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THE ASYMPTOTIC GEOMETRY OF TEICHMULLER SPACE

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INTRODUCTION

TEICHMULLER SPACE is the space of conformal structures on a topological surface M_g of genus g where two are equivalent if there is a conformal map between them which is homotopic to the identity. This space will be denoted by T_g. Teichmuller proved that when $g \geq 2$ T_g is homeomorphic to an open $6g$–6 dimensional ball. Moreover, his proof showed that this homeomorphism could be realized by the radial map along geodesic rays from a fixed base point.

In particular, this homeomorphism gives a natural way to compactify T_g by putting the endpoints on the rays. We denote the resulting closed $6g$–6 dimensional ball by $\bar{T}_g$. The immediate question to ask is to what extent $\bar{T}_g$ depends on the base point from which T_g was compactified. In this paper we show that the geometry along certain rays depends strongly on their base points.

Any diffeomorphism of M_g induces an isometry of T_g. The group of isometries of T_g induced by the group of diffeomorphisms of M_g is called the modular group and is denoted by Mod (g). Since $\bar{T}_g$ is defined in terms of geodesics, a natural question to pose is whether or not the action of Mod (g) extends continuously to $\bar{T}_g$.

There are several reasons to be interested in this questions. First, a continuous map on a closed ball is easier to understand than one on an open ball since it always has a fixed point. This fact has been successfully used by Thurston who compactified T_g in such a way that the action of Mod (g) extended continuously. By examining the fixed points of elements of Mod (g), he gave a geometric description of a canonical element of each connected component of Diff M. Thurston's compactification, ${T_g}^T$, is also homeomorphic to a closed $6g$–6 dimensional ball so it is reasonable to ask if his and Teichmuller's compactifications are the same; i.e., whether the identity map on the interiors extends to a homeomorphism from $\bar{T}_g$ to ${T_g}^T$. There was some evidence that this was true. (See [5] and Theorem 3 below.) However we show (Theorem 2) that the compactifications are distinct.

Secondly, compactification by geodesic rays have been used extensively by Mostow (and by numerous others) to study complete hyperbolic manifolds. The covering translations of such a manifold, acting on its universal cover, hyperbolic n-space, H^n, extend to the closure $\bar{H}^n$. The boundary sphere of $\bar{H}^n$ (the "sphere at infinity") is naturally identified with the space of rays through any interior point of H^n. By studying the action of the fundamental group on the sphere at infinity, Mostow proved his well-known rigidity theorem.

The allusion to hyperbolic manifolds is not pure whimsy; the Teichmuller space for the torus is isometric to H^2 and Mod (g) (which is isomorphic to $SL(2, \mathbf{Z})$) extends continuously to its closure. Moreover, T_g was thought to have negative curvature for several years. However, Linch[8] found a mistake in the proof and Masur[9] later showed that T_g is, in fact, not negatively curved. Thus the question of the extension of Mod (g) to $\bar{T}_g$ can be thought of both as a question of generalizing a result which is true for genus one and as a question of the extent to which T_g possess the properties of a negatively curved manifold.

The main result of this paper is:

THEOREM 1. *For $g \geq 2$ there is no continuous extension of Mod (g) to the closure $\bar{T}_g$ of Teichmuller space.*

Since the action of Mod (g) does extend continuously to $T_g{}^T$, we have the immediate corollary:

THEOREM 2. *Thurston's and Teichmuller's compactifications of T_g are distinct.*

Section 1 of this paper contains the necessary background results and definitions together with re-interpretations of the questions raised in this introduction. Section 2 provides a description of the geodesics to be considered here along with an outline of the proof. Section 3 is devoted to a discussion of extremal length. There are some new results there; in particular;

THEOREM 4. *The Teichmuller distance between two points M, M' in T_g is equal to $1/2 \log (\sup_{\gamma \in S} (E_{M'}(\gamma)/E_M(\gamma)))$ where $E_M(\ )$ denotes the extremal length of ( ) in M and γ ranges over all simple closed curves.*

We also give a short proof of a Hodge-like theorem, due to Hubbard and Masur, for measured foliations.

THEOREM 3. *Given a measured foliation F and a Riemann surface M, there is exactly one quadratic differential on M whose horizontal foliation is measure equivalent to F.*

Section 4 uses Theorem 4 to prove the propositions stated in §2.

§1.

(a) This section describes the necessary background material from Teichmuller theory. For more details on Teichmuller's theorems see Bers[3]; for details on metrics, isometries, and geodesics in T_g see Royden[10] and Kravetz[7].

Let M be a Riemann surface. A *quadrilateral* Q in M is an embedded closed disk with four distinguished points on its boundary. Q is conformally equivalent to a Euclidean rectangle which is unique up to scale change. The length divided by the width of this rectangle is called the modulus of Q. If $f: M \to M'$ is a homeomorphism of Riemann surfaces, then $f(Q)$ is a quadrilateral for every quadrilateral Q in M. If $K = \sup_{Q \subset M}$ (modulus $f(Q)$/modulus Q) is finite, (where Q runs over all quadrilaterals in M), then f is called *K-quasi-conformal* or just *quasi-conformal.*

If f is K-quasi-conformal, then it is differentiable almost everywhere. We can measure the deviation of f from conformality at a differentiable point p by the ratio $K_p(f) \geq 1$ of the axes of the infinitesimal ellipse at $f(p)$ which is the image of an infinitesimal circle centered at p. Note that $K_p(f)$ is invariant under change of scale. Define $K(f)$ to be the essential supremum of $K_p(f)$ over all $p \in M$. Then $K(f) = K$.

Teichmuller considered the problem of minimizing $K(f')$ over all f' homotopic to f. Note that a point in T_g is a Riemann surface, together with a homotopy class of homeomorphisms from a fixed surface. Thus all homeomorphisms between two points in T_g are homotopic by definition, and the problem above can be rephrased as minimizing $K(f)$ over all quasi-conformal maps between two given points in T_g. This problem was solved earlier by Grotzsch for the case of two rectangles. The minimizing map turned out to be the natural affine map between the rectangles. Teichmuller's solution is really a generalization of Grotzsch's. First a flat structure (with singularities) is defined on M; then the solution is an "affine" map between M and M' with respect to this structure.

Specifically, if θ is an analytic quadratic differential on M, then it is locally of the form $\theta(z)\,dz^2$ where $\theta(z)$ is holomorphic. The *horizontal line field* defined locally by $\theta(z)\,dz^2 > 0$ ($\theta(z) \neq 0$) is invariant under co-ordinate change and extends easily to a singular one where $\theta(z) = 0$. (See Fig. 1.) Similarly $\theta(z)\,dz^2 < 0$ defines the *vertical line field*. There is a metric g_θ naturally associated with θ; locally it is just $|\theta(z)|^{1/2}|dz|$. g_θ defines a (singular) flat structure on M. Away from the singularities of θ there is a natural parameter $w = x + iy$ where $dw^2 = \theta(z)\,dz^2$. With respect to this parameter the horizontal and vertical line fields are described by $x = \text{constant}$ and $y = \text{constant}$ respectively.

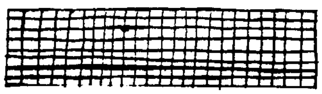
For every real number $K \geq 1$ we can define the (K, θ) *stretch map*, $f_{K,\theta}$ on M to be the identity on the underlying topological surface and be described locally by $x \to K^{-1/2}x$, $y \to K^{1/2}y$ with respect to the parameter w. Thus $f_{K,\theta}$ defines a new conformal structure, i.e., a new point M' in T_g, and a K-quasi-conformal map from M to M'. (It extends easily over the zeroes of θ.) With this terminology Teichmuller's theorem can be stated as follows:

THEOREM 1.1. (Teichmuller) *For any two points M, M' in T_g there is a (K, θ) stretch map $f_{K,\theta}$ from M to M' for which $K(f_{K,\theta}) < K(f)$ for all other quasi-conformal maps f from M to M'. K is unique and θ is unique up to multiplication by a positive real number.*

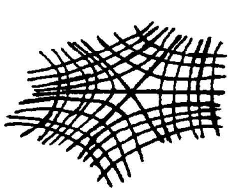
Definition. $f_{\theta,K}$ will be called the *Teichmuller map* from M to M'.

Definition. The *Teichmuller distance* from M to M', $d(M, M')$, is equal to $1/2 \log K$ where $f_{K,\theta}$ is the Teichmuller map from M to M'.

