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Finiteness of semialgebraic types of polynomial functions

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Introduction

Triangulation theorems have been proved for sets of increasing order of generality (semianalytic, subanalytic, Whitney stratified, etc.). In semialgebraic geometry, we have the much stronger result that triangulations of semialgebraic sets can be obtained in an effective way. In contrast to the triangulation of sets, the triangulation of mappings is often a more difficult problem (see [S]). The results of the present work are based on an effective triangulation theorem for semialgebraic mappings (Theorem 7).

To be a little more precise let us introduce some notation. We write $X \in S(n, d, r)$ if $X \subset \mathbb{R}^n$ is a semialgebraic set with a given presentation of the form:

$$X = \bigcup_{i=1}^{k} \bigcap_{j=1}^{s_i} \{f_{ij} *_{ij} 0\}$$

where for each i and j, f_{ij} is a polynomial function on $\mathbb{R}^n$ of degree $\leqq d$ and $*_{ij}$ means " = " or " > ", and $s_1 + \ldots + s_k \leqq r$. Similarly $f \in S(n, n', d, r)$ means that f is a continuous semialgebraic map from a semialgebraic set $X \subset \mathbb{R}^n$ to another $Y \subset \mathbb{R}^{n'}$ with graph $\Gamma_f \in S(n+n', d, r)$.

The *effectiveness* of *semialgebraic triangulation* of semialgebraic sets implies that there exists an algorithm which, starting from any couple $((n, d, r), X) \in \mathbb{N}^3 \times S(n, d, r)$ with compact X, produces

(a) a triangulation of X, $\tau\colon |K| \to X$, where τ is a semialgebraic homeomorphism and K is a finite simplicial complex in $\mathbb{R}^n$, and
(b) a function $\mathbb{N}^3 \ni (n, d, r) \to (D, R, h) \in \mathbb{N}^3$

such that $\tau \in S(n, n, D, R)$ and $\# K$, the number of simplexes of K, is not larger than h. For such an algorithm see [D-K], [B-C-R] and [B-R]. It is ultimately based on Tarski-Seidenberg theorem, that is, on what we could call the "projection method", it actually holds over any real closed field R.

Note that an immediate consequence of this effective triangulation theorem is that the number of semialgebraic types of semialgebraic sets in $S(n, d, r)$ is finite and effectively bounded in terms of (n, d, r) (the compactness assumption can be removed by the use of some standard compactification, as is shown later).

In [F] Fukuda proved that the number of *topological types* of polynomial functions on $\mathbb{R}^n$ of degree $\leqq d$ is finite. The proof is not elementary and is based on the theory of Whitney stratification and Thom's isotopy lemma. The aim of the present short paper is to improve Fukuda's result, showing that the number of *semialgebraic types* of such polynomial functions is finite and effectively bounded in terms of (n, d). Our proof will actually be a corollary of an *effective triangulation theorem for semialgebraic functions*; as in the case of sets the proof is elementary and based on the projection method and works over any real closed field; note that we shall use also the notions of C^∞ functions and manifolds which make sense on every such a field R (see [B-C-R, 2.9]). One of the author ([S]) already obtained the triangulation theorem for functions. Here we have simply to take into account the effectiveness of the construction.

It has been known since Thom's work (see [T]) that finiteness of topological types fails for general polynomial maps. The basic source of the lack of finiteness stems from the existence of "explosions". On the other hand Thom himself conjectured (and the question is still open) that maps without explosion can be triangulated. (The condition is necessary since PL maps have no explosions.) We believe that, in the semialgebraic case, Thom's conjecture should be strengthenned by stating that semialgebraic maps without explosion can be effectively triangulated.

1 Statement of the effective triangulation of semialgebraic functions

Let R denote a real closed field. Let us extend from $\mathbb{R}$ to R the notations $X \in S(n, d, r)$ and $f \in S(n, n', d, r)$ stated in the introduction. Let X be a bounded closed semialgebraic set in R^n and let $f\colon X \to R$ be a semialgebraic function. A *triangulation* of f is realized by a semialgebraic homeomorphism $\tau\colon |K| \to X$, where K is a finite simplicial complex in $R^{n+n'}$, for some n', such that the restriction of $f \circ \tau$ to every simplex of K is linear.

