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A review on some contributions to perturbation theory, singular limits and well-posedness

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Abstract

In the beginning of the 1990s we devoted a sequence of papers to perturbation theory, singular limits and well-posedness problems. In particular, the *strong* well-posedness of the initial–boundary value problem for the compressible Euler equations was demonstrate for the first time. Our method also allowed singular limit results in the *strong* norm, even under assumptions weaker than the current ones in the literature (where the strong norm is not reached). It is worth noting that, until now, the above method and results have not been substantially improved. Hence an introduction to it still looks timely. Actually, in a forthcoming paper, by returning to this method, we improve (in a very substantial way) some important results recently appeared in the literature.

Keywords: Perturbation theory; Well-posedness; Singular limits; Compressible non-viscous fluids; Initial–boundary value problems

1. Introduction

Following T. Kato, by *perturbation theory* we mean the study of the dependence of solutions to *linear* systems of equations on the coefficients of the operators. *Well-posedness*, in Hadamard's sense, means the continuous dependence of solutions to evolution problems on the initial data. Clearly, a main point in the above context is to define "continuity," i.e., the choice of the topologies. It is worth noting that *sufficiently strong* perturbation theorems for linear systems (like (2.10) below) lead, without difficulty, to well-posedness results for related non-linear systems (like (2.1) below). This fact was already remarked by T. Kato in the introduction to his work [29], where he points out that the results obtained for the abstract quasi-linear equation, denoted by (Q) in reference [29], are based on his previous results for the linear "hyperbolic" equation (L), see [29]. More generally, singular limit problems for non-linear equations (for instance, incompressible limit problems for compressible fluids), can be brought back to related perturbation theorems, as shown in our papers.

In the following we present a brief overview of our contribution to the above theories, in a deliberately very simplified framework. We are mainly driven by an interest in solutions to compressible non-viscous flows. By taking into account real physical problems, our efforts were basically directed to the study of initial–boundary value prob-

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lems. Clearly, the results may be proved for the corresponding Cauchy problems, in a much simpler way, by obvious simplifications in the original proofs. Nevertheless, even for the Cauchy problem, the results were new in some cases.

Having in mind compressible non-viscous fluids, we deal with strong, local in time solutions, that live in (for instance) $H = H^k(\Omega)$-type spaces, for sufficiently large values of k. Furthermore, we work in the framework of functional spaces that also take into account time derivatives of the solutions $u(t)$. For instance, one may look for solutions $u(t)$ of systems like (2.1) or (2.10) such that

$$\big(u, u', u'', \ldots, u^{(r)}\big) \in C\big([0,T]; \mathcal{H}\big), \tag{1.1}$$

where

$$\mathcal{H} = H^k(\Omega) \times H^{k-1}(\Omega) \times H^{k-2}(\Omega) \times \cdots \times H^{k-r}(\Omega), \tag{1.2}$$

and $r \leqslant k$. More precise notation will be introduced below.

A fundamental task in the theory of evolution partial differential equations is the extension to this field of the basic results that hold for ordinary differential equations, namely existence, uniqueness and continuous dependence theorems (or to show their failure). In considering evolution partial differential equations, a finite dimensional space, typical in the O.D.E. theories, is mostly replaced by a suitable infinite dimensional space (in the sequel, a Hilbert space $\mathcal{H}$), and the ordinary differential system of equations is replaced by an equation of the type $u' + B(u) = F$, where B is an unbounded operator in $\mathcal{H}$. Proving, in correspondence to each initial data $u_0 \in H$ ($H = H^k$ in the previous example), the existence of a solution $u(t)$, *continuous* in some interval $[0, T]$ with values in $\mathcal{H}$ is still non-trivial, in particular for initial–boundary value problems. Further, if this result holds, there still remains the task to show that if

$$\lim_{\nu\to\infty} \big\| u_0^\nu - u_0 \big\|_H = 0, \tag{1.3}$$

then

$$\lim_{\nu\to\infty} \| u_\nu - u \|_{C([0,T];\mathcal{H})} = 0, \tag{1.4}$$

where $u_\nu(t)$ is the solution with initial data u_0^ν. Roughly speaking, results weaker than (1.4) may be obtained in the following way. Typically, the existence theorems exhibit an $L^\infty(0,T;\mathcal{H})$ estimate of the norm of the solution, where T depends only on the norm of u_0 in H. Hence, by assuming (1.3), all the above solutions exist in a given interval $[0, T]$. By a well-known compactness argument, "suitable subsequences" of u_ν converge in the above space with respect to the weak-$*$ topology. This convergence implies, in particular, convergence in norm-topologies like $C([0,T]; H^{k-\alpha}(\Omega))$, $\alpha > 0$. The strong convergence of the derivatives present in the equations shows at once that limits of the above convergent subsequences are already solutions, with initial data u_0. Since these limits belong to quite regular spaces, uniqueness of the solution guarantees the weak-$*$ convergence of the full sequence u_ν to u in $L^\infty(0,T;\mathcal{H})$ and the strong convergence in spaces like $C([0,T]; H^{k-\alpha}(\Omega))$. Nevertheless, these significant results are not totally satisfactory. Since all the solutions belong to $C([0,T];\mathcal{H})$, this is the natural space for proving continuous dependence on the data. The aim of our work was to reach this result, as well as extensions to more general problems.

The strong, uniform continuous, dependence result (1.4) is, in general, not very difficult to prove for parabolic problems, thanks to the regularity of the solutions in terms of data and coefficients. On the contrary, in the hyperbolic case, in particular for initial–boundary value problems related to fluid mechanics, the problem is particularly difficult. As remarked by Kato and Lai in reference [32], where the Euler *incompressible* equation is studied, "the continuous dependence in the "strong" topology of the solution on the data is the most difficulty part in a theory dealing with nonlinear equations of evolution." At that time, even the existence problem for the Euler compressible equations was completely open (see Section 5.2 below).

Reference [14] is the leitmotif for Part I below, even if (for simplicity) we mostly consider the Cauchy (and not the initial–boundary value) problem. We study the Cauchy problem for the linear system (2.10) and for the non-linear system (2.1). The assumptions on the coefficients $A(t,x)$, $A^\nu(t,x)$ and $A(u)$, made here, come from (5.2), a situation that does not allow convenient choices: The topology under which one assumes that $A^\nu(t,x) \to A(t,x)$ must be weaker or (at most) equivalent to the topology under which, in the non-linear problem, $A(u_\nu) \to A(u)$ as $u_\nu \to u$ in $C([0,T]; H)$.