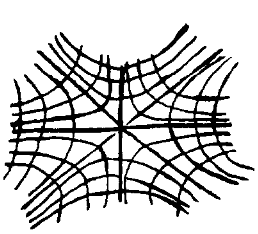
The Teichmuller distance is not induced by a Riemannian metric, but it is induced



Non-singular co-ordinates



A simple zero



A double zero

Fig. 1.

by a Finsler metric (i.e. a norm defined on the tangent space at each point, varying continuously with the point). Thus it makes sense to talk about geodesics in the Teichmuller metric. Between any two points in T_g there is precisely one geodesic; i.e., there are no conjugate points. The geodesic consists of the one parameter family of surfaces to which M is mapped by the (t, θ) stretch maps, $1 \le t \le K$. Fixing M and θ and letting K go to infinity describes an infinite geodesic which will be called a *ray* (*in the direction* θ) and denoted by $r(\theta)$ (or just θ when the meaning is clear from context).

For a fixed Riemann surface M, the Riemann–Roch Theorem implies that the space of quadratic differentials on M, $Q(M)$, is a $6g-6$ dimensional real vector space. Denote by $M_{K,\theta}$ the point in T_g to which M is mapped by $f_{K,\theta}$, $\theta \in Q(M)$. Then $M_{K,\theta} = M_{K,\theta'}$ if θ' is a positive real multiple of θ and $M_{1,\theta} = M$, $\forall \theta \in Q(M)$. Denote by $SQ(M)$ the space of projective equivalence classes of quadratic differentials where two are equivalent if they are positive real multiples of each other. $SQ(M)$ can be embedded in $Q(M)$ as the space of quadratic differentials θ whose metric g_θ has area 1. There is a map Ω_M from the open ball B^{6g-6} with polar co-ordinates (k, θ), $0 \le k < 1$, $\theta \in SQ(M)$, to T_g defined by $\Omega_M(k, \theta) = M_{K,\theta}$, $K = 1 + k/1 - k$.

THEOREM 1.2. (Teichmuller) Ω_M: $B^{6g-6} \to T_g$ *is a homeomorphism.*

Extending Ω_M to be a homeomorphism of the closed ball defines the compactification $\bar{T}_g$ of T_g.

Denote by $\mathrm{Mod}(g)$ the modular group of genus g. For $g \ge 3$ it is equal to $\mathrm{Out}(\pi_1 M_g)$, the group of outer automorphisms of $\pi_1 M_g$, M_g a topological surface of genus g. For $g = 2$, $\mathrm{Mod}(2) = \mathrm{Out}(\pi_1 M_2)/Z_2$, where Z_2 is the center of $\mathrm{Out}(\pi_1 M_2)$, generated by the hyperelliptic involution. Recall that $\mathrm{Out}(\pi_1 M_g)$ is naturally isomorphic to $\pi_0 \mathrm{Diff}\, M_g$, the group of isotopy classes of diffeomorphisms of M_g and to $\pi_0 \mathrm{Heq}(g)$, the group of homotopy equivalences of M_g. $\mathrm{Mod}(g)$ acts on T_g by taking a Riemann surface to the same Riemann surface with a new marking induced by the automorphism (i.e. with a new homotopy class of maps from the fixed surface). $\mathrm{Mod}(g)$ is precisely the group of isometries of T_g with the Teichmuller metric for $g \ge 2$.

In particular, $\mathrm{Mod}(g)$, acting on T_g, take rays to rays. However, most elements of $\mathrm{Mod}(g)$ do not fix the base point, M_0, so that the question of a continuous extension of $\mathrm{Mod}(g)$ to $\bar{T}_g$ is a question of the compatibility of rays based at M_0 with those based at $\varphi(M_0)$, $\varphi \in \mathrm{Mod}(g)$. Answering this question requires an understanding of the asymptotic behavior of rays in T_g. That is the main task of this paper.

(b) Measured foliations will be used throughout; we give a quick description of them here. For more details about measured foliations, especially about their relation to Teichmuller space (see Thurston[12] and Bers[2]).

A measured foliation F on a topological surface is a foliation (with a finite number of singularities) with an invariant transverse measure. This means that if the local charts send leaves of F to horizontal arcs in $\mathbf{R}^2$, the transition functions on $\mathbf{R}^2$ are of the form $\phi_{ij} = (f(x, y),\ c \pm y)$ where c is a constant. The singularities which are allowed are "p-pronged saddles", $p \ge 3$. These are topologically the result of glueing p rectangles together along their horizontal edges. (See Fig. 2.) Note that these singularities are the same topologically as those that occur at $z = 0$ in the line field $z^{p-2}\, dz^2 > 0$.

If F is a measured foliation, γ a simple closed curve, then $\int_\gamma F$ is defined to be the total variation of γ in the "y-direction", i.e. the integral of γ with respect to the transverse measure. $i(\gamma, F)$ is the infinum of $\int_{\gamma'} F$ where γ' ranges over all simple

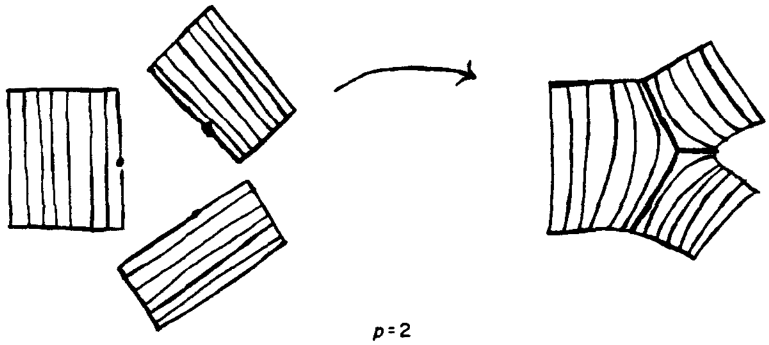


Fig. 2.

closed curves free homotopic to γ. $i(\gamma, F)$ is called the ***intersection number*** of γ with F. Two measured foliations F, F' are ***measured equivalent*** if, for all simple closed curves γ, $i(\gamma, F) = i(\gamma, F')$. We denote the space of measure classes of foliations by MF. F and F' are ***projectively equivalent*** if there is a constant b such that $i(\gamma, F) = bi(\gamma, F')$ for all γ. The space of projective equivalence classes of foliations is denoted by PF.

THEOREM 1.3. (Thurston) *MF is homeomorphic to a $6g-6$ dimensional ball; PF is homeomorphic to a $6g-7$ dimensional sphere.*

There is a special class of measured foliations that have the property that the complement of the critical leaves (those passing through singularities) is homeomorphic to a cylinder. The leaves of the foliation on the cylinder are all freely homotopic to a single simple closed curve γ. Such a foliation is completely determined as a point in MF by the height r of the cylinder ($r = i(A, F)$, the infinum of $\int_A F$ for arcs A with endpoints on the boundary of the cylinder) and the isotopy class of γ. We will denote such a foliation by $F_{\gamma,r}$.

Let S denote the set of isotopy classes of simple closed curves on a surface of fixed genus. The ***geometric intersection number*** $i(\varphi, \gamma)$ of φ, $\gamma \in S$ is defined to be the infinum of the number of points of intersection of φ' and γ' where φ' and γ' range over curves isotopic to φ and γ respectively. Then $r\, i(\varphi, \gamma) = i(\varphi, F_{\gamma,r})\ \forall \varphi \in S$.

THEOREM 1.4. (Thurston) *There is an embedding e: $S \times \mathbf{R}_+ \to MF$ that sends (γ, r) to $F_{\gamma,r}$. The image of e is dense in MF. Similarly the image of S in PF is dense.*

Recall that, by the Riemann mapping theorem, T_g, $g \geq 2$, can be identified with the space of hyperbolic metrics (metrics with constant sectional curvature -1) on M_g, where two are considered equivalent if they are isometric by an isometry isotopic to the identity. For a fixed hyperbolic structure there is a unique simple, closed geodesic in the free homotopy the class of every non-trivial $\gamma \in S$. Its length, $1(\gamma)$, is called the hyperbolic length of γ. The lengths (in fact a finite subset of them) $1(\gamma)$, $\forall \gamma \in S$, determine the metric up to isotopy so they determine the corresponding point in T_g ([4a]).

Thurston shows that as one goes to infinity in T_g the hyperbolic lengths of curves in S are approximated well by their intersection numbers with a measured foliation. This allows a compactification, T_g^T, of T_g to be defined in terms of ratios of hyperbolic lengths; the boundary of T_g^T is the space PF. Since the compactification of T_g is defined in terms of an intrinsic quantity invariant under isotopy, Mod (g) naturally extends to T_g^T.

As discussed in §1(a) a quadratic differential θ on a Riemann surface M defines a flat metric g_θ on M with canonical coordinates coming from two perpendicular foliations, the horizontal one, $F_{\theta,h}$, $(\theta(z)\,dz^2 > 0)$ and the vertical one, $F_{\theta,v}$, $(\theta(z)\,dz^2 < 0)$. These have an invariant measure induced by the Euclidean distance on local charts. The (K, θ) stretch map from M to $M_{K,\theta}$ multiplies the measure of $F_{\theta,h}$ by $K^{1/2}$ and that of $F_{\theta,v}$ by $K^{-1/2}$. Thus along the ray in the direction θ there is a one parameter family of flat (singular) matrics $d_K(\theta)$ where the distance in the "y-direction" is getting large and the distance in the "x-direction" is getting small. Let $\bar{l}_K(\gamma)$ denote the infinum of the lengths of simple closed curves freely homotopic to γ when measured with respect to $d_K(\theta)$. It is not hard to see that $\lim_{K\to\infty} \bar{l}_K(\gamma)/K^{1/2} = i(\gamma, F_{\theta,h})$.