Theorem 1 *We can define an algorithm which, starting from any couple* $((n, d, r), f\colon X \to R) \in \mathbb{N}^3 \times S(n, 1, d, r)$ *with* X *bounded and closed in* R^n, *produces*

(a) *a triangulation of* f, $\tau\colon |K| \to X$, *and*
(b) *a function* $\mathbb{N}^3 \ni (n, d, r) \to (D, C, h) \in \mathbb{N}^3$

such that $\tau \in S(n+1, n, D, C)$ *and* $\# K \leqq h$.

Note that we claim, in particular, that K can be realized in R^{n+1}.

Remark. We can refine the theorem by replacing the data $((n, d, r), f)$ by $((n, d, r), f\colon X \to R, X^1, \ldots, X^n) \in \mathbb{N}^3 \times S(n, 1, d, r) \times S(n, d, r)^n$, where $X^1, \ldots, X^n$ are closed subsets of X, and requiring τ to satisfy the condition that each $\tau^{-1}(X^i)$ is the underlying polyhedron of a subcomplex of K.

We postpone the proofs to § 3.

2 Finiteness of semialgebraic types of functions

Let f_i: $X_i \to R$, $i=1,2$, be semialgebraic functions. We say that f_1 and f_2 are *semialgebraically equivalent* if there are semialgebraic homeomorphisms π_1: $X_2 \to X_1$ and π_2: $R \to R$ such that $f_1 \circ \pi_1 = \pi_2 \circ f_2$. If we simply require that π_1 and π_2 are C^0 homeomorphisms we have the notion of *topological equivalence*. If f_1 and f_2 are simplicial functions defined on complexes K_1 and K_2, respectively, then we get the *simplicial equivalence* by requiring π_1: $K_2 \to K_1$ to be a simplicial isomorphism and π_2: $R \to R$ to be a PL homeomorphism.

Fukuda result in [F] concerned finiteness of the number of topological equivalence classes of polynomial functions. We want to improve it as follows.

Theorem 2 *There exists a computable function* ψ: $\mathbb{N}^2 \to \mathbb{N}$ *such that the number of semialgebraic equivalence classes of polynomial functions on* R^n *of degree* $\leqq m$, *is bounded by* $\psi(n, m)$.

Actually this is a particular case of the following more general statement.

Theorem 3 *There exists a computable function* ϕ: $\mathbb{N}^3 \to \mathbb{N}$ *such that the number of semialgebraic equivalence classes of all semialgebraic functions in* $S(n, 1, p, q)$ *is bounded by* $\phi(n, p, q)$.

Theorem 3 is an immediate corollary of Theorem 1 together with the following simple lemmas. We need a little preparation.

Fix a bounded semialgebraic embedding θ: $R \to R$ (for example, $\theta(t)=t/(1+|t|)$). Let $]b^-, b^+[=\theta(R)$. For each n, define θ_n: $R^n \to R^n$ by $\theta_n(x_1, \ldots, x_n) = (\theta(x_1), \ldots, \theta(x_n))$. We want to define a map

$$\begin{aligned} S(n,1,p,q) \ni (f: X \to R) \to ((\hat{f}: \hat{X} \to R), X^1, \ldots, X^n) \in S(n+1, 1, p', q') \\ \times S(n+1, p', q') \times \ldots \times S(n+1, p', q') \end{aligned}$$

so that p' and q' turn out to be computable functions of (p, q), $\hat{X}, X^1, \ldots, X^n$ is a decreasing sequence of bounded closed sets in $R^n \times R$ and the following lemma holds true.

