Fibre Bundles
and
Chern-Weil Theory

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Johan Dupont

Aarhus Universitet

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Preface

These lecture notes are based on some lectures given at Aarhus University in 2002 and repeated in 2003. The course prerequisite was a previous course on de Rham cohomology following the first 10 chapters of the book [MT]. Occasionally we have also referred to [D], [S] and [W] for background material.

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Johan Dupont

1. Introduction

Let us start considering a linear system of differential equations of order 1 on an interval $I \subseteq \mathbb{R}$.

$$\frac{dx}{dt} = -Ax, \qquad t \in I \tag{1.1}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $A = A(t)$ is an $n \times n$ real matrix.

It is well-known that given $x_0 = (x_{10}, \dots, x_{n0}) \in \mathbb{R}^n$ and $t_0 \in I$, there is a unique solution $x(t) = (x_1(t), \dots, x_n(t))$ to (1.1) such that $x(t_0) = x_0$. The simplest case is of course if $A \equiv 0$ and in fact we can change (1.1) into this by *change of gauge*

$$y = gx \tag{1.2}$$

where $g = g(t) \in \mathrm{GL}(n, \mathbb{R})$. To see this notice that we want to find g such that

$$\frac{dy}{dt} = \frac{dg}{dt} \cdot x + g \cdot \frac{dx}{dt} = \frac{dg}{dt} \cdot x - gAx = 0.$$

That is we want

$$\frac{dg}{dt} = gA \tag{1.3}$$

or by transposing

$$\frac{dg^t}{dt} = A^t g^t.$$

That is, the rows of g are solutions to (1.1) with A replaced by $-A^t$. Hence we just choose $g_0 = g(t_0) \in \mathrm{GL}(n, \mathbb{R})$ arbitrarily and find the unique solution corresponding to each row, i.e., we find the unique solution to (1.3) with $g(t_0) = g_0$. To see that g is in fact invertible we solve similarily the equation of matrices

$$\frac{dh}{dt} = -Ah$$

with $h(t_0) = g_0^{-1}$ and observe that

$$\frac{d}{dt}(gh) = \frac{dg}{dt} \cdot h + g \cdot \frac{dh}{dt} = gAh - gAh = 0,$$

thus by uniqueness $g \cdot h = 1$. Hence we have proved the following proposition.

Proposition 1.1. *Consider a system of equations (1.1) on an interval $I \subseteq \mathbb{R}$. Let $t_0 \in \mathbb{R}$ and $g_0 \in \mathrm{GL}(n, \mathbb{R})$. Then there is a unique* gauge transformation $g = g(t)$, *such that (1.1) is equivalent to*

$$\frac{dy}{dt} = 0, \qquad \textit{for } y = gx \textit{ and } g(t_0) = g_0.$$

Remark. We shall think of g as a family of linear transformations parametrised by I, i.e. given by a map $\bar{g}$ in the commutative diagram

$$\begin{tikzcd} I \times \mathbb{R}^n \arrow[rr, "\bar{g}"] \arrow[dr, "\mathrm{proj}"'] & & I \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}"] \\ & I & \end{tikzcd}$$

with $\bar{g}(t, x) = (t, g(t)x)$.

We want to generalize this to the case where I is replaced by an open set $U \subseteq \mathbb{R}^m$ or more generally by any *differentiable manifold* $M = M^m$. For that purpose it is convenient to rewrite (1.1) as an equation of differential forms:

$$dx = -(Adt)x.$$

Absorbing dt into the matrix we shall in general consider a matrix A of differential 1-forms on M and we want to solve the equation

$$dx = -Ax \tag{1.4}$$

for $x = (x_1, \ldots, x_n)$ a vector of functions on M. Again a gauge transformation is a smooth family af non-singular linear maps $g = g(t) \in \mathrm{GL}(n, \mathbb{R})$, $t \in M$, or equivalently, a smooth map $\bar{g}$ in the commutative diagram

$$\begin{tikzcd} M \times \mathbb{R}^n \arrow[rr, "\bar{g}"] \arrow[dr, "\mathrm{proj}"'] & & M \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}"] \\ & M & \end{tikzcd}$$

such that $\bar{g}(t, x) = (t, g(t)x)$ defines a non-singular linear map $g(t)$ for each $t \in M$. Again putting $y = gx$ the equation (1.4) changes into the *gauge equivalent* equation

$$dy = -A^g y$$

with $A^g = gAg^{-1} - (dg)g^{-1}$. In particular we can transform the equation into the *trivial equation*

$$dy = 0$$

(which has the obvious solution $y =$ constant) if and only if we can find g satisfying

$$dg = gA \qquad \text{or} \qquad g^{-1}dg = A. \tag{1.5}$$

Example 1.2. Let $M = \mathbb{R}^2$ with variables (t^1, t^2). Consider the equation (1.4) with $A = -t^2 dt^1$, that is

$$dx = (t^2 dt^1)x,$$

or equivalently, the partial differential equations

$$\frac{\partial x}{\partial t^1} = t^2 x, \qquad \frac{\partial x}{\partial t^2} = 0. \tag{1.6}$$

But this implies

$$0 = \frac{\partial^2 x}{\partial t^1 \partial t^2} = \frac{\partial}{\partial t^2}(t^2 x) = x.$$

Hence only $x \equiv 0$ is a solution to (1.6) whereas the equation $dy = 0$ has other (constant) solutions as well. Therefore they are *not* gauge equivalent.

More systematically let us find a necessary condition for solving (1.5): Suppose g is a solution; then

$$0 = ddg = d(gA) = (dg) \wedge A + gdA = gA \wedge A + gdA$$

and since g is invertible we obtain

$$F_A = A \wedge A + dA = 0. \tag{1.7}$$

This is called the *integrability condition* for the equation (1.4) and F_A is called the *curvature*. We have thus proved the first statement of the following proposition.

Proposition 1.3. *For the equation (1.4) to be gauge equivalent to the trivial equation* $dy = 0$ *a necessary condition is that the curvature* $F_A = 0$*.* Locally *this is also sufficient.*

Proof. For the proof of the second statement it suffices to take $M = B^m \subseteq \mathbb{R}^m$ the open ball of radius 1, that is $B^m = \{u = (u_1, \ldots, u_n) \mid |u| < 1\}$. Let S^{m-1} be the sphere $S^{m-1} = \{u \mid |u| = 1\}$ and define $g \colon B^m \longrightarrow \mathrm{GL}(n, \mathbb{R})$ by solving the equation

$$\frac{\partial g}{\partial r} = gA\left(\frac{\partial}{\partial r}\right), \qquad g(0) = 1,$$

along the radial lines $\{ru \mid 0 \leq r \leq 1\}$ for each $u \in S^{m-1}$. Now choose a local coordinate system $(v^1, \ldots, v^{m-1})$ for S^{m-1} so that we get polar coordinates $(r, v^1, \ldots, v^{m-1})$ on B^m. By construction the equation (1.5) holds when evaluated on the tangent vector $\partial/\partial r$. We need to evaluate on $\partial/\partial v^i$, $i = 1, \ldots, m-1$, as well, that is, we must prove

$$\frac{\partial g}{\partial v^i} = gA\left(\frac{\partial}{\partial v^i}\right), \qquad i = 1, \ldots, m-1. \tag{1.8}$$

Notice that by construction (1.5) and hence also (1.8) holds at $u = 0$. Let us calculate $\partial/\partial r$ of the difference using the assumption

$$\begin{aligned}
0 &= F_A\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial v^i}\right) \\
&= \frac{\partial}{\partial r} A\left(\frac{\partial}{\partial v^i}\right) - \frac{\partial}{\partial v^i} A\left(\frac{\partial}{\partial r}\right) \\
&\quad + A\left(\frac{\partial}{\partial r}\right) A\left(\frac{\partial}{\partial v^i}\right) - A\left(\frac{\partial}{\partial v^i}\right) A\left(\frac{\partial}{\partial r}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{\partial}{\partial r}\left(\frac{\partial g}{\partial v^i} - gA\left(\frac{\partial}{\partial v^i}\right)\right) \\
&\quad = \frac{\partial^2 g}{\partial r \partial v^i} - \frac{\partial g}{\partial r} \cdot A\left(\frac{\partial}{\partial v^i}\right) - g\frac{\partial}{\partial r} A\left(\frac{\partial}{\partial v^i}\right) \\
&\quad = \frac{\partial g}{\partial v^i} A\left(\frac{\partial}{\partial r}\right) + g\frac{\partial}{\partial v^i} A\left(\frac{\partial}{\partial r}\right) \\
&\qquad - gA\left(\frac{\partial}{\partial r}\right) A\left(\frac{\partial}{\partial v^i}\right) - g\frac{\partial}{\partial r} A\left(\frac{\partial}{\partial v^i}\right) \\
&\quad = \frac{\partial g}{\partial v^i} A\left(\frac{\partial}{\partial r}\right) - gA\left(\frac{\partial}{\partial v^i}\right) A\left(\frac{\partial}{\partial r}\right) \\
&\quad = \left(\frac{\partial g}{\partial v^i} - gA\left(\frac{\partial}{\partial v^i}\right)\right) A\left(\frac{\partial}{\partial r}\right).
\end{aligned}$$

By uniqueness of the solution to the equation

$$\frac{\partial x}{\partial r} = xA\left(\frac{\partial}{\partial r}\right)$$

along a radial we conclude that

$$\frac{\partial g}{\partial v^i} - gA\left(\frac{\partial}{\partial v^i}\right) \equiv 0$$

which was to be proven. □

In the global case there are obvious difficulties even for M of dimension 1.

Example 1.4. Let $M = S^1 = \{(\cos 2\pi t, \sin 2\pi t) \mid t \in [0,1]\}$ and suppose $A = A_0 dt$ for a constant matrix A_0. Then, on the interval $[0,1)$, the unique gauge transformation to the trivial equation with $g(0) = 1$ is given by

$$g(t) = \exp(tA_0) = \sum_{n=0}^{\infty} \frac{t^n A_0^n}{n!}$$

so that $g(1) = \exp(A_0)$. Hence we have a globally defined gauge transformation to the trivial equation if and only if $g_1 = \exp(A_0) = 1$. If $g_1 \neq 1$, on the other hand, we can overcome this difficulty if we replace $S^1 \times \mathbb{R}^n$ with the vector bundle obtained from $[0,1] \times \mathbb{R}^n$ by identifying $(1, v)$ with $(0, g_1 v)$.

This example suggests that we should (and will) generalize the problem to vector bundles (real or complex). However, as seen above, it is really the Lie group $G = \mathrm{GL}(n, \mathbb{R})$ (or $G = \mathrm{GL}(n, \mathbb{C})$) which enters in the question of gauge transformations. This gives rise to the notion of a *principal G-bundle* with a *connection* where locally the connection is given by the matrix A of 1-forms occuring in the equation (1.4). Again we will encounter the notion of *curvature* as in (1.7) which is the starting point for the definition of *characteristic forms* and *characteristic classes.*

2. Vector Bundles and Frame Bundles

In this chapter we shall introduce the notion of a vector bundle and the associated frame bundle. Unless otherwise specified all vector spaces are real, but we could of course use complex vector spaces instead.

Definition 2.1. An n-dimensional differentiable (real) vector bundle is a triple (V, π, M) where $\pi: V \longrightarrow M$ is a differentiable mapping of smooth manifolds with the following extra structure:

(1) For every $p \in M, V_p = \pi^{-1}(p)$ has the structure of a real vector space of dimension n, satisfying the following condition:

(2) Every point in M has an open neighborhood U with a diffeomorphism

$$f: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

such that the diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[rr, "f"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\mathrm{proj}"] \\
& U &
\end{tikzcd}$$

commutes, ie., $f(V_p) \subseteq p \times \mathbb{R}^n$ for all $p \in U$; and $f_p = f|_{V_p}: V_p \longrightarrow p \times \mathbb{R}^n$ is an isomorphism of vector spaces for each $p \in U$.

Notation. (V, π, M) is called a vector bundle over M, V is called the *total space*, M is the *base space*, and π is the projection. We shall often write V instead of (V, π, M). The diffeomorphism f in definition 2.1 is called a *local trivialization* of V over U. If f exists over all of M then we call V a *trivial bundle.*

Remark. If (V, π, M) is an n-dimensional vector bundle, then the total space V is a manifold of dimension $n + m$, where $m = \dim M$.

Example 2.2. The product bundle $M \times \mathbb{R}^n$ with $\pi = \mathrm{proj}\colon M \times \mathbb{R}^n \longrightarrow M$ is obviously a trivial vector bundle.

Exercise 2.3. For $g \in \mathrm{GL}(n, \mathbb{R})$ show that the quotient space of $\mathbb{R} \times \mathbb{R}^n$ by the identification $(t, x) \sim (t + 1, gx)$ where $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, defines an n-dimensional vector bundle over $\mathbb{R}/\mathbb{Z} \cong S^1$.

Example 2.4. The tangent bundle of a differentiable manifold $M = M^m$, that is, the disjoint union of tangent spaces

$$TM = \bigsqcup_{p \in M} T_pM$$

is in a natural way the total space in an m-dimensional vector bundle with the projection $\pi\colon TM \longrightarrow M$ given by $\pi(v) = p$ for $v \in T_pM$. If $(U, x) = (U, x^1, \dots, x^m)$ is a local coordinate system for M then

$$\pi^{-1}(U) = \bigsqcup_{p \in U} T_pM$$

and we have a local trivialization

$$x_*\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$$

defined by

$$x_*\left(\sum_{i=1}^{m} v^i \cdot \frac{\partial}{\partial x^i}\bigg|_p\right) = (p, v^1, \dots, v^m).$$

Sections

For a vector bundle (V, π, M) it is useful to study its sections.

Definition 2.5. A (differentiable) *section* σ in (V, π, M) is a differentiable mapping

$$\sigma\colon M \longrightarrow V$$

such that $\pi \circ \sigma = \mathrm{id}_M$, that is, $\sigma(p) \in V_p = \pi^{-1}(p)$ for all $p \in M$.

Notation. The set of differentiable sections in V is often denoted $\Gamma(V)$.

Example 2.6. Every vector bundle V has a zero section, i.e., the section $\sigma(p) = 0 \in V_p$, for all $p \in M$. The zero section is differentiable and in fact $\Gamma(V)$ is in a natural way a vector space with the zero section as the zero vector.

Example 2.7. A (differentiable) section in the tangent bundle of a manifold M is the same thing as a (differentiable) vector field on M.

Example 2.8. For the product bundle $V = M \times \mathbb{R}^n$ (Example 2.2) a section σ in V has the form

$$\sigma(p) = (p, s^1(p), \ldots, s^n(p)) = (p, s(p))$$

and σ is differentiable if and only if $s\colon M \longrightarrow \mathbb{R}^n$ is differentiable. Thus we have a 1-1 correspondence between differentiable sections in V and differentiable functions $s\colon M \longrightarrow \mathbb{R}^n$.

The Frame Bundle

By the last example the notion of a section in a vector bundle V generalizes the notion of a function on it. But we can actually do even better: We can consider a section of V as a function defined on a different manifold, the so called frame bundle $F(V)$ for V.

First consider a single n-dimensional real vector space V and define

$$F(V) = \mathrm{Iso}(\mathbb{R}^n, V) = \{\text{linear isomorphisms } x\colon \mathbb{R}^n \longrightarrow V\}.$$

An element $x \in F(V)$ is determined by the n linearly independent vectors

$$x_1 = x(e_1), \ldots, x_n = x(e_n)$$

where $\{e_1, \ldots, e_n\}$ is the standard basis in $\mathbb{R}^n$. The element $x \in F(V)$ is called an n-frame in V. Notice that a choice of basis in V gives an identification of $F(V)$ with

$$\mathrm{Iso}(\mathbb{R}^n, \mathbb{R}^n) = \mathrm{GL}(n, \mathbb{R})$$

which is an open set in the set $M(n,\mathbb{R}) = \mathbb{R}^{n^2}$ of all $n \times n$ matrices.

Now let us return to (V, π, M) a differentiable vector bundle over the manifold M. We shall make the disjoint union

$$F(V) = \bigsqcup_{p\in M} F(V_p) = \bigsqcup_{p\in M} \mathrm{Iso}(\mathbb{R}^n, V_p)$$

into a differentiable manifold such that $\bar{\pi}: F(V) \longrightarrow M$ given by $\bar{\pi}(x) = p$ for $x \in F(V_p)$ is differentiable. Thus suppose we have a local trivialization of V

$$f: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n.$$

Then there is a natural bijection

$$\bar{f}: \bar{\pi}^{-1}(U) \longrightarrow U \times \mathrm{GL}(n,\mathbb{R}) \tag{2.1}$$

defined by

$$x \longmapsto (p, f_p \circ x), \qquad \text{for } x \in F(V_p).$$

where $f_p: V_p \longrightarrow \mathbb{R}^n$ is the restriction of f to V_p.

Proposition 2.9. *There is a natural topology and differentiable structure on $F(V)$ satisfying:*

(1) *$F(V)$ is a differentiable manifold of dimension $m + n^2$.*

(2) *The bijections $\bar{f}$ defined by (2.1) are diffeomorphisms for all local trivializations f.*

(3) *The mapping $\bar{\pi}$ is differentiable and locally we have a commutative diagram*

$$\begin{tikzcd} \bar{\pi}^{-1}(U) \arrow[rr, "\bar{f}"] \arrow[dr, "\bar{\pi}"'] & & U \times \mathrm{GL}(n,\mathbb{R}) \arrow[dl, "\mathrm{proj}"] \\ & U & \end{tikzcd}$$

(4) *We have a differentiable right group action*

$$F(V) \times \mathrm{GL}(n,\mathbb{R}) \longrightarrow F(V) \tag{2.2}$$

given by

$$x \cdot g = x \circ g, \qquad \textit{for } x \in F(V_p),\ g \in \mathrm{GL}(n, \mathbb{R}).$$

Exercise 2.10. Prove Proposition 2.9.

Remark. (1) The proof of the first statement is similar to the construction of the differentiable structure on the tangent bundle of a manifold.

(2) That the mapping in (2.2) is a differentiable right group action means that it is given by a differentiable mapping and that it satisfies

$$(x \cdot g) \cdot g' = x \cdot (gg') \qquad \text{for all } x \in F(V),\ g, g' \in \mathrm{GL}(n, \mathbb{R}),$$

and

$$x \cdot 1 = x \qquad \text{for all } x \in F(V).$$

Notice also that each $F(V_p)$ is an orbit , that is $F(V_p) = x \cdot \mathrm{GL}(n, \mathbb{R})$ for any $x \in F(V_p)$, and we can identify M with the orbit space $F(V)/\,\mathrm{GL}(n, \mathbb{R})$.

Notation. The triple $(F(V), \bar{\pi}, M)$ is called the bundle of n-frames of V or for short, the frame bundle of V.

Now we can interpret the set of sections of V in the following way:

Proposition 2.11. *There is a natural 1-1 correspondence between the vector space $\Gamma(V)$ and the space of equivariant functions on $F(V)$ with values in $\mathbb{R}^n$, ie., the set of differentiable functions $\tilde{s}\colon F(V) \longrightarrow \mathbb{R}^n$ satisfying*

$$\tilde{s}(x \cdot g) = g^{-1}\tilde{s}(x), \qquad \textit{for all } x \in F(V),\ g \in \mathrm{GL}(n, \mathbb{R}). \tag{2.3}$$

Proof. Let $s \in \Gamma(V)$ and define $\tilde{s}$ by

$$\tilde{s}(x) = x^{-1}(s(p)), \qquad \text{for } x \in F(V_p) = \mathrm{Iso}(\mathbb{R}^n, V_p).$$

Then it is straightforward to check that $\tilde{s}$ satisfies (2.3). Also, using the local triviality in (2.1), it follows that $\tilde{s}$ is differentiable if and only if s is. On the other hand given $\tilde{s}\colon F(V) \longrightarrow \mathbb{R}^n$ satisfying (2.3) it is easy to see that $\bar{s}\colon F(V) \longrightarrow V$ given by

$$\bar{s}(x) = x(\tilde{s}(x))$$

is constant on every orbit $F(V_p)$, and so $\bar{s}$ defines a function $s\colon M \longrightarrow V$ such that the diagram

$$\begin{tikzcd} F(V) \arrow[r, "\bar{s}"] \arrow[d, "\bar{\pi}"'] & V \\ M \arrow[ur, "s"'] & \end{tikzcd}$$

commutes. Again s is seen to be differentiable (provided $\tilde{s}$ is) using the local trivialization in (2.1). □

We next study homomorphisms between vector bundles with the same base M, that is, vector bundles (V, π, M) and (V', π', M).

Definition 2.12. A *homomorphism* $\varphi: V \longrightarrow V'$ is a differentiable mapping of total spaces such that the following holds:

(1) The diagram

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & V' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes, that is, $\varphi_p = \varphi|_{V_p}$ maps V_p to V'_p.

(2) $\varphi_p: V_p \longrightarrow V'_p$ is a linear mapping for each $p \in M$.

An *isomorphism* $\varphi: V \longrightarrow V'$ is a bijective map where both φ and φ^{-1} are homomorphisms.

Example 2.13. A trivialization $f: V \longrightarrow M \times \mathbb{R}^n$ is an isomorphism to the product bundle.

Remark. It follows from the definition that a homomorphism $\varphi: V \longrightarrow V'$ is an isomorphism if and only if φ is a diffeomorphism of total spaces such that $\varphi_p: V_p \longrightarrow V'_p$ is an isomorphism of vector spaces for every $p \in M$. We can improve this:

Proposition 2.14. *A homomorphism $\varphi: V \longrightarrow V'$ is an isomorphism if and only if $\varphi_p: V_p \longrightarrow V'_p$ is an isomorphism of vector spaces for every $p \in M$.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) We must show that φ is a diffeomorphism. Since φ is clearly bijective it suffices to show that φ^{-1} is differentiable. This however is a local problem, so we can assume $V = M \times \mathbb{R}^n$, $V' = M \times \mathbb{R}^n$. In that case $\varphi: M \times \mathbb{R}^n \longrightarrow M \times \mathbb{R}^n$ has the form

$$\varphi(p, v) = (p, \varphi_p(v))$$

where $\varphi_p: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear isomorphism. It is easy to see using local coordinates for M that the Jacobi matrix for φ at every point (p, v) has the form

$$\begin{bmatrix} I & 0 \\ X & \varphi_p \end{bmatrix} \tag{2.4}$$

where φ_p is the matrix for $\varphi_p: \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Since φ_p is an isomorphism the matrix (2.4) is clearly invertible and hence the proposition follows from the Inverse Function Theorem. $\square$

Corollary 2.15. *A vector bundle (V, π, M) is trivial if and only if the associated frame bundle $(F(V), \bar{\pi}, M)$ has a section; i.e., if there is a differentiable mapping $\sigma: M \longrightarrow F(V)$ such that $\bar{\pi} \circ \sigma = \mathrm{id}_M$.*

Proof. ($\Rightarrow$) Let $f: V \longrightarrow M \times \mathbb{R}^n$ be a trivialization. Then we define

$$\sigma: M \longrightarrow F(V) \qquad \text{by} \qquad \sigma(p) = f_p^{-1} \in \mathrm{Iso}(\mathbb{R}^n, V_p).$$

By the definition of the differentiable structure in $F(V)$ σ is differentiable since

$$f \circ \sigma(p) = (p, 1) \in M \times \mathrm{GL}(n, \mathbb{R}^n).$$

($\Leftarrow$) Let $\sigma: M \longrightarrow F(V)$ be a differentiable section. Then we define a homomorphism $\varphi: M \times \mathbb{R}^n \longrightarrow V$ by

$$\varphi(p, v) = \sigma(p)(v), \qquad \text{for } (p, v) \in M \times \mathbb{R}^n.$$

(here $\sigma(p) \in \mathrm{Iso}(\mathbb{R}^n, V_p)$). It follows from Proposition 2.14 that φ is an isomorphism, hence $f = \varphi^{-1}: V \longrightarrow M \times \mathbb{R}^n$ is a trivialization. $\square$

Remark. A section σ in $F(V)$ is equivalent to a set of n sections $\{\sigma_1, \ldots, \sigma_n\}$ in V such that $\{\sigma_1(p), \ldots, \sigma_n(p)\} \subseteq V_p$ is a basis for V_p for every $p \in M$ (cf. the definition of $F(V_p)$). A section in $F(V)$ is also called a *moving frame* for V. Since every vector bundle is locally trivial it always has a local moving frame.

Riemannian Metrics

For the remainder of this chapter we shall study vector bundles with a Riemannian metric: First recall that on a single vector space V an inner product $\langle \cdot\,, \cdot \rangle$ is a symmetric, positive definite, bilinear form on V, that is, a function $\langle \cdot\,, \cdot \rangle \colon V \times V \longrightarrow \mathbb{R}$ such that

(1) $\langle v, w \rangle$, $v, w \in V$, is linear in both v and w,

(2) $\langle v, w \rangle = \langle w, v \rangle$, for all $v, w \in V$,

(3) $\langle v, v \rangle \geq 0$, for all $v \in V$, and

(4) $\langle v, v \rangle = 0$ if and only if $v = 0$.

Now return to V a vector bundle over M.

Definition 2.16. A *Riemannian metric* on a vector bundle V over M is a collection of inner products $\langle \cdot\,, \cdot \rangle_p$ on V_p, $p \in M$, which is differentiable in the following sense: For $s_1, s_2 \in \Gamma(V)$ the function $\langle s_1, s_2 \rangle$ given by $\langle s_1, s_2 \rangle(p) = \langle s_1(p), s_2(p) \rangle_p$ is differentiable.

Notation. We shall often just write $\langle v_1, v_2 \rangle = \langle v_1, v_2 \rangle_p$ for $v_1, v_2 \in V_p$.

Example 2.17. The product bundle $V = M \times \mathbb{R}^n$ has the standard inner product given by the usual inner product in $\mathbb{R}^n$:

$$\langle v, w \rangle = \sum_{i=1}^{n} v^i w^i, \qquad \text{for } v = (v^1, \ldots, v^n), w = (w^1, \ldots, w^n).$$

Proposition 2.18. *Every vector bundle has a Riemannian metric.*

Exercise 2.19. Prove Proposition 2.18. Hint: Use a partition of unity.

Now suppose (V, π, M) is an n-dimensional vector bundle with Riemannian metric $\langle \cdot\,, \cdot \rangle$.