One way to re-interpret the question of extending Mod (g) to $\bar{T}_g$ is to ask whether or not the metrics $d_K(\theta)$ are natural in some sense. Specifically, we can ask if the lengths $\bar{l}_K(\gamma)$ on $M_{K,\theta}$ are asymptotic to some intrinsic lengths such as the hyperbolic lengths $l_K(\gamma)$.

Question. Do there exist constants C_K such that for all $\gamma \in S$,

$$\lim_{K\to\infty} l_K(\gamma)/C_K = i(\gamma, F_{\theta,h})?$$

An affirmative answer would have implied that T_g^T and $\bar{T}_g$ were equal.

Remark. Actually, $\bar{l}_K$ is more closely related to $L_K(\gamma)$, the square root of the extremal length of γ on $M_{K,\theta}$ than the hyperbolic length. In fact, by comparing hyperbolic length to extremal length along certain rays, it is possible to prove Theorem 2 directly. However, Theorem 1 implies that the answer to the question above with $l_K(\gamma)$ replaced by $L_K(\gamma)$ is still "no".

§2

In this section we will state some results about the asymptotic behavior of rays in Teichmuller space. These results will be sufficient to prove Theorem 1. The proofs are postponed until §4.

Given a geodesic ray $r_0(\theta)$ from the base point M_0 of T_g, the closure of $r_0(\theta)$ in $\bar{T}_g$ is equal to the ray plus a single point on $\partial\bar{T}_g$. Call this point the endpoint of $r_0(\theta)$ and denote it by $P(r_0(\theta))$. Consider the closure $\overline{r_M(\theta)}$ of a ray $r_M(\theta)$ from some other point M. If $\overline{r_M(\theta)} - r_M(\theta)$ is a single point $P(r_M(\theta))$ then $P(r_M(\theta)) = P(r_0(\theta'))$ for some $\theta' \in SQ(M_0)$. In this case we say that $r_M(\theta)$ *converges* and that the rays $r_M(\theta)$ and $r_0(\theta')$ are *convergent.*

Remark. It is unknown whether or not a general ray from M converges. It is possible, using techniques similar to those used to prove Proposition 1 below, to show that most rays from M converge. However, in view of Theorem 1 it seems likely that some non-convergent rays exist.

Suppose that Mod (g) acts continuously on $\bar{T}_g$ and that $M = \varphi(M_0) \neq M_0$ for some $\varphi \in$ Mod (g). φ^{-1} maps $SQ(M)$ to $SQ(M_0)$ homeomorphically since it is an isometry, and the map P_0 from $SQ(M_0)$ to $\partial\bar{T}_g$ which takes a ray to its endpoint is a homeomorphism by definition. If φ is a homeomorphism on $\partial\bar{T}_g$, it follows that all rays through M converge and that the map P_M from $SQ(M)$ to $\partial\bar{T}_g$ which sends a ray from M to its end point is a homeomorphism. The proof of Theorem 1 is by contradiction; we will show that P_M is discontinuous, contradicting the assumption that φ acts continuously on $\partial \bar{T}_g$.

Consider the set C of measured foliations with the property that the complement of the critical leaves is a set of p cylinders $C_1, \ldots, C_p$, $1 \le p \le 3g$-3. The case $p = 1$ is the image of $S \times \mathbf{R}_+$ in MF (see Theorem 1.4). All the leaves in the cylinder C_i are freely homotopic to a single simple closed curve σ_i. Let A_i denote the homotopy class of arcs in C_i connecting the two boundary components of C_i. Then the measure class of $F \in C$ is completely determined by the isotopy class of $\sigma_i \in S$ and the intersection numbers $i(A_i, F)$.

If $F \in C$ is the horizontal foliation of a quadratic differential θ, then θ induces a flat metric on the C_i. The heights h_i of the C_i are equal to $i(A_i, F) = i(A_i, F_{\theta,h})$ and the lengths (circumferences) l_i are equal to $i(\sigma_i, F_{\theta,v})$. Define $h_i/l_i = m_i$ to be the *modulus* of C_i and say that two quadratic differentials (possibly on different points in T_g) θ, θ' such that $F_{\theta,h}$, $F_{\theta',h} \in C$ are *modularly equivalent* iff $\sigma_i = \sigma_i'$ $\forall i$ (as points in S) and there is a constant C such that $Cm_i = m_i'$, $\forall i$. θ and θ' are called *projectively equivalent* iff $F_{\theta,h}$ is projectively equivalent to $F_{\theta',h}$. Note that $m_i = m_i'$ if θ' is a positive real multiple of θ' so that both projective equivalence and modular equivalence are invariant by multiplication $\mathbf{R}_+$.

Jenkins and Strebel studied quadratic differentials whose horizontal foliations belong to C. These differentials will be called *J–S differentials*; their horizontal foliations *J–S foliations*. The same names will be used for the respective equivalence classes in SQ and PF.

THEOREM 2.1. (Jenkins[6], Strebel[11]). *Fix a Riemann surface M.*

(i) *There is exactly one J–S differential in SQ(M) in each projective equivalence class.*

(ii) *There is exactly one J–S differential in each modular equivalence class.*

The relationship between the modular equivalence class and the projective equivalence class of a J–S differential depends heavily on M, the point in Teichmuller space. J–S differentials with the same core curves and cylinders of the same height will, in general, be in different modular equivalence classes if they are defined on different points in T_g. This fact will play a crucial role in the proof.

Let $\gamma \in S$. Then γ corresponds to a single point in PF by Theorem 1.4 and thus by Theorem 2.1 to a single J–S differential in $SQ(M)$ for every $M \in T_g$. Let $[\gamma]_M \in SQ(M)$ denote this differential and the ray from M corresponding to it.

PROPOSITION 1. *For every* $M \in T_g$, $[\gamma]_M$ *is convergent to* $[\gamma]_{M_0}$.

Let σ denote a fixed system of $3g$–3 disjoint, distinct elements of S, σ_i, $1 \le i \le 3g$–3, and let $[\sigma]_M$ denote any J–S differential (and corresponding ray) in $SQ(M)$ with core curves σ. The corresponding horizontal foliations will be called *σ-foliations*.

Two rays θ, θ' are called asymptotic iff $\lim_{t\to\infty} \inf_{M\in\theta'} \mathrm{d}(M, \theta(t)) = 0$, where d(,) denotes the Teichmuller distance.

PROPOSITION 2. *For any* $M \in T_g$, $[\sigma]_M$ *is asymptotic to* $[\sigma]_{M_0}$ *iff it is modularly equivalent to* $[\sigma]_{M_0}$.

COROLLARY 1. $[\sigma]_{M_0}$ *and* $[\sigma]_M$ *are convergent iff they are modularly equivalent.*

Propositions 1 and 2 show that convergence for J–S differentials with one cylinder depends only on the horizontal measured foliation while for those with $3g$–3 cylinders it depends on both horizontal and vertical foliations. Below, using the fact that those with one cylinder are dense in PF we arrive at a contradiction.

Consider a simple closed curve γ^n which wraps around the surface n times in each "direction of σ" (see Fig. 3). In other words, as $n \to \infty$, γ^n converges as a projective measured foliation to a σ-foliation with cylinders of equal heights.

Let $PF_M : SQ(M) \to PF$ be the map which sends a quadratic differential to the projective class of its horizontal measured foliation. Since the metrics $g_{\theta'}$ corresponding to quadratic differentials θ' in a small neighborhood of a differential θ in $SQ(M)$ are close to g_θ, their canonical co-ordinates will be close to those of g_θ. The intersection number of $\gamma \in S$ with $F_{\theta,h}$ is just the infinum of the y-variation with respect to g_θ of curves isotopic to γ. It follows that the intersection numbers of γ with $F_{\theta',h}$ for θ' near θ will be near $i(\gamma, F_{\theta,h})$. Therefore PF_M is continuous.

Theorem 2.1 implies that PF_M is l–l and onto for J–S differentials. Since the curves γ^n converge in PF to a σ-foliation with equal heights and PF_M is continuous, the pre-images $[\gamma]_M$ converge to a J–S differential, $[\sigma]_M^\varepsilon$, with core curves σ whose cylinders have equal heights. However, the lengths of the curves σ_i in σ when measured with respect to the metric of $[\sigma]_M^\varepsilon$ depend on the surface M. In particular, the modular equivalence class of $[\sigma]_M^\varepsilon$ depends on M.