Lemma 4 (compactification lemma) *Let* f_1, $f_2 \in S(n, 1, p, q)$. *Assume* $\hat{f}_1$: $\hat{X}_1 \to R$ *and* $\hat{f}_2$: $\hat{X}_2 \to R \in S(n+1, 1, p' q')$ *are semialgebraically equivalent. Let* π_1: $\hat{X}_2 \to \hat{X}_1$, *the semialgebraic homeomorphism realizing the equivalence, satisfy* $\pi_1(X_2^i) = X_1^i$, $i=1, \ldots, n$. *Then* f_1 *and* f_2 *are semialgebraically equivalent.*

In order to define the promised map $f \to \hat{f}$, let $\tilde{X}$ be the closure in $R^n \times R$ of the graph of $(\theta \circ f \circ \theta_n^{-1})|_{\theta_n(X)}$. Let $\tilde{f}$ be the restriction to $\tilde{X}$ of the projection of $R^n \times R$ onto R, and set

$$\begin{aligned} X^1 = \overline{\tilde{X}\text{-graph}}, \quad X^2 = \overline{X^1 - (\tilde{X}\text{-graph})}, \ldots, \\ X^n = \overline{X^{n-1} - (X^{n-2} - (\cdots(\tilde{X}\text{-graph})\cdots))}. \end{aligned}$$

One might expect to be able to use $\tilde{f}$ directly, but for technical reasons (which will be clear in a moment) it is better to complete $(\tilde{X}, \tilde{f})$ as follows. Let $a^+, a^- \in R$ be points outside $\overline{\theta(R)}$. Set $a_n^{\pm} = (a^{\pm}, \ldots, a^{\pm}) \in R^n$. Then set $\hat{X} = \tilde{X} \cup \{(a_n^{\pm}, b^{\pm})\}$ and define $\hat{f}, X^1, \ldots, X^n$ as before.

Proof of Lemma 4 Note that we could use $\tilde{f}_1$ and $\tilde{f}_2$ instead of $\hat{f}_1$ and $\hat{f}_2$ provided that the equivalence should be realized by π_1: $\tilde{X}_2 \to \tilde{X}_1$ and π_2: $R \to R$ such that $\pi_2(\theta(R))=\theta(R)$. The introduction of extra points $(a_n^{\pm}, b^{\pm})$ in the definition of $\hat{X}$ forces the equivalence of $\hat{f}_1$ and $\hat{f}_2$ to be realized by π_1: $\hat{X}_2 \to \hat{X}_1$ and π_2: $R \to R$ with that condition $\pi_2(\theta(R))=\theta(R)$. The condition $\pi_1(X_2^i)=X_1^i$ implies that π_1 is invariant on graph $(\theta \circ f \circ \theta_n^{-1})|_{\theta_n(X)}$ because

$$\text{graph}=(\tilde{X}-X^1)\cup(X^2-X^3)\cup\cdots.$$

Hence the lemma is clear. □

Let us denote by $T(r)$ the set of simplicial complexes having at most r simplexes. Let $T(n,r)$ denote the set of all pairs of sequences $K \supset K^1 \supset \ldots \supset K^n$ in $T(r)$ and simplicial functions f on K. We say $(K_i, K_i^1, \ldots, K_i^n, f_i) \in T(n,r)$, $i=1,2$, are *simplicially equivalent* if f_1 and f_2 are simplicially equivalent by a simplicial homeomorphism which maps each K_2^i onto K_1^i.

Lemma 5 *The number of simplicial equivalence classes of $T(n,r)$ is a computable function in (n,r).*

Proof. This is an immediate consequence of the fact that a simplicial function is determined by its behaviour on the vertexes of the complex. □

Lemma 6 *Let $X_i \supset X_i^1 \supset \ldots \supset X_i^n$, $i=1,2$, be sequences of bounded semialgebraic sets in R^n, and let f_i, $i=1,2$, be semialgebraic functions on X_i. Let $\tau_i\colon |K_i| \to X_i$ be triangulations of f_i such that the sets $\tau_i^{-1}(X_i^j)$ are the underlying polyhedra of subcomplexes K_i^j of K_i. If $(K_1, K_1^1, \ldots, K_1^n, f_1 \circ \tau_1)$ and $(K_2, K_2^1, \ldots, K_2^n, f_2 \circ \tau_2)$ are simplicially equivalent then f_1 and f_2 are semialgebraically equivalent by a semialgebraic homeomorphism π_1 which maps each X_2^i onto X_1^i.*

Proof. Trivial. □

3 Proof of Theorem 1 and its remark

Replace f: $X \to R \in S(n,1,d,r)$ in Theorem 1 by $\hat{f}\colon \Gamma_f \to R \in S(n+1,1,d,r+1)$, where Γ_f is the graph of f and $\hat{f}$ is the restriction to Γ_f of the projection of $R^n \times R$ onto R. Then it is enough to prove the following statement.