Proposition 2.20. *Every point in M has a neighborhood U and a trivialization $f\colon \pi^{-1}(V) \longrightarrow U \times \mathbb{R}^n$ such that $f_p\colon V_p \longrightarrow \mathbb{R}^n$ is a linear isometry for every $p \in M$ (with the metric in $U \times \mathbb{R}^n$ given by Example 2.17).*

Proof. By the remark following Corollary 2.15 every point in M has a neighborhood U with a local *moving frame*, i.e. a set of sections $\{s_1, \dots, s_n\}$ of $(\pi^{-1}(U), \pi, U)$ such that $\{s_1(p), \dots, s_n(p)\}$ is a linearly independent set for each $p \in U$. By means of the Gram-Schmidt process we can replace this set by $\{\sigma_1, \dots, \sigma_n\}$ such that $\{\sigma_1(p), \dots, \sigma_n(p)\}$ is an orthonormal basis for V_p for every $p \in U$. Again $\sigma_1, \dots, \sigma_n$ are all differentiable. As in the proof of Corollary 2.15 we consider $\{\sigma_1, \dots, \sigma_n\}$ as a section of the frame bundle $F(V)$ over U, i.e. we obtain an isomorphism $\varphi\colon U \times \mathbb{R}^n \longrightarrow \pi^{-1}(U)$ given by

$$\varphi(p, v) = \sigma(p)(v), \qquad \text{for } (p, v) \in U \times \mathbb{R}^n.$$

Since $\varphi_p(e_i) = \sigma(p)(e_i) = \sigma_i(p)$ and since $\{\sigma_1(p), \dots, \sigma_n(p)\}$ is an orthonormal basis for V_p, it follows that $\varphi_p\colon \mathbb{R}^n \longrightarrow V_p$ is a linear isometry for each $p \in U$. Hence $f = \varphi^{-1}\colon \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$ has the desired properties. □

We can now define the orthogonal frame bundle for a vector bundle with a Riemannian metric. For a single vector space V with inner product $\langle \cdot\,, \cdot \rangle$ we let $F_{\mathrm{O}}(V) \subseteq F(V)$ be the set

$$F_{\mathrm{O}}(V) = \mathrm{Isom}(\mathbb{R}^n, V) = \{\text{linear isometries } x\colon \mathbb{R}^n \longrightarrow V\},$$

that is, $x \in F_{\mathrm{O}}(V)$ if and only if the vectors

$$x_1 = x(e_1), \dots, x_n = x(e_n)$$

constitute an orthonormal basis for $(V, \langle \cdot\,, \cdot \rangle)$. We will call x an *orthogonal n-frame* in V. With respect to a given orthonormal basis for V we get an identification of $F_{\mathrm{O}}(V)$ with the orthogonal group $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$, which is an $n(n-1)/2$-dimensional submanifold in $\mathrm{GL}(n, \mathbb{R})$.

Now return to (V, π, M) a vector bundle with a Riemannian metric $\langle \cdot\,, \cdot \rangle$ and we define the *orthogonal frame bundle* as the subset

$$F_{\mathrm{O}}(V) = \bigsqcup_{p \in M} F_{\mathrm{O}}(V_p) \subseteq F(V).$$

Proposition 2.21. (1) *$F_{\mathrm{O}}(V) \subseteq F(V)$ is a submanifold and*

$$\bar{\pi}|_{F_{\mathrm{O}}(V)}: F_{\mathrm{O}}(V) \longrightarrow M$$

is differentiable.

(2) *There is a differentiable right* $\mathrm{O}(n)$*-action*

$$F_{\mathrm{O}}(V) \times \mathrm{O}(n) \longrightarrow F_{\mathrm{O}}(V)$$

such that the orbits are the sets $F_{\mathrm{O}}(V_p)$, $p \in M$.

(3) *There are local diffeomorphisms*

$$\bar{f}: \bar{\pi}^{-1}(U) \cap F_{\mathrm{O}}(V) \longrightarrow U \times \mathrm{O}(n)$$

such that $\bar{f}_p = f|_{F_{\mathrm{O}}(V_p)}$ maps $F_{\mathrm{O}}(V_p)$ to $p \times \mathrm{O}(n)$ for every $p \in U$, and also

$$\bar{f}(x \cdot g) = \bar{f}(x) \cdot g, \qquad \text{for all } x \in F_{\mathrm{O}}(V_p),\ g \in \mathrm{O}(n).$$

Proof. Choose a local trivialization

$$f: \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^n$$

as in Proposition 2.20. Then the corresponding local diffeomorphism for the frame bundle $(F(V), \bar{\pi}, M)$

$$\bar{f}: \bar{\pi}^{-1}(U) \longrightarrow U \times \mathrm{GL}(n, \mathbb{R})$$

is given on $F_p(V)$ by

$$\bar{f}(x) = (p, f_p \circ x), \qquad x \in F_p(V)$$

where f_p is the restriction of f to V_p. Since $f_p: V_p \longrightarrow \mathbb{R}^n$ is an isometry it follows that $\bar{f}$ maps $F_{\mathrm{O}}(V_p)$ to $p \times \mathrm{O}(n)$, that is,

$$\bar{f}: \bar{\pi}^{-1}(U) \cap F_{\mathrm{O}}(V) \longrightarrow U \times \mathrm{O}(n)$$

is a bijection. Since $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$ is a submanifold it follows that $F_{\mathrm{O}}(V) \subseteq F(V)$ is also a submanifold and all the statements in the proposition are now straightforward. We leave the details to the reader. $\square$

Example 2.22. The real projective n-space $\mathbb{R}P^n$ is defined as the quotient space $(\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$, that is, $x = (x_1, \ldots, x_n)$ is equivalent to $y = (y_1, \ldots, y_n)$ if and only if $y = tx$ for some $t \in \mathbb{R}\setminus\{0\}$. Let $\eta\colon (\mathbb{R}^{n+1} \setminus \{0\}) \longrightarrow \mathbb{R}P^n$ be the natural projection, that is,

$$\eta(x) = [x] = [x_1 \colon \ldots \colon x_{n+1}]$$

and these $(n+1)$-tuples are called the *homogenous coordinates*. $\mathbb{R}P^n$ is an n-dimensional differentiable manifold with coordinate systems (U_i, ζ_i), $i = 1, \ldots, n+1$, given by

$$U_i = \{[x] \in \mathbb{R}P^n \mid x_i \neq 0\}$$

and $\zeta_i \colon U_i \longrightarrow \mathbb{R}^n$ defined by

$$\zeta_i[x_1, \ldots, x_{n+1}] = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right).$$

Notice that the inclusion of the unit n-sphere $i\colon S^n \subseteq (\mathbb{R}^{n+1} \setminus \{0\})$ gives rise to a commutative diagram

$$\begin{tikzcd}
S^n \arrow[r, "i"] \arrow[d] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[d, "\eta"] \\
S^n/\{\pm 1\} \arrow[r, "\bar{\imath}", "\cong"'] & \mathbb{R}P^n
\end{tikzcd}$$

Here $\bar{\imath}$ is a homeomorphism; hence $\mathbb{R}P^n$ is compact.

We shall now construct a 1-dimensional vector bundle with $\mathbb{R}P^n$ as basis. This is called the *real Hopf-bundle* or the *canonical line bundle*. The total space $H \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$ is the subset

$$H = \{([x], v) \mid v \in \operatorname{span}\{x\}\}$$

and the projection $\pi\colon H \longrightarrow \mathbb{R}P^n$ is the restriction of the projection onto the first component. There are local trivializations

$$h_i \colon \pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}, \qquad i = 1, \ldots, n+1,$$

given by

$$h_i([x], v) = ([x], v_i).$$

Theorem 2.23. (1) *$H \subseteq \mathbb{R}P^n \times \mathbb{R}^{n+1}$ is an embedded submanifold and $\pi\colon H \longrightarrow \mathbb{R}P^n$ is a 1-dimensional vector bundle with local trivializations h_i as above.*

(2) *The associated frame bundle is $\eta\colon (\mathbb{R}^{n+1} \setminus \{0\}) \longrightarrow \mathbb{R}P^n$. Here the action by $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R} \setminus \{0\}$ is just given by the usual scalar multiplication.*

Proof. (1) As before $\langle \cdot\,, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^{n+1}$, that is, $\langle x, y\rangle = \sum_{i=1}^{n+1} x_i y_i$ for $x, y \in \mathbb{R}^{n+1}$. For $x \in \mathbb{R}^{n+1} \setminus \{0\}$ let P_x denote the orthogonal projection onto $\operatorname{span}\{x\}$, that is,

$$P_x(y) = \frac{\langle y, x\rangle}{\langle x, x\rangle} x$$

and denote the projection onto the orthogonal complement by $P_x^\perp = \mathrm{id} - P_x$. Similarily, for $i = 1, \ldots, n+1$, let

$$P_i\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}, \qquad P_i^\perp\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$$

be the projections

$$P_i(x) = x_i, \qquad P_i^\perp(x) = (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$$

where the hat indicates that the term is left out. For $i = 1, \ldots, n+1$, define $k_i\colon U_i \times \mathbb{R}^{n+1} \longrightarrow U_i \times \mathbb{R} \times \mathbb{R}^n$ by

$$k_i([x], v) = ([x], P_i \circ P_x(v), P_i^\perp \circ P_x^\perp(v)).$$

It is easy to see that k_i is a homomorphism between the two product bundles and that it is injective (and hence bijective) on each fibre. By Proposition 2.14 k_i is therefore an isomorphism, hence in particular a diffeomorphism. Since

$$k_i(H \cap \pi^{-1}(U_i)) = U_i \times \mathbb{R} \times 0$$

it follows that H is embedded in $\mathbb{R}P^n \times \mathbb{R}^{n+1}$, and since $k_i|_{H\cap\pi^{-1}(U_i)} = h_i \times 0$ we have shown (1).

(2) By definition the frame bundle for H is given by

$$F(H) = H_0 = \{([x], v) \mid v \in \operatorname{span}\{x\}, v \neq 0\}$$

and $\bar{\pi} = \pi|_{H_0}$. Now projecting on the second component in $H_0 \subseteq \mathbb{R}P^n \times (\mathbb{R}^{n+1} \setminus \{0\})$ gives a diffeomorphism l in the commutative diagram

$$\begin{tikzcd} H_0 \arrow[rr, "l", "\cong"'] \arrow[dr, "\bar{\pi}"'] & & \mathbb{R}^{n+1} \setminus \{0\} \arrow[dl, "\eta"] \\ & \mathbb{R}P^n & \end{tikzcd}$$

In fact the inverse l^{-1} is given by

$$l^{-1}(v) = ([v], v), \qquad \text{for } v \in \mathbb{R}^{n-1} \setminus \{0\}.$$

Also l clearly respects the action of $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R} \setminus \{0\}$ since in both cases it is given by the scalar multiplication. $\square$

Corollary 2.24. *H is a non-trivial vector bundle for $n > 0$.*

Proof. If H is trivial then by Corollary 2.15 the frame bundle $F(H) \longrightarrow \mathbb{R}P^n$ has a section. That is we have a differentiable map

$$\sigma: \mathbb{R}P^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\},$$

such that $\eta \circ \sigma = \mathrm{id}$. For $x \in S^n \subseteq \mathbb{R}^{n+1}$ define $f(x) \in \mathbb{R}$ by

$$\sigma([x]) = f(x)x, \qquad x \in S^n \subseteq \mathbb{R}^{n+1} \setminus \{0\}.$$

Then $f: S^n \longrightarrow \mathbb{R} \setminus \{0\}$ is differentiable. But $f(-x)(-x) = \sigma([-x]) = \sigma([x]) = f(x)$, which implies that $f(-x) = -f(x)$. Hence f takes both values in $\mathbb{R}_+$ and $\mathbb{R}_-$. Since f is continuous and S^n is connected this is a contradiction. $\square$

3. Principal G-bundles

The frame bundle for a vector bundle is the special case of a principal G-bundle for the Lie group $G = \mathrm{GL}(n, \mathbb{R})$. In the following G denotes an arbitrary Lie group.

Definition 3.1. A *principal G-bundle* is a triple (E, π, M) in which $\pi \colon E \longrightarrow M$ is a differentiable mapping of differentiable manifolds. Furthermore E is given a differentiable right G-action $E \times G \longrightarrow E$ such that the following holds.

(1) $E_p = \pi^{-1}(p)$, $p \in M$ are the orbits for the G-action.

(2) (Local trivialization) Every point in M has a neighborhood U and a diffeomorphism $\varphi \colon \pi^{-1}(U) \longrightarrow U \times G$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\mathrm{proj}"] \\ & U & \end{tikzcd}$$

commutes, i.e. $\varphi_p = \varphi|_{E_p}$ maps E_p to $p \times G$; and φ is *equivariant*, i.e.,

$$\varphi(xg) = \varphi(x)g \qquad \forall x \in \pi^{-1}(U), g \in G$$

where G acts on $U \times G$ by $(p, g')g = (p, g'g)$

Notation. E is called the *total space*, M the *base space* and $E_p = \pi^{-1}(p)$ the *fibre* at $p \in M$. Often we shall just denote the G-bundle (E, π, M) by E.

Remark. (1) π is surjective and *open*.

(2) The orbit space E/G is homeomorphic to M.

(3) The G-action is *free*, i.e.,

$$xg = x \quad \text{implies} \quad g = 1 \qquad \text{for all } x \in E,\ g \in G.$$

(4) For each $x \in E$ the mapping $G \longrightarrow E_p$ given by $g \longmapsto x \cdot g$, is a diffeomorphism.

(5) If $N \subseteq M$ is a submanifold (e.g. if N is an open subset) then the *restriction* to N

$$E|_N = (\pi^{-1}(N), \pi, N)$$

is again a principal G-bundle with base space N.

Example 3.2. (1) For (V, π, M) an n-dimensional vector bundle the associated frame bundle $(F(V), \bar{\pi}, M)$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle.

(2) If V is equipped with a Riemannian metric then $(F_O(V), \bar{\pi}, M)$ is a principal $\mathrm{O}(n)$-bundle.

(3) Let G be any Lie group and M a manifold. Then $(M \times G, \pi, M)$, with π the projection onto the first factor, is a principal G-bundle called the *product bundle.*

Definition 3.3. Let (E, π, M), (F, π', M) be two principal G-bundles over the same base space M. An *isomorphism* $\varphi\colon E \longrightarrow F$ is a diffeomorphism of the total spaces such that

(1) The diagram

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes, i.e. $\varphi_p = \varphi|_{E_p}$ maps E_p to F_p.

(2) φ is *equivariant*, i.e.

$$\varphi(xg) = \varphi(x)g \qquad \text{for all } x \in E,\ g \in G.$$

Remark. In this case $\varphi_p\colon E_p \longrightarrow F_p$ is also a diffeomorphism for each $p \in M$.

Definition 3.4. A principal G-bundle (E, π, M) is called *trivial* if there is an isomorphism $\varphi: E \longrightarrow M \times G$ and φ is called a *trivialization.*

Remark. It follows from definition 3.1 that every principal G-bundle E has *local trivializations*

$$\varphi: E|_U \xrightarrow{\cong} U \times G.$$

Lemma 3.5. *Every isomorphism $\varphi: M \times G \longrightarrow M \times G$ has the form*

$$\varphi(p, a) = (p, g(p) \cdot a) \qquad p \in M, a \in G \tag{3.1}$$

where $g: M \longrightarrow G$ is a differentiable mapping.

Proof. It is easy to see that φ defined by (3.1) is an isomorphism with inverse φ^{-1} given by

$$\varphi^{-1}(p, b) = (p, g(p)^{-1} \cdot b) \qquad p \in M, b \in G. \tag{3.2}$$

conversely let $\varphi: M \times G \longrightarrow M \times G$ be an arbitrary isomorphism and let $g: M \longrightarrow G$ be the mapping defined by

$$\varphi(p, 1) = (p, g(p)) \qquad p \in M.$$

then g is clearly differentiable and since φ is equivariant we obtain for $p \in M$, $a \in G$:

$$\varphi(p, a) = \varphi((p, 1)a) = (\varphi(p, 1))a = (p, g(p) \cdot a)$$

that is, (3.1) holds. □

Now for an arbitrary G-bundle (E, π, M) choose an open covering of M, $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Sigma}$, and trivializations

$$\varphi_\alpha: E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G.$$

For $U_\alpha \cap U_\beta \neq \emptyset$ we consider the isomorphism

$$\varphi_\beta \circ \varphi_\alpha^{-1}: U_\alpha \cap U_\beta \times G \longrightarrow U_\alpha \cap U_\beta \times G$$

and by Lemma 3.5 this has the form

$$\varphi_\beta \circ \varphi_\alpha^{-1}(p, a) = (p, g_{\beta\alpha}(p) \cdot a) \tag{3.3}$$

where $a \in G, p \in U_\alpha \cap U_\beta$ and $g_{\beta\alpha}: U_\alpha \cap U_\beta \longrightarrow G$ is a differentiable mapping.

Notation. The collection $\{g_{\beta\alpha}\}_{\alpha,\beta\in\Sigma}$ are called the *transition functions* for E with respect to the covering $\mathcal{U}$ (and trivilizations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$).

Remark. For $\alpha, \beta, \gamma \in \Sigma$ such that $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ the following *cocycle condition* holds

$$\begin{aligned} &g_{\gamma\beta}(p) \cdot g_{\beta\alpha}(p) = g_{\gamma\alpha}(p) \qquad \text{for all } p \in U_\alpha \cap U_\beta \cap U_\gamma, \\ &g_{\alpha\alpha}(p) = 1 \qquad \text{for all } p \in U_\alpha. \end{aligned} \tag{3.4}$$

Conversely we have the following proposition.

Proposition 3.6. *Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of a manifold M and suppose $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ is a system of differentiable mappings $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$ satisfying the cocycle condition. Then there is a principal G-bundle (E, π, M) and trivializations $\varphi_\alpha\colon E_{|U_\alpha} \longrightarrow U_\alpha \times G$, $\alpha \in \Sigma$, such that $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ is the associated system of transition functions.*

Proof. The total space E is the quotient space

$$E = \Big(\bigsqcup_{\alpha\in\Sigma} U_\alpha \times G\Big) \Big/ \!\sim$$

of the disjoint union of all $U_\alpha \times G$, $\alpha \in \Sigma$, for the equivalence relation $\sim$ defined by

$$(p, a)_\alpha \sim (q, b)_\beta \quad \text{if and only if} \quad p = q \text{ and } b = g_{\beta\alpha}(p)a$$

where $(p, a)_\alpha \in U_\alpha \times G$ and $(q, b)_\beta \in U_\beta \times G$. The cocycle condition ensures that $\sim$ is an equivalence relation. Furthermore the projections $U_\alpha \times G \longrightarrow U_\alpha$ give a well-defined continous mapping $\pi\colon E \longrightarrow M$ and we also have obvious bijections

$$\varphi_\alpha\colon \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times G$$

given by $\varphi_\alpha((p, a)_\alpha) = (p, a)$. It is now straight forward to define a differentiable structure on E such that the maps φ_α become diffeomorphisms. Furthermore one checks that (E, π, M) is a principal G-bundle and by construction $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ are trivializations with $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ the associated system of transition functions. $\square$

Exercise 3.7. Show that the bundle constructed in Proposition 3.6 is trivial if and only if there is a system of differentiable mappings

$$h_\alpha \colon U_\alpha \longrightarrow G, \qquad \alpha \in \Sigma,$$

such that

$$g_{\beta\alpha}(p) = h_\beta(p) h_\alpha(p)^{-1}, \qquad \text{for all } p \in U_\alpha \cap U_\beta.$$

In the previous chapter we associated to a vector bundle (V, π, M) the frame bundle $(F(V), \bar{\pi}, M)$ which is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. We shall now show how to reconstruct the vector bundle V from the principal bundle using the natural action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$. In general for a Lie group G and a principal G-bundle (E, π, M) we consider a manifold N with a *left G-action* $G \times N \longrightarrow N$ and we shall associate to this a *fibre bundle* (E_N, π_N, M) with *fibre* N. For this we define the *total space* E_N as the *orbit space*

$$E_N = E \times_G N = (E \times N)/G$$

for the G-action on $E \times N$ given by

$$(x, u) \cdot g = (xg, g^{-1}u), \qquad x \in E,\ u \in N,\ g \in G$$

so that E_N is the quotient space for the equivalence relation $\sim$, where $(x, u) \sim (y, v)$ if and only if there exists $g \in G$ such that $y = xg$ and $u = gv$. Furthermore let $\pi_N \colon E_N \longrightarrow M$ be induced by the composite mapping

$$E \times N \xrightarrow{\text{proj}} E \xrightarrow{\pi} M.$$

Then we have the following proposition.

Proposition 3.8. (1) *E_N is in a natural way a differentiable manifold and $\pi_N \colon E_N \longrightarrow M$ is differentiable.*

(2) *There are* local trivializations, *i.e. every point in M has a neighborhood U and a diffeomorphism $f \colon \pi_N^{-1}(U) \longrightarrow U \times N$ such that the diagram*

$$\begin{tikzcd} \pi_N^{-1}(U) \arrow[rr, "f"] \arrow[dr, "\pi_N"'] & & U \times N \arrow[dl, "\text{proj}"] \\ & U & \end{tikzcd}$$

commutes.

Exercise 3.9. Prove Proposition 3.8.

Notation. The triple (E_N, π_N, M) is called the *fibre bundle with fibre N associated* to the principal G-bundle (E, π, M).

Example 3.10. Let (V, π, M) be a vector bundle and $(F(V), \bar{\pi}, M)$ the corresponding frame bundle. Then the associated fibre bundle with fibre $\mathbb{R}^n$ $(F(V)_{\mathbb{R}^n}, \bar{\pi}_{\mathbb{R}^n}, M)$ is in a natural way a vector bundle isomorphic to (V, π, M). In fact there is a natural isomorphism φ

$$\begin{tikzcd} F(V) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n \arrow[rr, "\varphi"] \arrow[dr, "\bar{\pi}_{\mathbb{R}^n}"'] & & V \arrow[dl, "\pi"] \\ & M & \end{tikzcd}$$

given by

$$\varphi(x, v) = x(v), \qquad x \in F_p(V) = \mathrm{Iso}(\mathbb{R}^n, V_p).$$

Exercise 3.11. Let $(E, \tilde{\pi}, M)$ be a principal $\mathrm{GL}(n, \mathbb{R})$ bundle and let (V, π, M) be the associated fibre bundle with fibre $\mathbb{R}^n$ using the natural action of $\mathrm{GL}(n, \mathbb{R}^n)$ on $\mathbb{R}^n$. Show that V is in a natural way a vector bundle and that the corresponding frame bundle $(F(V), \bar{\pi}, M)$ is isomorphic to $(E, \tilde{\pi}, M)$.

Finally let us consider bundles over different base spaces: Suppose (E', π', M') and (E, π, M) are principal G-bundles.

Definition 3.12. A *bundle map* from E' to E is a pair of differentiable mappings $(\bar{f}, f)$ in the commutative diagram

$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

such that $\bar{f}$ is G-equivariant, i.e.

$$\bar{f}(x \cdot g) = \bar{f}(x) \cdot g, \qquad \text{for all } x \in E',\ g \in G.$$

Example 3.13. (1) A bundle isomorphism is by definition a bundle map of the form

$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$

(2) If $N \subseteq M$ is a submanifold and (E, π, M) is a principal G-bundle, then the inclusion maps in the diagram

$$\begin{tikzcd} E|_N \arrow[r, hook] \arrow[d, "\pi|_{E|N}"'] & E \arrow[d, "\pi"] \\ N \arrow[r, hook] & M \end{tikzcd}$$

is a bundle map.

(3) In the product bundle $M \times G$ the projection π_2 on the second factor defines a bundle map of the form

$$\begin{tikzcd} M \times G \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & G \arrow[d] \\ M \arrow[r] & \mathrm{pt} \end{tikzcd}$$

Given a differentiable mapping $f\colon M' \longrightarrow M$ and a principal G-bundle (E, π, M) we can construct a G-bundle called the *pull-back* of E by f over M', denoted $f^*(E) = (f^*(E), \pi', M')$, and a bundle map $(\bar{f}, f)\colon f^*(E) \longrightarrow E$. That is, we construct a commutative diagram

$$\begin{tikzcd} f^*(E) \arrow[r, "\bar{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

To do this we let $f^*(E) \subseteq M' \times E$ be the subset

$$f^*(E) = \{(p, x) \in M' \times E \mid f(p) = \pi(x)\}$$

and let π' and $\bar{f}$ be defined by the restriction of the projections to M' and E respectively. Then we have

Proposition 3.14. *$f^*(E) = (f^*(E), \pi', M')$ is in a natural way a principal G-bundle and $(\bar{f}, f)$ is a bundle map.*

Proof. G clearly acts on $f^*(E)$, and for each $p \in M'$

$$(\pi')^{-1}(p) = \{(p, x) \mid x \in E_{f(p)}\}$$

is mapped bijectively by $\bar{f}$ to $E_{f(p)}$ which is a G-orbit. Hence it suffices to show that $f^*(E)$ is a locally trivial G-bundle. for this we can assume E to be a product bundle $E = M \times G$. In this case

$$f^*(E) = \{(p, q, g) \in M' \times M \times G \mid f(p) = q\} \cong M' \times G$$

by the map $(p, q, g) \longmapsto (p, g)$. Via this isomorphism $\bar{f}$ is furthermore given by the map $(p, g) \longmapsto (f(p), g)$ which shows that $\bar{f}$ is a bundle map. $\square$

Exercise 3.15. (1) Show that if $(\tilde{f}, f)\colon (E', \pi', M') \longrightarrow (E, \pi, M)$ is a bundle map then there is a canonical factorization $\tilde{f} = \bar{f} \circ \varphi$, where $\varphi\colon E' \longrightarrow f^*(E)$ is an isomorphism and $(\bar{f}, f)$ is the bundle map in Proposition 3.14.

(2) Show that this provides a 1-1 correspondence between the set of bundle maps with fixed map $f\colon M' \longrightarrow M$ of base spaces, and the set of isomorphisms $E' \longrightarrow f^*(E)$.

(3) In particular there is a 1-1 correspondence between the set of trivializations of a bundle (E, π, M) and the set of bundle maps to the trivial G-bundle over a point (G, π, pt).

