \to 3$ *flattening transit* $(T, b, w, f) \to (T', b', w', f')$ if, for each common edge $e \in T \cap T'$ we have the following relations between the associated log-branches:

$$\sum_{a \in \epsilon_T^{-1}(e)} * \,\mathrm{l}(a) = \sum_{a' \in \epsilon_{T'}^{-1}(e)} * \,\mathrm{l}'(a') \tag{3.4}$$

where $* = \pm 1$ according to the b-orientation of the tetrahedron that contains a (resp. a').
The sum of values of l' about the new edge of T' is equal to zero. So the flattening transits for the inverse $3 \to 2$ moves are defined in exactly the same way, except that we also require that this last condition holds.

A $2 \to 3$ branched move $(T, b, c) \to (T', b', c')$ on charged tetrahedra is a *integral charge transit* if, for each common edge $e \in T \cap T'$ we have the following relations between the associated charges:

$$\sum_{a \in \epsilon_T^{-1}(e)} c(a) = \sum_{a' \in \epsilon_{T'}^{-1}(e)} c'(a'). \tag{3.5}$$

These conditions implies that the sum of the charges around the new edge of T' is equal to 2. So we have to impose this last condition in order to define the inverse $3 \to 2$ transits.

Finally the *flat/charged $\mathcal{I}$-transits* are defined by assembling the above definitions.

Assume that we are in a flat/charged $\mathcal{I}$-transit configuration, and let us associate to each abstract tetrahedron the corresponding $\mathcal{R}_N(\Delta, b, w, f, c)$.

A *state* of (T, b, w, f, c) is a function which associate to every triangle of the 2-skeleton of T a value in $\{0, \dots, N-1\}$. So, every state determines indeed a matrix element of each matrix dilogarithm. As two tetrahedra induce opposite orientation on any common face, our formal identification rule

$$\mathcal{R}_N(\Delta, b, w, f, c) = \mathcal{R}_N(w_0, f, c)^{*_b}$$

implies that a common index at such a common face actually is "down" for one while it is "up" for the other. By applying Einstein rule of "summing on repeated indices", we get the contraction (or the *trace*) of these patterns of tensors. We denote this trace by

$$\prod_{\Delta \subset T} \mathcal{R}_N(\Delta, b, w, f, c) \ .$$

Do similarly for (T', b', w', f', c').
As any flat/charged $\mathcal{I}$-transit is, in particular, a branching transit, then the traces of the two patterns of dilogarithms are tensors of the same type. Finally we can formally state the main result about the five terms identities.

Theorem 3.1. *For every odd $N \geq 1$, for any $2 \to 3$ flat/charge $\mathcal{I}$-transit*

$$(T, b, w, f, c) \to (T', b', w', f', c')$$

the traces of the two patterns of associated matrix dilogarithms lead to the same tensor, possibly up to a sign and multiplication by Nth roots of unity. In formula

$$\prod_{\Delta \subset T} \mathcal{R}_N(\Delta, b, w, f, c) \equiv_N \pm \prod_{\Delta' \subset T'} \mathcal{R}_N(\Delta', b', w', f', c') \tag{3.6}$$

where $\equiv_N$ means equality up to multiplication by Nth roots of unity.

In fact, these matrix dilogarithms $\mathcal{R}_N$, as well as the elaborated extra decoration on $\mathcal{I}$-tetrahedra, arise from the solution of a *symmetrization problem* for a family of *basic* matrix dilogarithms $\mathcal{L}_N(\Delta, b, w)$ which only satisfy one peculiar five terms identity (called *matrix Schaeffer's identity*), with determined geometric constraints on the cross ratio moduli. It corresponds to the $\mathcal{I}$-transit of Fig. 3.3, with the convexity constraints discussed above.
In the classical case $N = 1$ the basic dilogarithm coincides with (the exponential of) the classical *Rogers dilogarithm* (see [22, 23, 24]); in fact the matrix Schaeffer's identity is modeled on the classical one verified by the Rogers dilogarithm.
The quantum ($N > 1$) basic dilogarithms are derived from the $6j$-symbols for the cyclic representation theory of a Borel quantum subalgebra $\mathcal{B}_\zeta$ of $U_\zeta(sl(2, \mathbb{C}))$, where $\zeta = \exp(2i\pi/N)$ (see [25, 26, 29]).
The solution of the symmetrization problem involves a *uniformization* of the basic dilogarithms, and the study of their behaviour w.r.t. the *tetrahedral symmetries.*