Assume that the surfaces M_0 and M, $M = \varphi(M_0)$, $\varphi \in \mathrm{Mod}(g)$, are chosen so that $[\sigma]_{M_0}^\varepsilon$ and $[\sigma]_M^\varepsilon$ are not modularly equivalent. By Proposition 1, the rays $[\gamma^n]_M$ and $[\gamma^n]_{M_0}$ converge for all n. However, by Proposition 2, the ray $[\sigma]_M^\varepsilon$ does not converge to $[\sigma]_{M_0}^\varepsilon$ but to some other J–S ray $[\sigma]_{M_0}^u$ based at M_0 which is in the same modular equivalence class as $[\sigma]_M^\varepsilon$. Since $\lim_{n\to\infty} [\gamma^n]_M = [\sigma]_M^\varepsilon$ while

$$\lim_{n\to\infty} P([\gamma^n]_M) = P([\sigma]_{M_0}^u) \neq P([\sigma]_M^\varepsilon)$$

this implies that the map P_M from $SQ(M)$ to $\partial \bar{T}_g$ is discontinuous. Thus $\mathrm{Mod}(g)$ cannot act continuously on $\bar{T}_g$ and Theorem 1 is proved.

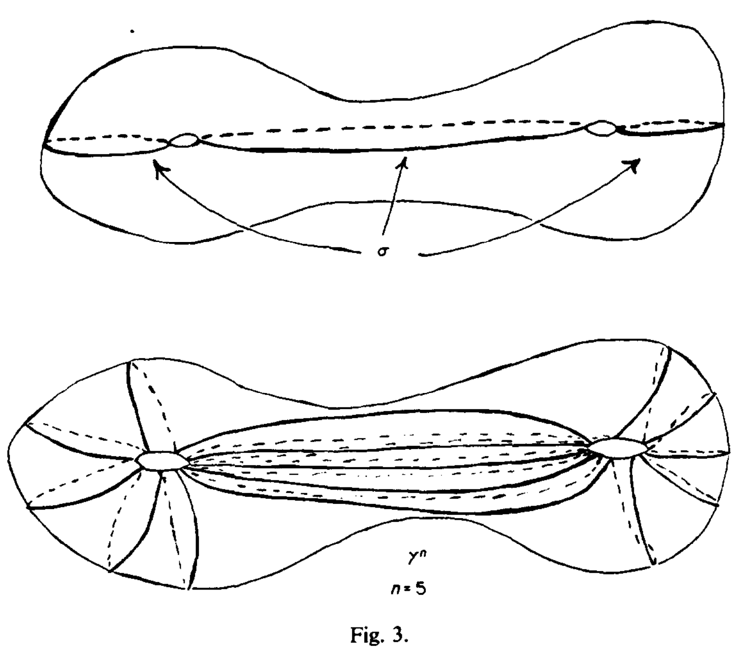


Fig. 3.

Remark 1. To see that it is possible to choose $\varphi \in \mathrm{Mod}(g)$ such that $[\sigma]^{\epsilon}_{M_0}$ and $[\sigma]_M^{\epsilon}$ $(\varphi(M_0) = M)$ are not modularly equivalent, note that it follows from Jenkins–Strebel theory that if the extremal length in M of one curve in σ, say σ_1, is much larger than that of σ_{3g-3}, then the modulus of the cylinder C_1 in $[\sigma]_M^{\epsilon}$ will be much smaller than that of C_{3g-3}. By taking $\varphi = \alpha^m$ for some large m, where α is a Dehn twist around a simple closed curve δ, intersecting σ_1 but not σ_{3g-3}, (see Fig. 4), it is possible to make the ratio of the moduli m_1 to m_{3g-3} in M smaller than the corresponding ratio in M_0.

Remark 2. The set $\{[\sigma]_M\}$ is an open $3g$–4 dimensional simplex Δ_M in $SQ(M)$. Theorem 2.1 (ii) and Corollary 1 determine a 1–1 correspondence between $P_M(\Delta_M)$ and $P_{M'}(\Delta_{M'})$ for all $M, M' \in T_g$. However, as we have seen above, this correspondence is not compatible with the convergence of J–S differentials with one cylinder.

Note that for a fixed σ all of the σ-foliations are topologically conjugate (allowing also the collapsing and pulling apart of leaves joining singularities). Let Top (θ) denote the subset of ∂T_g corresponding to rays whose horizontal foliations are topologically conjugate (in the extended sense) to that of θ. By following the proof of Proposition 1, it is possible to show that for every ray, $\overline{r_M(\theta)} - r_M(\theta)$ lies in Top (θ). Therefore, if, for each θ, Top (θ) is identified to a point, all rays would converge. The modular group would extend continuously to this new compactification ($\bar{T}_g$/Top) of T_g.

Since the open $3g$–4 simplex of σ-differentials is identified to a point, $\bar{T}_g$/Top is necessarily non-Hausdorff. The $(i-1)$-faces of the simplex correspond to J–S differentials with i cylinders; in particular, the vertices correspond to J–S differentials with one cylinder. Any two such differentials can be joined by a path lying in a sequence of closed $3g$–4 simplices (corresponding to a sequence of collections of $3g$–3 simple closed curves). Since the differentials with one cylinder are dense in the boundary, if we attempted to rectify the non-Hausdorff property of $\bar{T}_g$/Top by collapsing the *closed* simplices to a point, the entire boundary would have to be collapsed to a point.

It is reasonable to conjecture that the set of rays through M such that Top (θ) is a single point has full measure in $SQ(M)$. A closely related conjecture has been made in the context of interval exchange maps (see Keane [6a]). If the conjecture were true, then it would follow that there is a measurable extension of Mod (g) to $\bar{T}_g$.

§3

Extremal length is a conformal invariant of an isotopy class of simple closed curves. It was introduced by Beurling and developed by Ahlfors and him. It has both

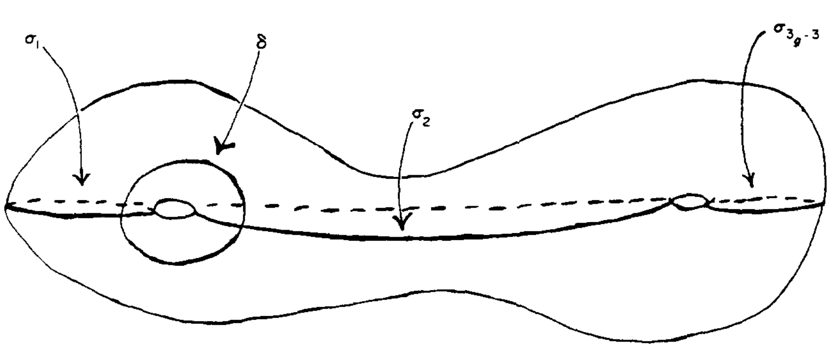


Fig. 4.

an analytic and a geometric definition; the interplay between them will be heavily exploited here. For further details see Ahlfors [1], Strebel [11], and Jenkins [6].

Fix a Riemann surface M. Then a conformal metric on M is any metric (possibly with singularities) on M which is locally of the form $\rho(z)|dz|$, where ρ is a non-negative real valued function. For $\gamma \in S$ let $l_\rho(\gamma)$ denote the infinum of lengths of simple closed curves isotopic to γ when measured with respect to ρ. Let A_ρ equal the area of M with respect to ρ. Then $l_\rho(\gamma)^2/A_\rho$ is invariant under change of scale.

Definition (analytic). The *extremal length* of γ in M, $E_M(\gamma)$, is equal to $\sup_{\rho'} l_{\rho'}(\gamma)^2/A_{\rho'}$, where ρ' ranges over all conformal metrics with $0 < A_{\rho'} < \infty$.

Any cylinder C embedded in M has a conformal structure induced from that of M. C is conformally equivalent to exactly one flat cylinder up to change of scale. The *modulus* of C is defined to be the modulus of this flat cylinder ($= h/1$; see p. 35).

Definition (geometric). The *extremal length* $E_M(\gamma)$ of $\gamma \in S$ is equal to $1/\mathrm{mod}(\gamma)$ where $\mathrm{mod}(\gamma)$ is the supremum of the moduli of all cylinders embedded in M with core curve isotopic to γ.

Remark. The analytic definition of E_M is most useful for finding lower bounds for E_M while the geometric definition is best for upper bounds.

Not only are the two definitions of E_M compatible, but the (unique) solutions to the problems they pose are equal.

THEOREM 3.1. (Jenkins [6], Strebel [11]). *Let $[\gamma]_M$ be the J–S differential in $SQ(M)$ with a single cylinder with core curve $\gamma \in S$. The metric g_γ induced by $[\gamma]_M$ is the metric that maximizes $l_\rho(\gamma)^2/A_\rho$. The cylinder with core curve γ which has the largest modulus is the complement of the critical leaves of the horizontal foliation of $[\gamma]_M$.*

Remark. If the extremal length of $n\gamma$, $n \in \mathbf{Z}_+$ is interpreted as the extremal length of n copies of γ, it is clear from the analytic definition of E_M that $E_M(n\gamma) = n^2 E_M(\gamma)$. To get a quantity that is linear over $\mathbf{Z}_+$ we define $L_M(\gamma)$ to be the square root of $E_M(\gamma)$.