In Section 2.1 we introduce some notation and state Theorems 2.1, 2.2 and 2.3. In Section 2.2 we discuss some substantial (non-technical) obstacles that impose the introduction of suitable devices. In Section 3 we prove the very basic result, namely Theorem 2.2 (see Remark 1.2). Finally, in Section 4 we consider the initial–boundary value problem under the assumption (4.3).

Theorem 2.3, for the *Cauchy problem*, was proved by T. Kato by means of his perturbation theory for linear evolution equations, see [27,28], and has been applied by him to prove Theorem 2.1 for a large class of systems. See also Hughes, Kato and Marsden [26]. Successively, this same theory has been extended to a class of initial–boundary value problems, see [31], in particular to non-linear wave equations, but not to the Euler compressible equations.

In Part II we touch on references [13,15], by describing related, but simpler, situations. It is dedicated, at least ideally, to the Euler compressible equations (5.1). This system is the very center of our concern, in spite of the fact that in Section 8 (the core of Part II) we deal with the simplified system (8.1). The complete system (5.1) will be considered only at the level of the existence theory, in Section 6. It is worth noting that the system (5.2) does not verify the assumption (4.3). We overcome this obstacle by finding an (apparently more complicated) system of equations, equivalent to the system (5.1), for which the assumption (4.3) holds. More precisely, by appealing to the classical curl and div differential operators, we show that the system (5.1) is equivalent to the system consisting of the first-order elliptic system (6.4), the Euler type transport equation (6.5), and the second-order hyperbolic equation (6.6) (plus (6.7)). This decomposition is very helpful, since the boundary value associated with (6.4) is well known; Eq. (6.5) has no boundary conditions; the boundary value problem (6.6) satisfies (4.3). The system (6.6) carries out the main hyperbolic features of (5.1). So, in order to simplify our exposition, we could limit oneself to considering the system (6.6). Actually, we simplify even more our framework, by considering the system (8.1), a very drastic simplification of (6.6). In this way we avoid further strong obstacles (compare these two last systems). In Section 8, by appealing to (8.1), we try to explain the real motivations that lead to our approach to the perturbation theory. A preliminary overview on the existence theory for the complete compressible Euler equations (see Section 6) has the pretension of giving a glimpse of some of the obstacles that cannot be overcome in the framework of the continuous dependence theory, in a short presentation. This is the real motivation for Section 6.

Finally, in Part III, we state and briefly discuss some singular limit results, see [21], in particular strong convergence of solutions for *compressible* Navier–Stokes equations to solutions of the Euler *incompressible* equations, as (simultaneously, with independent rates) the Mach number λ^{-1} goes to 0, the viscosity coefficient μ goes to zero and the viscosity coefficient ζ remains bounded.

Remark 1.1. Another way to overcome the lack of (4.3) could be to extend the method described in Section 4, in such a way as to cover the initial–boundary value problem (5.1). We have shown (not published) that this can be done at least in the presence of flat boundaries, and for $n = k = 3$ (we take this opportunity to point out that non-flat boundaries are much harder to handle).

Remark 1.2. In Section 2.1 the crucial result is Theorem 2.2, which is the basis of the theory. As a matter of fact, Theorems 2.1 and 2.3 are corollaries. Similarly, in Section 4, where we consider the initial–boundary value problem, the basic result is the estimate (4.13). In Part II (where, for convenience, $k = 3$) the situation is similar. See (8.14). The reader should compare (8.14) with (4.11), since in Part II we will not repeat the last argument of the proof.

Remark 1.3. Note that, in the hyperbolic case, the continuity of the map $S(t)$ from H to H, where $S(t)u_0 = u(t)$, for a fixed $t > 0$, can not be replaced by a better result (as, for instance, Hölder or Lipschitz continuity). A very simple example comes from the equation $\partial_t u + u\partial_x u = 0$ in $H^k(\mathbb{R})$. See Subsection 5.3 in reference [30]. Hence, in Eulerian coordinates, C^0-continuous dependence on the initial data is the most we may expect, even for trivial Cauchy problems. On the contrary, in Lagrangian coordinates dependence could be much stronger, and does not necessarily imply continuous dependence in Eulerian coordinates. See the illuminating remark in [24, Section 6, p. 483] (C^1-dependence results, in Lagrangian coordinates, for small initial data are also announced).

Finally, we also refer the reader to our strictly related papers [7,11,16,17,19], not quoted elsewhere in this work.

Part I. First-order hyperbolic systems

2. The Cauchy problem

2.1. Notation and main results

The well-posedness problem, for non-linear systems, is strictly related to the more general *structural stability problem*, for linear systems. This means here sharp continuous dependence of the solution in terms of the coefficients of the operators. For instance, consider a non-linear Cauchy problem described by the system

$$\begin{cases} \partial_t u + A(u)\partial_x u = F & \text{in } Q_T, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $Q_T = [0,T] \times \mathbb{R}^n$, $\partial_i = \partial_{x_i}$, $i = 1, \dots, n$, $u = (u_1, \dots, u_m)$,

$$A(u)\partial_x u = \sum_{i=1}^{n} A^{(i)}(u)\partial_i u. \tag{2.2}$$

$A^{(i)}(u)$ are $m \times m$ symmetric matrices with coefficients $a^{(i)}_{q,l}(\cdot)$ of class $C^k(\mathbb{R}^m;\mathbb{R})$, $i = 1, \dots, n$, $q, l = 1, \dots, m$. We assume that $k > 1 + \frac{n}{2}$. Further, $u_0 \in H^k(\mathbb{R}^n)$ and $F \in \mathcal{L}^2_{T_0}(H^k)$, see below. We denote by $\|\cdot\|_l$ the canonical norm in $H^l(\mathbb{R}^n)$, moreover,

$$\||u|\|_l^2 = \sum_{j=0}^{l} \|\partial_t^j u\|_{l-j}^2.$$

Other main notations are

$$\mathcal{C}_T(H^l) = \bigcap_{j=0}^{l} C^j\big([0,T]; H^{l-j}\big), \qquad \mathcal{L}^2_T(H^l) = \bigcap_{j=0}^{l} H^j\big(0,T; H^{l-j}\big),$$

and

$$\||u|\|^2_{l,T} = \sup_{0 \leqslant t \leqslant T} \big\||u(t)|\big\|_l^2; \qquad [u]^2_{l,T} = \int_0^T \|u(t)\|_l^2\, dt; \qquad |[u]|^2_{l,T} = \int_0^T \big\||u(t)|\big\|_l^2\, dt.$$

Below, we also mention spaces

$$\mathcal{L}^p_T(H^l) = \bigcap_{j=0}^{l} W^{j,p}\big(0,T; H^{l-j}\big),$$

where $1 < p \leqslant \infty$, even if their role here is negligible.