A first step towards global results.

This first step consists of the *idealization* of so called $\mathcal{D}$-triangulations of compact oriented 3-manifolds Y (possibly with non empty boundary), equipped with a flat connection ρ on a principal $PSL(2, \mathbb{C})$-bundle on Y.
Let T be a quasi-regular triangulation of Y (i.e. every edge of T has two distinct end-points), equipped with a global branching b (induced, for example, by a total ordering on the vertices of T), and with a *generic* $PSL(2, \mathbb{C})$-valued 1-cocycle z, representing ρ (up to gauge equivalence).
Then, for every tetrahedron (Δ, b, z) of T (as usual, we write $z_j = z(e_j)$ and $z'_j = z(e'_j)$)

$$u_0 = 0,\ u_1 = z_0(0),\ u_2 = z_0 z_1(0),\ u_3 = z_0 z_1 z'_0(0)$$

are 4 distinct points on $\mathbb{C} \subset \mathbb{CP}^1 = \partial\bar{\mathbb{H}}^3$. So, we can associate to e_j and e'_j the same cross-ratio modulus $w_j \in \mathbb{C} \setminus \{0, 1\}$ of the hyperbolic ideal tetrahedron spanned by (u_0, u_1, u_2, u_3), getting an $\mathcal{I}$-tetrahedron. The union of these $\mathcal{I}$-tetrahedra actually makes an $\mathcal{I}$-*triangulation* of Y, that is, at each *interior* edge of T, it is satisfied the following *edge compatibility condition*:

$$\prod_{a \in \epsilon_T^{-1}(e)} w^j(a)^{*_{b^j}} = 1$$

where $*_{b^j} = \pm 1$ according to the b^j-orientation of the tetrahedron Δ^j that contains a. This condition is a natural one to impose, in order to have a class of triangulations which is *stable* for the $\mathcal{I}$-transits.

But this is only the beginning. We have to get *global* flat/charge, to deduce *full invariance* results from the simple *transit invariance*, for the contractions of patterns of matrix dilogarithms associated to globally flat/charged triangulations, and so on. The interested reader is addressed to the papers quoted in the references.

References

[1] S. Baseilhac, R. Benedetti, *QHI, 3-manifold scissors congruence classes and the volume conjecture.* Geometry & Topology Monographs, Volume 4: Invariants of Knots and 3-manifolds, Kyoto, 2001, 13–28.

[2] S. Baseilhac, R. Benedetti, *Quantum hyperbolic invariants of 3-manifolds with* $PSL(2,\mathbb{C})$*-characters.* arXiv math.GT/02110161, to appear in Topology.

[3] S. Baseilhac, R. Benedetti, *Classical and Quantum Dilogarithmic Invariants of* 3*-Manifolds with Flat* $PSL(2,\mathbb{C})$*-bundles.* arXiv math.GT/0306280.

[4] S. Baseilhac, R. Benedetti, *3D Dilogarithmic Quantum Field Theory.* In preparation.

[5] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry.* Springer, 1992.

[6] R. Benedetti and E. Guadagnini, *Cosmological time in (2+1)-gravity.* Nuclear Phys. B **613** (2001), 330–352.

[7] R. Benedetti and E. Guadagnini, *Geometric cone surfaces and* $(2+1)$*-gravity coupled to particles.* Nuclear Phys. B **588** (2000), no. 1-2, 436–450.