The rest of this section will be devoted to extending some known results for simple closed curves to PF (and MF) using the fact that S is dense in PF. The basic geometric idea is to consider the J–S differential $[\gamma]_M$ for $\gamma \in S$ and to estimate the lengths in g_γ of other simple closed curves in terms of their intersection numbers with $F_{\gamma,v}$, the vertical foliation of $[\gamma]_M$.

Suppose $\varphi, \varphi' \in S$ are "close" in PF, i.e. there is a constant c such that $c\varphi$ and φ' are close in MF (e.g. see Fig. 5). We would like to say that $i(\varphi', F_{\varphi,v})/ci(\varphi, F_{\varphi,v})$ is close to one. This is ture, but it is necessary to be careful because it is not true that $i(\varphi', F)/ci(\varphi, F)$ is close to one for all $F \in MF$. For example, the curve δ intersects φ' once but φ zero times (see Fig. 5). The point here is that φ and δ (and φ' and δ) intersect a small number of times compared to their lengths (measured in any reasonable Riemannian metric).

With this example in mind we make the following definition:

Definition. If $\varphi, \varphi' \in S$ and ρ is any Riemannian metric (with isolated zeroes allowed), then the *transversality coefficient* of φ and φ' (with respect to ρ), $t^\rho(\varphi, \varphi')$, equals $l_\rho(\varphi)l_\rho(\varphi')/i(\varphi, \varphi')$. It is not hard to see that $t^\rho(\varphi, \varphi')$ is bounded from below for

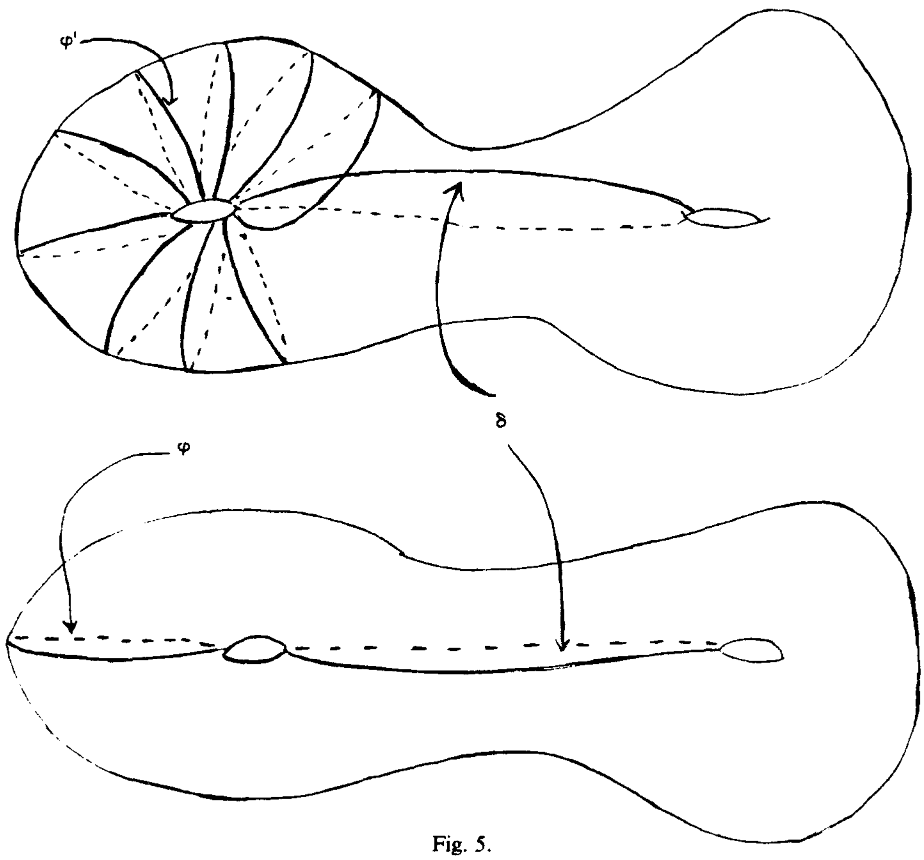


Fig. 5.

a fixed ρ. Note that $t^\rho(r\varphi, s\varphi') = t^\rho(\varphi, \varphi')$, r, $s \in \mathbf{R}_+$. t^ρ can be extended over all $(F, F') \in PF \times PF$ by taking the infinum over sequences $(\varphi, \varphi') \in S \times S$ converging to (F, F') in PF. It can be extended over $MF - 0 \times MF - 0$ but is singular at zero.

Example. If $\rho = g_\varphi$, the metric of area 1 associated to $[\varphi]_M$, then $l_{g_\varphi}(\varphi) = L_M(\varphi)$ and $\lim_{i\to\infty} c_i l_{g_\varphi}(\delta_i) = 1$ where $(c_i, \delta_i) \in R_\times \times S$ converges to $F_{\varphi,v}$, the vertical foliation of $[\varphi]_M$. Since $i(\varphi, F_{\varphi,v}) = L_M(\varphi)$, $t^\rho(\varphi, F_{\varphi,v}) = 1$.

Now it is possible to make the notion of closeness in PF precise. Fix a Riemannian metric ρ on the topological surface M_g and denote t^ρ by t.

Definition. Let $\varphi, \varphi' \in S$. Then φ' is ϵ-close to φ iff there exists a constant c such that $\forall F \in MF$-0

$$|(i(\varphi', F)/ci(\varphi, F)) - 1| \leq t(\varphi, F)\epsilon.$$

We will also say that φ' is ϵ-close to $c\varphi$ to designate the dependence of the definition on the constant c.

The definition of ϵ-close extends to all of PF by taking limits of elements in S. Since the ratios of intersection numbers with a finite set of curves in S serve as co-ordinates for PF, ϵ-closeness is compatible with the topology on PF. For small ϵ the sets $B_\epsilon(\varphi)$ of points ϵ-close to φ form a (closed) basis for a neighborhood of φ.

Now it is possible to make some estimates of extremal lengths. Fix a Riemann surface M and suppose that φ' is ϵ-close to $c\varphi$, where $\varphi', \varphi \in S$. Then, by definition, $|(i(\varphi', F_v)/i(\varphi, F_v)c) - 1| \leq t(\varphi, F_v)\epsilon$ where F_v is the vertical foliation of $[\varphi]_M$. From the example above $t^{\rho'}(\varphi, F_v) = 1$ for the metric $\rho' = g_\varphi$. The set of metrics of area 1 corresponding to quadratic differentials on M is compact so there is an upper bound B to the ratio $l_\rho(\gamma)/l_{\rho'}(\gamma)$ where γ ranges over S and ρ' ranges over all these flat metrics. Thus $t(\varphi, F_v) \leq B^2$ and

$$|1-(i(\varphi', F_v)/ci(\varphi, F_v))| \leq B^2\epsilon \tag{1}$$

where B depends only on M. Let $B^2\epsilon = \bar{\epsilon}$.

Note that $i(\varphi, F_v) = L_M(\varphi) = E_M(\varphi)^{1/2}$. Denote $L_M(\varphi)$ and $L_M(\varphi')$ by L and L' respectively. Since the length of φ' when measured with respect to g_φ is at least as large as its total variation in the "x-direction" and, by definition, the variation in the "x-direction" is at least $i(\varphi', F_v)$, $l_{g_\varphi}(\varphi') \geq i(\varphi', F_v)$. Since g_φ has area 1, $L' \geq i(\varphi', F_v)$ by the analytic definition of $E_M(\varphi')$. Then (1) becomes

$$L'/cL + \bar{\epsilon} \geq 1. \tag{2}$$

Similarly, by considering the differential corresponding to the extremal length problem for φ', we find that

$$cL/L' + \bar{\epsilon} \geq 1. \tag{2'}$$

Putting (2) and (2′) together gives

$$1/1-\bar{\epsilon} \geq L'/cL \geq 1-\bar{\epsilon}. \tag{3}$$

PROPOSITION 3. *There is a unique continuous extension of the extremal length function from S to MF-0 where $E(r\cdot) = r^2E(\cdot)$, $r \in \mathbf{R}_+$.*

Proof. Suppose that (φ_i, c_i) is a sequence in $S \times \mathbf{R}_+$ converging to a measured foliation F. Define $E(F) = \lim_{i\to\infty} c_i^2E(\varphi_i) = \lim_{i\to\infty} (c_iL(\varphi_i))^2$. Since, for any ϵ there is an N such that for all $i, j > N$, φ_i is ϵ-close to $(c_j/c_i)\varphi_j$, (3) implies that $1/1-\bar{\epsilon} \geq c_iL_i/c_jL_j \geq 1-\bar{\epsilon}$, where $L_i = L(\varphi_i)$. Thus the limit exists.