(4) Show that if $f\colon M' \longrightarrow M$ is a differentiable map and if $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ is a cocycle of transition functions for a G-bundle E over M with covering $\{U_\alpha\}_{\alpha\in\Sigma}$ then $\{g_{\alpha\beta}\circ f\}$ is a cocycle of transition functions for $f^*(E)$ over M' with covering $\{f^{-1}(U_\alpha)\}_{\alpha\in\Sigma}$.

4. Extension and reduction of principal bundles

We will now examine the relation between principal bundles with different structure groups. In the following let G and H be two Lie groups and $\alpha\colon H \longrightarrow G$ a Lie group homomorphism. Typically α is the inclusion of a closed Lie subgroup. Now suppose (F, π, M) is a principal H-bundle and (E, ξ, M) is a principal G-bundle.

Definition 4.1. Let $\varphi\colon F \longrightarrow E$ be a differentiable mapping of the total spaces such that the following holds.

(1) The diagram

$$\begin{tikzcd} F \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E \arrow[dl, "\xi"] \\ & M & \end{tikzcd}$$

commutes, ie. $\varphi_p = \varphi|_{F_p}$ maps F_p into E_p for all $p \in M$.

(2) The map φ is α-equivariant, ie.,

$$\varphi(x \cdot h) = \varphi(x) \cdot \alpha(h) \qquad \text{for all } x \in F,\ h \in H.$$

Then $\varphi\colon F \longrightarrow E$ is called an *extension* of F to G relative to α and is also called a *reduction* of E to H relative to α

Notation. (1) We will often omit the term "relative to α" if α is clear from the context, e.g. when α is the inclusion of a Lie subgroup.

(2) When α is surjective with non-trivial kernel one usually calls a reduction a *lifting* of the bundle E to H.

(3) Often the extension is just denoted by the target E and similarly a reduction is denoted by the domain F. But it should be kept in mind that φ is part of the structure. This is important when talking about equivalences of extensions resp. reductions (liftings).

Definition 4.2. (1) Two extensions $\varphi_1\colon F \longrightarrow E_1$ and $\varphi_2\colon F \longrightarrow E_2$ are equivalent if there is an isomorphism ψ in the commutative diagram

$$\begin{tikzcd} & E_1 \arrow[dd, "\psi"] \\ F \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\ & E_2 \end{tikzcd}$$

(2) Two reductions (liftings) $\varphi_1\colon F_1 \longrightarrow E$ and $\varphi_2\colon F_2 \longrightarrow E$ are equivalent if there is an isomorphism ψ in the commutative diagram

$$\begin{tikzcd} F_1 \arrow[dd, "\psi"'] \arrow[dr, "\varphi_1"] & \\ & E \\ F_2 \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

Example 4.3. (1) Let (V, π, M) be a vector bundle with a Riemannian metric. Then the inclusion $F_O(V) \subset F(V)$ of the orthogonal frame bundle into the frame bundle is an extension relative to the inclusion $\mathrm{O}(n) \subseteq \mathrm{GL}(n, \mathbb{R})$. Thus the Riemanian metric defines a reduction of the principal $\mathrm{GL}(n, \mathbb{R})$-bundle $F(V)$ to $\mathrm{O}(n)$. Furthermore there is a 1-1 correspondence between the set of Riemannian metrics on V and the set of equivalence classes of reductions.

(2) Let $\mathrm{GL}(n, \mathbb{R})^+ \subseteq \mathrm{GL}(n, \mathbb{R})$ be the subgroup of non-singular matrices with positive determinant. By definition a vector bundle (V, π, M) is called *orientable* if the frame bundle $F(V)$ has a reduction to $\mathrm{GL}(n, \mathbb{R})^+$ and a choice of equivalence class of reductions is called an *orientation* of V (if orientable).

Proposition 4.4. *Let $\alpha\colon H \longrightarrow G$ be a Lie group homomorphism and let (F, π, M) be a principal H-bundle. Then there is an extension of F to G relative to α and any two extensions are equivalent.*

Proof. There is a left H-action on G defined by $h \cdot g = \alpha(h)g$ for $h \in H$ and $g \in G$. Consider the associated fibre bundle with fibre G, i.e. the bundle (F_G, π_G, M) where $F_G = F \times_G G$ and $\pi_G(x, g) = \pi(x)$ for $x \in F, g \in G$. Here F_G has a natural right G-action given by

$$(x, g)g' = (x, gg'), \qquad x \in F,\ g, g' \in G.$$

It now follows from Proposition 3.8 that (F_G, π_G, M) is a principal G-bundle. Furthermore the natural mapping $\varphi: F \longrightarrow F \times_G G$ defined by $\varphi(x) = (x, 1)$ makes F_G an extension of F to G. If $\varphi': F \longrightarrow E'$ is any other extension then there is a natural isomorphism $\psi: F_G \longrightarrow E'$ given by

$$\psi(x, g) = \varphi'(x) \cdot g, \qquad x \in F,\ g \in G$$

and clearly the diagram

$$\begin{tikzcd} & F_G \arrow[dd, "\psi"] \\ F \arrow[ur, "\varphi"] \arrow[dr, "\varphi'"'] & \\ & E' \end{tikzcd}$$

commutes. Hence $\varphi': F \longrightarrow E'$ is equivalent to $\varphi: F \longrightarrow F_G$. □

Hence extensions exist and are unique up to equivalence. Reductions (or liftings) do however not always exist, and if they do, they are usually not unique.

Exercise 4.5. Let $\alpha: H \longrightarrow G$ be a Lie group homomorphism and (E, π, M) a principal G-bundle.

(1) Show that E has a reduction to H if and only if there is a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Sigma}$ of M and a set of transition functions for E of the form $\{\alpha \circ h_{\alpha\beta}\}_{\alpha,\beta \in \Sigma}$, where $h_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow H$ are smooth functions satisfying the cocycle condition (3.4)

(2) If $H \subseteq G$ is a closed embedded Lie subgroup and α is the inclusion, show that E has a reduction to H if and only if there is a covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Sigma}$ of M and a set of transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta \in \Sigma}$ for E with $\{g_{\alpha\beta}\}$ mapping into H.

Let us now restrict to the case where $H \subseteq G$ is a closed embedded Lie subgroup and α is the inclusion.

We need the following lemma.

Lemma 4.6. *The natural projection $\pi: G \longrightarrow G/H$ defines a principal H-bundle $(G, \pi, G/H)$.*

Proof. Let $U \subseteq G$ be a local cross section, that is, U is an embedded submanifold of G containing the identity element e, such that $\pi(U) = W$ is open in G/H and $\pi: U \longrightarrow W$ is a diffeomorphism. Let $s: W \longrightarrow U$ be the inverse. We now get a local trivialization $f: \pi^{-1}(W) \longrightarrow W \times H$ defined by

$$f(a) = (\pi(a), s(\pi(a))^{-1}a).$$

This is clearly smooth and so is the inverse

$$f^{-1}(w, h) = s(w) \cdot h.$$

Also f and f^{-1} are H-equivariant, hence f is a local trivialization . Similarly over the neighborhood gW we have the trivialization

$$f_g: \pi^{-1}(gW) \longrightarrow gW \times H$$

given by

$$f_g(a) = (\pi(a), s(\pi(g^{-1}a))^{-1}g^{-1}a),$$

with inverse

$$f_g^{-1}(u, h) = g \cdot s(g^{-1}u) \cdot h.$$

This shows that $(G, \pi, G/H)$ is a principal H-bundle. ☐

More generally we can now prove:

Theorem 4.7. *Let $H \subseteq G$ be a closed embedded Lie-subgroup and let (E, π, M) be a principal G-bundle.*

(1) *There is a natural homeomorphism k in the commutative diagram*

$$\begin{tikzcd} E/H \arrow[rr, "k", "\cong"'] \arrow[dr, "\bar\pi"'] & & E \times_G (G/H) \arrow[dl, "\pi_{G/H}"] \\ & M & \end{tikzcd}$$

where $\bar\pi$ is induced by π. In particular E/H has a natural differentiable structure induced by k.

(2) *Let $\tilde\pi\colon E \longrightarrow E/H$ be the natural projection. Then $(E, \tilde\pi, E/H)$ is a principal H-bundle.*

(3) *There is a 1-1 correspondence between the set of sections of the bundle $(E/H, \bar\pi, M)$ with fibre G/H and the set of equivalence classes of reductions of E to H.*

Proof. (1) The map k is just induced by the natural inclusion $E \longrightarrow E \times (G/H)$ sending x to $(x, [H])$ and the inverse is induced by the map $E \times G/H \longrightarrow E/H$ given by $k^{-1}(x, gH) = xgH$. Since $E \times_G (G/H) = E_{G/H}$ is the total space in the associated fibre bundle with fibre G/H, it has a differentiable structure as noted in Proposition 3.8.

(2) Since the differentiable structure on E/H is given via the homeomorphism k we have local trivializations of $(E/H, \bar\pi, M)$, that is, over suitable neighborhoods $U \subseteq M$ we have a commutative diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong"] \arrow[d, "\tilde\pi"'] & U \times G \arrow[d] \\ \bar\pi^{-1}(U) \arrow[r, "\cong"] \arrow[d] & U \times G/H \arrow[d, "\mathrm{proj}"] \\ U \arrow[r, "="] & U \end{tikzcd}$$

with the horizontal maps being diffeomorphisms.

By Lemma 4.6 $G \longrightarrow G/H$ is a locally trivial H-bundle; hence by the upper part of the diagram above $(E, \tilde\pi, E/H)$ is also locally trivial.

(3) Suppose we have a reduction $\varphi\colon F \longrightarrow E$, where (F, π_0, M) is a principal H-bundle. Then φ induces a natural map

$$s_\varphi\colon M = F/H \longrightarrow E/H$$

which is easily checked to be a smooth section of $(E/H, \bar{\pi}, M)$ using local trivializations. Also if $\varphi_1\colon F_1 \longrightarrow E$ and $\varphi_2\colon F_2 \longrightarrow E$ are equivalent reductions then clearly $s_{\varphi_1} = s_{\varphi_2}$.

On the other hand if $s\colon M \longrightarrow E/H$ is a section then we get a bundle map of H-bundles

$$\begin{tikzcd} s^*(E) \arrow[r, "\bar{s}"] \arrow[d] & E \arrow[d] \\ M \arrow[r, "s"'] & E/H \end{tikzcd}$$

and it follows that $\bar{s}\colon s^*(E) \longrightarrow E$ is a reduction of E to H. $\square$

Remark. (1) In particular E has a reduction to H if and only if $(E/H, \bar{\pi}, M)$ has a section.

(2) For $H = \{e\}$ Theorem 4.7 gives a 1-1 correspondence between trivializations of E and sections of E.

Exercise 4.8. (1) Let $G = H \cdot K$ be a semi-direct product of the two closed embedded Lie subgroups H and K, that is, H is invariant and the natural map $K \hookrightarrow G \longrightarrow G/H$ is an isomorphism of Lie groups. show that if (E, π, M) is a principal G-bundle then $(E/H, \bar{\pi}, M)$ is a principal K-bundle.

(2) For $k \leq n$ let $W_{n,k}$ be the manifold of k linearly independent vectors in $\mathbb{R}^n$, let

$$G_k(\mathbb{R}^n) = W_{n,k}/\operatorname{GL}(k, \mathbb{R})$$

be the *Grassmann manifold* of k-planes in $\mathbb{R}^n$, and let $\gamma_{n,k}\colon W_{n,k} \longrightarrow G_k(\mathbb{R}^n)$ be the natural projection. Show that $(W_{n,k}, \gamma_{n,k}, G_k(\mathbb{R}^n))$ is a principal $\operatorname{GL}(k, \mathbb{R})$-bundle.

(3) Similarly let $V_{n,k} \subseteq W_{n,k}$ be the *Stiefel manifold* of k orthogonal vectors in $\mathbb{R}^n$ with the usual inner product. Show that the inclusion $V_{n,k} \subseteq W_{n,k}$ defines a reduction of the bundle defined in (2) to the group $\mathrm{O}(k) \subseteq \mathrm{GL}(k, \mathbb{R})$

(4) Show that the natural map

$$l \colon W_k(\mathbb{R}^n) \times_{\mathrm{GL}(k,\mathbb{R})} \mathbb{R}^k \longrightarrow G_k(\mathbb{R}^n) \times \mathbb{R}^n$$

defined by

$$l(X, v) = ([X], Xv)$$

is an embedding. (Here $[X] = \gamma_{n,k}(X)$ is the subspace spanned by the column vectors in the matrix X, and Xv denotes usual matrix multiplication. Notice that l identifies the total space of the associated bundle with fibre $\mathbb{R}^k$ with the submanifold of $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ consisting of pairs $([X], w)$ where $w \in [X]$.)

5. Differential Forms with Values in a Vector Space

In the following M denotes a differentiable manifold and V a finite dimensional vector space. We shall consider differential forms with values in V, generalizing the usual real valued differential forms.

Definition 5.1. A differential form ω on M with values in V associates to k differentiable vector fields $X_1, \dots, X_k$ on M a differentiable function

$$\omega(X_1, \dots, X_k) \colon M \longrightarrow V$$

such that

(1) ω is multilinear and alternating.

(2) ω has the tensor property, ie.,

$$\omega(X_1, \dots, fX_i, \dots, X_k) = f\omega(X_1, \dots, X_k)$$

for all vector fields $X_1, \dots, X_k$ on M, $f \in C^\infty(M)$ and $i = 1, \dots, k$.

Remark. Alternatively ω is defined as a family ω_x, $x \in M$ of k-linear alternating maps

$$\omega_x \colon T_x(M) \times \cdots \times T_x(M) \longrightarrow V$$

such that for all k-tuples of differentiable vector fields $X_1, \dots, X_k$ the mapping

$$x \longmapsto \omega_x(X_1(x), \dots, X_k(x))$$

is differentiable.

Remark. If we choose a basis $\{e_1, \dots, e_n\}$ for V then we can write ω uniquely in the form

$$\omega = \omega_1 e_1 + \cdots + \omega_n e_n$$

where $\omega_1, \dots, \omega_n$ are usual differential forms on M. Hence relative to a choice of basis $\{e_1, \dots, e_n\}$, there is a 1-1 correspondence between differential forms with values in V and n-tuples of usual differential forms $\{\omega_1, \dots, \omega_n\}$. Note that we tacitly did so already in the introduction in the case of $V = \mathbb{R}^n$ or $V = M(n, \mathbb{R})$.

Notation. The set of differential k-forms on M with values in V is denoted by $\Omega^k(M, V)$. For $V = \mathbb{R}$ we have $\Omega^k(M) = \Omega^k(M, \mathbb{R})$.

Similar to the usual case we have an exterior differential

$$d\colon \Omega^k(M, V) \longrightarrow \Omega^{k+1}(M, V).$$

Relative to a choice of basis $\{e_1, \dots, e_n\}$ for V it is just defined for $\omega = \omega_1 e_1 + \cdots + \omega_n e_n$ by

$$d\omega = (d\omega_1)e_1 + \cdots + (d\omega_n)e_n$$

and it is easy to see that this equation is independent of basis. Furthermore we have the usual identities

$$\begin{aligned} &d(d\omega) = 0, \qquad \text{for all } \omega \in \Omega^k(M, V),\ k \in \mathbb{N}, \\ &(d\omega)(X_1, \dots, X_{k+1}) \\ &\quad = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, \dots, \widehat{X_i}, \dots, X_{k+1}) \\ &\qquad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}) \end{aligned} \tag{5.1}$$

for all differentiable vector fields $X_1, \dots, X_{k+1}$. These formulas follow easily from the corresponding ones for usual differential forms.

In order to generalize the wedge product of two differential forms we need the notion of the tensor product $V \otimes W$ of two finite dimensional vector spaces V and W. First let $\mathrm{Hom}^2(V \times W, \mathbb{R})$ denote the vector space of bilinear maps $V \times W \longrightarrow \mathbb{R}$ and then define $V \otimes W$ as the dual vector space

$$V \otimes W = \operatorname{Hom}(\operatorname{Hom}^2(V \times W, \mathbb{R}), \mathbb{R}).$$

For $v \in V$ and $w \in W$ we define $v \otimes w \in V \otimes W$ by

$$\langle \varphi, v \otimes w \rangle = \varphi(v, w), \qquad \varphi \in \operatorname{Hom}^2(V \times W, \mathbb{R}).$$

We now have the following proposition.

Proposition 5.2.

(1) *The mapping* $\otimes \colon V \times W \longrightarrow V \otimes W$ *given by* $(v, w) \longmapsto v \otimes w$ *is bilinear.*

(2) *There is a bijection for any vector space* U

$$\operatorname{Hom}(V \otimes W, U) \xrightarrow{\cong} \operatorname{Hom}^2(V \times W, U)$$

given by $\varphi \longmapsto \varphi \circ \otimes$.

(3) $V \otimes W$ *is generated by the set of vectors of the form* $v \otimes w$*, where* $v \in V$ *and* $w \in W$.

(4) *If* $\{e_1, \dots, e_n\}$ *and* $\{f_1, \dots, f_m\}$ *are bases for* V *resp.* W *then* $\{e_i \otimes f_j\}$ *is a basis for* $V \otimes W$*. In particular*

$$\dim(V \otimes W) = \dim(V) \cdot \dim(W).$$

Exercise 5.3. Prove Proposition 5.2.

For $\omega_1 \in \Omega^k(M, V)$ and $\omega_2 \in \Omega^l(M, W)$ we can now define the wedge product $\omega_1 \wedge \omega_2 \in \Omega^{l+k}(M, V \otimes W)$ by the usual formula

$$\begin{aligned}
&(\omega_1 \wedge \omega_2)(X_1, \dots, X_{k+l}) \\
&\quad = \sum_{\sigma} \operatorname{sign}(\sigma) \omega_1(X_{\sigma(1)}, \dots, X_{\sigma(k)}) \otimes \omega_2(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})
\end{aligned}$$

where σ runs over all (k, l)-shuffles of $1, \dots, k + l$. As usual one has the formulas

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3,$$

for all $\omega_1 \in \Omega^k(M,U)$, $\omega_2 \in \Omega^l(M,V)$, $\omega_3 \in \Omega^m(M,W)$,

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (d\omega_2)$$

for all $\omega_1 \in \Omega^k(M,U)$, $\omega_2 \in \Omega^l(M,W)$. Furthermore for a linear mapping $P\colon V \longrightarrow W$ there is an induced mapping $P\colon \Omega^k(M,V) \longrightarrow \Omega^k(M,W)$ defined by

$$(P\omega)(X_1,\ldots,X_k) = P \circ \omega(X_1,\ldots,X_k)$$

and it is easy to see that

$$d(P\omega) = P(d\omega), \qquad \omega \in \Omega^k(M,V).$$

In particular let $T\colon V \otimes W \longrightarrow W \otimes V$ be the linear mapping given by $T(v \otimes w) = w \otimes v$. Then one has

$$\omega_2 \wedge \omega_1 = (-1)^{kl} T(\omega_1 \wedge \omega_2) \tag{5.2}$$

for all $\omega_1 \in \Omega^k(M,V)$, $\omega_2 \in \Omega^l(M,W)$. Finally for $f\colon M \longrightarrow N$ a differentiable mapping of differentiable manifolds we get as usual an induced mapping $f^*\colon \Omega^k(N,V) \longrightarrow \Omega^k(M,V)$, where for $\omega \in \Omega^k(N,V)$, $f^*(\omega)$ is defined pointwise by

$$f^*(\omega)_x(X_1,\ldots,X_k) = \omega_{f(x)}(f_*(X_1),\ldots,f_*(X_k))$$

for $X_1,\ldots,X_k \in T_x(M)$. Then one also has the formulas

$$\begin{aligned}
&f^*(\omega_1 \wedge \omega_2) = (f^*\omega_1) \wedge (f^*\omega_2),\\
&d(f^*(\omega)) = f^*(d\omega),\\
&f^*(P(\omega)) = P(f^*(\omega))
\end{aligned}$$

for $P\colon V \longrightarrow W$ a linear mapping and for all $\omega \in \Omega^k(N,V)$, $\omega_1 \in \Omega^k(N,V)$ and $\omega_2 \in \Omega^l(N,W)$.

Exercise 5.4. Prove all unproven statements in the above.

6. Connections in Principal G-bundles

We now come to our main topic which, as we shall se later, generalizes the differential systems considered in the introduction. This is the notion of a *connection*.

In general we consider a Lie group G with Lie algebra $\mathfrak{g} = T_eG$, and we let $\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g})$ be the adjoint representation, i.e., for $g \in G$ $\mathrm{Ad}(g)$ is the differential at the identity element e of the mapping $x \longmapsto gxg^{-1}$, $x \in G$.

Let (E, π, M) be a principal G-bundle. For a fixed $x \in E$ the mapping $G \longrightarrow E$ given by $g \longmapsto xg$, $g \in G$, induces an injective map $v_x\colon \mathfrak{g} \longrightarrow T_xE$ and the quotient space by the image of v_x is mapped isomorphically onto $T_{\pi(x)}M$ by the differential π_* of π. That is, we have an exact sequence of vectorspaces

$$0 \longrightarrow \mathfrak{g} \xrightarrow{v_x} T_xE \xrightarrow{\pi_*} T_{\pi(x)}M \longrightarrow 0$$

The vectors in $v_x(\mathfrak{g}) \subseteq T_xE$ are called *vertical tangent vectors* of E and we want to choose a complementary subspace $H_x \subseteq T_xE$ of *horizontal vectors*, i.e., H_x is mapped isomorphically onto $T_{\pi(x)}M$ by π_*. This choice is equivalent to a choice of linear mapping $\omega_x\colon T_xE \longrightarrow \mathfrak{g}$, such that

$$\omega_x \circ v_x = \mathrm{id}_{\mathfrak{g}} \tag{6.1}$$

and such that $H_x = \ker \omega_x$. Furthermore we shall require ω_x to vary differentially, i.e., $\{\omega_x \mid x \in E\}$ defines a differential 1-form with values in $\mathfrak{g}$, hence, $\omega \in \Omega^1(E, \mathfrak{g})$.

Example 6.1. Consider the trivial bundle $E = M \times G$, $\pi\colon M \times G \longrightarrow M$ the natural projection. We define $\omega_{\mathrm{MC}} \in \Omega^1(E, \mathfrak{g})$ as follows:

$$(\omega_{\mathrm{MC}})_{(p,g)} = (L_{g^{-1}} \circ \pi_2)_*, \qquad p \in M,\ g \in G \tag{6.2}$$

where $\pi_2\colon M \times G \longrightarrow G$ is the projection onto the second factor and $L_{g^{-1}}\colon G \longrightarrow G$ is left translation by g^{-1}. Let us show that $\omega_{\mathrm{MC}} \in \Omega^1(E, \mathfrak{g})$, that is, we shall show that ω_{MC} is differentiable. First notice that $\omega_{\mathrm{MC}} = \pi_2^*(\omega_0)$ where $\omega_0 \in \Omega^1(G, \mathfrak{g})$ is defined by

$$(\omega_0)_g = (L_{g^{-1}})_*. \tag{6.3}$$

Hence it suffices to show that ω_0 is differentiable. For this notice that if $X \in \mathfrak{g} = T_e G$ and $\widetilde{X}$ is the corresponding left invariant vector field then

$$\omega_0(\widetilde{X}) = X$$

is constant and hence differentiable. Since every differentiable vector field Y on G is a linear combination of left invariant ones with differentiable coefficients it follows that $\omega_0(Y)$ is differentiable, hence ω_0 is differentiable. Furthermore ω_{MC} satisfies (6.1): In fact for $x = (p, g)$, v_x is the differential of the map $G \longrightarrow M \times G$ given by $a \longmapsto (p, ga)$, an hence $(\omega_{\mathrm{MC}})_x \circ v_x = \mathrm{id}_* = \mathrm{id}$.

Remark. The form ω_0 on G is called the *Maurer-Cartan form.* For $G = \mathrm{GL}(n, \mathbb{R})$ it is just the form

$$\omega_0 = g^{-1} dg, \qquad g \in G.$$

The form ω_{MC} in example 6.1 satisfies another identity: For $g \in G$ let $R_g\colon E \longrightarrow E$ denote the right multiplication by g, that is $R_g(x) = xg$, $x \in E$. In the case $E = M \times G$ we just have

$$R_g(p, a) = (p, ag) \qquad p \in M,\ a \in G.$$

Lemma 6.2. *In $E = M \times G$ the form ω_{MC} defined by (6.2) satifies*

$$R_g^* \omega_{\mathrm{MC}} = \mathrm{Ad}(g^{-1}) \circ \omega_{\mathrm{MC}}, \qquad \text{for all } g \in G, \tag{6.4}$$

where $\mathrm{Ad}(g^{-1})\circ\colon \Omega^1(E, \mathfrak{g}) \longrightarrow \Omega^1(E, \mathfrak{g})$ *is induced by the linear map*

$$\mathrm{Ad}(g^{-1})\colon \mathfrak{g} \longrightarrow \mathfrak{g}.$$

Proof. Since $\omega_{\mathrm{MC}} = \pi_2^* \omega_0$, for ω_0 given by (6.3), and since

$$R_g^* \omega_{\mathrm{MC}} = R_g^* \circ \pi_2^* \omega_0 = \pi_2^* \circ R_g^* \omega_0$$

it suffices to prove (6.4) for $\omega = \omega_0$ on G. But here we have for $a \in G$:

$$\begin{aligned} R_g^*(\omega)_a &= \omega_{ag} \circ (R_g)_* \\ &= (L_{(ag)^{-1}})_* \circ (R_g)_* \\ &= (L_{g^{-1}})_* \circ (L_{a^{-1}})_* \circ (R_g)_* \\ &= \mathrm{Ad}(g^{-1}) \circ (L_{a^{-1}})_* \\ &= \mathrm{Ad}(g^{-1}) \circ \omega_a, \end{aligned}$$

completing the proof. □

With this as motivation we now make the following definition.

Definition 6.3. A *connection* in a principal G-bundle (E, π, M) is a 1-form $\omega \in \Omega^1(E, \mathfrak{g})$ satisfying

(1) $\omega_x \circ v_x = \mathrm{id}$ where $v_x\colon \mathfrak{g} \longrightarrow T_x E$ is the differential of the mapping $g \longmapsto xg$.

(2) $R_g^* \omega = \mathrm{Ad}(g^{-1}) \circ \omega$, for all $g \in G$, where $R_g\colon E \longrightarrow E$ is given by $R_g(x) = xg$.