[8] R. Benedetti and F. Bonsante, *Wick rotations in 3D gravity.* In preparation.

[9] E. Witten, *2+1 dimensional gravity as an exactly soluble system.* Nucl. Phys. B **311** (1989), 46.

[10] G. Mess, *Lorentz Spacetimes of Constant Curvature.* Preprint IHES/M/90/28, April 1990.

[11] F. Bonsante, *Flat Spacetimes with Compact Hyperbolic Cauchy Surfaces.* arXiv math.DG/0311019.

[12] L. Andersson, G.J. Galloway and R. Howards, *The cosmological time function.* Class. Quantum Grav. **15** (1998), 309–322.

[13] W. Thurston, *Geometry and topology of* 3*-manifolds.* Princeton University Lecture Notes, 1979.

[14] S. Francaviglia, *Hyperbolic volume of representations of fundamental groups of cusped* 3*-manifolds.* arXiv:math.GT/0305275.

[15] N. Dumfield, *Cyclic surgery, degree of maps of character curves, and volume rigidity for hyperbolic manifolds.* Inventiones Math. **136** (1999), 623–657.

[16] J-P. Otal , *The hyperbolization theorem for fibered 3-manifolds.* SMF/AMS Texts and Monographs, 7. American Mathematical Society, Providence, RI. Société Mathématique de France, Paris, 2001.

[17] F. Bonahon, *Geodesic laminations on surfaces.* Laminations and foliations in dynamics, geometry and topology (Stony Brook, NY, 1998), 1–37, Contemp. Math. **269** Amer. Math. Soc., Providence, RI, 2001.

[18] D.B.A. Epstein and A. Marden, *Convex hulls in hyperbolic space, a theorem of Sullivan and measured pleated surfaces.* London Math. Soc. Lectures Note Ser., Cambridge University Press **111** (1987), 113–253.

[19] F. Bonahon and J-P. Otal, *Laminations mesurées de plissage des varietés hyperboliques de dimension* 3. Preprint 2001.

[20] W.M. Goldman, *Projective structures with Fuchsian holonomy.* J. Differential Geom. **25** (1987), 297–326; **136** (1999), 623–657.

[21] S.V. Matveev, *Transformations of special spines and the Zeeman conjecture.* Mathematic USSR Izvestia **31** (1988), 423–434.

[22] J.L. Dupont, *The dilogarithm as a characteristic class for flat bundles.* Journal of Pure and Applied Algebra **44** (1987), 137–164.

[23] L.Lewin, *Dilogarithm and Associated Functions.* Mc Donald, 1958.

[24] W.D. Neumann, *Hilbert's 3rd problem and invariants of 3-manifolds.* Geometry & Topology Monographs Volume 1: The Epstein Birthday Schrift, Coventry, 1998, 383–411.

[25] L.D. Faddeev, R.M. Kashaev, *Quantum dilogarithm.* Modern Physics Letters A **9** (5) (1994), 427–434.

[26] R.M. Kashaev, *The algebraic nature of quantum dilogarithm.* Geometry and integrable models, Dubna, 1994, 32–51, World Scientific Publishing, River Edge, NJ, 1996.

[27] R.M. Kashaev, *A link invariant from quantum dilogarithm.* Modern Physics Letters A **10** (1995), 1409–1418.

[28] H. Murakami, J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot.* Acta Mathematica **186** (1) (2001), 85–104.

[29] C. Kassel, *Quantum Groups.* Graduate Texts in Mathematics 155, Springer, 1995.

Riccardo Benedetti
Dipartimento di Matematica
Università di Pisa
Via F. Buonarroti, 2
I-56127 Pisa, Italy
E-mail: benedett@dm.unipi.it

Lecture held in the *Seminario Matematico e Fisico* on November 24, 2003
Manuscript received April, 2004