Together with (2.1), we also consider a sequence of similar problems

$$\begin{cases} \partial_t u^\nu + A^\nu(u^\nu)\partial_x u^\nu = F^\nu, \\ u^\nu(0) = u_0^\nu, \end{cases} \tag{2.3}$$

where $A^\nu(\cdot)$, u_0^ν and F^ν are as $A(\cdot)$, u_0 and F above. Note that, in typical applications, $A^\nu = A$. Assume that

$$\lim_{\nu\to\infty} \|u_0^\nu - u_0\|_k = 0, \qquad \lim_{\nu\to\infty} \big|[F^\nu - F]\big|_{k,T_0} = 0, \tag{2.4}$$

and

$$\lim_{\nu\to\infty} A^\nu(\cdot) = A(\cdot) \quad \text{in } C^k, \tag{2.5}$$

on compact subsets of $\mathbb{R}^m$. Under the above hypotheses there are $T > 0$ and $C > 0$ such that the problem (2.1) has a unique solution $u \in \mathcal{C}_T(H^k)$ satisfying $\||u|\|^2_{k,T} \leqslant C$. Actually, it would be sufficient to assume that $u \in \mathcal{L}^\infty_T(H^k)$, since continuity follows then easily by appealing to estimates proved in the sequel. Upper bounds for T^{-1} and for C depend (non-decreasingly) on the norms $\|u_0\|_k$ and $|[F]|_{k,T_0}$, and on the C^k norms of the matrices $A^{(i)}(\cdot)$ on a fixed open, bounded subset of $\mathbb{R}^m$ that contains the closure of the set $\{u_0(x)\colon x \in \mathbb{R}^n\}$. By applying this result to the system (2.3), under the hypotheses (2.4), (2.5), it follows that the constants T and C may be chosen independently of ν. In particular

$$\big\| |u^\nu| \big\|_{k,T}^2 \leqslant C, \quad \forall \nu \in \mathbb{N}, \tag{2.6}$$

where $\mathbb{N}$ denotes the set of positive integers. It easily follows that if (2.4) and (2.5) hold, then $\lim_{\nu\to\infty} \|u^\nu - u\|_{0,T} = 0$. Furthermore, by interpolation,

$$\lim_{\nu\to 0} \big\| |u^\nu - u| \big\|_{k-\alpha,T} = 0, \tag{2.7}$$

for each (arbitrarily small) positive α. In bounded domains, one also has

$$u^\nu \to u, \quad \text{w.r.t. the weak-}{*}\text{ topology in } \mathcal{L}_T^\infty\big(H^k\big). \tag{2.8}$$

These well-posedness results are not completely satisfactory. In fact, since the solutions u and u^ν belong to $\mathcal{C}_T(H^k)$, the conclusive result should be (2.9) below.

Eq. (2.9) guarantees, in particular, that if a solution $u(t)$ exists on $[0, T^*]$, for some $T^* > 0$, then (for sufficiently large values of ν) the solutions $u^\nu(t)$ exist on $[0, T^*]$ and satisfy (2.9) in this same interval.

In the next sections we sketch the proof of (2.9) for the Cauchy linear and non-linear problems, see Theorems 2.1 and 2.3, and for the related initial–boundary value problems, see Theorems 4.1 and 4.2. The result proved for the Cauchy non-linear problem applies, in particular, to the Cauchy problem for the compressible Euler equation. However, since (4.3) fails, Theorem 4.2 does not apply (at least in a simple way) to the boundary value problem (5.1). See Remark 1.1.

One has the following results.

Theorem 2.1. *Assume that* (2.4) *and* (2.5) *hold. Then, the solutions to the non-linear problem* (2.3) *converge in $\mathcal{C}_T(H^k)$ to the solution u of problem* (2.1)*, i.e.,*

$$\lim_{\nu\to\infty} \big\| |u^\nu - u| \big\|_{k,T} = 0. \tag{2.9}$$

Next consider the following linear systems:

$$\begin{cases} \partial_t u + A(t,x)\partial_x u = F, \\ u(0) = u_0, \end{cases} \tag{2.10}$$

and

$$\begin{cases} \partial_t u^\nu + A_\nu(t,x)\partial_x u^\nu = F^\nu, \\ u^\nu(0) = u_0^\nu, \end{cases} \tag{2.11}$$

where notation follows that introduced above (now $A(u)$ is replaced by $A(t,x)$, $A^{(i)}(u)$ by $A^{(i)}(t,x)$, and so on). In this case we assume that $A, A_\nu \in \mathcal{L}_{T_0}^\infty(H^k)$. It is worth noting that if we want to relate the linear case to the non-linear one, we cannot assume more than $A, A_\nu \in \mathcal{C}_{T_0}(H^k)$.

We prove the following result (where we may replace everywhere $\mathcal{L}_{T_0}^2(H^k)$ and $\mathcal{L}_{T_0}^\infty(H^k)$ by $\mathcal{L}_{T_0}^p(H^k)$, for some $p > 1$).

Theorem 2.2. *Assume that $u_0, u_0^\nu \in H^k$, and that $F, F^\nu \in \mathcal{L}_{T_0}^2(H^k)$, with norms bounded by a some constant C, independent of ν. Let u and u^ν be the solutions to the systems* (2.10) *and* (2.11)*, respectively. Assume, moreover, that*

$$\| |A_\nu| \|_{k,T_0} \leqslant C, \quad \forall \nu \in \mathbb{N}. \tag{2.12}$$

Then, to each $\epsilon > 0$ it corresponds a positive $\Lambda(\epsilon)$, that depends only on ϵ, T_0, u_0, F, and A, such that

$$\big\| \big|(u^\nu - u)(t)\big| \big\|_k^2 \leqslant C\big\{\epsilon + \big\|u_0^\nu - u_0\big\|_k^2 + \big|\big[F^\nu - F\big]\big|_{k,T_0}^2 + |[A_\nu - A]|_{k,t}^2 + \Lambda(\epsilon)|[A_\nu - A]|_{k-1,t}^2\big\}, \tag{2.13}$$

for all $\nu \in \mathbb{N}$.