There is a more geometric formulation of (2): For a 1-form $\omega \in \Omega^1(E, \mathfrak{g})$ satisfying (1) in Definition 6.3 let $H_x \subseteq T_x E$, $x \in E$, be the subspace

$$H_x = \ker \omega_x.$$

Then as noted above $\pi_*\colon H_x \longrightarrow T_{\pi(x)} M$ is an isomorphism. Therefore $H_x \subseteq T_x E$ is called the *horizontal subspace* at x given by ω, and a vector in H_x is called a *horizontal tangent vector* in E.

Proposition 6.4. *For $\omega \in \Omega^1(E, \mathfrak{g})$ satisfying definition 6.3 (1), the requirement (2) is equivalent to*

(2′) $(R_g)_* H_x = H_{xg}$, *for all $x \in E$ and $g \in G$.*

That is, the horizontal vector spaces are permuted by the right action of G on TE.

Proof. (2) $\Rightarrow$ (2′). If $X \in H_x$, then we obtain

$$\omega_{xg}(R_{g*} X) = (R_g^* \omega)(X) = \mathrm{Ad}(g^{-1})(\omega(X)) = 0;$$

hence $R_{g*}X \in H_{xg}$.

$(2') \Rightarrow (2)$. To prove (2) notice that both sides are zero when evaluated on horizontal vectors. Hence it is enough the verify (2) when evaluated on a vertical vector $v_x(X)$, $X \in \mathfrak{g}$. But for $g \in G$ we have

$$R_{g*} \circ v_x = v_{xg} \circ \mathrm{Ad}(g^{-1})$$

since both sides are the differential of the mapping $G \longrightarrow E$ given by

$$a \longmapsto xag = xg(g^{-1}ag).$$

Hence by (1)

$$R_g^*(\omega)(v_x(X)) = \omega_{xg}(v_{xg} \circ \mathrm{Ad}(g^{-1})(X)) = \mathrm{Ad}(g^{-1})(X),$$

completing the proof. □

Remark. By Lemma 6.2 the form $\omega_{\mathrm{MC}} \in \Omega^1(M \times G)$ defined in example 6.1 is a connection in $M \times G$. This is often called the *Maurer-Cartan connection*, the *trivial connection* or the *flat connection*. Notice that in this case the horizontal subspace at $x = (p, g)$ is the tangent space to the submanifold $M \times \{g\} \subseteq M \times G$.

Proposition 6.5. (1) *Let $(\bar{f}, f)\colon (E', \pi', M') \longrightarrow (E, \pi, M)$ be a bundle map of principal G-bundles and let $\omega \in \Omega^1(E, \mathfrak{g})$ be a connection in E. Then $\bar{f}^*\omega$ is a connection in E'.*

(2) *In particular if $\varphi\colon F \longrightarrow E$ is an isomorphism of principal G-bundles over M and $\omega \in \Omega^1(E, \mathfrak{g})$ is a connection in E then $\omega^\varphi = \varphi^*\omega$ is a connection on F.*

(3) *Suppose $\omega_1, \dots, \omega_k \in \Omega^1(E, \mathfrak{g})$ are connections on the bundle (E, π, M) and $\lambda_1, \dots, \lambda_k \in C^\infty(M)$ satisfy $\sum_i \lambda_i = 1$ then the sum*

$$\omega = \sum_i \lambda_i \omega_i$$

is also a connection on E.

Exercise 6.6. Prove Proposition 6.5.

Corollary 6.7. *Every principal G-bundle has a connection.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Sigma}$ be a covering of M with trivializations

$$\varphi_\alpha : E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G.$$

By Proposition 6.5 the Maurer-Cartan connection in $U_\alpha \times G$ pulls back to a connection ω_α in $E|_{U_\alpha}$. Now choose a partition of unity $\{\lambda_\alpha\}_{\alpha\in\Sigma}$ and put

$$\omega = \sum_\alpha \lambda_\alpha \omega_\alpha .$$

Then ω is a well-defined 1-form on E, and, by Proposition 6.5 (3), it satisfies the requirements for a connection (since these are local conditions). $\square$

Next let us look at a *local description* of a connection, i.e., let us look at a general connection ω in a product bundle:

Proposition 6.8. *Let $E = M \times G$ be the product bundle with projection $\pi: E \longrightarrow M$, and let $i: M \longrightarrow E$ be the inclusion $i(p) = (p, e)$.*

(1) *The induced map $i^*: \Omega^1(E, \mathfrak{g}) \longrightarrow \Omega^1(M, \mathfrak{g})$ gives a 1-1 correspondence between connections in E and $\mathfrak{g}$-valued 1-forms on M.*

(2) *Let $\varphi: E \longrightarrow E$ be an isomorphism of the form $\varphi(p, a) = (p, g(p)a)$ for $g: M \longrightarrow G$ a differentiable map, and let $\omega \in \Omega^1(E, \mathfrak{g})$ be a connection in E with $i^*\omega = A \in \Omega^1(M, \mathfrak{g})$. Then $\omega^\varphi = \varphi^*\omega$ satisfies*

$$i^*\omega^\varphi = A^\varphi = \mathrm{Ad}(g^{-1}) \circ A + g^*(\omega_0) \tag{6.5}$$

where $\omega_0 \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form on G.

Proof. (1) First notice that given $A \in \Omega^1(M, \mathfrak{g})$ there is a unique form $\widetilde{A} \in \Omega^1(E, \mathfrak{g})$ with $i^*\widetilde{A} = A$ such that

(i) $\widetilde{A}(X) = 0$ for $X \in T_xE$ a vertical vector.

(ii) $R_g^*\widetilde{A} = \mathrm{Ad}(g^{-1}) \circ \widetilde{A}$ for all $g \in G$.

In fact, for $x = (p, e)$ $\widetilde{A}$ is determined by A and (i) since

$$T_xE = \ker(\pi_*) \oplus i_*(T_pM),$$

and for $y = (p, g) = R_g(x)$, (ii) implies that

$$\widetilde{A}_y = \mathrm{Ad}(g^{-1}) \circ \widetilde{A}_x$$

which proves uniqueness. On the other hand the form $\widetilde{A}$ given by

$$\widetilde{A}_{(p,g)} = \mathrm{Ad}(g^{-1}) \circ A_p \circ \pi_* \tag{6.6}$$

defines a form satisfying (i) and (ii). Now if ω_{MC} denotes the Maurer-Cartan connection om E then the correspondence

$$\widetilde{A} \longleftrightarrow \omega_{\mathrm{MC}} + \widetilde{A}$$

gives a 1-1 correspondence between 1-forms satisfying (i) and (ii), and the set of connections in E.

(2) As above we write

$$\omega = \omega_{\mathrm{MC}} + \widetilde{A}$$

where $A = i^*\omega$. Then

$$\omega^\varphi = \varphi^*\omega_{\mathrm{MC}} + \varphi^*\widetilde{A}$$

gives

$$A^\varphi = i^*\omega^\varphi = i^*\varphi^*\omega_{\mathrm{MC}} + i^*\varphi^*\widetilde{A}$$

where $\omega_{\mathrm{MC}} = \pi_2^*\omega_0$ with $\pi_2 \colon M \times G \longrightarrow G$ the projection. Now $\pi_2 \circ g \circ i = g$ so that

$$i^*\varphi^*\omega_{\mathrm{MC}} = i^*\varphi^*\pi_2^*\omega_0 = g^*\omega_0.$$

Also by (6.6)

$$\begin{aligned}(i^*\varphi^*\widetilde{A})_p &= \widetilde{A}_{(p,g(p))} \circ (\varphi \circ i)_* \\ &= \mathrm{Ad}(g^{-1}) \circ A_p \circ \pi_* \circ (\varphi \circ i)_* \\ &= \mathrm{Ad}(g^{-1}) \circ A_p\end{aligned}$$

since $\pi \circ \varphi \circ i = \mathrm{id}$. □

Remark. Notice that for the Maurer-Cartan connection ω_{MC}, $A = i^*\omega_{\mathrm{MC}} = 0$.

Corollary 6.9. *Let (E, π, M) be a principal G-bundle and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Sigma}$ be a covering af M with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ and transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$. Then there is a 1-1 correspondence between connections in E and collections of 1-forms $\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}_{\alpha\in\Sigma}$ satisfying*

$$A_\beta = \mathrm{Ad}(g_{\alpha\beta}^{-1}) \circ A_\alpha + g_{\alpha\beta}^* \omega_0 \tag{6.7}$$

on $U_\alpha \cap U_\beta$.

Proof. If ω is a connection in E the trivialization $\varphi_\alpha: E|_{U_\alpha} \longrightarrow U_\alpha \times G$ defines a connection in $U_\alpha \times G$ by

$$\omega_\alpha = (\varphi_\alpha^{-1})^* \omega.$$

For $\beta \in \Sigma$ we then have over $U_\alpha \cap U_\beta$ that

$$\omega_\beta = \psi_{\alpha\beta}^* \omega_\alpha \tag{6.8}$$

where $\psi_{\alpha\beta}: (U_\alpha \cap U_\beta) \times G \longrightarrow (U_\alpha \cap U_\beta) \times G$ is the isomorphism $\psi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$, so that

$$\psi_{\alpha\beta}(p, a) = (p, g_{\alpha\beta}(p)a).$$

Hence (6.7) follows from (6.8) and Proposition 6.8, (2).

On the other hand suppose $\{A_\alpha\}_{\alpha\in\Sigma}$ is given. Then there are corresponding connections $\{\omega_\alpha\}$ in $U_\alpha \times G$ as in Proposition 6.8, (1) and by Proposition 6.8, (2), (6.7) implies (6.8) or equivalently

$$\varphi_\alpha^* \omega_\alpha = \varphi_\beta^* \omega_\beta$$

on $E|_{U_\alpha \cap U_\beta}$. Hence we get a well-defined connection ω in E such that the restriction to $E|_{U_\alpha}$ is $\varphi_\alpha^* \omega_\alpha$. ☐

Notation. Often a connection is identified with the collection $\{A_\alpha\}_{\alpha\in\Sigma}$ of *local connection forms*. It is then denoted by A.

In the proof of Proposition 6.8 we encountered two important conditions ((i) and (ii)) on differential 1-forms on the total space E of a principal G-bundle (E, π, M). Let us state these for general k-forms on E with values in a finite dimensional vectorspace V.

Definition 6.10. (1) A differential k-form $\omega \in \Omega^k(E, V)$ is called *horizontal* if $\omega_X(X_1, \dots, X_k) = 0$ for all k-tuples of tangent vectors $X_1, \dots, X_k \in T_xE$ for which at least one is vertical.

(2) Let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be a representation of G on V. Then $\omega \in \Omega^k(E, V)$ is called *ρ-equivariant* if

$$R_g^*\omega = \rho(g^{-1}) \circ \omega, \qquad \text{for all } g \in G.$$

(3) if ρ in (2) is the trivial representation then a ρ-equivariant form is called *invariant.*

(4) If ω is both invariant and horizontal then it is called *basic.*

Proposition 6.11. *Let (E, π, M) be a principal G-bundle and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in\Sigma}$ be a covering of M with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ and transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$. Let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be a representation. Then there is a 1-1 correpondence between horizontal ρ-equivariant k-forms $\tilde{\omega}$ on E and collections of k-forms $\{\omega_\alpha \in \Omega^k(U_\alpha, V)\}_{\alpha\in\Sigma}$ satisfying*

$$\omega_\beta = \rho(g_{\beta\alpha}) \circ \omega_\alpha \qquad \text{on } U_\alpha \cap U_\beta. \tag{6.9}$$

Here ω_α is the pull-back of ω by the local section $U_\alpha \longrightarrow E|_{U_\alpha}$ sending $p \in U_\alpha$ to $\varphi_\alpha^{-1}(p, e)$.

Proof. This is proved exactly as in the proof of Corollary 6.9 using the following lemma. The details are left to the reader. □

Lemma 6.12. *Let $E = M \times G$ be the product bundle with projection $\pi\colon E \longrightarrow M$, and let $i\colon M \longrightarrow E$ be the inclusion $i(p) = (p, e)$. Let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be a representation.*

(1) *The induced map $i^*\colon \Omega^k(E, V) \longrightarrow \Omega^k(M, V)$ gives a 1-1 correspondence between horizontal ρ-equivariant k-forms on E and all V-valued k-forms on M.*

(2) *Let $\varphi\colon E \longrightarrow E$ be a isomorphism of the form $\varphi(p, a) = (p, g(p)a)$ for $g\colon M \longrightarrow G$ a differentiable map. If $\tilde{\omega} \in \Omega^k(E, V)$ is horizontal and equivariant and if we put $\omega = i^*\tilde{\omega}$ then*

$$i^*(\varphi^*\tilde{\omega}) = \rho(g^{-1}) \circ \omega. \tag{6.10}$$

Proof. Again the proof is similar to the proof of Proposition 6.8 and the details are left to the reader. We only note that given $\omega \in \Omega^k(M,V)$ the corresponding ρ-equivariant horizontal k-form $\tilde{\omega}$ on E is given by

$$\tilde{\omega}_{(p,g)}(X_1,\ldots,X_k) = \rho(g^{-1})(\omega_p(\pi_* X_1,\ldots,\pi_* X_k)) \tag{6.11}$$

for $X_1,\ldots,X_k \in T_{(p,g)}(E)$, $p \in M$, $g \in G$. □

Corollary 6.13. *Let (E,π,M) be any principal G-bundle and let V be a vector space. Then $\pi^*\colon \Omega^k(M,V) \longrightarrow \Omega^k(E,V)$ gives an isomorphism onto the basic forms on E.*

Proof. This follows immediately from Proposition 6.11 since the collection $\{\omega_\alpha\}_{\alpha\in\Sigma}$ in that case defines a well-defined form ω on M and since by (6.11) the corresponding horizontal invariant form $\tilde{\omega}$ on E is just the pull-back by π. □

Remark. Let $\rho\colon G \longrightarrow \mathrm{GL}(V)$ be any representation, and for (E,π,M) a principal G-bundle let (E_V,π_V,M) be the *associated vector bundle*, i.e., the associated fibre bundle with fibre V using the left action of G on V given by $gv = \rho(g)v$, $g \in G$, $v \in V$ (cf. Exercise 3.11). Then Proposition 6.11 states in particular for $k = 0$ that there is a 1-1 correspondence between the set of ρ-equivariant functions $\tilde{s}\colon E \longrightarrow V$ and the set of sections s of the vectorbundle E_V. This set is often denoted $\Gamma(M,E_V)$ (cf. Proposition 2.11). We shall denote the set of *ρ-equivariant horizontal k-forms* on E by $\Omega^k(M,E_V)$, so that in particular $\Omega^0(M,E_V) = \Gamma(M,E_V)$. Notice that $\Omega^k(M,E_V)$ is a real vector space.

Corollary 6.14. *Let (E,π,M) be a principal G-bundle. Then the set of connections in E is an affine space for the vector space $\Omega^1(M,E_{\mathfrak{g}})$. That is, given one connection ω_0 any other connection ω_1 has the form $\omega_0 + A$ for some $A \in \Omega^1(M,E_{\mathfrak{g}})$.*

Notation. The *set of connections* in (E,π,M) is denoted $\mathcal{A}(E)$ or just $\mathcal{A}$ when E is clear from the context.

Definition 6.15. (1) A *gauge transformation* φ of the principal G-bundle (E,π,M) is an automorphism of E, that is, a bundle isomorphism $\varphi\colon E \longrightarrow E$.

(2) Two connections $\omega_1, \omega_2 \in \mathcal{A}(E)$ are called *gauge-equivalent* if there exists a gauge transformation φ such that

$$\omega_2 = \omega_1^{\varphi} = \varphi^* \omega_1. \tag{6.12}$$

Remark. Notice that the *set of gauge transformations* $\mathcal{G} = \mathcal{G}(E)$ is a group and that

$$\begin{aligned} (\omega^{\varphi})^{\psi} &= \omega^{\varphi \circ \psi}, \qquad \text{for all } \varphi, \psi \in \mathcal{G}, \\ \omega^{\mathrm{id}} &= \omega. \end{aligned} \tag{6.13}$$

That is, $\mathcal{G}$ acts from the right on the set $\mathcal{A}$ and the set of gauge equivalence classes is just the orbit space $\mathcal{A}/\mathcal{G}$.

Exercise 6.16. (1) Show that there is a 1-1 correspondence between $\mathcal{G}$ and each of the following 3 sets.

(1.a) The set of differentiable maps $\tilde{\sigma}: E \longrightarrow G$ satisfying

$$\tilde{\sigma}(xg) = g^{-1}\tilde{\sigma}(x)g,\ x \in E,\ g \in G.$$

(1.b) The set of sections of the fibre bundle (E_{iG}, π_{iG}, M) associated to the action of G on itself by inner conjugation (that is, $g(a) = gag^{-1}$).

(1.c) Given $\{U_\alpha\}_{\alpha \in \Sigma}$ a covering of M and trivializations with transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta \in \Sigma}$, the set of families of differentiable maps $\{\sigma_\alpha : U_\alpha \longrightarrow G\}_{\alpha \in \Sigma}$ satisfying

$$g_{\alpha\beta}\sigma_\beta = \sigma_\alpha g_{\alpha\beta} \qquad \text{on } U_\alpha \cap U_\beta. \tag{6.14}$$

(2) In the above notation let $\varphi \in \mathcal{G}$ correspond to the family $\{\sigma_\alpha\}_{\alpha \in \Sigma}$ and let ω be a connection in E with corresponding local connection forms $\{A_\alpha\}_{\alpha \in \Sigma}$. Show that ω^{φ} has local connection forms $\{A_\alpha^{\sigma_\alpha}\}_{\alpha \in \Sigma}$ where

$$A_\alpha^{\sigma_\alpha} = \mathrm{Ad}(\sigma_\alpha^{-1}) \circ A_\alpha + \sigma_\alpha^* \omega_0 \tag{6.15}$$

an ω_0 is the Maurer-Cartan form.

Finally let us consider extension and reduction of connections.

Definition 6.17. Let $\alpha\colon H \longrightarrow G$ be a Lie group homomorphism and let F be an H-bundle and $\varphi\colon F \longrightarrow E$ an extension of F to G. Furthermore let ω_F be a connection in F and ω_E be a connection in E. Then ω_E is called an *extension* of ω_F (and ω_F a *reduction* of ω_E) if

$$\varphi^* \omega_E = \alpha_* \circ \omega_F. \tag{6.16}$$

Exercise 6.18. Let $\alpha\colon H \longrightarrow G$ and $\varphi\colon F \longrightarrow E$ be as above.

(1) Show that if ω_F is a connection in F and ω_E is an extension in E then φ_* maps the horizontal vector spaces in F isomorphically to the horizontal vector spaces in E.

(2) Show that if ω_F is a connection in F then there is a unique extension ω_E to E.

(3) Let $\{U_\alpha\}_{\alpha\in\Sigma}$ be a covering of M with trivializations of F respectively of E such that the transition functions are $\{h_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ for F respectively $\{\alpha \circ h_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ for E. Show that if ω_F has local connections forms $\{A_\alpha\}_{\alpha\in\Sigma}$ then the extension ω_E has local connection forms $\{\alpha_* \circ A_\alpha\}_{\alpha\in\Sigma}$.

Remark. In particular if H is a Lie subgroup and α is the inclusion then ω_F and ω_E have the same local connection form.

7. The Curvature Form

As before let G be a Lie group with Lie algebra $\mathfrak{g}$ and let (E, π, M) be a principal G-bundle with connection ω. We will now define the *curvature form* generalizing the form F_A in (1.7). Since $\omega \in \Omega^1(E, \mathfrak{g})$ we have $\omega \wedge \omega \in \Omega^2(E, \mathfrak{g} \otimes \mathfrak{g})$ and we define $[\omega, \omega] \in \Omega^2(E, \mathfrak{g})$ to be the image of $\omega \wedge \omega$ by the linear mapping

$$[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

determined by the Lie bracket, i.e. the mapping sending $X \otimes Y$ to $[X, Y]$, $X, Y \in \mathfrak{g}$.

Definition 7.1. The *curvature form* $F_\omega \in \Omega^2(E, \mathfrak{g})$ for the connection ω is defined by the equation (the structural equation)

$$d\omega = -\tfrac{1}{2}\,[\omega, \omega] + F_\omega. \tag{7.1}$$

In the above notation we have the following theorem.

Theorem 7.2. (1) *If $E = M \times G$ and $\omega = \omega_{\mathrm{MC}}$ is the Maurer-Cartan connection then $F_{\omega_{\mathrm{MC}}} = 0$, that is,*

$$d\omega_{\mathrm{MC}} = -\tfrac{1}{2}\,[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}]. \tag{7.2}$$

(2) *In general F_ω is horizontal and* Ad*-equivariant, i.e. defines a 2-form (also denoted) $F_\omega \in \Omega^2(M, E_{\mathfrak{g}})$.*

(3) *(The Bianchi identity) Furthermore*

$$dF_\omega = [F_\omega, \omega]. \tag{7.3}$$

In particular dF_ω vanishes on triples of horizontal vectors.

(4) *Suppose $\{U_\alpha\}_{\alpha \in \Sigma}$ is a covering of M with trivializations $\{\varphi_\alpha\}_{\alpha \in \Sigma}$ of $E|_{U_\alpha}$ and transition functions $\{g_{\alpha\beta}\}_{\alpha\beta \in \Sigma}$. Suppose ω has local connection*

forms $\{A_\alpha\}_{\alpha\in\Sigma}$. *Then the curvature form* F_ω *corresponds to the family* $F_{A_\alpha} \in \Omega^2(U_\alpha, \mathfrak{g})$ *given by*

$$F_{A_\alpha} = dA_\alpha + \tfrac{1}{2}\,[A_\alpha, A_\alpha]. \tag{7.4}$$

Notation. We will often denote the connection by $A = \{A_\alpha\}_{\alpha\in\Sigma}$ and in that case identify the curvature form with the collection $F_A = \{F_{A_\alpha}\}_{\alpha\in\Sigma}$ given by (7.4).

For the proof of Theorem 7.2 we need a few preparations. First note that given $A \in \mathfrak{g}$ there is an associated vector field A^* on E defined by $A^*_x = v_x(A)$, $x \in E$. Here as usual v_x is the differential of the map $G \longrightarrow E$ given by $g \longmapsto xg$, $g \in G$. Notice that for $A, B \in \mathfrak{g}$ we have

$$[A, B]^* = [A^*, B^*]. \tag{7.5}$$

To see this we observe that it is enough to show (7.5) locally, and hence we can assume that E is a product bundle $E = M \times G$. If $\pi_1\colon M \times G \longrightarrow M$ and $\pi_2\colon M\times G \longrightarrow G$ are the two projections then for $A \in \mathfrak{g}$, the vector field A^* is the unique vector field on E which is π_1-related to the zero vector field on M and is π_2-related to the left invariant vector field $\widetilde{A}$ on G. Since $\widetilde{[A,B]} = [\widetilde{A}, \widetilde{B}]$ for $A, B \in \mathfrak{g}$ it follows that $[A^*, B^*]$ is again π_1-related to zero and π_2-related to $\widetilde{[A,B]}$, hence (7.5) follows.

Next we observe that the vector field A^* generates a 1-parameter group of diffeomorphisms of E given by $t \longmapsto R_{g_t}$, $t \in \mathbb{R}$, with $g_t = \exp(tA)$. That is, we claim that for each $x \in E$, the curve $t \longmapsto xg_t$, $t \in \mathbb{R}$, is an integral curve for the vector field A^*, i.e. it satisfies the differential equation

$$\frac{\partial}{\partial t} R_{g_t}(x) = A^*_{R_{g_t}(x)}. \tag{7.6}$$

For $t = 0$ this follows from the definition of A^* and hence we have for t arbitrary:

$$\frac{\partial}{\partial t} R_{g_t}(x) = \frac{\partial}{\partial s} R_{g_{s+t}}(x)|_{s=0} = \frac{\partial}{\partial s} R_{g_s}(R_{g_t}(x))|_{s=0} = A^*_{R_{g_t}(x)}.$$

We now have the following lemma.

Lemma 7.3. *Let Y be a differentiable vector field on E and let $A \in \mathfrak{g}$ as above. Then*

$$[A^*, Y]_x = \lim_{t \longrightarrow 0} \frac{Y_x - Y_x^{g_t}}{t}$$

where $Y_x^{g_t} = (R_{g_t})_*(Y_{R_{g_t}^{-1}(x)}) \in T_x E$.

Proof. (For a more general result see e.g. [S, ch.V, Thm. 10] or [W, Proposition 2.25 (b)].)

Since E is locally trivial we can take $E = U \times G$ where $U \subseteq M$ has a local coordinate system $(x^1, \ldots, x^n)$. Also choose a basis $A_1, \ldots, A_k$ for $\mathfrak{g}$. Then every vector field Y on E has the form

$$Y = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{k} b^j A_j^*$$

where $a^i, b^j \in C^\infty(E)$. Also, since A^* is constant in the x^i-direction we have $[A^*, \partial/\partial x^i] = 0$, $i = 1, \ldots, n$, so that

$$[A^*, Y] = \sum_{i=1}^{n} A^*(a^i) \frac{\partial}{\partial x^i} + \sum_{j=1}^{k} (A^*(b^j) A_j^* + b^j [A, A_j]^*). \tag{7.7}$$

On the other hand

$$\begin{aligned} Y_x - Y_x^{g_t} = & \sum_{i=1}^{n} (a^i(x) - a^i(x g_t^{-1})) \frac{\partial}{\partial x^i} + \\ & + \sum_{j=1}^{k} ((b^j(x) A_j^*(x) - b^j(x g_t^{-1}) A_j^*(x))) + \\ & + \sum_{j=1}^{k} ((b^j(x g_t^{-1}) A_j^*(x) - b^j(x g_t^{-1}) R_{g_t *} A_j^*(x g_t^{-1}))). \end{aligned}$$

Hence, using (7.6), we have

$$\lim_{t\longrightarrow 0}\frac{Y_x - Y_x^{g_t}}{t} = \sum_{i=1}^{n} A_x^*(a^i)\frac{\partial}{\partial x^i} + \sum_{j=1}^{k} A^*(b^j)A_j^*(x) + \\ + \sum_{j=1}^{k} b^j(x)\lim_{t\longrightarrow 0}\frac{1}{t}(A_j^*(x) - R_{g_t\,*}A_j^*(xg_t^{-1})). \tag{7.8}$$

Here

$$R_{g_t\,*}A_j^*(xg_t^{-1}) = v_x(\mathrm{Ad}(g_t^{-1})(A_j));$$

hence

$$\lim_{t\longrightarrow 0}\frac{1}{t}(A_j^*(x) - R_{g_t\,*}A_j^*(xg_t^{-1})) = v_x(B_j)$$

with

$$B_j = \lim_{t\longrightarrow 0}\frac{1}{t}(A_j - \mathrm{Ad}(g_t^{-1})(A)) = -\,\mathrm{ad}(-A)(A_j) = [A, A_j].$$

Inserting this in (7.8) and comparing with (7.7) we obtain the formula in Lemma 7.3 □

Proof of Theorem 7.2. First notice that (1) follows from (2). In fact, as in the proof of Lemma 6.2, $\omega_{\mathrm{MC}} = \pi_2^*\omega_0$, where ω_0 is the Maurer-Cartan connection on the bundle $G \longrightarrow \mathrm{pt}$, that is, $F_{\omega_{\mathrm{MC}}} = \pi_2^* F_{\omega_0}$. But if F_{ω_0} is horizontal then it is clearly 0, and hence also $F_{\omega_{\mathrm{MC}}} = 0$.