Remark. $E(F)$ is equal to the area of the metric which comes from the quadratic differential whose horizontal foliation is measure equivalent to F. We show below that for a fixed M there is a unique such differential. Note that from this point of view it is clear that the extension $E(0) = 0$ is continuous.

THEOREM 3. *For a fixed Riemann surface M and a fixed measured foliation F, there is precisely one quadratic differential whose horizontal measured foliation is measure equivalent to F.*

This theorem is just Theorem 2.1(i) for $F \in C$ and was extended recently to the above form by Hubbard and Masur [5]. Below is a short proof, assuming the existence of J–S differentials with one cylinder.

Proof of Theorem 3. Consider the map PF: $SQ(M) \to PF$ which sends a quadratic differential of area 1 to the projective class of its horizontal measured foliation. It is continuous. There is a commutative triangle (below), where j takes γ to $[\gamma]_M$ and e is as in Theorem 1.4.

$$\begin{tikzcd} S \arrow[r, "j"] \arrow[dr, "e"'] & SQ \arrow[d, "PF"] \\ & PF \end{tikzcd}$$

The closure of $j(S)$ in $SQ(M)$, $\overline{j(S)}$, maps onto the closure of $e(S)$ in PF. Since $e(S)$ is dense in PF, the map PF is onto. Below we show that it is 1–1 on $\overline{j(S)}$ so that $\overline{j(S)}$ is homeomorphic to S^{6g-7}. Thus it is all of SQ and PF is a homeomorphism. (Note that this also proves that $j(S)$ is dense in SQ; this was first proved by Masur [9a]; see also Douady and Hubbard [4].) Multiplying by $\mathbf{R}_+$, it follows that MF: $Q(M) \to MF$ is a homeomorphism.

Suppose that $\varphi_i \in S$ converges to F in PF. Then, for any sufficiently small ϵ there is an N such that if $i, j > N$ then φ_i is ϵ-close to $c_j\varphi_j$ in PF for some c_j. Let $[\varphi_i]$ and $[\varphi_j]$ denote the J–S differentials with area 1 corresponding to φ_i and φ_j respectively. We claim that $[\varphi_i]$ and $[\varphi_j]$ are ϵ-close in $SQ(M) \subset Q(M)$ for some reasonable measurement of closeness in $Q(M)$. It follows that $[\varphi_i]$ converges and thus that PF is 1–1.

To establish the claim we need a slight generalization of Strebel's proof of uniqueness for the case of one cylinder. Let φ' be ϵ-close to $c\varphi$, and let L and L' denote the extremal length of φ and φ' respectively.

LEMMA 3.2. *Let f: $A \to M$ be a conformal embedding of a cylinder with area* 1 *and modulus* $1/L^2$ *into M with core curve φ. Let $[\varphi']$ denote the J–S differential with area* 1 *corresponding to φ', and let $w = \eta + i\zeta$ be the natural co-ordinate defined by $[\varphi']$. Let $z = x + iy$ be the co-ordinate of A such that the closed curves are horizontal. Then*

$$\int_0^{1/L} \int_0^L \zeta_x^2 \,\mathrm{d}x\,\mathrm{d}y \le 1 - (1 - \bar{\epsilon})^4. \tag{4}$$

This lemma shows that the total variation in the imaginary direction of the differential $[\varphi]$ from $[\varphi']$ is at most $1 - (1 - \bar{\epsilon})^4$. It follows easily that the two quadratic differentials are close and the claim is established.

Remark. If $\varphi = \varphi'$ then $\bar{\epsilon} = 0$ and $[\varphi] = [\varphi']$ almost everywhere; hence $[\varphi] = [\varphi']$. This was Strebel's proof of uniqueness.

Proof of Lemma 3.2. Since $[\varphi']$ has area 1 we have

$$1 = \int_0^{1/L} \int_0^L (\eta_x\zeta_y - \eta_y\zeta_x) \,\mathrm{d}x\,\mathrm{d}y. \tag{5}$$

Since f is conformal, $(\eta_x\zeta_y - \eta_y\zeta_x) = \eta_x^2 + \zeta_x^2$ so that

$$1 = \int_0^{1/L} \int_0^L \eta_x^2 + \zeta_x^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{6}$$

Applying Schwartz' inequality to $\int \eta_x^2 \,\mathrm{d}x$ gives

$$1 \ge \left(1/L\left(\int_0^{1/L} \left(\int_0^L |\eta_x| \,\mathrm{d}x\right)^2 \mathrm{d}y\right)\right) + \iint_A \zeta_x^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

However, $\int |\eta_x| \,\mathrm{d}x$ is just the η-variation of the core curve φ corresponding to a fixed y in A. Thus $\int |\eta_x| \,\mathrm{d}x \ge i(\varphi, F_v)$, where F_v is the vertical foliation of $[\varphi']$, and (7) becomes

$$1 \ge (1/L)^2 i(\varphi, F_v)^2 + \int\int \zeta_x^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{8}$$

$c\varphi$ and φ' are ϵ-close in PF so that by (1) and (2),

$$(1/L)^2 i(\varphi, F_v)^2 \geq c^{-2}(1-\bar{\epsilon})^2(L'/L)^2 \geq (1-\bar{\epsilon})^4. \tag{9}$$

Putting (8) and (9) together gives

$$1 \geq (1-\bar{\epsilon})^4 + \int\int \zeta_x^2 \, dx \, dy \tag{10}$$

which is what we wanted.

The following is needed for the proof of Proposition 2.

THEOREM 4. *The Teichmuller distance between two points in T_g is equal to $1/2 \log (\sup_{\gamma \in S} (E_{M'}(\gamma)/E_M(\gamma)))$ where $E_M(\ )$ denotes the extremal length of ( ) in M and γ ranges over all of S.*

Proof. Let $f: M \to M'$ be the Teichmuller map between M and M' and suppose it is e^d-quasiconformal. For any $\gamma \in S$ let $C_M(\gamma)$ denote the cylinder in M realizing the extremal length of γ. Then $\operatorname{mod}(C_M(\gamma)) = (E_M(\gamma))^{-1}$. Since f is e^d-quasiconformal, $\operatorname{mod}(f(C_M(\gamma))) \geq e^{-d} \operatorname{mod}(C_M(\gamma))$. This follows from the fact that the Teichmuller distance between two cylinders equals the log of the ratio of their moduli. But $f(C_M(\gamma))$ is a cylinder in M' with core curve γ so that

$$E_{M'}(\gamma) \leq (\operatorname{mod}(f(C_M(\gamma))))^{-1} \leq e^d E_M(\gamma).$$

If D denotes the supremum of $E_{M'}(\gamma)/E_M(\gamma)$, $\gamma \in S$, then $D \leq e^d$ since γ was arbitrary.

On the other hand, the extremal length of the vertical foliation defined in M by the quadratic differential associated to f is multiplied by e^d. Since extremal length extends continuously from simple closed curves to PF and S is dense in PF, there is a sequence of curves γ_n such that $\lim_{n\to\infty} E_{M'}(\gamma_n)/E_M(\gamma_n) = e^d$. Therefore $D \geq e^d$. Since $d/2 = d(M, M')$, we are done.

§4

This section contains the proofs of Propositions 1 and 2 and of Corollary 1.

PROPOSITION 1. *For every $M \in T_g$, $[\gamma]_M$ is convergent to $[\gamma]_{M_0}$.*

Proof. Denote the parametrized ray of $[\gamma]_M$ by M_t. Then in co-ordinates based at M_0, $M_t = (k_t, \theta_t)$ where $\lim_{t\to\infty} k_t = 1$. Take a convergent sequence of $\theta_t \in SQ(M_0)$ and denote its limit by θ. We will show that $\theta = [\gamma]_{M_0}$. Since the convergent subsequence was chosen arbitrarily, this shows that $[\gamma]_M$ is convergent and that it converges to $[\gamma]_{M_0}$.

Let d_t denote the flat metric with area 1 on M_t induced by the (K_t, θ_t) stretch map f_{K_t,θ_t}, where $K_t = 1 + k_t/1 - k_t$. The intersection number of a curve φ with the horizontal foliation of d_t is a lower bound for the length of $\varphi \in S$ in d_t. Since f_{K_t,θ_t} multiplies the measure of $F_{\theta_t,h}$ by $K_t^{1/2}$ and d_t has area 1, $L_{M_t}(\varphi) \geq K_t^{1/2} i(\varphi, F_{\theta_t,h})$ by the analytic definition of extremal length.