Theorem 2.3. *Under the assumptions of Theorem* 2.2 *if, moreover,* (2.4) *holds and*

$$\lim_{\nu\to\infty} |[A_\nu - A]|_{k,T_0} = 0, \tag{2.14}$$

then the solutions to the linear problem (2.11) *converge in* $\mathcal{C}_{T_0}(H^k)$ *to the solution* u *of problem* (2.10)*, i.e., Eq.* (2.9) *holds here.*

We point out that (ϵ,ν)-estimates like (2.13) are the very central point in our approach. It looks convenient to explain this very central point before going on to the proofs of the above theorems. This is the subject of the next subsection.

2.2. The ϵ-device and the (ϵ,ν)-estimates. Motivations

In this subsection we illustrate the main reasons that force us to introducing some crucial new devices in the next sections. For simplicity, we assume that $A^\nu(\cdot) = A(\cdot)$.

We only want to show to the reader the starting, necessarily imprecise, ideas that lead us to the decisive method of proof. In this scheme of things, "spaces" and "norms" are simply outlined by their main, expected, characteristics.

At a first glance, the more natural way to try to prove (2.9) seems appealing to energy estimates for the difference $u^\nu - u$, which solves the problem

$$\begin{cases} \partial_t\big(u-u^\nu\big) + A(u)\partial_x\big(u-u^\nu\big) = \big(A\big(u^\nu\big) - A(u)\big)\partial_x u^\nu + \big(F - F^\nu\big) & \text{in } Q_T, \\ \big(u-u^\nu\big)(0) = u_0 - u_0^\nu. \end{cases} \tag{2.15}$$

However, as the function $(A(u^\nu) - A(u))\partial_x u^\nu$ does not belong to H^k, but merely to H^{k-1}, we can not obtain from (2.15) an H^k estimate for $(u-u^\nu)(t)$. The obstacle here is the lack of regularity of the "pivot" function u^ν, in the right-hand side of (2.15). So we have tried the following idea: To single out, in correspondence to any $\epsilon > 0$, a positive integer $N(\epsilon)$ and a new "pivot" function u^ϵ characterized by

$$u^\epsilon \in L^\infty\big(0,T;H^{k+1}\big),$$

and such that

$$\big\|u - u^\epsilon\big\|_{k,T} < c\epsilon \quad \text{and} \quad \big\|u^\nu - u^\epsilon\big\|_{k,T} < c\epsilon \quad \text{for each } \nu > N(\epsilon),$$

where, in principle, norms are in $L^\infty(0,T;H^k)$. Here c should be independent of the other quantities involved in the above equations. Note that it is expected that the $L^\infty_T(H^{k+1})$-norm of u^ϵ blows up as ϵ goes to zero. We also hope to be able to estimate the term $A(u^\nu) - A(u)$ in a suitable way.

Our first attempt to individuate the u^ϵ was to consider the auxiliary system

$$\begin{cases} \partial_t u^\epsilon + A(u)\partial_x u^\epsilon = F^\epsilon, \\ u^\epsilon(0) = u_0^\epsilon, \end{cases} \tag{2.16}$$

where $u_0^\epsilon \in H^{k+1}$, $F^\epsilon \in \mathcal{L}^2_T(H^{k+1})$, and

$$\big\|u_0^\epsilon - u_0\big\|_k^2 \leqslant c\epsilon, \qquad \big|\big[F^\epsilon - F\big]\big|_{k,T} \leqslant c\epsilon. \tag{2.17}$$

Unfortunately, since $A(u) \notin H^{k+1}$, we cannot expect $u^\epsilon(t) \in H^{k+1}$, as desired. Hence, we need an additional idea to improve the above strategy. Before going to this point, we note the following.

Remark 2.1. An attempt to overcome that $A(u) \notin H^{k+1}$ could be to replace in (2.16) the coefficient $A(u)$ by $A(u^\epsilon)$. Note that this device requires an extra-regularity for the coefficients $A(\cdot)$, not suitable in applications. Nevertheless, even by assuming this extra-regularity (or, possibly, by replacing $A(u^\epsilon)$ by $A^\epsilon(u^\epsilon)$), we come up against new obstacles and even more technical situations. It looks out of place here to discuss, and compare, all the ways we have tried. We may just note that replacing the coefficient $A(u)$ by $A(u^\epsilon)$, leads to the need of showing that

$$\lim_{\epsilon\to 0}\big|\big[u^\epsilon - u\big]\big|^2_{k-1,t}\,\big\|\big|u^\epsilon\big|\big\|^2_{k+1,t} = 0.$$

We may prove this result for any Cauchy problem, and for some particular initial–boundary value problems that do not require compatibility conditions (like incompressible Euler, under the classical homogeneous slip boundary condition). We may also consider more general situations, but not the more general initial–boundary value problems for compressible Euler equations.

As shown in the next subsection, the above ϵ-device still works, if combined with a suitable new device. Let us briefly illustrate ideas, before going into the detailed description presented in the next subsection.

First of all, since $A(u) \in H^k$, we may look for suitable perturbations theorems in H^{k-1}, instead of H^k. On the other hand, differentiation of Eq. (2.1) with respect to x_j gives

$$\begin{cases} \partial_t(\partial_j u) + A(u)\partial_x(\partial_j u) = \partial_j F - \big(\partial_j A(u)\big)\partial_x u, \\ (\partial_j)u(0) = \partial_j u_0. \end{cases} \tag{2.18}$$

The coefficient $A(u)$ belongs to H^k, and the right-hand side of Eq. (2.18) belongs to H^{k-1}. So, we have high hopes of being able to prove suitable H^{k-1} estimates for each of the first-order derivatives $\partial_j(u - u^\nu)$, by still appealing to the above ϵ-device, now applied to the n systems (2.18), $j = 1, \dots, n$. The set of all these H^{k-1} estimates is equivalent to an H^k estimate for $u - u^\nu$. As we will see below, this is the winning strategy.