Theorem 7 *There exists an algorithm which, starting from data $((n,n',d,r), X, Y^1, \ldots, Y_{n'}) \in \mathbb{N}^4 \times S(n+1,d,r)^{n'+1}$-where $X, Y^1, \ldots, Y_{n'}$, are bounded closed sets in R^{n+1} and $X \supset Y_i$, $\forall i'$ – produces:*

(a) *a triangulation of X, $\tau\colon |K| \to X$, and*
(b) *a function $\mathbb{N}^4 \ni (n,n',d,r) \to (D,C,h) \in \mathbb{N}^3$*

such that $\tau \in S(n+1,n+1,D,C)$, $\# K \leqq h$, each $\tau^{-1}(Y_i)$ is the underlying polyhedron of a subcomplex of K and τ is of the form

$$\tau(x,t)=(\tau'(x,t),t) \quad \textit{for } (x,t) \in |K| \subset R^n \times R.$$

In the proof of the effective triangulation theorem for semialgebraic sets, τ did not have to be of the above form. A triangulation was constructed by induction on the dimension $n+1$ by means of a projection of $R^n \times R$ on a hyperplane in a good direction. In the present case we can not use this projection method directly. The following lemma, which provides a way of "bending" the direction of projection, must first be applied.

Let Z be an algebraic set in $R^n \times R$. Let p denote the projection $R^n \times R \to R^{n-1} \times R$ which forgets the first factor. We call p *good* for Z if $p^{-1}(y, t) \cap Z$ is of dimension 0 for every $(y, t) \in R^{n-1} \times R$.

Lemma 8 *There is an algorithm which, starting from an algebraic set Z in $R^n \times R$ which does not containing any set of the form $R^n \times c$, $c \in R$, produces an isomorphism π of $R^n \times R$ of the form*

$$\pi(x, y, t) = (x, y + \pi'(x), t) \quad \text{for } (x, y, t) = (x, y_1, \ldots, y_{n-1}, t) \in R^n \times R$$

such that p is good for $\pi(Z)$, where π' is a polynominal map $\pi': R \to R^{n-1}$.

Proof. We will find effectively points $a_1, \ldots, a_{n+1}$ in R and $b_1, \ldots, b_{n+1}$ in R^{n-1} so that if π' satisfies the condition $\pi'(a_i) = b_i$, $i = 1, \ldots, n+1$, then p is good for $\pi(Z)$.

Let $a_1, \ldots, a_{n+1}, b_1, \ldots, b_{n+1}$ and π' be given so that $\pi'(a_i) = b_i$, $i = 1, \ldots, n+1$, but p is not necessarily good for $\pi(Z)$. Set

$$W = \{(y, t) \in R^{n-1} \times R : \dim(R \times (y, t)) \cap \pi(Z) = 1\}.$$

By definition p is good for $\pi(Z)$ if W is empty. Let g be a polynomial function on $R^n \times R$ whose zero set is Z. Then

$$\pi(Z) = (g \circ \pi^{-1})^{-1}(0), \qquad g \circ \pi^{-1}(x, y, t) = g(x, y - \pi'(x), t).$$

For each $i = 1, \ldots, n+1$, let W_i denote the common zero set of $g(a_1, y - b_1, t), \ldots, g(a_i, y - b_i, t)$ in $R^{n-1} \times R$. Then $W_1 \supset W_2 \supset \ldots \supset W_{n+1} \supset W$ because W is an algebraic set. Thus it suffices to construct an algorithm which produces a_i and b_i from g so that the sequence n, $\dim W_1, \ldots, \dim W_{n+1}$ is strictly decreasing.