(2) Since ω is Ad-equivariant also $F_\omega = d\omega + \frac{1}{2}[\omega,\omega]$ is Ad-equivariant. We shall just show that it is horizontal, that is, for $X, Y \in T_xE$ we must show that if X is vertical then

$$(d\omega)(X,Y) = -\,\tfrac{1}{2}\,[\omega,\omega](X,Y). \tag{7.9}$$

Since

$$[\omega,\omega](X,Y) = [\omega(X),\omega(Y)] - [\omega(Y),\omega(X)] = 2[\omega(X),\omega(Y)]$$

(7.9) is equivalent to

$$(d\omega)(X,Y) = -[\omega(X),\omega(Y)]. \tag{7.10}$$

We have two cases: (1) Y is vertical; (2) Y is horizontal.

Case (1). Y is vertical. To show (7.10) for X and Y vertical we now take $X = A^*_x$, $Y = B^*_x$ for $A, B \in \mathfrak{g}$ and compute using (5.1):

$$\begin{aligned}(d\omega)(A^*, B^*) &= A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega([A^*, B^*]) \\ &= A^*(B) - B^*(A) - \omega([A, B]^*) \\ &= -[A, B] \\ &= -[\omega(A^*), \omega(B^*)]\end{aligned} \tag{7.11}$$

which is (7.10) in this case.

Case (2). Y is horizontal. Again we take $X = A^*_x$ for A^* the vector field associated to $A \in \mathfrak{g}$ as above. Also we extend Y to a vector field of horizontal vectors (also denoted by Y). This is possible since for an arbitrary vector field Z on E extending Y the vector field defined by

$$Y_y = Z_y - v_y \circ \omega_y(Z_y), \qquad y \in E$$

is horizontal. For the proof of (7.10) we thus have to prove for $A \in \mathfrak{g}$ and Y a horizontal vector field:

$$(d\omega)(A^*, Y) = 0. \tag{7.12}$$

Since $\omega(A^*) = A$ is constant and since $\omega(Y) = 0$ we get using (5.1):

$$(d\omega)(A^*, Y) = -\omega([A^*, Y]). \tag{7.13}$$

Since Y is horizontal we get

$$\omega(Y^{g_t}_x) = (R^*_{g_t}\omega)(Y_{R^{-1}_{g_t}(x)}) = \mathrm{Ad}(g_t^{-1}) \circ \omega(Y_{R^{-1}_{g_t}(x)}) = 0.$$

Hence

$$\omega([A^*, Y_x]) = \lim_{t \longrightarrow 0} \omega\left(\frac{Y_x - Y^{g_t}_x}{t}\right) = 0,$$

and by (7.13) we conclude

$$(d\omega)(A^*, Y) = 0$$

which proves (7.12) and hence (7.10) in case (2). This finishes the proof of Theorem 7.2 (2) and hence also of (1).

(3) Let us differentiate the equation (7.1):

$$\begin{aligned} 0 &= -\tfrac{1}{2}\,[d\omega,\omega] + \tfrac{1}{2}\,[\omega, d\omega] + dF_\omega \\ &= -[d\omega,\omega] + dF_\omega \\ &= \tfrac{1}{2}\,[[\omega,\omega],\omega] - [F_\omega,\omega] + dF_\omega \end{aligned} \tag{7.14}$$

where we have used (5.2) and (7.1). But

$$[[\omega,\omega],\omega] = 0 \tag{7.15}$$

since

$$\begin{aligned} &[[\omega,\omega],\omega](X,Y,Z) \\ &\quad = [[\omega,\omega](X,Y),\omega(Z)] - [[\omega,\omega](X,Z),\omega(Y)] + \\ &\qquad + [[\omega,\omega](Y,Z),\omega(X)] \\ &\quad = 2([[\omega(X),\omega(Y)],\omega(Z)] - [[\omega(X),\omega(Z)],\omega(Y)] + \\ &\qquad + [[\omega(Y),\omega(Z)],\omega(X)]) \\ &\quad = 0, \end{aligned}$$

by the Jacobi identity. By (7.14) and (7.15) we clearly have proved (7.3).

(4) This follows directly from (7.1) □

Remark. Let X and Y be two horizontal vector fields on E. Then by (7.1) we get

$$F_\omega(X,Y) = -\omega([X,Y]). \tag{7.16}$$

Remark. Suppose $(f,\bar{f})\colon (E',\pi',M') \longrightarrow (E,\pi,M)$ is bundle map and ω is a connection in E with curvature form F_ω. Then $\bar{f}^*\omega$ is a connection in E' with curvature form $\bar{f}^*F_\omega$. In particular if $\varphi\colon E' \longrightarrow E$ is a bundle isomorphism and ω connection in E then $\omega^\varphi = \varphi^*\omega$ has curvature $F_{\omega^\varphi} = \varphi^*F_\omega$.

Definition 7.4. A connection ω in a principal G-bundle is called *flat* if the curvature form vanishes identically, that is if $F_\omega \equiv 0$.

Theorem 7.5. *Let (E, π, M) be a principal G-bundle with connection ω. Then ω is flat if and only if around every point in M there is a neighborhood U with a trivialization $\varphi\colon \pi^{-1}(U) \longrightarrow U \times G$ such that ω restricted to $E|_U$ is induced by the Maurer-Cartan connection in $U \times G$, that is $\omega|_{E|_U} = \varphi^*\omega_{\mathrm{MC}}$.*

Proof. ($\Leftarrow$) This follows clearly from Theorem 7.2 and the above remark.

($\Rightarrow$) Assume $F_\omega \equiv 0$. For $x \in E$ let $H_x \subseteq T_xE$ be the subspace of horizontal vectors, that is $H_x = \ker \omega_x$, $x \in E$. This defines a distribution which is integrable; in fact, if X, Y are horizontal vector fields then by (7.16) we have

$$0 = F_\omega(X, Y) = -\omega([X, Y])$$

so that $[X, Y]$ is again a horizontal vector field. By the Frobenius Integrability Theorem there is a foliation $\mathcal{F} = \{\mathcal{F}_\alpha\}_{\alpha \in I}$ of E such that for each $x \in E$ H_x is the tangent space to the leaf through x. For $g \in G$ the diffeomorphism $R_g\colon E \longrightarrow E$ satisfies ${R_g}_* H_x = H_{xg}$, $\forall x \in E$, by Proposition 6.4; hence R_g maps the leaf through x diffeomorphically to the leaf through xg. Now fix $p \in M$, $x \in \pi^{-1}(p)$ and let $\mathcal{F}_\alpha \subseteq E$ be the leaf through x. Since $T_x\mathcal{F}_\alpha = H_x$ and $\pi_*\colon H_x \longrightarrow T_pM$ is an isomorphism we can apply the Inverse Function Theorem and we can choose neighborhoods U of p and $V \subseteq \mathcal{F}_\alpha$ of x such that $\pi\colon V \longrightarrow U$ is a diffeomorphism. The inverse mapping $s\colon U \longrightarrow V \subseteq \pi^{-1}(U)$ defines a differentiable section in the bundle $E|_U$ and hence a trivialization $\varphi = \psi^{-1}\colon \pi^{-1}(U) \longrightarrow U \times G$ whose inverse is defined by $\psi(q, g) = s(q)g$, $q \in U$, $g \in G$. The fact that φ is differentiable follows from the Inverse Function Theorem. Now let ω' be the connection in $E|_U$ induced by the Maurer-Cartan connection in $U \times G$, that is $\omega' = \varphi^*\omega_{\mathrm{MC}}$. By the remark following Proposition 6.4 the horizontal vector space at a point of the form yg, $y \in V$, $g \in G$ is

$$(R_g)_*(T_y(V)) = (R_g)_*H_y = H_{yg}$$

which is also the horizontal vector space for ω. Hence $\omega|_{E|_U}$ and ω' have the same horizontal subspaces and therefore they agree. $\square$

Corollary 7.6. *Let (E, π, M) be a principal G-bundle. Then the following are equivalent:*

(1) *E has a flat connection.*

(2) *There is an open covering* $\{U_\alpha\}_{\alpha\in\Sigma}$ *and trivializations* $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ *such that all transition functions* $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$ *are constant.*

Proof. (2) $\Rightarrow$ (1). Since $g_{\alpha\beta}$ is constant $g_{\alpha\beta}^*\omega_0 = 0$. Hence the collection $\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g}) \mid A_\alpha \equiv 0,\ \alpha \in \Sigma\}$ satisfies (6.7), and thus defines a connection ω in E by Corollary 6.9. For this $F_\omega = 0$ by Theorem 7.2 d) since clearly $F_{A_\alpha} = 0$, $\forall \alpha \in \Sigma$.

(1) $\Rightarrow$ (2). By Theorem 7.5 we can choose a covering $\{U_\alpha\}_{\alpha\in\Sigma}$ and trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ of M such that the given flat connection ω restricted to $E|_{U_\alpha}$ is induced by the Maurer-Cartan connection, ω_{MC} in $U_\alpha \times G$, $\alpha \in \Sigma$. Furthermore we can arrange that all intersections are connected (e.g. by choosing a Riemannian metric on M and choosing all U_α to be geodesically convex sets). Now by construction the local connection forms $\{A_\alpha\}_{\alpha\in\Sigma}$ for ω are all zero, hence by Corollary 6.9 we have $g_{\alpha\beta}^*\omega_0 = 0$ on $U_\alpha \cap U_\beta$ where ω_0 is the Maurer-Cartan form on G. It follows that $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow G$ has zero differential and since $U_\alpha \cap U_\beta$ is connected $g_{\alpha\beta}$ is constant. $\square$

Exercise 7.7. Let (E, π, M) be a principal G-bundle and let $\{U_\alpha\}_{\alpha\in\Sigma}$ be an open covering of M with trivializations $\{\varphi_\alpha\}_{\alpha\in\Sigma}$ and transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$. Let $\varphi\colon E \longrightarrow E$ be a gauge transformation corresponding to the family of differentiable maps $\sigma_\alpha\colon U_\alpha \longrightarrow G$ satisfying (6.14). Let ω be a connection in E with local connection forms $\{A_\alpha\}_{\alpha\in\Sigma}$.

(1) Show that the curvature form F_{ω^φ} for the connection $\omega^\varphi = \varphi^*\omega$ is given locally by

$$F_{A_\alpha^{\sigma_\alpha}} = \mathrm{Ad}({\sigma_\alpha}^{-1}) \circ F_{A_\alpha}, \qquad \text{for all } \alpha \in \Sigma$$

where F_ω is given locally by $\{F_{A_\alpha}\}_{\alpha\in\Sigma}$.

(2) Let $H \subseteq G$ be a Lie subgroup with Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$. Show that if there is a reduction of E and ω to H then F_ω satisfies the following: For all $x \in E$ there is a $g \in G$ such that $\mathrm{Ad}(g)(\mathfrak{h})$ contains the set $\{F_\omega(X, Y) \mid X, Y \in T_x E\}$.

(3) The connection is called *irreducible* if for all $x \in E$, $\mathfrak{g}$ is generated as a Lie algebra by the set $\{F_\omega(X, Y) \mid X, Y \in T_x E\}$, i.e., is spanned by all iterated Lie brackets of such elements. Show, that if G is connected, ω is irreducible and if $\varphi\colon E \longrightarrow E$ is a gauge transformation given by $\{\sigma_\alpha\}_{\alpha\in\Sigma}$ as above then $\omega^\varphi = \omega$ if and only if $\sigma_\alpha(p) \in Z(G)$, for all $p \in U_\alpha$, $\alpha \in \Sigma$, where $Z(G)$ is the center of G.

8. Linear Connections

Let us study in particular the case where $G = \mathrm{GL}(n, \mathbb{R})$. As usual the Lie algebra is $M(n, \mathbb{R})$, the set of $n \times n$ real matrices with Lie bracket

$$[A, B] = AB - BA, \qquad A, B \in M(n, \mathbb{R}), \tag{8.1}$$

and the adjoint representation

$$\mathrm{Ad}(g)(A) = gAg^{-1}, \qquad A \in M(n, \mathbb{R}),\ g \in \mathrm{GL}(n, \mathbb{R}). \tag{8.2}$$

Now consider an n-dimensional vector bundle V on a manifold M and let $E = F(V)$ be the frame bundle. A connection in this is therefore a 1-form $\omega \in \Omega^1(E, M(n))$, ie. a matrix of ordinary 1-forms

$$\omega = \begin{bmatrix} \omega_{11} & \dots & \omega_{1n} \\ \vdots & & \vdots \\ \omega_{n1} & \dots & \omega_{nn} \end{bmatrix}, \qquad \omega_{ij} \in \Omega^1(E).$$

Matrix multiplication defines a linear map

$$M(n) \otimes M(n) \longrightarrow M(n)$$

sending $A \otimes B \longmapsto AB$, and this induces a map

$$\Omega^2(E, M(n) \otimes M(n)) \longrightarrow \Omega^2(E, M(n)).$$

The image of $\omega \wedge \omega$ by this is also denoted $\omega \wedge \omega$; that is,

$$(\omega \wedge \omega)(X, Y) = \omega(X)\omega(Y) - \omega(Y)\omega(X) \tag{8.3}$$

and the components are given by

$$(\omega \wedge \omega)_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \tag{8.4}$$

It follows from (8.1) and (8.3) that

$$\begin{aligned}&[\omega,\omega](X,Y) = 2[\omega(X),\omega(Y)] = 2(\omega\wedge\omega)(X,Y),\\ &\tfrac{1}{2}\,[\omega,\omega] = \omega\wedge\omega.\end{aligned} \tag{8.5}$$

Hence the structural equation (7.1) becomes

$$d\omega = -\omega\wedge\omega + F_\omega, \tag{8.6}$$

where F_ω is a matrix of 2-forms on E, i.e.,

$$F_\omega = \begin{bmatrix} F_{11} & \dots & F_{1n}\\ \vdots & & \vdots\\ F_{n1} & \dots & F_{nn}\end{bmatrix}, \qquad F_{ij}\in\Omega^2(E).$$

Given a covering $\{U_\alpha\}_{\alpha\in\Sigma}$ and local trivializations of V we get corresponding local trivializations for E, and hence for each α the local connection form $A_\alpha\in\Omega^1(U_\alpha,M(n))$ is just a matrix of one forms as in (1.4) and the corresponding curvature form

$$F_{A_\alpha} = A_\alpha\wedge A_\alpha + dA_\alpha$$

is just the formula (1.7). We shall now interprete the connection and curvature in terms of a differential operator on the bundle V.

First observe that there is a natural isomorphism of V with the vector bundle $E_{\mathbb{R}^n}$ associated with E via ι, the identity representation (cf. 3.10 and the remark following Corollary 6.13). This gives another interpretation of $\Omega^k(M,V)$:

Lemma 8.1. *Let $\tilde\omega\in\Omega^k(E,\mathbb{R}^n)$ be a horizontal and ι-equivariant k-form Then $\tilde\omega$ defines for each $p\in M$ a k-linear map $\omega_p\colon T_pM\times\dots\times T_pM\longrightarrow V_p$, which is differentiable in the following sense: For $X_1,\dots,X_k$ differentiable vector fields on M $\omega(X_1,\dots,X_k)$ gives a differentiable section of V.*

Proof. Let $\pi\colon E\longrightarrow M$ be the projection in the frame bundle for V and for $p\in M$ choose $x\in E$ with $\pi(x)=p$. For $X_1,\dots,X_k\in T_pM$ choose $\widetilde{X}_1,\dots,\widetilde{X}_k\in T_xE$ with $\pi_*\widetilde{X}_i = X_i$, $i=1,\dots k$, and put

$$\omega_p(X_1,\dots,X_k) = x\circ\tilde\omega_x(\widetilde{X}_1,\dots,\widetilde{X}_k) \tag{8.7}$$

where we recall that $x \in \mathrm{Iso}(\mathbb{R}^n, V_p)$. This is well-defined since $\tilde{\omega}$ is horizontal and and since for $g \in \mathrm{GL}(n,\mathbb{R})$, $R_{g_*}\widetilde{X}_1, \ldots, R_{g_*}\widetilde{X}_k \in T_{xg}E$ also maps to $X_1, \ldots, X_k$ such that

$$\begin{aligned} xg(\tilde{\omega}_{xg}(R_{g_*}\widetilde{X}_1, \ldots, R_{g_*}\widetilde{X}_k)) &= xg(R_g^*\tilde{\omega})(\widetilde{X}_1, \ldots, \widetilde{X}_k) \\ &= xg(g^{-1}\tilde{\omega}(\widetilde{X}_1, \ldots, \widetilde{X}_k)) \\ &= x(\tilde{\omega}(\widetilde{X}_1, \ldots, \widetilde{X}_k)). \end{aligned}$$

We leave it to the reader to check differentiability. □

Remark. Conversely if $\omega_p \colon T_pM \times \cdots \times T_pM \longrightarrow V_p$ satifies the conditions of the lemma then $\tilde{\omega}$ defined by (8.7) defines a horizontal and ι-equivariant k-form on E. Hence $\Omega^k(M,V)$ is the set of such ω's.

Now suppose $\omega \in \Omega^1(E, M(n))$ is a connection. We shall construct a differential operator called the *covariant derivative*

$$\nabla \colon \Gamma(M,V) \longrightarrow \Omega^1(M,V). \tag{8.8}$$

For this consider $s \in \Gamma(M,V)$ and let $\tilde{s} \colon E \longrightarrow \mathbb{R}^n$ be the corresponding ι-equivariant map. Then we define $\widetilde{\nabla}(\tilde{s}) \in \Omega^1(E, \mathbb{R}^n)$ by the formula

$$\widetilde{\nabla}(\tilde{s}) = d\tilde{s} + \omega\tilde{s}, \tag{8.9}$$

where the multiplikation is the usual matrix multiplikation of the matrix ω and the column vector $\tilde{s}$.

Proposition 8.2. *For $s \in \Gamma(M,V)$ corresponding to the ι-equivariant map $\tilde{s} \colon E \longrightarrow \mathbb{R}^n$, the 1-form $\widetilde{\nabla}(\tilde{s}) \in \Omega^1(E, \mathbb{R}^n)$ is also ι-equivariant and horizontal, hence defines a form $\nabla(s) \in \Omega^1(M,V)$.*

Proof. Let us first show ι-equivariance:

$$\begin{aligned} R_g^*\widetilde{\nabla}(\tilde{s}) &= R_g^* d\tilde{s} + R_g^*(\omega)(\tilde{s} \circ R_g) \\ &= d(\tilde{s} \circ R_g) + (\mathrm{Ad}(g^{-1}) \circ \omega)(\tilde{s} \circ R_g) \\ &= d(g^{-1}\tilde{s}) + (g^{-1}\omega g)g^{-1}\tilde{s} \\ &= g^{-1}d\tilde{s} + g^{-1}\omega\tilde{s} \\ &= g^{-1}\widetilde{\nabla}(\tilde{s}). \end{aligned}$$

Next let us show that $\widetilde{\nabla}(\tilde{s})$ is horizontal. For this we first show that for $x \in E$ we have

$$(d\tilde{s})(v_x A) = -A\tilde{s}, \qquad A \in M(n). \tag{8.10}$$

In fact $(d\tilde{s}) \circ v_x$ is the differential of the map $G \longrightarrow E$ sending g to $\tilde{s}(xg) = g^{-1}\tilde{s}(x)$, hence (8.10) follows from the fact that the differential of $g \longmapsto g^{-1}$ is given by multiplication by -1. It follows that for $A \in M(n)$ we have

$$\begin{aligned}\widetilde{\nabla}(\tilde{s})(v_x A) &= (d\tilde{s})(v_x A) + \omega_x(v_x A)\tilde{s}(x) \\ &= -A\tilde{s}(x) + A\tilde{s}(x) \\ &= 0,\end{aligned}$$

which proves the proposition. □

Notation. By Lemma 8.1, $\nabla(s)$ gives us for each $p \in M$ a linear map $\nabla(s)_p\colon T_pM \longrightarrow V_p$. For $X \in T_pM$ we shall write

$$\nabla_X(s) = \nabla(s)(X) \in V_p \tag{8.11}$$

and this is called the *covariant derivative* of s in the direction X.

Proposition 8.3. *The covariant derivative satifies*

(1) $\nabla_{X+Y}(s) = \nabla_X(s) + \nabla_Y(s)$

(2) $\nabla_X(s + s') = \nabla_X(s) + \nabla_X(s')$

(3) $\nabla_{\lambda X}(s) = \nabla_X(\lambda s) = \lambda\nabla_X(s)$

(4) $\nabla_X(fs) = X(f)s + f(p)\nabla_X(s)$

for all $X, Y \in T_pM$, $s \in \Gamma(M, V)$, $\lambda \in \mathbb{R}$ *and* $f \in C^\infty(M)$.

Exercise 8.4. Prove Proposition 8.3.

Next let us express the curvature form of the connection ω on the frame bundle E in terms of the covariant derivative ∇. First let us introduce the notation

$$\mathrm{End}(V) = E_{M(n)} \tag{8.12}$$

for the vector bundle associated with the adjoint representation (8.2) of $\mathrm{GL}(n, \mathbb{R})$ on $M(n)$. This is justified by the following lemma.

Lemma 8.5. *There is a $1-1$ correspondence between horizontal* Ad-*equivariant k-forms $\widetilde{\Theta} \in \Omega^k(E, M(n))$ and families of k-linear maps*

$$\Theta_p \colon T_pM \times \cdots \times T_pM \longrightarrow \operatorname{End}(V)_p$$

which vary differentiably in the sense that k vector fields $X_1, \ldots, X_k$ and a section $s \in \Gamma(M, V)$ give rise to another differentiable section $\Theta(X_1, \ldots, X_k)(s)$ given by

$$\Theta(X_1, \ldots, X_k)(s)(p) = \Theta_p(X_1(p), \ldots, X_k(p))(s(p)) \tag{8.13}$$

Proof. This is analogous to the proof of Lemma 8.1 and the proof is left to the reader. We only note that for $x \in \pi^{-1}(p)$ and $\widetilde{X}_1, \ldots, \widetilde{X}_k \in T_xE$ with $\pi_*\widetilde{X}_i = X_i$, $i = 1, \ldots, k$, $\widetilde{\Theta}$ and Θ is related by

$$\Theta_p(X_1, \ldots, X_k) = x \circ \widetilde{\Theta}_x(\widetilde{X}_1, \ldots, \widetilde{X}_k) \circ x^{-1}, \tag{8.14}$$

where $x \in E_p = \operatorname{Iso}(\mathbb{R}^n, V_p)$. □

Proposition 8.6. *Let $\widetilde{F}_\omega \in \Omega^2(E, M(n))$ be the curvature form for the connection ω and let $F_\omega \in \Omega^2(M, \operatorname{End}(V))$ be the corresponding 2-form as in Lemma 8.5. Let ∇ denote the covariant derivation associated to ω. Then for X, Y differentiable vector fields on M and s a section of V we have*

$$F_\omega(X, Y)(s) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})(s). \tag{8.15}$$

Proof. Since it is enough to prove (8.15) locally we can assume V and hence E to be trivial; hence we can choose $\widetilde{X}$ and $\widetilde{Y}$ on $E = M \times \operatorname{GL}(n, \mathbb{R})$ such that $\widetilde{X}$ respectively $\widetilde{Y}$ are π-related to X respectively Y (that is $\pi_*\widetilde{X}_x = X_{\pi(x)}$ and $\pi_*\widetilde{Y}_x = Y_{\pi(x)}$, $\forall x \in E$). Also, as in the proof of Theorem 7.2 we can assume $\widetilde{X}$ and $\widetilde{Y}$ to be horizontal. Furthermore for $s \in \Gamma(M, V)$ let $\tilde{s} \colon E \longrightarrow \mathbb{R}^n$ be the corresponding ι-equivariant function. Then by (8.14) and (7.16) we have for $\pi(x) = p$:

$$\begin{aligned} F_\omega(X_p, Y_p)(s(p)) &= x \circ \widetilde{F}_\omega(\widetilde{X}_x, \widetilde{Y}_x) \circ x^{-1}(s(p)) \\ &= -x \circ \omega_x([\widetilde{X}, \widetilde{Y}]_x)(\tilde{s}(x)). \end{aligned} \tag{8.16}$$

On the other hand by (8.7) $\nabla_{X_p}(s) = x \circ \widetilde{\nabla}(\tilde{s})(\widetilde{X}_x)$, and hence

$$\nabla_{Y_p}(\nabla_{X_p}) = x \circ \widetilde{\nabla}(\widetilde{\nabla}(\tilde{s})(\widetilde{X}))(\widetilde{Y}_x).$$

Therefore (8.15) is equivalent to

$$\begin{aligned}-\omega([\widetilde{X},\widetilde{Y}])(\tilde{s}) = \widetilde{\nabla}(\widetilde{\nabla}(\tilde{s})(\widetilde{Y}))(\widetilde{X}) - \widetilde{\nabla}(\widetilde{\nabla}(\tilde{s})(\widetilde{X}))(\widetilde{Y})\\ - \widetilde{\nabla}(\tilde{s})([\widetilde{X},\widetilde{Y}]),\end{aligned} \tag{8.17}$$

where we have used that $[\widetilde{X},\widetilde{Y}]$ is π-related to $[X,Y]$. But since $\widetilde{X}$ and $\widetilde{Y}$ are horizontal we have

$$\widetilde{\nabla}(\widetilde{\nabla}(\tilde{s}(\widetilde{X})))(\widetilde{Y}) = d(d(\tilde{s})(\widetilde{X}))(\widetilde{Y}) = \widetilde{Y}(\widetilde{X}(\tilde{s})),$$

and similarly for $\widetilde{X}$ and $\widetilde{Y}$ interchanged. Hence the right hand side of (8.17) becomes

$$\widetilde{X}(\widetilde{Y}(\tilde{s})) - \widetilde{Y}(\widetilde{X}(\tilde{s})) - [\widetilde{X},\widetilde{Y}](\tilde{s}) - \omega([\widetilde{X},\widetilde{Y}])(\tilde{s}) = -\omega([\widetilde{X},\widetilde{Y}])(\tilde{s}),$$

which was to be proved. □

Next let us express ∇ in terms of local trivializations. Suppose $\{U_\alpha\}_{\alpha\in\Sigma}$ is a covering of M with local trivializations $f_\alpha\colon V|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{R}^n$, and associated transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \longrightarrow \mathrm{GL}(n,\mathbb{R})$. Then there is a $1-1$ correspondence between sections s of V and families $\{s_\alpha\}_{\alpha\in\Sigma}$ of differentiable functions $s_\alpha\colon U_\alpha \longrightarrow \mathbb{R}^n$, satisfying for $\alpha,\beta\in\Sigma$:

$$s_\beta(p) = g_{\beta\alpha}(p)s_\alpha(p), \qquad p \in U_\alpha \cap U_\beta. \tag{8.18}$$

In fact s and $\{s_\alpha\}$ are related by

$$f_\alpha \circ s(p) = (p, s_\alpha(p)), \qquad p \in U_\alpha. \tag{8.19}$$

Now $\{g_{\alpha\beta}\}_{\alpha,\beta\in\Sigma}$ are also transition functions for the frame bundle $E = F(V)$ corresponding to the local trivializations $\{\varphi_\alpha = \bar{f}_\alpha\}_{\alpha\in\Sigma}$ corresponding to $\{f_\alpha\}_{\alpha\in\Sigma}$, as in (2.1). In the notation of Proposition 6.11 we have the following proposition.