However, since $[\gamma]_M$ defines a cylinder with core γ whose modulus goes to infinity along the $[\gamma]_M$, $\lim_{t\to\infty} L_{M_t}(\gamma) = 0$. Since $K_t \to \infty$ as $t \to \infty$,

$$\lim_{t\to\infty} i(\gamma, F_{\theta_t,h}) = i(\gamma, F_{\theta,h}) = 0.$$

The extremal length of any curve $\delta \in S$ which does not intersect γ stays bounded along the ray $[\gamma]_M$ so $i(\delta, F_{\theta,h}) = 0$ also. The only measured foliations with this property are those which define a single cylinder with core curve γ. Thus $\theta = [\gamma]_{M_0}$.

PROPOSITION 2. *For any $M \in T_g$, $[\sigma]_M$ is asymptotic to $[\sigma]_{M_0}$ iff it is modularly equivalent to $[\sigma]_{M_0}$.*

Proof. The idea of the proof is to estimate extremal lengths on the surfaces M_K which are distance $1/2 \log K$ from M along the ray of a σ-differential. The estimates depend, up to a factor coming from M, only on K and the moduli of the $3g-3$ cylinders defined by $[\sigma]_M$. As K goes to infinity the error factor becomes negligible and we can apply Theorem 4 to prove Proposition 2.

It is not hard to estimate the extremal lengths of the curves σ_i belonging to σ. Let the moduli of the cylinders determined by $[\sigma]_M$ be denoted by m_i, $1 \le i \le 3g-3$. The quadratic differential on M_K induced by $[\sigma]_M$ defines cylinders with core curves σ_i and moduli Km_i, $1 \le i \le 3g-3$. By the geometric definition of extremal length $E_{M_K}(\sigma_i) \le 1/Km_i$. On the other hand it is possible to define a metric on M_K whose support is contained in a small neighborhood of the ith cylinder C_i with the following properties:

(i) It has total area less than $Km_i + A$, A a constant depending only on M;

(ii) It is the standard flat metric on C_i with area Km_i (i.e., the leaves in C_i have length 1 and C_i has height Km_i);

(iii) No curve freely homotopic to σ_i has length less than 1. It follows from the analytic defintion of extremal length that $E_{M_K}(\sigma_i) \ge 1/(Km_i + A)$. Together with the first estimate this gives:

$$1/Km_i \ge E_{M_K}(\sigma_i) \ge 1/(Km_i + A). \tag{1}$$

The only if half of Proposition 2 follows immediately from (1) and Theorem 4.

Now think of M as $3g-3$ cylinders (defined by $[\sigma]_M$) joined together in threes. Take small neighborhoods of the intersections of these cylinders and call them the *body* of M. The body will consist of $2g-2$ "pairs of pants" (homeomorphic to S^2–3 disks). Assume that the boundary curves are leaves of the horizontal foliation of $[\sigma]_M$ and that the moduli of the $3g-3$ cylinders C_i in the complement of the body are equal to $m_i - \bar{A}$ for a fixed constant $\bar{A}$. Along the σ-ray corresponding to $[\sigma]_M$ the cylinders attached in the body get longer and longer but the way they are attached stays fixed (see Fig. 6).

Define the body of M_K to be neighborhoods of the intersections of the corresponding cylinders in M_K, (with boundary curves equal to leaves of the induced horizontal foliation on M_K), that intersect each cylinder in a subcylinder of fixed modulus $\bar{A}$. Thus the conformal structure of the body remains fixed along the σ-ray and depends only on the constant $\bar{A}$ and the base surface M. On the other hand the complement of the body in M_K is a set of cylinders $\tilde{C}_i$ with moduli $Km_i - \bar{A}$ ($=K\tilde{m}_i$ from now on) that go to infinity as K does. The contribution of the body to the extremal length of any fixed simple closed curve is bounded above along the σ-ray

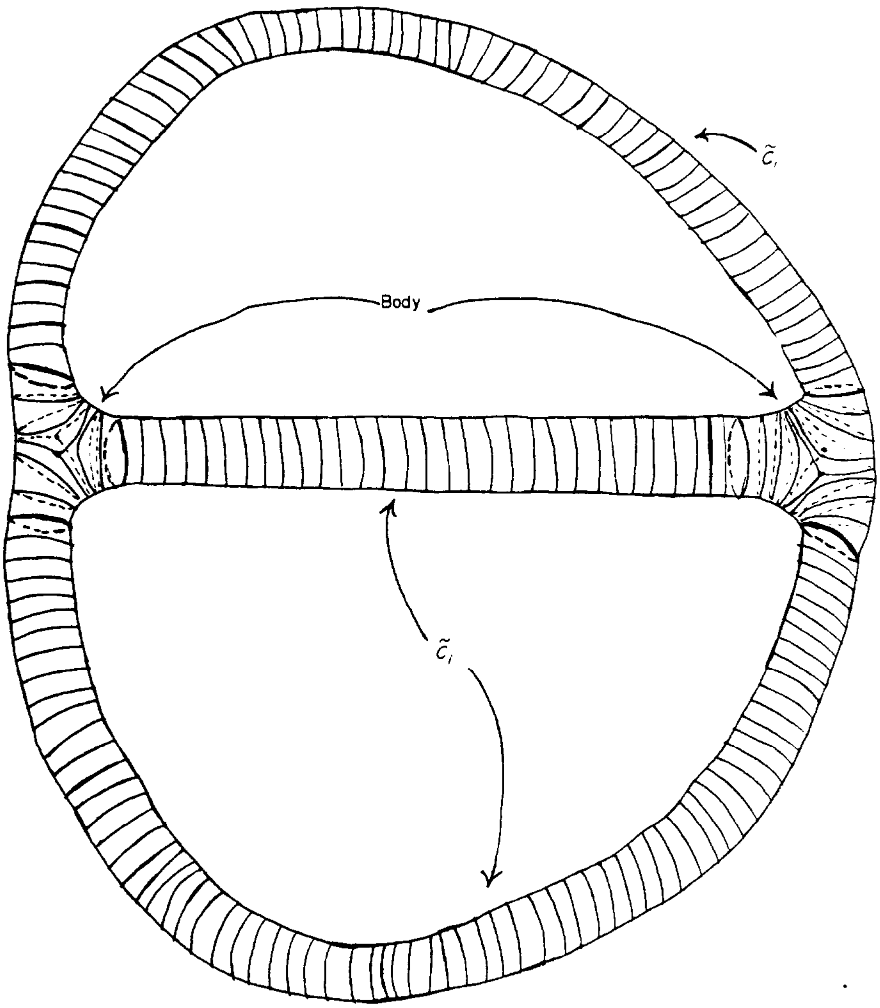


Fig. 6.

(i.e. the bound is independent of K). We will see below that this is sufficient to show that the extremal length for all simple closed curves is approximated uniformly well by the contribution from the $\tilde{C}_i$. This will allow us to apply Theorem 4.

Any $\gamma \in S$ not isotopic to one of the σ_i has non-zero intersection number with at least one of the σ_i. Thus it must travel through the ith cylinder n_i times, where $n_i = i(\gamma, \sigma_i)$. To estimate the extremal length of γ, it is necessary to estimate the contribution of each arc passing through these cylinders. It is possible to put γ in a canonical position with respect to the induced flat metrics on the subcylinders $\tilde{C}_i$. Isotope γ so that the n_i strands travel through $\tilde{C}_i$ at a constant angle, wrapping around $\tilde{C}_i$ some t_i times and intersecting $\partial\tilde{C}_i$ at equally spaced points. The strands are then connected in the body of M_K, wrapping around it at most once (see Fig. 7).

The n_i arcs are geodesics in the flat metric on $\tilde{C}_i$ and have length $(n_i^2(K^2\tilde{m}_i^2 + t_i^2))^{1/2}$ if the metric is normalized so that the length of σ_i is 1. A metric can be defined on the body of M_K so that it agrees with the metrics on the $\partial\tilde{C}_i$ and so that it has area at most B, where B is a constant depending only on M (and $\bar{A}$).