In the case of *initial–boundary value problems* the situation is (as usual) much more difficult to treat, in particular in the case of compressible Euler equations. However this is the real physical situation. We take a glance at this problem in Section 4 below, where we consider the linear system (4.1), instead of the non-linear system (2.1), endowed with the same boundary conditions. The situation is very similar in both cases, since we do not assume "additional" regularity for the coefficients $A(t, x)$. This means here that the regularity assumed for these coefficients is not greater than that of the coefficients $A(u(t, x))$, when $u(t, x)$ has the regularity furnished by the existence theorems. In this situation the proofs are essentially the same in both cases, as shown below for the Cauchy problem.

In order to simplify the argument, in Section 4 we will assume a flat boundary (say, $\Omega = \mathbb{R}^n_+$, where x_n is the coordinate in normal direction to the boundary). In principle, for $j \neq n$, the system (2.18) can be endowed with suitable boundary conditions obtained by differentiating the (given) boundary conditions with respect to the (tangential) coordinate x_j (see, for instance, the first $n-1$ equations (4.7) bellow). However, for $j = n$, this is false in general. Hence we replace differentiation with respect to the normal coordinate x_n by differentiation with respect to t. Clearly, this leads to more technical proofs. In particular, the pivot auxiliary functions u^ϵ must satisfy the boundary conditions and the compatibility conditions. Now, a main point is to obtain sharp estimates for $\partial_n(u - u^\nu)$ by appealing to the prior estimates proved for the derivatives $\partial_j(u - u^\nu)$, $j < n$, and for $\partial_t(u - u^\nu)$. This must be done by appealing directly to the equations.

In the case of the compressible Euler equations the above strategy seems difficult to apply, since (4.3) is false for this system. Actually, the proof of Theorem 4.2 can be improved in such a way as to cover this problem, at least for $k = n = 3$, and flat boundaries. However, the more "physical" approach described in Section 6.2 below, allows us to prove the key well-posedness result [15, Theorem 1.4]. In Section 6.2 below we give an introduction to this subject.

3. Proof of Theorem 2.2

In this section we apply the strategy described in the previous subsection to the Cauchy problems (2.10) and (2.11). Differentiation of (2.10) with respect to x_j, $j = 1, \dots, n$, yields

$$\begin{cases} \partial_t(\partial_j u) + A(t,x)\partial_x(\partial_j u) = \partial_j F - \big(\partial_j A(t,x)\big)\partial_x u, \\ (\partial_j)u(0) = \partial_j u_0. \end{cases} \tag{3.1}$$

By setting $U = (\partial_1 u, \dots, \partial_n u)$, $\phi \equiv \partial_x u_0 = (\partial_1 u_0, \dots, \partial_n u_0)$, and $\widehat{A}^{(l)} \equiv$ diagonal bloc matrix $(A^{(l)}, \dots, A^{(l)})$, where the matrix $A^{(l)}$ is repeated n times, one shows that U, which belongs to $\mathcal{C}_{T_0}(H^{k-1})$, solves, by the construction, the system

$$\begin{cases} \partial_t U + \widehat{A}(t,x)\partial_x U = \Phi, \\ U(0) = \phi, \end{cases} \tag{3.2}$$

where $\Phi = \partial_x F - (\partial_x A(t,x))\partial_x u$. Note that $\phi \in H^{k-1}$ and $\Phi \in \mathcal{L}^2_{T_0}(H^{k-1})$. We do not write a detailed expression of $(\partial_x A(t,x))\partial_x u$ since it is superfluous here.

Similarly, one gets from (2.11)

$$\begin{cases} \partial_t U^\nu + \widehat{A}_\nu(t,x)\partial_x U^\nu = \Phi^\nu, \\ U^\nu(0) = \phi^\nu, \end{cases} \tag{3.3}$$

where

$$\phi^\nu \equiv \partial_x u_0^\nu, \qquad \Phi^\nu = \partial_x F^\nu - \big(\partial_x A_\nu(t,x)\big)\partial_x u^\nu,$$

and $\widehat{A}_\nu^{(l)} \equiv$ diagonal bloc matrix $(A_\nu^{(l)}, \dots, A_\nu^{(l)})$.

For each $\epsilon > 0$ we fix $\phi^\epsilon \in H^k$ and $\Phi^\epsilon \in \mathcal{L}^2_{T_0}(H^k)$ such that

$$\|\phi^\epsilon - \phi\|^2_{k-1} \leqslant \epsilon, \qquad \big|\big[\Phi^\epsilon - \Phi\big]\big|^2_{k-1,T_0} \leqslant \epsilon, \tag{3.4}$$

and we consider the solutions U^ϵ of problems

$$\begin{cases} \partial_t U^\epsilon + \widehat{A}(t,x)\partial_x U^\epsilon = \Phi^\epsilon, \\ U^\epsilon(0) = \phi^\epsilon. \end{cases} \tag{3.5}$$

Since $\widehat{A} \in \mathcal{L}^\infty_{T_0}(H^k)$ it follows that $U^\epsilon \in \mathcal{C}_{T_0}(H^k)$. Note that an upper bound for the norm $\||U^\epsilon|\|_{k,T_0}$ depends only on ϵ and T_0, and on the given functions ϕ, Φ and $\widehat{A}(t,x)$. Hence the above norm depends only on ϵ, T_0, u_0, F, and A. We write, for convenience,

$$\big\||U^\epsilon|\big\|_{k,T_0} \leqslant C(\epsilon, T_0; u_0, F, A) \equiv \Lambda(\epsilon). \tag{3.6}$$

By taking the difference, side by side, between Eqs. (3.3) and (3.5), we get

$$\begin{cases} \partial_t\big(U^\nu - U^\epsilon\big) + \widehat{A}_\nu(t,x)\partial_x\big(U^\nu - U^\epsilon\big) = \big(\Phi^\nu - \Phi^\epsilon\big) + (\widehat{A} - \widehat{A}_\nu)\partial_x U^\epsilon, \\ \big(U^\nu - U^\epsilon\big)(0) = \phi^\nu - \phi^\epsilon. \end{cases} \tag{3.7}$$

The classical H^{k-1}-energy estimate gives

$$\big\|\big|\big(U^\nu - U^\epsilon\big)(t)\big|\big\|^2_{k-1} \leqslant C\big\{\|\phi^\nu - \phi^\epsilon\|^2_{k-1} + \big|\big[\Phi^\nu - \Phi^\epsilon\big]\big|^2_{k-1,t} + \big|\big[(\widehat{A} - \widehat{A}_\nu)\partial_x U^\epsilon\big]\big|^2_{k-1,t}\big\},$$

where C depends on T_0. Hence,

$$\big\|\big|\big(U^\nu - U^\epsilon\big)(t)\big|\big\|^2_{k-1} \leqslant C\big\{\epsilon + \|\phi^\nu - \phi\|^2_{k-1} + \big|\big[\Phi^\nu - \Phi\big]\big|^2_{k-1,t} + |[\widehat{A} - \widehat{A}_\nu]|^2_{k-1,t}\big\||\partial_x U^\epsilon|\big\|^2_{k-1,t}\big\}. \tag{3.8}$$

Note the changing of ϕ^ϵ and Φ^ϵ by ϕ and Φ, respectively.