Proposition 8.7. *Let $s \in \Gamma(M,V)$ correspond to the family $\{s_\alpha\}_{\alpha\in\Sigma}$ as above, then $\nabla(s) \in \Omega^1(M,V)$ corresponds to the family of 1-forms $\{\nabla(s_\alpha)\}_{\alpha\in\Sigma}$ given by*

$$\nabla(s_\alpha) = ds_\alpha + A_\alpha s_\alpha \tag{8.20}$$

where $\{A_\alpha\}_{\alpha\in\Sigma}$ are the local connection forms for ω defining ∇.

Proof. Notice that for $\alpha \in \Sigma$ the trivializations $\varphi_\alpha \colon E|_{U_\alpha} \longrightarrow U_\alpha \times \mathrm{GL}(n,\mathbb{R})$ corresponding to f_α has the inverse defined by $\varphi_\alpha^{-1}(p,g) = (f_\alpha)_p^{-1} \circ g$. and hence the section $\sigma_\alpha \colon U_\alpha \longrightarrow E|_{U_\alpha}$ used in Proposition 6.11 is given by

$$\sigma_\alpha(p) = \varphi_\alpha^{-1}(p, \mathrm{id}) = (f_\alpha)_p^{-1} \in \mathrm{Iso}(\mathbb{R}^n, V_p) = E_p.$$

For $s \in \Gamma(M,V)$ the coresponding ι-equivariant function $\tilde{s} \colon E \longrightarrow \mathbb{R}^n$ is related to $\{s_\alpha\}_{\alpha\in\Sigma}$ by

$$s_\alpha = \tilde{s} \circ \sigma_\alpha$$

and hence the local 1-form $\nabla(s_\alpha)$ corresponding to the horizontal ι-equivariant 1-form is given by

$$\begin{aligned}
\nabla(s_\alpha) &= (\sigma_\alpha)^*(\widetilde{\nabla}(\tilde{s})) \\
&= \sigma_\alpha^*(d\tilde{s} + \omega\tilde{s}) \\
&= d(\tilde{s} \circ \sigma_\alpha) + (\sigma_\alpha^*\omega)(\tilde{s} \circ \sigma_\alpha) \\
&= d(s_\alpha) + A_\alpha s_\alpha,
\end{aligned}$$

where $A_\alpha = \sigma_\alpha^*\omega$ by the proof of Corollary 6.9. $\square$

Exercise 8.8. Let G be an arbitrary Lie group and $\rho \colon G \longrightarrow \mathrm{GL}(n,\mathbb{R})$ a representation. Let (E,π,M) be a principal G-bundle. and let V be the associated vector bundle. Let ω be a connection in E.

(1) Show that the frame bundle of V is the extension of E to $\mathrm{GL}(n,\mathbb{R})$ via ρ, and conclude that ω has a unique extension ω_ρ to $F(V)$, cf. Exercise 6.18.

(2) Now choose a covering $\{U_\alpha\}_{\alpha\in\Sigma}$ of M and local trivializations of E and hence also V and $F(V)$. Show that the covariant derivative ∇ corresponding to ω_ρ is given in terms of the local trivializations by

$$\nabla(s_\alpha) = ds_\alpha + \rho_*(A_\alpha)s_\alpha \tag{8.21}$$

where ω corresponds to $\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}$, $\rho_* \colon \mathfrak{g} \longrightarrow M(n)$ is the differential of ρ, and $\{s_\alpha\}_{\alpha\in\Sigma}$ defines a section in V.

Again return to the case of a vector bundle V on M, $E = F(V)$ the frame bundle and let ∇ be the covariant derivative.

Definition 8.9. A section $s \in \Gamma(M, V)$ is called *parallel* if $\nabla_X(s) = 0$ for all tangent vectors $X \in T_pM$ and for all $p \in M$.

In the notation of Proposition 8.7 we have the following corollary.

Corollary 8.10. *Let $s \in \Gamma(M, V)$ correspond to $\{s_\alpha : U_\alpha \longrightarrow \mathbb{R}^n\}_{\alpha \in \Sigma}$. Then s is parallel if and only if*

$$ds_\alpha = -A_\alpha s_\alpha, \qquad \alpha \in \Sigma. \tag{8.22}$$

That is, locally a parallel section satisfies a differential system as in (1.4).

Proposition 8.11. *The following are equivalent:*

(1) *Every point of M has a neighborhood U and parallel sections $\sigma_1, \ldots, \sigma_n \in \Gamma(U, V)$ such that $\{\sigma_1(p), \ldots, \sigma_n(p)\}$ is a basis for V_p for all $p \in U$.*

(2) *The connection is flat, ie., $F_\omega = 0$*

Proof. As in the proof of Corollary 2.15 a set of sections $\sigma_1, \ldots, \sigma_n \in \Gamma(U, V)$ defining a local frame for $V|_U$ gives a trivialization of

$$V|_U,\ f : V|_U \longrightarrow U \times \mathbb{R}^n,$$

such that

$$f \circ \sigma_i = (p, e_i), \qquad i = 1, \ldots, n,$$

where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. Hence by (8.20) $\nabla(\sigma_i)$ is given in terms of this trivialization by $\nabla(\sigma_i) = Ae_i$, where A is the local connection form for ω. It follows that $\sigma_1, \ldots, \sigma_n$ are parallel if and only if $A = 0$, that is, $\omega|_{E|_U}$ is induced by the Maurer-Cartan connection via the trivalization (2.1) of $E|_U$ corresponding to f. The proposition now follows from Theorem 7.2 (1). ☐

Exercise 8.12. We consider the $\mathrm{GL}(k, \mathbb{R})$-bundle $(W_{n,k}, \gamma_{n,k}, G_k(\mathbb{R}^n))$, for $k \leq n$, and we consider $W(n, k) \subseteq M(n, k) = \mathbb{R}^{nk}$ as the open set of $n \times k$ matrices X of rank k (cf. exercise 4.8).

(1) Show that the 1-form

$$\omega = (X^tX)^{-1}X^t dX \in \Omega^1(W_{n,k}, M(k)), \tag{8.23}$$

gives a connection in the above bundle.

(2) Show that the curvature form is given by

$$F_\omega = (X^tX)^{-1}dX^t \wedge dX - (X^tX)^{-1}dX^t \wedge X(X^tX)^{-1}X^tdX. \tag{8.24}$$

(3) Show that the restriction of ω to the Stiefel manifold $V_{n,k} \subseteq W_{n,k}$ is given by

$$\omega = X^tdX \tag{8.25}$$

and that this defines a reduction to the $\mathrm{O}(k)$-bundle $(V_{n,k}, \gamma_{n,k}, G_k(\mathbb{R}^n))$ with curvature form

$$\begin{aligned} F_\omega &= dX^t \wedge dX - dX^t \wedge XX^tdX \\ &= dX^t \wedge dX + X^tdX \wedge X^tdX. \end{aligned} \tag{8.26}$$

(4) For $k = 1$ the connection ω in the $\mathbb{R}^*$-bundle $(\mathbb{R}^n \setminus \{0\}, \eta, \mathbb{R}P^{n-1})$ is given by

$$\omega = \frac{x^t}{|x|^2}dx, \qquad x \in \mathbb{R}^n \setminus \{0\}.$$

Show that this connection is flat, ie. $F_\omega = 0$, and conclude that the real Hopf-bundle locally has a non-vanishing parallel section. Notice that this does not exist globally by Corollary 2.24.

9. The Chern-Weil Homomorphism

We can now associate to a principal G-bundle (E, π, M) with a connection ω some closed differential forms on M, and as we shall see, the corresponding classes in de Rham cohomology do not depend on ω but only on the isomorphism class of the G-bundle.

First some linear algebra: Let V be a finite dimensional real vector space. For $k \geq 1$ let $S^k(V^*)$ denote the vector space of *symmetric* k-linear functions

$$P: V \times V \times \ldots \times V \longrightarrow \mathbb{R}.$$

We shall identify P with with the corresponding linear map

$$P: V \otimes V \otimes \ldots \otimes V \longrightarrow \mathbb{R}$$

which is invariant under the action of the symmetric group acting on $V \otimes V \otimes \ldots \otimes V$. That is

$$P(v_{\sigma_1} \otimes \ldots \otimes v_{\sigma_k}) = P(v_1 \otimes \ldots \otimes v_k)$$

for every permutation σ of $1, \ldots, k$. We also have a product

$$\bigcirc: S^k(V^*) \otimes S^l(V^*) \longrightarrow S^{k+l}(V^*)$$

defined by

$$\begin{aligned} P \bigcirc Q(v_1, \ldots, v_{k+l}) \\ = \frac{1}{(k+l)!} \sum_{\sigma} P(v_{\sigma_1}, \ldots, v_{\sigma_k}) Q(v_{\sigma_{k+1}}, \ldots, v_{\sigma_{k+l}}) \end{aligned} \tag{9.1}$$

where σ runs over all permutations of $1, \dots, k+l$. We also put $S^0(V^*) = \mathbb{R}$ and then

$$S^*(V^*) = \bigoplus_{k=0}^{\infty} S^k(V^*)$$

becomes a commutative ring with unit $1 \in \mathbb{R} = S^0(V^*)$. The associativity can be shown directly from (9.1), but it also follows from the proposition below. Choose a basis $\{e_1, \dots, e_n\}$ for V and let $\mathbb{R}[x_1, \dots, x_n]$ denote the polynomium ring in the variables $x_1, \dots, x_n$. Let $\mathbb{R}[x_1, \dots, x_n]^k$ denote the subset of homogeneous polynomials of degree k, and define a mapping

$$S^k(V^*) \xrightarrow{\sim} \mathbb{R}[x_1, \dots, x_n]^k$$

by

$$\widetilde{P}(x_1, \dots, x_n) = P(v, \dots, v), \qquad v = \sum_{i=1}^{n} x_i e_i \tag{9.2}$$

for $P \in S^k(V^*)$. We then have

Proposition 9.1. (1) *The mapping $P \longmapsto \widetilde{P}$ is an isomorphism of vector spaces*

$$S^k(V^*) \cong \mathbb{R}[x_1, \dots, x_n]^k. \tag{9.3}$$

(2) $(P \bigcirc Q)^{\sim} = \widetilde{P}\widetilde{Q}$*, hence (9.3) gives an isomorphism of rings*

$$S^*(V^*) \xrightarrow{\cong} \mathbb{R}[x_1, \dots, x_n]$$

Proof. (2) Clearly follows from (9.1) and (1).

(1) We first show injectivity. For this notice that the coefficient to $x_1^{i_1} \cdots x_n^{i_n}$ is a positive multiple of

$$P(e_1, \dots, e_1, e_2, \dots, e_2, \dots, e_n, \dots, e_n) = a_{i_1 \dots i_n},$$

where e_j is repeated i_j times, $j = 1, \dots, n$. Hence if $\widetilde{P} = 0$ then $a_{i_1 \dots i_n} = 0$ for all $\{i_1, \dots, i_n\}$ satisfying $i_1 + \cdots + i_n = k$. By the symmetry of P we conclude that

$$P(e_{j_1}, \dots, e_{j_k}) = 0, \qquad \text{for all } j_1, \dots, j_k \in \{1, \dots, n\}.$$

Hence by the multi-linearity of P,

$$P(v_1, \ldots, v_k) = 0, \qquad \text{for all } v_1, \ldots, v_k \in V,$$

that is, $P = 0$. It follows that $S^*(V^*)$ is isomorphic to a subring of $\mathbb{R}[x_1, \ldots, x_n]$. But since

$$S^1(V^*) = \operatorname{Hom}(V, \mathbb{R}) \xrightarrow{\ \sim\ } \mathbb{R}[x_1, \ldots, x_n]^1$$

is clearly an isomorphism, the subring has to contain $x_1, \ldots, x_n$ and hence is the full polynomial ring. □

Remark. We have thus shown that $P \in S^k(V^*)$ is determined by the function $\widetilde{P}\colon v \longmapsto P(v, \ldots, v)$. This is called a *polynomial function* for V. A choice of basis $\{e_i, \ldots, e_n\}$ for V gives an identification of the ring of polynomial functions with the usual polynomial ring in n variables. The inverse operation which to a polynomial function $\widetilde{P}$ associates a symmetric multilinear map P is often called *polarization*.

We now let $V = \mathfrak{g}$ be the Lie algebra of a Lie group G. Then the adjoint representation of G on the Lie algebra $\mathfrak{g}$ induces an action of G on $S^k(\mathfrak{g}^*)$ for every k:

$$(gP)(X_1, \ldots, X_k) = P(\operatorname{Ad}(g^{-1})X_1, \ldots, \operatorname{Ad}(g^{-1})X_k),$$

where $X_1, \ldots, X_k \in \mathfrak{g}$ and $g \in G$.

Definition 9.2. (1) $P \in S^k(\mathfrak{g}^*)$ is called *invariant* if $gP = P$, $\forall g \in G$.

(2) The set of invariant elements in $S^*(\mathfrak{g}^*)$ is denoted $I^*(G)$ and $P \in I^k(G)$ is called an *invariant polynomial* (although it is a k-linear function).

Remark. $I^*(G) = \bigoplus_{k=0}^{\infty} I^k(G)$ is a subring of $S^*(\mathfrak{g}^*)$.

We now return to the situation of a principal G-bundle (E, π, M) with connection ω. Let $F_\omega \in \Omega^2(E, \mathfrak{g})$ be the associated curvature form. For $k \geq 1$ we have

$$F_\omega^k = F_\omega \wedge \ldots \wedge F_\omega \in \Omega^{2k}(E, \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}) \tag{9.4}$$

and since $P \in I^k(G)$ defines a linear map

$$P\colon \mathfrak{g} \otimes \ldots \otimes \mathfrak{g} \longrightarrow \mathbb{R}$$

we obtain a $2k$-form $P(F_\omega^k) \in \Omega^{2k}(E)$. Since F_ω is Ad-equivariant and horizontal (by Theorem 7.2) and since P is invariant, it follows that $P(F_\omega^k)$ is invariant and horizontal i.e. a basic $2k$-form. Hence by Corollary 6.13 there is a unique $2k$-form on M which pulls back to $P(F_\omega^k)$ by $\pi^*\colon \Omega^{2k}(M) \longrightarrow \Omega^{2k}(E)$.

Notation. The form on M corresponding to $P(F_\omega^k) \in \Omega^{2k}(E)$ is also denoted by $P(F_\omega^k)$, and is called the *characteristic form* corresponding to P.

Remark. In the notation of Theorem 7.2 the characteristic form is given in U_α by $P(F_{A_\alpha}^k) \in \Omega^{2k}(U_\alpha)$ where $A = \{A_\alpha\}_{\alpha\in\Sigma}$ are the local connection forms. In these terms we shall write $P(F_A^k) \in \Omega^{2k}(M)$ for the globally defined characteristic form.

Proposition 9.3.

(1) *$P(F_\omega^k) \in \Omega^{2k}(M)$ is a closed form, that is, $d(P(F_\omega^k)) = 0$.*

(2) *For $P \in I^k(G)$ and $Q \in I^l(G)$ we have*

$$P \bigcirc Q(F_\omega^{k+l}) = P(F_\omega^k) \wedge Q(F_\omega^l).$$

(3) *Consider a bundle map $(\bar f, f)$:*

$$\begin{tikzcd} E' \arrow[r, "\bar f"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Then for ω a connection in E and $\omega' = \bar f^*\omega$ the induced connection in E' we have

$$P(F_{\omega'}^k) = f^*(P(F_\omega^k)).$$

Proof. (1) Since $\pi^*\colon \Omega^k(M) \longrightarrow \Omega^k(E)$ is injective it suffices to prove that $d(P(F_\omega^k)) = 0$ in $\Omega^k(E)$. By Bianchi's identity we get

$$d(P(F_\omega^k)) = kP(dF_\omega \wedge F_\omega^{k-1}) = kP([F_\omega, \omega] \wedge F_\omega^{k-1}) \tag{9.5}$$

using the symmetry of P. But since the form in (9.5) is horizontal it is enough to show that it vanishes on sets of horizontal vectors. But this is obvious since $[F_\omega, \omega]$ vanishes on sets of horizontal vectors.

(2) For σ a permutation of $1, \ldots, k+l$ let T_σ denote the endomorphism of $\mathfrak{g}^{\otimes k+l}$ given by

$$T_\sigma(X_1 \otimes \ldots \otimes X_{k+l}) = X_{\sigma(1)} \otimes \ldots \otimes X_{\sigma(k+l)}, \qquad X_1, \ldots, X_{k+l} \in \mathfrak{g}.$$

Then by (5.2)

$$F_\omega^{k+l} = T_\sigma \circ F_\omega^{k+l} = T_\sigma \circ (F_\omega^k \wedge F_\omega^l)$$

since F_ω has degree two. Hence by (9.1)

$$\begin{aligned} (P \bigcirc Q)(F_\omega^{k+l}) &= \frac{1}{(k+l)!} \sum_\sigma (P \otimes Q) \circ T_\sigma \circ (F_\omega^{k+l}) \\ &= \frac{1}{(k+l)!} \sum_\sigma (P(F_\omega^k) \wedge Q(F_\omega^l)) \\ &= P(F_\omega^k) \wedge Q(F_\omega^l). \end{aligned}$$

(3) Since $F_{\omega'} = \bar{f}^*(F_\omega)$ we clearly have in $\Omega^*(E')$:

$$P(F_{\omega'}^k) = \bar{f}^* P(F_\omega^k).$$

Hence the statement follows from the injectivity of

$$\pi'^*\colon \Omega^{2k}(M') \longrightarrow \Omega^{2k}(E').$$

$\square$

Remark. In particular if $\varphi\colon E' \longrightarrow E$ is a bundle isomorphism and ω is a connection in E then for $\omega^\varphi = \varphi^*(\omega)$ we have

$$P(F_{\omega^\varphi}^k) = P(F_\omega^k)$$

It follows that gauge equivalent connections have the same characteristic forms.

Definition 9.4. Let (E, π, M) be a principal G-bundle with connection ω. For $P \in I^k(G)$ let

$$w(E; P) = [P(F_\omega^k)] \in H^{2k}(\Omega^*(M)) = H^{2k}_{\mathrm{dR}}(M)$$

denote the cohomology class of $P(F_\omega^k)$.

Notation. The mapping $w(E; -)\colon I^k(G) \longrightarrow H^{2k}(M)$ is called the *Chern-Weil homomorphism.* It is often just denoted by $w(-)$ if the bundle E is clear from the context. For $P \in I^k(G)$, $w(E; P) \in H^{2k}_{\mathrm{dR}}(M)$ is called the *characteristic class* for E corresponding to P.

Theorem 9.5. (1) *The cohomology class $w(E, P) \in H^{2k}_{\mathrm{dR}}(M)$ does not depend on the connection ω, and depends only on the isomorphism class of E.*

(2) $w(E; -)\colon (I^*(G), \bigcirc) \longrightarrow (H^*_{\mathrm{dR}}(M), \wedge)$ *is a ring homomorphism.*

(3) *For a bundle map*

$$\begin{tikzcd} E' \arrow[r, "\bar{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

we have $w(E', P) = f^* w(E, P)$.

For the proof we need the following version of the Poincaré Lemma which we will state without proof.

Let $h\colon \Omega^k(M \times \mathbb{R}) \longrightarrow \Omega^{k-1}(M)$ be the operator defined as follows: For $\omega \in \Omega^k(M \times \mathbb{R})$, $k \geq 1$ write $\omega = ds \wedge \alpha + \beta$, where s is the variable in $\mathbb{R}$, and put

$$h(\omega) = \int_{s=0}^{1} \alpha, \quad \text{and} \quad h(\omega) = 0, \quad \text{for} \quad \omega \in \Omega^0(M \times \mathbb{R}).$$

Lemma 9.6. *Let $i_0, i_1\colon M \longrightarrow M \times \mathbb{R}$ be the inclusions $i_0(p) = (p, 0)$, $i_1(p) = (p, 1)$, $p \in M$. Then*

$$dh(\omega) + h(d\omega) = i_1^*\omega - i_0^*\omega, \quad \textit{for} \quad \omega \in \Omega^*(M \times \mathbb{R}).$$

Proof of Theorem 9.5. We only have to show that $[P(F^k_\omega)]$ is independent of the choice of connection. Then the remaining statements follow from Proposition 9.3.

Let ω_0 and ω_1 be the two connections in E and consider the principal bundle $(E \times \mathbb{R}, \pi \times \mathrm{id}, M \times \mathbb{R})$. This has connection $\tilde{\omega} \in \Omega^1(E \times \mathbb{R}, \mathfrak{g})$ defined by

$$\tilde{\omega}_{(x,s)} = (1-s)\omega_{0x} + s\omega_{1x}, \qquad (x,s) \in E \times \mathbb{R}.$$

This is a connection by Proposition 6.5, and clearly $i^*_\nu(\tilde{\omega}) = \omega_\nu$, $\nu = 0,1$. Hence

$$i^*_\nu(F_{\tilde{\omega}}) = F_{\omega_\nu}, \qquad \nu = 0,1$$

and we obtain from Lemma 9.6:

$$\begin{aligned} dh(P(F^k_{\tilde{\omega}})) &= i^*_1 P(F^k_{\tilde{\omega}}) - i^*_0 P(F^k_{\tilde{\omega}}) \\ &= P(F^k_{\omega_1}) - P(F^k_{\omega_0}), \end{aligned} \tag{9.6}$$

since $d(P(F^k_{\tilde{\omega}})) = 0$ by Proposition 9.3. Hence by (9.6) $P(F_{\omega_1})$ and $P(F_{\omega_0})$ represent the same cohomology class in the de Rham complex. □

Motivated by Theorem 9.5 let us introduce the following:

Definition 9.7. A *characteristic class* c (with $\mathbb{R}$ coefficients) for a principal G-bundle associates to every principal G-bundle (E,π,M) a cohomology class $c(E) \in H^*_{\mathrm{dR}}(M)$ such that for any bundle map $(\bar{f}, f)\colon (E',\pi',M') \longrightarrow (E,\pi,M)$ we have

$$c(E') = f^*(c(E)).$$

If $c(E) \in H^l_{\mathrm{dR}}(M)$ then c is said to have degree l.

Remark. The set of characteristic classes (with $\mathbb{R}$ coefficients) is a ring denoted H^*_G (or $H^*_G(\mathbb{R})$).

Corollary 9.8. *Let $P \in I^k(G)$. Then $E \longmapsto w(E;P)$ defines a characteristic class $w(-,P) = w(P)$.*

One can prove the following theorem:

Theorem 9.9 (H. Cartan). *Let G be a compact Lie group. Then*

$$w: I^*(G) \longrightarrow H^*_G$$

is an isomorphism.

Remark. (1) Theorem 9.9 also determines the ring of characteristic classes in the case G is an arbitrary Lie group with finitely many connected components. In that case there is a maximal compact subgroup $K \subseteq G$ and one shows that the inclusion gives an isomorphism $H^*_G \cong H^*_K$.

(2) We can also define the ring of complex valued G-invariant polynomials on the Lie algebra $\mathfrak{g}$, using multi-linear functions (over $\mathbb{R}$) $P: \mathfrak{g} \times \cdots \times \mathfrak{g} \longrightarrow \mathbb{C}$. The ring of these is denoted $I^*_{\mathbb{C}}(G)$. The constructions in this chapter go through and we can thus define a Chern-Weil homomorphism for (E, π, M) a G-bundle with connection

$$w_{\mathbb{C}}(E, -): I^*_{\mathbb{C}}(G) \longrightarrow H(\Omega^*(M, \mathbb{C})) = H^*_{\mathrm{dR}}(M, \mathbb{C}).$$

Finally let us consider the behaviour of the Chern-Weil homomorphism in connection with extensions and reductions. For simplicity we restrict to the case where $H \subseteq G$ is the inclusion of a Lie subgroup and $\mathfrak{h} \subseteq \mathfrak{g}$ the corresponding inclusion of Lie algebras. Let (E, π, M) be a principal G-bundle, and suppose $F \subseteq E$ is a submanifold, so that $(F, \pi|_F, M)$ is a principal H-bundle. Then F is a reduction of E to H (cf. Chapter 4). The following lemma is a special case of Exercise 6.18.