We choose a_i and b_i by induction on i. Choose a point $(a_1, y_0, t_0) \in R \times R^{n-1} \times R - Z$ and set $b_1 = 0$. Then

$$W_1 = g(a_1, y, t)^{-1}(0) = (a_1 \times R^{n-1} \times R) \cap Z$$

is of dimension $< n$. Assume $a_1, \ldots, a_{i-1}, b_1, \ldots, b_{i-1}$ have been chosen. Choose effectively a set Y_{i-1} containing exactly one point from each semialgebraically connected component of the C^∞ smooth point set of W_{i-1}. We want to choose a_i and b_i so that $g(a_i, y - b_i, t)$ does not vanish on Y_{i-1}. For such a_i and b_i, W_i does not contain Y_{i-1} and hence is of dimension $< \dim W_{i-1}$. Let a_i be a point of R such that Z does not contain any set of the form $a_i \times R^{n-1} \times c$, $(b, c) \in Y_{i-1}$. The existence of a_i follows from the hypothesis in the lemma. This implies that the y-function $g(a_i, y, c)$ for any $(b, c) \in Y_{i-1}$ is not identically zero. Choose b_i so that $g(a_i, b - b_i, c)$ does not vanish for any $(b, c) \in Y_{i-1}$. Thus (a_i, b_i) satisfies the above requirements. □

Proof of Theorem 7 We write a point of $R^n \times R$ as $(x, y, t) = (x, y_1, \ldots, y_{n-1}, t)$. We prove the theorem by induction on n. The case of $n = 0$ is trivial. Hence assume the theorem for $n - 1$. Let X and $\{Y_i\}$ be given by non-zero polynomial functions g_j, $j = 1, \ldots, r$, on $R^n \times R$. We can assume that $g_1, \ldots, g_{r'}$ are of the form $t - c$, $c \in R$, and $g_{r'+1}, \ldots, g_r$ are not divisible by polynomials of this form.

This is possible because of the following effectiveness of unique factorization of polynomial functions.

Regard a polynomial function $f(x_1, \ldots, x_n)$ of degree $\leqq d$ as a point of R^N for some integer N by the map $f=\sum a_\alpha x^\alpha \to (\ldots, a_\alpha, \ldots)\in R^N$. Moreover, regard a point of R^N as a semialgebraic set in R^N. Then there exists an algorithm which produces the following correspondence.

$$\begin{aligned}&(n, d, f)\in\{(n, d, f)\in\mathbb{N}^2\times R[x_1, x_2, \ldots]\colon \deg f\leqq d, f=f(x_1, \ldots, x_n)\}\\ &\downarrow\\ &(d', N, D, R, X)\in\mathbb{N}^4\times S(N, D, R),\end{aligned}$$

where X consists of d' points which represent polynomial functions $f_1, \ldots, f_{d'}$ such that $\prod f_i$ is a unique factorization of f into prime polynomial functions.

We prove this statement as follows. Let f be a polynomial function of degree d in the variables $x_1, \ldots, x_n$. After changing linearly the coordinate system we can assume that f is monic in x_1, that is, of the form

$$f(x)=x_1^d+g_1(x')x_1^{d-1}+\ldots+g_d(x'), \qquad x'=(x_2, \ldots, x_n).$$

Let $R_u[x_1, \ldots, x_n]$ denote the set of polynomial functions in the variables $x_1, \ldots, x_n$ and monic in x_1. Then it suffices to find an algorithm which produces the correspondence

$$\begin{aligned}&(n, d, f)\in\{(n, d, f)\in\mathbb{N}^2\times R_u[x_1, x_2, \ldots]\colon \deg f\leqq d, f=f(x_1, \ldots, x_n)\}\\ &\downarrow\\ &(d', f_1, f_2, \ldots)\in\mathbb{N}\times R_u[x_1, x_2, \ldots]\times R_u[x_1, x_2, \ldots]\times\ldots\end{aligned}$$

such that $\prod_{i=1}^{d'} f_i$ is a unique factorization of f into prime polynomials. Here we regard an element of $R_u[x_1, x_2, \ldots]$ as a semialgebraic subset of R^N for some integer N as above. Let us consider a map