On the other hand (recall that $k - 1 > \frac{n}{2}$),

$$\big|\big[\Phi^\nu - \Phi\big]\big|^2_{k-1,t} \leqslant \big|\big[F^\nu - F\big]\big|^2_{k,t} + \big|\big[\partial_x(A - A_\nu)\big]\big|^2_{k-1,t}\||\partial_x u|\|^2_{k-1,t} + \||\partial_x A_\nu|\|^2_{k-1,t}\big|\big[\partial_x(u - u_\nu)\big]\big|^2_{k-1,t}.$$

Hence,

$$\begin{aligned} &\big\|\big|\big(U^\nu - U^\epsilon\big)(t)\big|\big\|^2_{k-1} \\ &\quad \leqslant C\big\{\epsilon + \|u_0^\nu - u_0\|^2_k + \big|\big[F^\nu - F\big]\big|^2_{k,T_0} + |[A_\nu - A]|^2_{k,t} + \big|\big[u^\nu - u\big]\big|^2_{k,t} + \Lambda(\epsilon)|[A_\nu - A]|^2_{k-1,t}\big\}. \end{aligned} \tag{3.9}$$

By replacing in the above calculations the system (3.3) simply by (3.2), one gets

$$\big\|\big|\big(U - U^\epsilon\big)(t)\big|\big\|^2_{k-1} \leqslant C\epsilon. \tag{3.10}$$

Hence $\||(U^\nu - U)(t)|\|^2_{k-1}$ is bounded by the right-hand side of (3.9). Since this quantity is equivalent to $\||(u^\nu - u)(t)|\|^2_k$, Eq. (2.13) follows, for each $t \in [0, T]$. The term $|[u^\nu - u]|^2_{k,t}$ was previously dropped by appealing to Gronwall's lemma. This proves Theorem 2.3. In fact, given $\sigma > 0$ we fix, in Eq. (2.13), $\epsilon_0 = \epsilon_0(\sigma)$ in such a way that $C\epsilon < \frac{\sigma}{2}$. Since $\Lambda(\epsilon_0)$ is a fixed quantity, the thesis follows.

3.1. *Proof of Theorem* 2.1

As already remarked, we assume that $A^\nu(\cdot) = A(\cdot)$, for each $\nu \in \mathbb{N}$, leaving to the interested reader the proof when this assumption is not fulfilled. Define $A(t,x) = A(u(t,x))$, $A_\nu(t,x) = A(u^\nu(t,x))$. Due to (2.6), the estimate (2.12) holds if T_0 is replaced by T. Furthermore, $|[A(u^\nu) - A(u)]|^2_{l,t} \leqslant C|[u^\nu - u]|^2_{l,t}$, for $l \leqslant k$, since $A(\cdot)$ is of class C^k on compact sets. It follows that

$$\big\|\big|\big(u^\nu - u\big)(t)\big|\big\|_k^2 \leqslant C\big\{\epsilon + \big\|u_0^\nu - u_0\big\|_k^2 + \big|\big[F^\nu - F\big]\big|^2_{k,T_0} + \Lambda(\epsilon)\big|\big[u^\nu - u\big]\big|^2_{k-1,t}\big\}, \tag{3.11}$$

where the term $[u^\nu - u]^2_{k,t}$ was dropped by appealing to Gronwall's lemma. Since $A(t,x) = A(u(t,x))$, dependence of $\Lambda(\epsilon)$ on $A(t,x)$ becomes dependence on u, hence dependence on the fixed elements u_0, F and $A(u)$. Now the desired result follows trivially from (3.11), as in Theorem 2.3. Note that $|[u^\nu - u]|^2_{k-1,T} \to 0$ as $\nu \to 0$.

4. The initial–boundary value problem

4.1. *Notation and main results*

Since our aim here is to emphasize the basic features in our method, we consider the half-space case $\Omega = \mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n\colon\ x_n > 0\}$ and assume a boundary condition $Mu = 0$ on Σ_T, where the $p \times m$ matrix M $(p \leqslant m)$ has constant coefficients and rank p. Notation is that used in the previous sections, by replacing $\mathbb{R}^n$ by $\mathbb{R}^n_+$. We set $\Gamma \equiv \{x \in \mathbb{R}^n\colon$ $x_n = 0\}$, $\Sigma_T = [0,T] \times \Gamma$.

The main difference between the proofs for the Cauchy and the initial–boundary value problems follows from the fact that for initial–boundary value problems the system (3.1) is not closed. However, differentiation of the boundary conditions with respect to x_j, $j = 1,\dots,n-1$, gives boundary conditions on $\partial_j x$, that make the corresponding system complete. This argument fails for the normal direction x_n. However, it works for the t direction. Furthermore, we show that suitable estimates for $\partial_1 u,\dots,\partial_{n-1}u$, $\partial_t u$ lead to estimates for $\partial_n u$, at least if the matrix $A^{(n)}$, recall (2.2), is non-singular on the boundary Σ_T. Clearly, the proofs are now more involved.

A main technical difference between the two problems is also due to the compatibility conditions for the initial–boundary value problem. In particular, the construction of the couples $(\phi^\epsilon, \Phi^\epsilon)$ must be done very carefully. We will not touch this argument.