Lemma 9.10. *If $\omega_F \in \Omega^1(F, \mathfrak{h})$ is a connection in F then there is a unique connection $\omega_E \in \Omega^1(E, \mathfrak{g})$, such that $\omega_{E|F} = \omega_F$.*

Proof. Let $\{U_\alpha\}_{\alpha \in \Sigma}$ be a covering of M with trivializations of $F|_{U_\alpha} \longrightarrow U_\alpha \times H$ with transition functions $\{h_{\alpha\beta}\}$, $h_{\alpha\beta}: U_\alpha \cap U_\beta \longrightarrow H$. Then $\{h_{\alpha\beta}\}$ is also a set of transition functions for E. The form ω_F is determined by $\{A_\alpha\}_{\alpha \in \Sigma}$, $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ such that $A_\beta = \mathrm{Ad}(h^{-1}_{\alpha\beta}) \circ A_\beta + h^*_{\alpha\beta}\omega^H_0$. Since $h_{\alpha\beta}$ maps to $H \subseteq G$ and $\omega^G_{0|H} = \omega^H_0$ we have $h^*_{\alpha\beta}\omega^H_0 = h^*_{\alpha\beta}\omega^G_0$. Hence $\{A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})\}$ determines a unique connection on E by Corollary 6.9 □

The proof of the following proposition is straight-forward and is left as an exercise.

Proposition 9.11. *Let (E, π, M) be a G-bundle with a reduction $(F, \pi|_F, M)$ and suppose $\omega \in \Omega^1(E, \mathfrak{g})$ is induced from a connection $\omega_F \in \Omega^1(F, \mathfrak{h})$ as above. Then for $P \in I^k(G)$ we have*

$$P(F^k_{\omega_F}) = P(F^k_{\omega_E}) \quad \text{in} \quad \Omega^{2k}(M).$$

In particular we have the following commutative diagram

$$\begin{tikzcd}
I^*(G) \arrow[rr, "\mathrm{res}"] \arrow[dr, "w(E,-)"'] & & I^*(H) \arrow[dl, "w(F,-)"] \\
& H^*_{\mathrm{dR}}(M) &
\end{tikzcd}$$

where res *denotes the restriction map of polynomials on the Lie algebra $\mathfrak{g}$ to the Lie algebra $\mathfrak{h}$.*

10. Examples of Invariant Polynomials and Characteristic Classes

We now give some examples of invariant polynomials for some classical groups. In all cases we shall exhibit the polynomial function $v \longmapsto P(v, \ldots, v)$, $v \in \mathfrak{g}$ for each $P \in I^k(G)$.

Example 10.1. $G = \mathrm{GL}(n, \mathbb{R})$. The Lie algebra is $\mathfrak{g} = M(n, \mathbb{R})$, the set of $n \times n$ real matrices with Lie bracket

$$[A, B] = AB - BA, \qquad A, B \in M(n, \mathbb{R}),$$

and the adjoint representation

$$\mathrm{Ad}(g)(A) = gAg^{-1}, \qquad A \in M(n, \mathbb{R}),\ g \in \mathrm{GL}(n, \mathbb{R}).$$

For k a positive integer we let $P_{k/2}$ denote the homogeneous polynomial of degree k which is the coefficient of λ^{n-k} for the polynomial in λ given by

$$\det\left(\lambda I - \frac{1}{2\pi}A\right) = \sum_{k=0}^{n} P_{k/2}(A, \ldots, A)\lambda^{n-k}, \qquad A \in M(n, \mathbb{R}).$$

The polynomial $P_{k/2}$, called the $k/2$-th *Pontrjagin polynomial*, is clearly Ad-invariant. For (E, π, M) a principal $\mathrm{GL}(n, \mathbb{R})$-bundle

$$p_{k/2}(E) = w(E, P_{k/2}) \in H^{2k}_{\mathrm{dR}}(M)$$

is called the *Pontrjagin class* for E. For V an n-dimensional real vector bundle on M we write

$$p_{k/2}(V) = p_{k/2}(F(V))$$

where $F(V)$ is the frame bundle.

Example 10.2. $G = \mathrm{O}(n) \subseteq \mathrm{GL}(n,\mathbb{R})$ the orthogonal group af matrices g satisfying $g^t g = I$. The Lie algebra is $\mathfrak{o}(n) \subseteq M(n,\mathbb{R})$ of skew-symmetric matrices, i.e.

$$\mathfrak{o}(n) = \{A \in M(n,\mathbb{R}) \mid A + A^t = 0\}. \tag{10.1}$$

hence, by transposing

$$\det\left(\lambda I + \frac{1}{2\pi}A\right) = \det\left(\lambda I - \frac{1}{2\pi}A\right), \qquad A \in \mathfrak{o}(n).$$

and it follows that the restriction of $P_{k/2}$ to $\mathfrak{o}(n)$ vanishes for k odd. We shall therefore only consider $P_l \in I^{2l}(\mathrm{O}(n))$ for $l = 0,\dots,[n/2]$. Notice that since every vector bundle can be given a Riemannian metric it follows that the frame bundle has a reduction to $\mathrm{O}(n)$. Therefore also for any $\mathrm{GL}(n,\mathbb{R})$-bundle E we have $p_{k/2}(E) = 0$ for k odd, although the representing characteristic form is not necessarily equal to zero.

Example 10.3. $G = \mathrm{SO}(2m) \subseteq \mathrm{O}(2m)$, the subgroup of orthogonal matrices satisfying $\det(g) = 1$. The Lie algebra is $\mathfrak{so}(2m) = \mathfrak{o}(2m)$ given by (10.1). Hence the Pontrjagin polynomials

$$P_l \in I^{2l}(\mathrm{SO}(2m)),\, l = 0, 1, \dots, m$$

are also invariant polynomials in this case. But there is another homogeneous polynomial Pf called the "Pfaffian polynomial" of degree m given by

$$\begin{aligned}&\mathrm{Pf}(A,\dots,A)\\ &\quad= \frac{1}{2^{2m}\pi^m m!}\sum_\sigma (\operatorname{sgn}\sigma) a_{\sigma1\sigma2}a_{\sigma3\sigma4}\cdots a_{\sigma(2m-1)\sigma(2m)},\end{aligned} \tag{10.2}$$

where σ runs through the set of permutations of $1,\dots 2m$, and where $A = (a_{ij})$ satifies $a_{ij} = -a_{ji}$.

Let us show that Pf is Ad-invariant: Let $g = (g_{ij}) \in \mathrm{O}(2m)$ and put

$$gAg^{-1} = gAg^t = A' = (a'_{ij})$$

that is

$$a'_{ij} = \sum_{k_1k_2} g_{ik_1}a_{k_1k_2}g_{jk_2}.$$

Then for $\mathrm{Pf}' = 2^{2m}\pi^m m!\,\mathrm{Pf}$ we have

$$\begin{aligned}
&\mathrm{Pf}'\,(A', \dots, A') \\
&\quad = \sum_{k_1,\dots,k_{2m}} a_{k_1k_2}\cdots a_{k_{2m-1}k_{2m}} \sum_\sigma \mathrm{sgn}(\sigma) g_{\sigma 1 k_1}\cdots g_{\sigma(2m)k_{2m}}.
\end{aligned}$$

In this sum the coefficient to $a_{k_1k_2}\cdots a_{k_{2m-1}k_{2m}}$ is just the determinant of (g_{ik_j}). This determinant is clearly zero unless $(k_1,\dots,k_{2m})$ is a permutation of $(1,\dots,2m)$. Hence

$$\begin{aligned}
\mathrm{Pf}'\,(A', \dots, A') &= \sum_\sigma \det(g_{i\sigma j}) a_{\sigma 1\sigma 2}\cdots a_{\sigma(2m-1)\sigma(2m)} \\
&= \det(g_{ij}) \sum_\sigma \mathrm{sgn}(\sigma) a_{\sigma 1\sigma 2}\cdots a_{\sigma(2m-1)\sigma(2m)} \\
&= \det(g)\,\mathrm{Pf}'\,(A,\dots,A).
\end{aligned}$$

That is

$$\mathrm{Pf}(A', \dots, A') = \det(g)\,\mathrm{Pf}(A, \dots A) \tag{10.3}$$

for all $A \in \mathfrak{o}(2m)$, $g \in \mathrm{O}(2m)$. In particular Pf is an invariant polynomial for $\mathrm{SO}(2m)$, but it is not an invariant polynomial for $\mathrm{O}(2m)$ since

$$\mathrm{Pf}(A', \dots, A') = -\,\mathrm{Pf}(A, \dots A)$$

if $\det(g) = -1$.

For an $\mathrm{SO}(2m)$-bundle (E, π, M) the characteristic class

$$e(E) = w(E, \mathrm{Pf}) \in H^{2m}_{\mathrm{dR}}(M) \tag{10.4}$$

is called the *Euler class* for E. If E is the reduction to $\mathrm{SO}(2m)$ of the frame bundle of an oriented vector bundle V on M then $e(V) = e(E)$ is called the Euler class of V. One can show that for a compact oriented Riemannian manifold of dimension $2m$ the Euler-Poincaré characteristic $\chi(M)$ satisfies

$$\chi(M) = \langle e(TM), [M]\rangle = \int_M \mathrm{Pf}(F_\omega^m), \tag{10.5}$$

where TM is the tangent bundle of M and ω is the connection for the reduction to $\mathrm{SO}(2m)$ of the orthogonal frame bundle of TM determined by the orientation. (Usually the connection is chosen to be the Levi-Civita connection). The formula (10.5) is the higher-dimensional *Gauss-Bonnet Theorem.*

Example 10.4. $G = \mathrm{GL}(n, \mathbb{C})$. The Lie algebra is $M(n, \mathbb{C})$, the set of $n \times n$ complex matrices. As in the remark at the end of chapter 9 we shall consider the polynomials with complex values C_k, $k = 0, 1, \ldots, n$, given as the coefficients to λ^{n-k} in the polynomial in λ:

$$\det\left(\lambda I - \frac{1}{2\pi i} A\right) = \sum_k C_k(A, \ldots, A)\lambda^{n-k}, \qquad A \in M(n, \mathbb{C}), \tag{10.6}$$

with $i = \sqrt{-1}$. For (E, π, M) a principal $\mathrm{GL}(n, \mathbb{C})$-bundle we thus obtain characteristic classes in de Rham cohomology

$$c_k(E) = w(E, C_k) \in H^{2k}_{\mathrm{dR}}(M, \mathbb{C}) \tag{10.7}$$

where the differential forms have complex values. The polynomials C_k are called the *Chern polynomials* and the classes in (10.7) are called the *Chern classes* of E. Again for V a complex vector bundle over M we have an associated frame bundle of complex frames $F(V)$ which is a principal $\mathrm{GL}(n, \mathbb{C})$-bundle and we define

$$c_k(V) = c_k(F(V))$$

the Chern classes of a complex vector bundle V. Notice that the restriction of C_k to $M(n, \mathbb{R})$ satisfies

$$i^k C_k(A, \ldots, A) = P_{k/2}(A, \ldots, A), \qquad A \in M(n, \mathbb{R}).$$

Hence if we extend a principal $\mathrm{GL}(n, \mathbb{R})$-bundle E to a principal $\mathrm{GL}(n, \mathbb{C})$-bundle $E_{\mathbb{C}}$ then for $k = 2l$ we have in $H^{4l}_{\mathrm{dR}}(M, \mathbb{C})$:

$$(-1)^l c_{2l}(E_{\mathbb{C}}) = p_l(E). \tag{10.8}$$

Example 10.5. $G = \mathrm{U}(n) \subseteq \mathrm{GL}(n, \mathbb{C})$ the subgroup of unitary matrices, ie. the complex matrices g satisfying $g\bar{g}^t = I$ where $\bar{g}$ denotes the complex conjugate of g. The Lie algebra is $\mathfrak{u}(n) \subseteq M(n, \mathbb{C})$ of skew-Hermitian matrices, ie.

$$\mathfrak{u}(n) = \{A \in M(n, \mathbb{C}) \mid A + \overline{A}^t = 0\}$$

In particular we have for $A \in \mathfrak{u}(n)$

$$\det\left(\lambda I - \frac{1}{2\pi i}A\right) = \det\left(\lambda I + \frac{1}{2\pi i}\bar{A}^t\right) = \overline{\det\left(\bar{\lambda} I - \frac{1}{2\pi i}A\right)}.$$

That is, $C_k(A, \ldots, A) \in \mathbb{R}$ for $A \in \mathfrak{u}(n)$, hence C_k is a real-valued polynomial on $\mathfrak{u}(n)$. For (E, π, M) a $\mathrm{U}(n)$-bundle the Chern classes are therefore real cohomology classes.

Since every complex vector bundle can be given a Hermitian metric it follows that any principal $\mathrm{GL}(n, \mathbb{C})$-bundle has a reduction to $\mathrm{U}(n)$ and hence all Chern classes for a $\mathrm{GL}(n, \mathbb{C})$-bundle are real cohomology classes.

Finally let us calculate the first Chern class in a non-trivial case.

Example 10.6. Consider the complex Hopf-bundle over $\mathbb{C}P^n$: This is the $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$-bundle $(H_{\mathbb{C}}, \gamma, \mathbb{C}P^n)$ with total space $H_{\mathbb{C}} = \mathbb{C}^{n+1} \setminus \{0\}$ and where γ is the natural projection map

$$\gamma(z_0, \ldots, z_n) = [z_0, \ldots, z_n], \qquad (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}.$$

Notice that the Lie algebra of $\mathbb{C}^*$ is just the abelian Lie algebra $\mathbb{C}$ so that a connection in $H_{\mathbb{C}}$ is a complex valued 1-form on $\mathbb{C}^{n+1} \setminus \{0\}$. We claim that

$$\omega = \frac{\bar{z}^t dz}{|z|^2}$$

is a connection. That is, we must check (1) and (2) in the definition of a connection.

(1) For fixed $z \in \mathbb{C}^{n+1} \setminus \{0\}$ the map $v_z \colon \mathbb{C} \longrightarrow \mathbb{C}^{n+1}$ given by $v_z(\lambda) = z\lambda$ satisfies

$$\omega_z \circ v_z(\lambda) = \frac{\bar{z} dz(z\lambda)}{|z|^2} = \lambda$$

as required.

(2) For fixed $\lambda \in \mathbb{C}^*$ we have

$$R_\lambda^* \omega = \frac{\bar{z}^t dz \bar{\lambda}\lambda}{|z\lambda|^2} = \omega$$

as required since $\mathbb{C}^*$ is abelian.

Hence ω is a connection. Again since $\mathbb{C}^*$ is abelian the curvature is given by

$$F_\omega = d\omega = \frac{1}{|z|^2} d\bar{z}^t \wedge dz + d\left(\frac{1}{|z|^2}\right) \wedge \bar{z}^t dz$$

which is also a basic form, i.e. the pullback of a form on $\mathbb{C}P^n$. Now the inclusion $S^{2n+1} \subset \mathbb{C}^{n+1} \backslash \{0\}$ induces a diffeomorphism $S^{2n+1}/\mathrm{U}(1) \cong \mathbb{C}P^n$ and since $|z|^2 = 1$ on S^{2n+1} we conclude that as a form on $S^{2n+1}/\mathrm{U}(1)$ the curvature is given by

$$F_\omega = d\bar{z}^t \wedge dz.$$

Notice also that

$$\overline{F}_\omega = dz^t \wedge d\bar{z} = -F_\omega$$

so F_ω takes on purely imaginary values. Finally F_ω is invariant under the action of $\mathrm{U}(n+1)$ on $\mathbb{C}P^n$ since

$$g^* F_\omega = (d\bar{z}^t)\bar{g}^t \wedge g dz = d\bar{z}^t \wedge dz, \qquad g \in \mathrm{U}(n+1).$$

In the notation of Example A15 $F_\omega = i \operatorname{Im} F_\omega$ now corresponds to the $\mathrm{U}(n)$ invariant real alternating 2-form $2i\kappa$ defined on $\mathbb{C}^n = \operatorname{span}\{e_1, \ldots, e_n\}$ since

$$d\bar{z}^t \wedge dz(e_1, ie_1) = i - (-i) = 2i.$$

Now the first Chern polynomial on $\mathfrak{gl}(1, \mathbb{C}) = \mathbb{C}$ is given by

$$C_1(A) = -\frac{1}{2\pi i} A, \qquad A \in \mathbb{C}.$$

It follows that

$$\begin{aligned} c_1(H_\mathbb{C}) &= \left[-\frac{1}{2\pi i} d\bar{z}^t \wedge dz\right] \in H^2_{\mathrm{dR}}(\mathbb{C}P^n) \\ &= -\frac{1}{\pi}\kappa \in \mathrm{Alt}^2_\mathbb{R}(\mathbb{C}^n)^{\mathrm{U}(n)} \end{aligned}$$

so in particular $c_1(H_{\mathbb{C}}) \neq 0$. The constant in $C_1(A)$ is chosen such that for $n = 1$

$$\int_{\mathbb{C}P^1} c_1(F_\omega) = -1. \tag{10.9}$$

In order to see this we first observe that in the principal U(1)-bundle $\gamma\colon S^3 \longrightarrow \mathbb{C}P^1$ we have

$$\int_{\gamma^{-1}[z]} \omega = \int_{|\lambda|=1} \frac{d\lambda}{\lambda} = 2\pi i \qquad \forall z \in S^3$$

so that by Fubini's Theorem

$$\int_{S^3} \omega \wedge F_\omega = 2\pi i \int_{\mathbb{C}P^1} F_\omega.$$

On the other hand $\omega \wedge F_\omega$ is a U(3)-invariant 3-form on S^3 whose value at the tangent frame $\{ie_0, e_1, ie_1\} \subseteq T_{e_0}S^3$ is

$$(\omega \wedge F_\omega)(ie_0, e_1, ie_1) = (dz_0 \wedge d\overline{z_1} \wedge dz_1)(ie_0, e, ie_1) = -2.$$

Hence $\omega \wedge F_\omega$ is -2 times the volume form on S^3 so that

$$2\pi i \int_{\mathbb{C}P^1} F_\omega = -2\,\mathrm{vol}(S^3) = -4\pi^2$$

i.e.

$$\int_{\mathbb{C}P^1} F_\omega = 2\pi i$$

which proves (10.9).

A. Cohomology of Homogeneous Spaces

In this section we show how to calculate de Rham cohomology of compact homogeneous spaces. For notation see [D].

In the following G is a Lie group and $H \subseteq G$ is a closed Lie subgroup, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ respectively. We shall study the homogeneous manifold $M = G/H$ with projection $\pi\colon G \longrightarrow M$.

Proposition A1. (1) *The following sequence of vector spaces*

$$0 \longrightarrow \mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{\pi_*} T_{eH}(G/H) \longrightarrow 0$$

is exact.

(2) *Given $h \in H$ we have the following commutative diagram.*

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{h} \arrow[r] \arrow[d, "\mathrm{Ad}(h)"'] & \mathfrak{g} \arrow[r, "\pi_*"] \arrow[d, "\mathrm{Ad}(h)"] & T_{eH}(G/H) \arrow[r] \arrow[d, "L_{h*}"] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{g} \arrow[r, "\pi_*"] & T_{eH}(G/H) \arrow[r] & 0
\end{tikzcd}$$

Proof. (1) The differentiable structure on G/H is defined as follows: For $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $U_{\mathfrak{m}} \subseteq \mathfrak{m}$ a suitable neighborhood of 0, the composite mapping

$$U_{\mathfrak{m}} \xrightarrow{\exp} G \xrightarrow{\pi} G/H$$

is a diffeomorphism onto an open neighborhood of eH (see e.g. [D, Theorem 9.43]. Consequently $\pi_*\colon \mathfrak{m} \longrightarrow T_{eH}(G/H)$ is an isomorphism, this shows the claim.

(2) It is clear that $\mathrm{Ad}(h)$ on $\mathfrak{h}$ is the restriction of $\mathrm{Ad}(h)$ on $\mathfrak{g}$. The commutativity of the square to the right follows from the following commutative diagram

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\sigma_h"'] & G/H \arrow[d, "L_h"] \\ G \arrow[r, "\pi"] & G/H \end{tikzcd}$$

where $\sigma_h(x) = hxh^{-1}$ and $L_h(xH) = hxH$. □

Remark. We see from the above proposition that π_* induces an isomorphism

$$\pi_*\colon \mathfrak{g}/\mathfrak{h} \xrightarrow{\cong} T_{eH}(G/H)$$

and that the isotropy representation of H on $T_{eH}(G/H)$, that is, $h \longmapsto L_{h*}$, is identified with the adjoint representation on $\mathfrak{g}/\mathfrak{h}$.

Notation. For V a finite dimensional vector space let $\mathrm{Alt}^k(V)$, $k \in \mathbb{N}$, denote the vector space of alternating k-linear forms on V. For $H \subseteq G$ as above with Lie algebras $\mathfrak{h} \subseteq \mathfrak{g}$ let

$$\mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H \subseteq \mathrm{Alt}^k(\mathfrak{g})$$

be the subspace consisting of $\alpha \in \mathrm{Alt}^k(\mathfrak{g})$ satisfying

(1) $\alpha(v_1, \ldots, v_k) = 0$ if at least one $v_i \in \mathfrak{h}$

(2) $\alpha(\mathrm{Ad}(h)v_1, \ldots, \mathrm{Ad}(h)v_k) = \alpha(v_1, \ldots, v_k)$ for all $h \in H$.

Finally for a manifold M let $\Omega^k(M)$, $k \in \mathbb{N}$ denote the vector space of differential forms of degree k on M. If $M = G/H$ as above then

$$\Omega^k(M)^G \subseteq \Omega^k(M),$$

denotes the subspace of G-invariant forms, that is, forms ω satisfying

$$L_g^*\omega = \omega \qquad \text{for all } g \in G,$$

where $L_g\colon M \longrightarrow M$ is given by $L_g(xH) = gxH$, for all $x \in G$.

Proposition A2. *Let $H \subseteq G$ and $M = G/H$ as above.*

(1) *There is a natural isomorphism of vector spaces*

$$\iota\colon \Omega^k(M)^G \xrightarrow{\cong} \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H.$$

(2) *Furthermore ι is a chain map, where the differential on the right hand side is defined by*

$$(d\alpha)(v_1,\dots,v_{k+1}) = \sum_{i<j}(-1)^{i+j}\alpha([v_i,v_j],v_1,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_{k+1}),$$

$$\alpha \in \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H, \quad v_1,\dots,v_{k+1} \in \mathfrak{g}.$$

Proof. (1) As before we let $\pi\colon G \longrightarrow G/H$ be the canonical projection map and let $\omega \in \Omega^k(M)^G$ be a G-invariant form. We define $\iota(\omega)$ by

$$\iota(\omega) = (\pi^*\omega)_e \in \mathrm{Alt}^k(\mathfrak{g})$$

and we clearly have that $\iota(\omega) \in \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H$. Furthermore it is clear that ι is injective because $\omega \in \Omega^k(M)^G$ satisfies

$$\omega_{gH}(v_1,\dots,v_k) = \omega_{eH}((L_{g^{-1}})_*v_1,\dots,(L_{g^{-1}})_*v_k),$$

for all $g \in G$ and $v_1,\dots,v_k \in T_{gH}M$. For the surjectivity of ι let $\alpha \in \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H$ be given and we define $\omega_{gH} \in \mathrm{Alt}^k(T_{gH}M)$ by

$$\omega_{gH}(v_1,\dots,v_k) = \alpha((L_{g^{-1}})_*v_1,\dots,(L_{g^{-1}})_*v_k),$$

$$g \in G \quad v_1,\dots,v_k \in T_{gH}M.$$

Since $\alpha \in \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H$ it easily follows that ω_{gH} is independent of the choice of $g \in G$ and of the choice of representatives for $(L_{g^{-1}})_*v_i$ in $\mathfrak{g}$. To see that ω is differentiable, we notice that because ω by definition is G-invariant, it is enough to show that ω is differentiable in a neighborhood of eH. But in this neighborhood we can find a local cross section, that is, a submanifold $U \subseteq G$ with $e \in U$ such that $\pi\colon U \longrightarrow \pi(U)$ is a diffeomorphism. Thus it is enough to see that $\pi^*(\omega)$ is differentiable. For this we choose a basis $\{X_1,\dots,X_n\}$ for $\mathfrak{g}$ and let $\{\widetilde{X}_1,\dots,\widetilde{X}_n\}$ be the corresponding left invariant vector fields on G. Because these are smooth, the 1-forms $\{\gamma_1,\dots,\gamma_n\}$ on G defined by

$$\gamma_i(\widetilde{X}_j) = \delta_{ij}$$

are also smooth. But $\pi^*(\omega)$ is G-invariant so it can be written as a linear combination of the forms

$$\{\gamma_{i_1}, \wedge \ldots \wedge \gamma_{i_k} \mid 1 \leq i_1 \leq \ldots \leq i_k \leq n\}$$

with constant coefficients. It follows that $\pi^*(\omega)$ is smooth.