$$\begin{aligned}\xi\colon\{(f_1, f_2, \ldots)\in R_u[x_1, x_2, \ldots]\times\ldots\colon f_1\neq 1, \ldots, f_\ell\neq 1, f_{\ell+1}=f_{\ell+2}\\ =\ldots=1 \quad \text{for some } \ell\}\to R_u[x_1, \ldots]\end{aligned}$$

given by $\xi(f_1, f_2, \ldots)=\prod f_i$. Then ξ is a finite-to-one map. For each $f\in R_u[x_1, \ldots, x_n]$, let $(f_1, f_2, \ldots, f_\ell, 1, 1, \ldots)$ be an element of $\xi^{-1}(f)$ such that $f_\ell\neq 1$ and if $(g_1, \ldots, g_{\ell'}, 1, 1, \ldots)\in\xi^{-1}(f)$ with $g_{\ell'}\neq 1$ then $\ell'\leqq\ell$. Then this correspondence is computable, and $\prod_{i=1}^{\ell} f_i$ is a unique factorization of f. □

Let Z denote the union of the zero sets of $g_{r'+1}, \ldots, g_r$, and let C denote the set of points c of R where $t-c=g_j$ for some j. Then by Lemma 4 we can assume that the projection $p\colon R^n\times R\to R^{n-1}\times R$ which forgets the first factor is good for Z.

Apply the démontage theorem 2.3.1 in [B-C-R] to $g_{r'+1}, \ldots, g_r$. Then we obtain effectively a finite partition of $R^{n-1}\times R$ into semialgebraic sets $\{A_k\}$ and, for each k, semialgebraic functions $\xi_{k,1}<\ldots<\xi_{k,\ell_k}$ on A_k such that for each (y, t) of A_k, $\{\xi_{k,1}(y, t), \ldots, \xi_{k,\ell_k}(y, t)\}$ is the set of roots of the x-polynomials $g_j(x, y, t)$, $j=r'+1, \ldots, r$. Here we note that $g_j(x, y, t)$ are not identically zero as x-polynomials by the goodness of p. Subdivide effectively the partition $\{A_k\}$ so that each member is semialgebraically connected (see 2.4 in [B-C-R]) and $R^{n-1} \times c$ is a member for each c of C. Then the family

$$B_{k,k'} = \{(x, y, t) \in R^n \times R : (y, t) \in A_k, x = \xi_{k,k'}(y, t)\},$$
$$B'_{k,k'} = \{(x, y, t) \in R^n \times R : (y, t) \in A_k, \xi_{k,k'}(y, t) < x < \xi_{k,k'+1}(y, t)\},$$

for all possible k, k', is compatible with X and $\{Y_i\}$, and the union of members of the family included X. These are clear because X is bounded, each $R^{n-1} \times c, c \in C$, is a member of $\{A_k\}$, and because A_k are semialgebraically connected. It also is clear that $p(X)$ is a union of some A_k's. Let κ denote the index set of these A_k. For each $k \in \kappa$, define index sets λ_k and μ_k so that

$$X = \bigcup_{k \in \kappa} \Big(\bigcup_{k' \in \lambda_k} B_{k,k'} \Big) \cup \Big(\bigcup_{k' \in \mu_k} B'_{k,k'} \Big).$$

Subdivide effectively $\{A_k\}_{k \in \kappa}$ so that each $(\overline{A_k}, A_k)$ is semialgebraically homeomorphic to (a simplex, its interior) (for example, a semialgebraic triangulation compatible with $\{A_k\}_{k \in \kappa}$). We want to show that for each $k \in \kappa$ and $k' \in \lambda_k$, $\xi_{k,k'}$ is extensible to $\overline{A_k}$. For this it suffices to see the following fact. Let (y_0, t_0) be a point of $\overline{A_k} - A_k$. Then