We start by considering the linear systems

$$\begin{cases} \partial_t u + A(t,x)\partial_x u = F, \\ Mu|_{\Sigma_T} = 0, \\ u(0) = u_0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \partial_t u^\nu + A_\nu(t,x)\partial_x u^\nu = F^\nu, \\ Mu^\nu|_{\Sigma_T} = 0, \\ u^\nu(0) = u_0^\nu, \end{cases} \tag{4.2}$$

where $A, A_\nu, u_0 u_0^\nu, F F^\nu$ are as in Section 2, provided that $\mathbb{R}^n$ is replaced by $\mathbb{R}^n_+$ in all assumptions, definitions and equations. We suppose that the matrices $A^{(i)}$, $A^{(i)}_\nu$ are symmetric; that there is a positive constant σ such that

$$\big|\det A^{(n)}\big| > \sigma \quad \text{and} \quad \big|\det A^{(n)}_\nu\big| > \sigma, \quad \forall \nu \in \mathbb{N}, \tag{4.3}$$

on Σ_{T_0}; and that the set $\widetilde{\mathbb{N}} \equiv \{v \in \mathbb{R}^m\colon\ Mv = 0\}$ is maximal non-positive with respect to $A^{(n)}(t,x)$ and $A^{(n)}_\nu(t,x)$, for each $(t,x) \in \Sigma_{T_0}$. These assumptions are done here just for convenience. In fact, the only essential assumption is the existence of regular solutions $u \in \mathcal{L}^\infty_{T_0}(H^l)$ satisfying the classical H^l-energy estimates for $l = k-1, k$. Finally we assume that the couples (u_0, F) and (u_0^ν, F^ν) satisfy the compatibility conditions up to order $k-1$ with respect to the systems (4.1) and (4.2), respectively.

Since $H^k \subset C^{1,\alpha}$, for some $\alpha > 0$, it readily follows from (2.12) and (4.3) that $|\det A^n| > \frac{\sigma}{2}$ and $|\det A^n_\nu| > \frac{\sigma}{2}$, in a neighborhood S_{T_0} of Σ_{T_0}, independent of ν. This leads to consider a cut-off function $\theta = \theta(x_n)$, $x_n \geqslant 0$, depending only on x_n, equal to 1 in a neighborhood of $x_n = 0$ and vanishing far from the boundary. Since the main point is the regularity up to the boundary, there is no inconvenient in assuming that (4.3) holds on the whole of $Q_{T_0} \equiv [0, T_0] \times \Omega$.

One has the following results.

Theorem 4.1. *Assume that* (2.12) *and* (4.3) *hold, and that the couples* (u_0, F) *and* (u_0^ν, F^ν) *belong to* $H^k \times \mathcal{L}^2_{T_0}(H^k)$, *and satisfy the compatibility conditions up to order* $k-1$ *for the systems* (4.1) *and* (4.2), *respectively. Let* u *and* u^ν *be the solutions of these linear systems. Then* (4.13) *holds. In particular, if the assumptions* (2.4) *and* (2.14) *are satisfied, then* (2.9) *holds.*

Theorem 4.2. *Assume that* $A(\cdot)$ *and* $A^\nu(\cdot)$ *are as in Section* 2, *and* M *is as above. Assume, moreover, that the boundary matrices* $A^{(n)}(v)$ *and* $A^{\nu,(n)}(v)$ *are non-singular for each* $v \in \widetilde{\mathbb{N}}$, *and that the set* $\widetilde{\mathbb{N}}$ *is maximal non-positive with respect to* $A^{(n)}(v)$ *and* $A^{\nu,(n)}(v)$, *for each* $v \in \widetilde{\mathbb{N}}$. *Furthermore, the couples* (u_0, F) *and* (u_0^ν, F^ν) *belong to* $H^k \times \mathcal{L}^2_{T_0}(H^k)$, *and satisfy the compatibility conditions up to order* $k-1$ *for the systems*

$$\begin{cases} \partial_t u + A(u)\partial_x u = F \quad \text{in } Q_T, \\ Mu|_{\Sigma_T} = 0, \\ u(0) = u_0, \end{cases} \tag{4.4}$$

and

$$\begin{cases} \partial_t u^\nu + A^\nu\big(u^\nu\big)\partial_x u^\nu = F^\nu, \\ Mu^\nu|_{\Sigma_T} = 0, \\ u^\nu(0) = u_0^\nu, \end{cases} \tag{4.5}$$

respectively. Finally, assume that (2.4) *and* (2.5) *are satisfied.*

Then Eq. (2.9) *holds, where* u *and* u^ν *denote, respectively, the solutions of the systems* (4.1) *and* (4.2).

4.2. *Proof of Theorems* 4.1 *and* 4.2

Let us go into the proof of Theorem 4.1 below. Together with Eqs. (3.1) for $j = 1, \ldots, n-1$, we also consider the equations

$$\begin{cases} \partial_t(\partial_t u) + A(t,x)\partial_x(\partial_t u) = \partial_t F - \big(\partial_t A(t,x)\big)\partial_x u, \\ (\partial_t u)(0) = \partial_t u_0 \equiv F(0) - A(0,x)\partial_x u_0, \end{cases} \tag{4.6}$$

obtained from (4.1) by differentiation with respect to t. Note the formal definition of the quantity $\partial_t u$, introduced just for notational convenience.

Differentiation of the boundary condition that appears in (4.1) gives

$$M(\partial_j u)|_{\Sigma_{T_0}} = 0, \quad j = 1, \ldots, n-1, \qquad M(\partial_t u)|_{\Sigma_{T_0}} = 0. \tag{4.7}$$

Set, for convenience, $\partial_\tau \equiv (\partial_1, \ldots, \partial_{n-1}, \partial_t)$, and define $U = \partial_\tau u$, $\phi = \partial_\tau u_0$ and

$$\Phi = \partial_\tau F - \big(\partial_\tau A(t,x)\big)\partial_x u. \tag{4.8}$$

Moreover, denote by $\widehat{M}$ the diagonal bloc matrix $(M, \ldots, M)$, repeated n times. Eqs. (3.1) for $j = 1, \ldots, n-1$, (4.6) and (4.7) can be written in the abbreviate form

$$\begin{cases} \partial_t U + \widehat{A}(t,x)\partial_x U = \Phi, \\ \widehat{M}U|_{\Sigma_{T_0}} = 0, \\ U(0) = \phi, \end{cases} \tag{4.9}$$

that corresponds to (3.2). By replacing in the above arguments the system (4.1) by (4.2), we get

$$\begin{cases} \partial_t U^\nu + \widehat{A}_\nu(t,x)\partial_x U^\nu = \Phi^\nu, \\ \widehat{M}U^\nu|_{\Sigma_{T_0}} = 0, \\ U^\nu(0) = \phi^\nu, \end{cases} \tag{4.10}$$

where U^ν, Φ^ν and ϕ^ν are defined in the obvious way.