Fix a curve $\gamma \in S$ and denote by 1_i the quantity $[n_i^2(K^2\tilde{m}_i^2 + t_i^2)]/K\tilde{m}_i$. This is the extremal length of the n_i arcs in $\tilde{C}_i$. Let $L = \sum_{i=1}^{3g-3} 1_i$ and let m be the minimum of the $\tilde{m}_i$. Defined a metric which, on the $\tilde{C}_i$, is weighted according to 1_i. That is, let the metric be

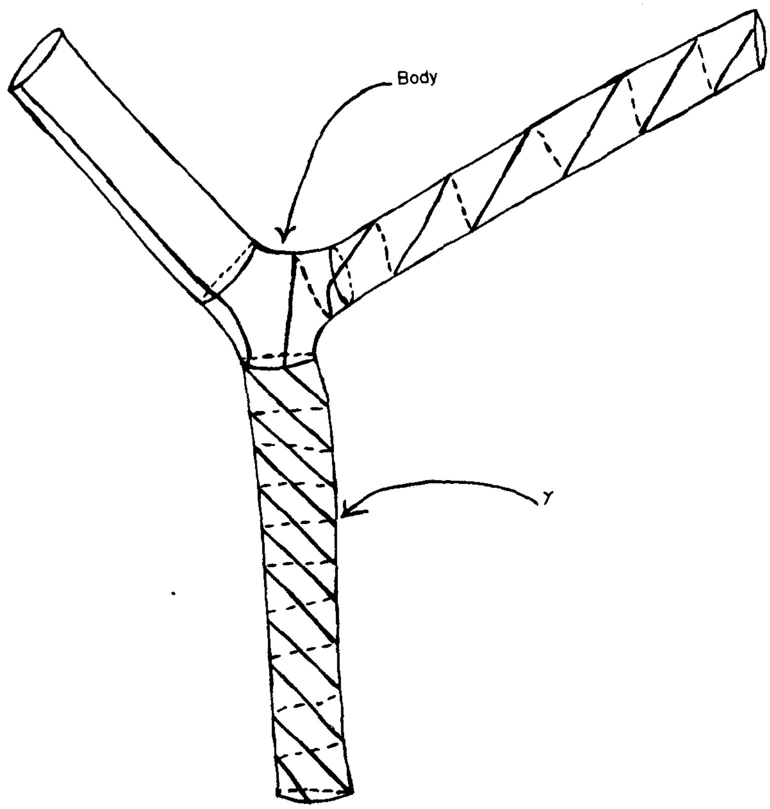


Fig. 7.

the standard one on $\tilde{C}_i$ with area 1_i. Then the length of the boundary curves σ_i is $(1_i/Km_i)^{1/2}$ so that the metric extends over all of M_K with the area of the body at most $B(L/Km)$. Therefore the area of this metric is at most $L(1+B/Km)$ and the length of γ is at least L. This gives the estimate:

$$E_{M_K}(\gamma) \geq L^2/L(1+B/Km). \tag{2}$$

In the same way a cylinder C_γ with core curve γ can be constructed so that it has constant width in the flat metrics of the $\tilde{C}_i$ and then connects up across the body. Call the inverse of its modulus the extremal length of C_γ. Since the conformal structure of the body is fixed for all K, the contribution to the extremal length of C_γ coming from n_i strands in one component of the body is bounded above by $\bar{B}n_i^2$ for some fixed constant $\bar{B}$ independent of K. Thus the total contribution of the connecting strands is at most $\Sigma\, \bar{B}n_i^2$, where the summation runs from 1 to $3g$–3. The contribution from the cylinder $\tilde{C}_i$ is 1_i. By the geometric definition of extremal length, the extremal length of C_γ is greater than or equal to the extremal length of γ. Thus we have:

$$\sum_{i=1}^{3g-3} n_i^2(\tilde{m}_iK + (t_i^2/\tilde{m}_iK) + \bar{B}) \geq E_{M_K}(\gamma). \tag{3}$$

Proposition 2 follows from (1), (2), (3) and Theorem 4.

To complete the proof of Theorem 1 we need to prove the corollary:

COROLLARY 1: $[\sigma]_M$ *and* $[\sigma]_{M_0}$ *are convergent iff they are modularly equivalent.*

The proof of this corollary uses a slight modification of Bers' proof of the uniqueness half of Teichmuller's theorem (Bers [3]). Let M_0 and M' be two Riemann surfaces, f a map from M_0 to M', and ϕ and ψ, two quadratic differentials on M_0 and M' respectively. Then, following Bers, let $j_{f,\phi,\psi}$ denote the Jacobian of f with respect to the metrics induced by ϕ and ψ. Similarly, let $\lambda_{f,\phi,\psi}$ be the infinitesimal change of the length of a vector which is vertical with respect to ϕ. In other words, if $x+iy$, $x'+iy'$ are local coordinates of ϕ and ψ away from the singularities, $\lambda_{f,\phi,\psi} = (x_y'^2 + y_y'^2)^{1/2}$ and $j_{f,\phi,\psi} = (x_x'y_y' - x_y'y_x')$. Bers shows ([3], Lemma B, p. 110) that for any M', $h : M' \to M'$ isotopic to the identity, one has the inequality:

$$\iint_{M'} \mathrm{d}A_\psi \le \iint_{M'} \lambda^2_{h,\psi,\psi}\, \mathrm{d}A_\psi. \tag{4}$$

($\mathrm{d}A_\psi$ is the area element induced by ψ.)

Consider the following situation. There is a (K_0, ϕ) stretch map f_0 from M_0 to M'. Let ψ be the differential on M' induced by ϕ and f_0. Assume that both ϕ and ψ have area 1. Suppose that there is another map $f : M_0 \to M'$. Then $h = ff_0^{-1} : M' \to M'$ is isotopic to the identity so that (4) applies. Since f_0^{-1} is area preserving and shrinks vertical (with respect to ψ) vectors by a constant factor of $K_0^{-1/2}$, it follows that

$$1 \le \iint_{M'} \lambda^2_{h,\psi,\psi}\, \mathrm{d}A_\psi = 1/K_0 \iint_{M_0} \lambda^2_{f,\phi,\psi}\, \mathrm{d}A_\phi. \tag{5}$$

We note that the quantity $\lambda^2_{f,\phi,\psi}/j_{f,\phi,\psi}$ is invariant under conformal change of co-ordinates and metric. It measures the deviation of f from conformality in the vertical direction. In particular, it is always less than or equal to K if f is K-quasi-conformal, with equality if and only if the maximal stretching takes place at that point and in the vertical direction. Suppose that f is composed of two maps, $f = s_\epsilon \circ g_0$, where g_0 is a (K, θ) stretch map and s_ϵ is e^ϵ-quasi-conformal. Since g_0 stretches by $K^{1/2}$ in the direction vertical with respect to θ and s_ϵ stretches by at most e^ϵ in any direction, we have that

$$\lambda^2_{f,\phi,\psi}/j_{f,\phi,\psi} \le e^\epsilon K(\cos^2\xi + 1/K^2 \sin^2\xi) \tag{6}$$

where ξ is the angle between the vertical vectors of θ and ϕ in M_0 away from the singularities. (ξ is defined only up to addition of $k\pi$ because of the lack of a global framing but this does not affect the formula.)

Putting (5) and (6) together gives

$$1 \le e^\epsilon K/K_0 \iint_{M_0} (\cos^2\xi + (1/K^2)\sin^2\xi) j_{f,\phi,\psi}\, \mathrm{d}A_\phi. \tag{7}$$

Since the area of ψ is one, this implies that:

$$1 \le e^\epsilon K/K_0 - \left(\iint_{M_0} 1 - (\cos^2\xi + (1/K^2)\sin^2\xi) j_{f,\phi,\psi}\, \mathrm{d}A_\phi\right). \tag{8}$$

Now we are in a position to prove Corollary 1.

Proof of Corollary 1. Suppose that $[\sigma]_{M_0}$ and $[\sigma]_M$ are modularly equivalent. Then by Proposition 2 they are asymptotic. We want to show that they are convergent. In other words, if $\sigma_{M_0}(t) = (K_t, \theta)$, $\sigma_M(s) = (K_s, \theta_s)$ in co-ordinates based at M_0, we wish to show that $\theta_s \to \theta$ as $s \to \infty$. Let $\bar{M}$ be the surface (K_t, θ) for some large t, g_0 the (K_t, θ) stretch map from M_0 to $\bar{M}$; let $M' = (K_s, \theta_s)$ be some surface on the ray $I\sigma I_M$ distance ϵ from $\bar{M}$, f_0 the (K_s, θ_s) stretch map from M_0 to M'; and finally, let s_ϵ be the stretch map from $\bar{M}$ to M'.

This is the situation described above so that (8) holds, where $K_s = K_0$, $K_t = K$, and $\theta_s = \phi$. By the triangle inequality for the Teichmuller metric, $K_t/K_s \leq e^\epsilon$ so that (8) implies that

$$e^{2\epsilon} - 1 \geq \iint_{M_0} 1 - (\cos^2 \xi + (1/K_t^2) \sin^2 \xi) j_{f,\phi,\psi}\, dA_\phi. \tag{9}$$

As K_s and K_t go to infinity, ξ goes to 0 since the rays are asymptotic. Thus $\xi \to 0$ almost everywhere as K_s, $K_t \to \infty$. This implies that $\lim_{s\to\infty} \theta_s$ agrees with θ almost everywhere. Two quadratic differentials on M_0 agreeing almost everywhere are the same so that $\lim_{s\to\infty} \theta_s = \theta$.

The "*if*" direction follows immediately from the "only if"direction since $[\sigma]_M$ converges to some ray in the direction of some other σ-differential M_0 and, by definition, a ray can converge to only one ray through M_0.

This completes the proof of the corollary and hence of Theorem 1.

Remark. The proof of Corollary 1 also shows that no two distinct rays through M_0 are asymptotic. Thus T_g has no conjugate points "at infinity".

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