$$D = \overline{B_{k,k'}} \cap R \times (y_0, t_0)$$

consists of a point. This set is not empty by Proposition 2.5.3 in [B-C-R] and of dimension 0 since p is good for Z. Assume that D contains two points (x_1, y_0, t_0) and (x_2, y_0, t_0) with $x_1 < x_2$. Let U be a small semialgebraic neighborhood of (y_0, t_0) in $\overline{A_k}$ such that $U \cap A_k$ is semialgebraically connected. (The existence of U follows if we regard $(\overline{A_k}, A_k)$ as (a simplex, its interior).) As $\xi_{k,k'}(U \cap A_k)$ is semialgebraically connected, $\overline{\xi_{k,k'}(U \cap A_k)}$ is also semialgebraically connected and hence contains the interval $[x_1, x_2]$. Therefore D includes $[x_1, x_2] \times y_0 \times t_0$, which contradicts the fact D is of dimension 0. Thus $\xi_{k,k'}$ is extensible to $\overline{A_k}$. Keep the same notation $\xi_{k,k'}$ for the extension.

Consider $p(X)$ and $\{A_k\}_{k \in \kappa}$. By the induction hypothesis we can assume that $p(X)$ is the underlying polyhedron of a simplicial complex L the family of whose open simplexes is compatible with $\{A_k\}_{k \in \kappa}$. Here replace L by its barycentric subdivision if necessary. Then we can suppose, moreover, for $\sigma \in L$ and for $k'_1 \neq k'_2 \in \lambda_k$ with $A_k \supset \operatorname{Int} \sigma$, ξ_{k,k'_1} and ξ_{k,k'_2} take distinct values at one, at least, of the vertexes of σ.

Now we can define the underlying polyhedron of the required simplicial complex K as the union of the following sets. For each $\sigma \in$ L and $k' \in \lambda_k$ with $A_k \supset \operatorname{Int} \sigma$, let $F_{\sigma,k'}$ denote the simplex spanned by $(\xi_{k,k'}(y, t), y, t), (y, t) \in \sigma^0$ (= the vertexes of σ), and for each $\sigma \in L$ and $k' \in \mu_k$ with $A_k \supset \operatorname{Int} \sigma$, let $F'_{\sigma,k'}$ denote the cell lying between $F_{\sigma,k'}$ and $F_{\sigma,k'+1}$. Define P as the union of all $F_{\sigma,k'}$ and $F'_{\sigma,k'}$. Then we can define effectively a semialgebraic homeomorphism $\tau: P \to X$ of the form

$$\tau(x, y, t) = (\tau''(x, y, t), y, t) \qquad \text{for } (x, y, t) \in P,$$

for some function τ'', so that

$$\tau(F_{\sigma,k'}) = \overline{B_{k,k'}} \cap p^{-1}(\sigma),$$
$$\tau(F'_{\sigma,k'}) = \overline{B'_{k,k'}} \cap p^{-1}(\sigma),$$

and τ'' is linear on $F'_{\sigma,k'} \cap R \times (y,t)$ for each $F'_{\sigma,k'}$ and $(y,t) \in \sigma$ for the following reason.

Clearly on $F_{\sigma,k'}$ and $F'_{\sigma,k'} \cap p^{-1}(\operatorname{Int}\sigma)\tau$ is well-defined. Moreover τ is so on $F'_{\sigma,k'} \cap p^{-1}(\partial\sigma)$ if for each $(y,t) \in \partial\sigma$ and F_{σ_1,k'_1}

$$F'_{\sigma,k'} \cap F'_{\sigma_1,k'_1} \cap p^{-1}(y,t) = F'_{\sigma,k'} \cap p^{-1}(y,t) \quad \text{or} = \phi.$$

This condition holds true if we add to $\{g_{r'+1}, \ldots, g_r\}$ their derivatives with respect to the variable x. Indeed in this case the union of the zero sets of $g_{r'+1}, \ldots, g_r$ includes a dense C^∞ submanifold to which the restriction of p is a local diffeomorphism, and hence $\bigcup_{\dim\sigma = n} F_{\sigma,k'} = \cup F_{\sigma,k'}$.

Let K be a simplicial complex with underlying polyhedron P such that $F_{\sigma,k'}$ and $F'_{\sigma,k'}$ are unions of some simplexes of K. Note that a canonical construction of K exist (see 2.9 in [R-S]). Then (K,τ) fulfills the requirements in the theorem. □

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