(2) Because $d\pi^* = \pi^* d$ and $\pi^* \colon \Omega^*(M)^G \longrightarrow \Omega^*(G)^G$ is injective it is enough to show the formula on G. As before let $\{\widetilde{X}_1, \ldots, \widetilde{X}_n\}$ be a basis for the left invariant vector fields on G, so that $\{\widetilde{X}_{1g}, \ldots, \widetilde{X}_{ng}\}$ is a basis for $T_g G$ for every $g \in G$. It suffices to show the formula

$$\begin{aligned}(d\omega)&(\widetilde{X}_{l_1}, \ldots, \widetilde{X}_{l_{k+1}}) \\ &= \sum_{i<j} (-1)^{i+j} \omega([\widetilde{X}_{l_i}, \widetilde{X}_{l_j}], \ldots, \widehat{\widetilde{X}}_{l_i}, \ldots, \widehat{\widetilde{X}}_{l_j}, \ldots \widetilde{X}_{l_{k+1}})\end{aligned}$$

for all tuples $(l_1, \ldots, l_{k+1})$ satisfying $1 \leq l_{i_1} \leq \ldots \leq l_{i_{k+1}} \leq n$. This follows from the formula for $d\omega$ because the remaining terms in the formula are directional derivatives of functions on the form

$$\omega(\widetilde{X}_{l_1}, \ldots, \widehat{\widetilde{X}}_{l_j}, \ldots, \widetilde{X}_{l_{k+1}})$$

which are clearly constant when ω and $\widetilde{X}_j$ are G-invariant. ☐

Remark. It follows that the formula in (2) defines a form in $\mathrm{Alt}^{k+1}(\mathfrak{g}/\mathfrak{h})^H$

Exercise A3. Show the claim in the remark above directly. (Hint: First show that for $\alpha \in \mathrm{Alt}^k(\mathfrak{g}/\mathfrak{h})^H$ and $X \in \mathfrak{h}$ we have

$$\sum_{i=1}^{k} \alpha(v_1, \ldots, v_{i-1}, [X, v_i], v_{i+1}, \ldots, v_k) = 0, \quad \text{for all } v_1, \ldots, v_k \in \mathfrak{g}.$$

Corollary A4. *If H is compact and connected then $M = G/H$ has a G-invariant volume form, that is, a nonzero element $v_M \in \Omega^n(M)^G$, $n = \dim(M)$, and v_M is unique up to a scalar multiple.*

Proof. Since $\mathrm{Alt}^n(\mathfrak{g}/\mathfrak{h})$ is 1-dimensional we have $\dim(\mathrm{Alt}^n(\mathfrak{g}/\mathfrak{h}))^H \leq 1$ with equality if and only if $\det(\mathrm{Ad}(h)) = 1$ for all $h \in H$, where $\mathrm{Ad}(h) \colon \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h}$ is induced by the diagram in Proposition A1 (2). But $\lambda(h) = \det(\mathrm{Ad}(h))$

defines a continuous homomorphism $\lambda\colon H \longrightarrow \mathbb{R}\setminus\{0\}$, and since H is compact $\lambda(h) = \pm 1$, for all $h \in H$. Hence λ is locally constant and since H is connected $\lambda \equiv 1$. $\square$

Example A5. If we identify $\mathrm{SO}(n-1)$ with the subgroup of matrices in $\mathrm{SO}(n)$ of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & h \end{bmatrix}, \qquad h \in \mathrm{SO}(n-1),$$

there is a natural diffeomorphism

$$\pi\colon \mathrm{SO}(n)/\mathrm{SO}(n-1) \xrightarrow{\cong} S^{n-1} \subseteq \mathbb{R}^n$$

given by $\pi(g\,\mathrm{SO}(n-1)) = ge_1$, $g \in \mathrm{SO}(n)$, where $e_1 = (1, 0\ldots, 0)$ (cf. [DG, Exercise 9.45]). Notice that π commutes with the $\mathrm{SO}(n)$ action, and the $\mathrm{SO}(n)$ action on $\mathbb{R}^n$ is given by matrix multiplication. It follows from Corollary A4 that S^{n-1} has a $\mathrm{SO}(n)$-invariant volume form $v_{S^{n-1}} \in \Omega^n(S^{n-1})$.

Exercise A6. Given the coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ show that

$$v_{S^{n-1}} = c\sum_{i=1}^{n}(-1)^i x_i dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_n, \qquad c \in \mathbb{R}\setminus\{0\}.$$

Proposition A7. *Let G be a compact Lie group and $\rho\colon G \longrightarrow \mathrm{GL}(V)$ a representation on a finite dimensional vector space (that is, ρ is a Lie group homomorphism). Then there exists an inner product on V such that $\rho(g)$ is orthogonal for all $g \in G$, that is,*

$$\langle \rho(g)v, \rho(g)w\rangle = \langle v, w\rangle \qquad \text{for all } v, w \in V,\ g \in G.$$

Proof. Pick an arbitrary inner product $\langle\cdot,\cdot\rangle$ on V and choose an orientation on G. It follows from Corollary A4 with $H = \{e\}$ that there is a unique volume form v_G on G that satisfies

$$\int_G v_G = 1.$$

In fact, for $v'_G \in \Omega^n(G)^G$ a volume form we have (as G is compact) that $\mathrm{Vol} = \int_G v'_G > 0$ and it follows that $v_G = v'_G/\mathrm{Vol}$. We now define the inner product on V by

$$\langle v, w\rangle = \int_G \langle \rho(g^{-1})v, \rho(g^{-1})w\rangle v_G.$$

Then, for an arbritrary $g' \in G$ we have for $v, w \in V$:

$$
\begin{aligned}
\langle \rho({g'}^{-1})v, \rho({g'}^{-1})w\rangle &= \int_G \langle \rho((g'g)^{-1})v, \rho((g'g)^{-1})w\rangle v_G \\
&= \int_G \langle \rho(g^{-1})v, \rho(g^{-1})w\rangle (L_{{g'}^{-1}})^* v_G \\
&= \int_G \langle \rho(g^{-1})v, \rho(g^{-1})w\rangle v_G \\
&= \langle v, w\rangle .
\end{aligned}
$$

□

Remark. Let $\mathrm{O}(V) \subseteq \mathrm{GL}(V)$ denote the group of orthogonal transformations with respect to the inner product $\langle \cdot, \cdot \rangle$. Then $\mathrm{O}(V)$ is compact and therefore a closed Lie subgroup in $\mathrm{GL}(V)$. It follows that $\rho: G \longrightarrow \mathrm{O}(V)$ is a Lie group homomorphism. If G is connected $\rho(g) \in \mathrm{SO}(V)$ for all $g \in G$, where $\mathrm{SO}(V) \subseteq \mathrm{O}(V)$ is the subgroup of orthogonal transformations with determinant 1.

Theorem A8. *Let $M = G/H$, with G and $H \subseteq G$ as above and G compact and connected. We then have the following.*

(1) *The inclusion $i: \Omega^*(M)^G \longrightarrow \Omega^*(M)$ induces an isomorphism on cohomology.*

(2) *There is a natural isomorphism*

$$H^*_{\mathrm{dR}}(M) \cong H^*(\mathrm{Alt}^*(\mathfrak{g}/\mathfrak{h})^H)$$

Proof. Clearly (2) follows from (1) and Proposition A2.

To show (1) we pick, according to Proposition A7, an inner product on the Lie algebra $\mathfrak{g}$ of G which is invariant under the adjoint representation $\mathrm{Ad}: G \longrightarrow \mathrm{GL}(\mathfrak{g})$, and since G is assumed to be connected we have $\mathrm{Ad}(g) \in \mathrm{SO}(\mathfrak{g})$ for all $g \in G$. By multiplying the inner product by a positive scalar we can assume that the open unit disc

$$B = \{X \in \mathfrak{g} \mid \langle X, X\rangle < 1\}$$

is mapped diffeomorphically under the exponential map onto an open neighborhood U of $e \in G$. As in the proof of Proposition A7 we let v_G

be the unique left invariant volume form on G that satisfies

$$\int_G v_G = 1.$$

Finally for $g \in G$ we let $L_g\colon M \longrightarrow M$ denote the diffeomorphism given by left multiplication with $g \in G$.

For $\omega \in \Omega^k(M)$ we will define a G-invariant k-form $\bar{\omega} \in \Omega^k(M)^G$, in the following way: Let $p \in M$ and $v_1, \ldots, v_k \in T_pM$ and notice that

$$g \longmapsto (L_g^*\omega)_p(v_1, \ldots, v_k) = \omega_{gp}(L_{g*}v_1, \ldots, L_{g*}v_k)$$

defines a differentiable function on G. We now put

$$\bar{\omega}_p(v_1, \ldots, v_k) = \int_{g \in G} \omega_{gp}(L_{g*}v_1, \ldots, L_{g*}v_k) v_G$$

or for short

$$\bar{\omega} = \int_{g \in G} (L_g^*\omega) v_G.$$

We shall show the following three claims:

(i) $\bar{\omega}$ is a G-invariant differential k-form.

(ii) If $\omega \in \Omega^k(M)^G$ then $\bar{\omega} = \omega$.

(iii) There are linear operators

$$S^0 = 0, S^k\colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \qquad k = 1, 2, \ldots$$

satisfying

$$dS^k\omega + S^{k+1}d\omega = \bar{\omega} - \omega \qquad \text{for all } \omega \in \Omega^k(M),\ k = 0, 1, 2, \ldots.$$

The theorem then follows from (i)–(iii).

(i) That $\bar{\omega}$ is differentiable is seen the same way as in the proof of Proposition A2. To see that $\bar{\omega}$ is G-invariant we first notice that v_G is a right G-invariant form on G, that is $R_g^*(v_G) = v_G$ for all $g \in G$. This follows from the fact that $R_{g'}^* v_G$ again is left invariant so that

$$(R_{g'})^* v_G = \lambda(g') v_G, \qquad \lambda(g') \in \mathbb{R} \setminus \{0\},$$

where $\lambda\colon G \longrightarrow \mathbb{R}\backslash\{0\}$ is a differentiable homomorphism hence is constantly equal to 1 since G is compact and connected. We now get

$$\begin{aligned}
(L_{g'})^*\bar{\omega} &= \int_{g\in G} (L_{g'}^* L_g^*\omega)v_G \\
&= \int_{g\in G} ((L_{gg'})^*\omega)v_G \\
&= \int_{x\in G} (L_x^*\omega)((R_{g'^{-1}})^* v_G) \\
&= \int_{x\in G} (L_x^*\omega)v_G = \bar{\omega},
\end{aligned}$$

where $x = gg'$ that is $g = x(g')^{-1} = R_{g'^{-1}}(x)$, it follows that $L_{g'^{-1}}^*(\bar{\omega}) = \bar{\omega}$, which was to be proven.

(ii) For $\omega \in \Omega(M)^G$ we have

$$\bar{\omega} = \int_{g\in G} L_g^*(\omega)v_G = \int_{g\in G} \omega v_G = \omega \int_G v_G = \omega$$

(iii) For this we use the neighborhood $U = \exp B$. We first notice that

$$\langle U\rangle = \bigcup_i U^i = \bigcup_{i=1}^{\infty} \underbrace{U\cdots U}_{i}$$

is an open subgroup in G and since G is connected we have $\langle U\rangle = G$ (because the complement consists of cosets which are also open).

As $U^i \subseteq U^{i+1}$ and G is compact, there is a $j \in \mathbb{N}$ such that $G = U^j$. We get that $\{L_g U \mid g \in U^j\}$ is an open covering of G hence by compactness we obtain $G = \bigcup_{i=1}^{l} g_i U$. Now define inductively $W_1, \dots, W_l$ by

$$W_1 = g_1 U, \dots, W_{i+1} = g_{i+1}U - \Big(\bigcup_{\nu=1}^{i} W_\nu\Big).$$

Then $G = W_1 \cup \ldots \cup W_l$ is a disjoint union and

$$V = \mathrm{int}(W_1) \cup \ldots \cup \mathrm{int}(W_l)$$

is a open subset such that $\partial V \subseteq \bigcup_{i=1}^{l} g_i \partial U$, is a union of $(m-1)$-dimensional submanifolds in G, where $m = \dim G$. For f an arbitrary continuous function on G we thus have

$$\int_G f v_G = \int_V f v_G = \sum_{i=1}^{l} \int_{\mathrm{int}(W_i)} f v_G.$$

For $g \in W_i \subseteq g_i U$, $g_i \in U^j$, we write g in the form

$$g = \exp(X_1) \cdots \exp(X_j) \exp(X)$$

and define a curve $\gamma(t,g) \in G$, $t \in [0,1]$ by

$$\gamma(t,g) = \exp(tX_1) \cdots \exp(tX_j) \exp(tX).$$

Then $\gamma(0,g) = e$ and $\gamma(1,g) = g$. From γ we get a homotopy $L_{\gamma(t,g)} \colon M \longrightarrow M$, $t \in [0,1]$, where $L_{\gamma(0,g)} = \mathrm{id}$ and $L_{\gamma(1,g)} = L_g$. As in the proof of the Poincaré Lemma (see [MT]) we get an operator

$$S_g^k \colon \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$$

such that

$$dS_g^k \omega + S_g^{k+1} d\omega = L_g^* \omega - \omega.$$

Here $S_g^k \omega$ is explicitly given by the formula

$$S_g^k(\omega)_p(v_1, \ldots, v_{k-1}) = \int_0^1 ((L_{\gamma(-,g)})^* \omega)_p \left(\frac{d}{dt}, v_1, \ldots, v_{k-1} \right) dt,$$

$$v_1, \ldots, v_k \in T_p M$$

This is differentiable in g for $g \in \mathrm{int}(W_i)$ and we may define

$$S^k(\omega) = \int_{g \in V} (S_g^k \omega) v_G.$$

By integration of the formula above we get

$$\begin{aligned} dS^k \omega + S^{k+1} d\omega &= \int_{g \in V} (L_g^* \omega) v_G - \int_{g \in V} (\omega) v_G \\ &= \int_{g \in G} (L_g^* \omega) v_G - \int_{g \in G} (\omega) v_G \\ &= \bar{\omega} - \omega. \end{aligned}$$

$\square$

Exercise A9. Let $M = G/H$, where G is a compact Lie group with identity component $G_0 \subseteq G$.

(1) Show that the number of components of M is $[G\colon G_0H]$, and that each component is diffeomorphic to $M_0 = G_0/G_0 \cap H$.

(2) Conclude that

$$H^*_{\mathrm{dR}}(M) \cong H^*(\mathrm{Alt}^*(\mathfrak{g}/\mathfrak{h})^{G_0 \cap H})^{[G:G_0H]}.$$

Exercise A10. Let $M^n = G/H$, where G is a compact connected Lie group. Show that the following statements are equivalent:

(1) M is orientable.

(2) $H^n_{\mathrm{dR}}(M) \neq 0$.

(3) For all $h \in H$, $\det(\mathrm{Ad}(h)) = 1$ where $\mathrm{Ad}(h)\colon \mathfrak{g}/\mathfrak{h} \longrightarrow \mathfrak{g}/\mathfrak{h}$.

(4) There is a volume form $v_M \in \Omega^n(M)^G$ with $\int_M v_M = 1$.

Exercise A11. Let G be a compact Lie group and $\rho\colon G \longrightarrow \mathrm{GL}(V)$ a representation on a finite dimensional vector space V. We write $gv = \rho(g)v$ for $g \in G, v \in V$. Now define the dual representation on $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ by

$$(gv^*)(v) = v^*(g^{-1}v), \qquad v^* \in V^*, v \in V, g \in G$$

Let $V^G = \{v \in V \mid gv = v\}$ and $(V^*)^G = \{v^* \in V^* \mid gv^* = v^*\}$.

(1) Show that the natural mapping given by restriction

$$(V^*)^G \longrightarrow (V^G)^*$$

is an isomorphism.

(2) Now let V and V' be two finite dimensional vector spaces with G-representations as above and let $B\colon V \times V' \longrightarrow \mathbb{R}$ be a non-degenerate G-invariant bilinear function. (B is non-degenerate if the mapping $B^\sharp\colon V' \longrightarrow V^*$ given by $B^\sharp(v')(v) = B(v, v')$, $v \in V, v' \in V'$, is an isomorphism. B is G-invariant if $B(gv, gv') = B(v, v')$ for all $v \in V$, $v' \in V'$, $g \in G$.)

Show that B by restriction gives a non-degenerate bilinear function

$$B\colon V^G \times {V'}^G \longrightarrow \mathbb{R}.$$

Exercise A12 (Poincaré duality). Let G be a compact connected Lie group, $H \subseteq G$ a closed Lie subgroup and put $M = G/H$. Assume that M is oriented and for $k = 0, \dots, n$ define the bilinear function

$$\Omega^k(M)^G \times \Omega^{n-k}(M)^G \longrightarrow \mathbb{R}$$

by

$$B(\alpha, \beta) = \int_M \alpha \wedge \beta, \qquad \alpha \in \Omega^k(M)^G, \quad \beta \in \Omega^{n-k}(M)^G.$$

(1) Show that B is non-degenerate (cf. Exercise A11).

(2) Show the formula

$$B(d\alpha, \beta) = (-1)^k B(\alpha, d\beta), \text{ for all } \alpha \in \Omega^{k-1}(M)^G,\ \beta \in \Omega^{n-k}(M)^G.$$

(3) Show that B induces a non-degenerate bilinear function

$$\overline{B}\colon H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \longrightarrow \mathbb{R} \quad k = 0, \dots, n.$$

That is $\overline{B}^\sharp\colon H^{n-k}_{\mathrm{dR}} \longrightarrow H^k_{\mathrm{dR}}(M)^*$ is an isomorphism.

Hints to (1): First show the following lemma: For V an n-dimensional vector space and $v \in \mathrm{Alt}^n(V)$ a generator, the bilinear function

$$B\colon \mathrm{Alt}^k(V) \times \mathrm{Alt}^{n-k}(V) \longrightarrow \mathbb{R}$$

given by

$$\alpha \wedge \beta = B(\alpha, \beta) v, \qquad \alpha \in \mathrm{Alt}^k(V), \beta \in \mathrm{Alt}^{n-k}(V)$$

is non-degenerate.

We conclude the appendix with a few examples.

Example A13. $M = S^n$ (cf. Example A5). Let $\mathbb{R}^{n+1}$ have the coordinates $(x_0, \dots, x_n)$ and consider the diffeomorphism

$$\pi\colon \mathrm{SO}(n+1)/\,\mathrm{SO}(n) \xrightarrow{\cong} S^n \subseteq \mathbb{R}^{n+1}$$

given by $\pi(g\,\mathrm{SO}(n)) = ge_0$, $g \in \mathrm{SO}(n+1)$, where $e_0 = (1, 0, \ldots, 0)$. The differential

$$\pi_*\colon \mathfrak{so}(n+1)/\mathfrak{so}(n) \longrightarrow \mathbb{R}^n = \mathrm{span}(e_0)^\perp$$

satisfies

$$\pi_*(\mathrm{Ad}(h)X) = h\pi_*(X) \qquad X \in \mathfrak{so}(n+1), \quad h \in \mathrm{SO}(n)$$

and thereby induces an isomorphism

$$\mathrm{Alt}^k(\mathbb{R}^n)^{\mathrm{SO}(n)} \xrightarrow{\pi^*} \mathrm{Alt}^k(\mathfrak{so}(n+1)/\mathfrak{so}(n))^{\mathrm{SO}(n)} \qquad k = 0, ..., n,$$

where

$$\mathrm{Alt}^k(\mathbb{R}^n)^{\mathrm{SO}(n)} = \{\alpha \in \mathrm{Alt}^k(\mathbb{R}^n) \mid \alpha(hv_1, \ldots, hv_k) = \alpha(v_1, \ldots, v_k)\},$$

for all $v_1, \ldots, v_k \in \mathbb{R}^n$ and $h \in \mathrm{SO}(n)$. We now have

$$\mathrm{Alt}^k(\mathbb{R}^n)^{\mathrm{SO}(n)} = \begin{cases} \mathbb{R}\det & k = n \\ 0 & k = 1, \ldots, n-1 \end{cases}$$

where "det" is defined by the determinant. It is clear that "det" is $\mathrm{SO}(n)$-invariant (by definition). So let us show that every $\alpha \in \mathrm{Alt}^k(\mathbb{R}^n)^{\mathrm{SO}(n)}$, $k < n$, is 0:

Let $\{e_1, \ldots, e_n\}$ be the canonical basis for $\mathbb{R}^n$, and notice that α is determined by $\{\alpha(e_{i_1}, \ldots, e_{i_k}) \mid 1 \le i_1 < \ldots < i_k \le n\}$. But for $k < n$ there are $h, h' \in \mathrm{SO}(n)$ such that

$$h(e_1) = e_{i_1}, \ldots, h(e_k) = e_{i_k}$$

and

$$h'(e_1) = -e_{i_1}, h'(e_2) = e_{i_2}, \ldots, h'(e_k) = e_{i_k}$$

hence

$$\alpha(e_{i_1}, \ldots, e_{i_k}) = \alpha(e_1, \ldots, e_k) = -\alpha(e_{i_1}, \ldots, e_{i_k}),$$

that is $\alpha(e_{i_1}, \ldots, e_{i_k}) = 0$. We have thus shown that

$$H^k_{\mathrm{dR}}(S^n) = \begin{cases} \mathbb{R}[v_{S^n}] & k = n \\ 0 & k = 1, \ldots, n-1 \\ \mathbb{R} & k = 0 \end{cases}$$

where $v_{S^n} \in \Omega^n(S^n)$ is the volume form from Example A5.

Exercise A14. For $M = \mathbb{R}P^n$ the n-dimensional projective space, show that

$$H^k_{\mathrm{dR}}(\mathbb{R}P^n) = \begin{cases} \mathbb{R} & k = 0 \text{ and } k = n \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Example A15. $M = \mathbb{C}P^n = \mathrm{U}(n+1)/(\mathrm{U}(1) \times \mathrm{U}(n))$. $\mathbb{C}^{n+1}$ has the coordinates $(z_0, \ldots, z_n)$ and $H = \mathrm{U}(1) \times \mathrm{U}(n) \subseteq \mathrm{U}(n+1)$ is the subgroup of matrices on the form

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & h \end{pmatrix} \qquad \lambda \in \mathrm{U}(1), h \in \mathrm{U}(n).$$

It is easy to show that the mapping

$$\pi_* \colon \mathfrak{u}(n+1)/\mathfrak{u}(n) \times \mathfrak{u}(1) \longrightarrow \mathbb{C}^n$$

given by

$$\pi_*(X) = \begin{bmatrix} x_{n,1} \\ \vdots \\ x_{n,n} \end{bmatrix}$$

is an isomorphism, and that

$$\pi_*(\mathrm{Ad}(g)X) = \bar{\lambda} h \pi_*(X), \qquad \text{for } X \in \mathfrak{u}(n+1) \subseteq M(n+1)$$

and $g \in H$ as above. That is we get an isomorphism

$$\mathrm{Alt}^k_{\mathbb{R}}(\mathbb{C}^n)^{\mathrm{U}(n)} \xrightarrow{\pi^*} \mathrm{Alt}^k_{\mathbb{R}}(\mathfrak{u}(n+1)/(\mathfrak{u}(1) \times \mathfrak{u}(n)))^H$$

where

$$\mathrm{Alt}^k_{\mathbb{R}}(\mathbb{C}^n)^{\mathrm{U}(n)} = \{\alpha \in \mathrm{Alt}^k_{\mathbb{R}}(\mathbb{C}^n) \mid \alpha(hv_1, \ldots, hv_k) = \alpha(v_1, \ldots, v_k)\},$$

for all $v_1, \ldots, v_k \in \mathbb{C}^n$ and $h \in \mathrm{U}(n)$. Notice that $\kappa \in \mathrm{Alt}^2_{\mathbb{R}}(\mathbb{C}^n)$ given by

$$\kappa(v, w) = -\mathrm{Im}(\overline{w}^t v)$$

is $\mathrm{U}(n)$ invariant and that $\kappa(e_1, ie_1) = 1$ so that

$$\mathrm{Alt}^2_{\mathbb{R}}(\mathbb{C}^n)^{\mathrm{U}(n)} \neq 0.$$

We show that

$$\mathrm{Alt}^k_{\mathbb{R}}(\mathbb{C}^n)^{\mathrm{U}(n)} = \begin{cases} \mathbb{R}\kappa^{k/2} & 0 \le k \le 2n, k \text{ even} \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

Since $\mathbb{C}^n = \mathrm{span}_{\mathbb{R}}\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$, $\alpha \in \mathrm{Alt}^k(\mathbb{C}^n)^{U(n)}$ is determined by the values

$$\alpha(e_{i_1}, \dots, e_{i_p}, ie_{j_1}, \dots, ie_{j_q}),$$

where $1 \le i_1 < \dots < i_p \le n$, $1 \le j_1 < \dots < j_p \le n$ and $p + q = k$. But if there is a $j_s \notin \{i_1, \dots, i_p\}$ then the mapping

$$e_l \longmapsto \begin{cases} -e_l & l = j_s \\ e_l & l \ne j_s \end{cases}$$

is unitary so that

$$\alpha(e_{i_1}, \dots, e_{i_p}, ie_{j_1}, \dots, ie_{j_q}) = -\alpha(e_{i_1}, \dots, e_{i_p}, ie_{j_1}, \dots, ie_{j_q}),$$

that is $\alpha(e_{i_1}, \dots, e_{i_p},\ ie_{j_1}, \dots, ie_{j_q}) = 0$. In the same way we show that if $i_t \notin \{j_1, \dots, j_q\}$ then $\alpha(e_{i_1}, \dots, e_{i_p}, ie_{j_1}, \dots, ie_{j_q}) = 0$. That is, $\alpha = 0$ if $p \ne q$ and for $k = 2p$, α is determined by the values

$$\alpha(e_{i_1}, \dots, e_{i_p}, ie_{i_1}, \dots, ie_{i_p}).$$

There is an $h \in \mathrm{U}(n)$ such that $h(e_l) = e_{i_l}$, $l = 1, \dots, p$, hence α is determined by $\alpha(e_1, \dots, e_p, ie_1, \dots, ie_p)$.

We conclude that $\mathrm{Alt}^{2p}_{\mathbb{R}}(\mathbb{C}^n)^{\mathrm{U}(n)}$ is at most 1-dimensional, so it is enough to show that $\kappa^p \ne 0$. An easy calculation shows that

$$\kappa^p(e_1, \dots, e_p, ie_1, \dots, ie_p) = p!\,, \qquad p \le n,$$

which shows (A.1). This also shows that

$$H^k_{\mathrm{dR}}(\mathbb{C}P^n) = \begin{cases} \mathbb{R}\kappa^{k/2} & 0 \le k \le 2n, k \text{ even} \\ 0 & \text{otherwise.} \end{cases} \tag{A.2}$$

Exercise A16 (Symmetric spaces). Let $M = G/H$, where G is a compact connected Lie group and $H \subseteq G$ is a closed Lie subgroup. Let $\mathfrak{h} \subseteq \mathfrak{g}$

be the Lie algebras for H and G respectively, and assume that there is a complement $\mathfrak{m} \subseteq \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and such that the following is satisfied:

(1) $\mathrm{Ad}(h)(X) \in \mathfrak{m}$ for all $h \in H$, $X \in \mathfrak{m}$.

(2) $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{m}$.

Show that there is a natural isomorphism

$$H^k_{\mathrm{dR}}(M) \cong \mathrm{Alt}^k(\mathfrak{m})^H, \quad k = 0, 1, 2, \ldots$$

where the action of H on $\mathrm{Alt}^k(\mathfrak{m})$ is induced by the adjoint action of H on $\mathfrak{m}$ given by (1).

Exercise A17. Let G be a compact connected Lie group. As G acts on the Lie algebra $\mathfrak{g}$ by the adjoint representation, show that there is a natural isomorphism

$$H^k_{\mathrm{dR}}(G) \cong \mathrm{Alt}^k(\mathfrak{g})^G.$$

Hint: Use Exercise A16 for the case $H \subseteq G \times G$, $H = \{(g, g) \mid g \in G\}$.

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