By the construction, the couples (ϕ, Φ) and (ϕ^ν, Φ^ν) satisfy the compatibility conditions up to order $k-2$ for the systems (4.9) and (4.10). Next we prove (3.9) and (3.10), where now $U = \partial_\tau u \equiv (\partial_1 u, \dots, \partial_{n-1} u, \partial_t u)$ instead of $U = \partial_\tau u \equiv (\partial_1 u, \dots, \partial_{n-1} u, \partial_n u)$, and similarly for U^ν and U^ϵ. Basically, the argument is an adaptation of that shown in Section 2. Here, the construction of the couples

$$\left(\phi^\epsilon, \Phi^\epsilon\right) \in H^k \times \mathcal{L}^2_{T_0}\left(H^k\right)$$

must be done in a more careful way. In fact, besides (3.4), each couple must satisfy the compatibility conditions up to order $k-1$ for the system (3.5) endowed with the boundary condition $\widehat{M} U^\epsilon|_{\Sigma_{T_0}} = 0$. See Proposition 4.1 in reference [14].

Equations similar to (3.9) and (3.10) lead to

$$\begin{aligned}&\left\|\left|\partial_t\left(u^\nu - u\right)(t)\right|\right\|^2_{k-1} + \sum_{j=1}^{n-1} \left\|\left|\partial_j\left(u^\nu - u\right)(t)\right|\right\|^2_{k-1}\\ &\quad \leqslant C\left\{\epsilon + \left\|u^\nu_0 - u_0\right\|^2_k + \left|\left[F^\nu - F\right]\right|^2_{k,T_0} + |[A_\nu - A]|^2_{k,t} + \left|\left[u^\nu - u\right]\right|^2_{k,t} + \Lambda(\epsilon)|[A_\nu - A]|^2_{k-1,t}\right\},\end{aligned} \tag{4.11}$$

for all $\nu \in \mathbb{N}$.

Finally, we appeal to Eqs. (4.1) and (4.2) to express $\partial_n(u^\nu - u)$ in terms of the other n first-order derivatives of $u^\nu - u$. This is done here by taking into account (4.3). One gets

$$\partial_n u = \left(A^{(n)}\right)^{-1}\left(\sum_{j=1}^{n-1} A^{(j)} \partial_j u - \partial_t u - F\right),$$

and similarly for $\partial_n u^\nu$. It readily follows, from the expression of $\partial_n(u^\nu - u)$, that

$$\begin{aligned}\left\|\left|\partial_n\left(u^\nu - u\right)(t)\right|\right\|^2_{k-1} \leqslant C\Bigg(&\left\|\left|\partial_t\left(u^\nu - u\right)(t)\right|\right\|^2_{k-1} + \sum_{j=1}^{n-1}\left\|\left|\partial_j\left(u^\nu - u\right)(t)\right|\right\|^2_{k-1}\\ &+ \left\|\left|\left(A^\nu - A\right)(t)\right|\right\|^2_{k-1} + \left\|\left|\left(F^\nu - F\right)(t)\right|\right\|^2_{k-1} + \left\|\left|\left(u^\nu - u\right)(t)\right|\right\|^2_{k-1}\Bigg).\end{aligned} \tag{4.12}$$

Hence, by (4.11),

$$\left\|\left|\left(u^\nu - u\right)(t)\right|\right\|^2_k \leqslant C_{T_0}\left\{\epsilon + \left\|u^\nu_0 - u_0\right\|^2_k + \left|\left[F^\nu - F\right]\right|^2_{k,T_0} + |[A_\nu - A]|^2_{k,t} + \Lambda(\epsilon)|[A_\nu - A]|^2_{k-1,t}\right\}, \tag{4.13}$$

where the term $|[u_\nu - u]|^2_{k,t}$ was dropped by appealing to Gronwall's lemma. Note that, formally, the estimate (4.13) is just the estimate (2.13), obtained above for the Cauchy problem. By appealing to (4.13), and by arguing as for proving Theorems 2.3 and 2.1, one proves Theorems 4.1 and 4.2.

Part II. The compressible Euler equations

5. Introduction

5.1. Strong continuous dependence. Incompressible and compressible fluids

Let us start Part II by quoting once again the following remark from the introduction of Kato and Lai's paper [32], concerning the Euler incompressible equations: "A remark is in order regarding the continuous dependence in "strong" topology of the solution on the data. It is the most difficulty part in a theory dealing with nonlinear equations of evolution. As far as we know, [25] is the only place where continuous dependence (in the strong sense) has been proved for the Euler equation in a bounded domain." In [25], the continuous dependence, in bounded domains, for the Euler *incompressible* equation, was proved by appealing to Riemannian Geometry in infinite dimensional manifolds. The authors consider an infinite dimensional manifold consisting of measure preserving endomorphisms of the spatial domain Ω, and introduce in this manifold a suitable Riemannian metric. Each solution is identified with a trajectory (parameterized by time) in this manifold. An analytical proof of the same result was given by Kato and Lai in reference [32], in the framework of Hilbert spaces $W^{m,2}$, without however appealing to the perturbation theory of the first author (at that time, T.Kato mentioned to me that his approach to the Euler incompressible equation in [32] would probably apply in the framework of $W^{m,p}$ spaces, $p \neq 2$). Let us quote again from the introduction of [32]: "The general theory developed in [29] by one of the authors for quasi-linear equations is unfortunately not applicable, since it is difficult to find the operator S with the required properties in the case of a bounded domain." It seems quite interesting that (by introducing a suitable new idea) we succeed in proving, see [10], that Kato's general perturbation theory does apply to the above problem, even in the framework of $W^{m,p}$ spaces. This is essentially due to the fact that the Euler *incompressible* equations do not require compatibility conditions. This fact brings the initial–boundary value problem near the Cauchy problem, in the case of the Euler incompressible equation. In the mathematical theory of initial–boundary value problems for non-viscous fluids, it is definitely necessary to distinguish between the compressible and the incompressible cases.

Let us turn back to the compressible Euler equations. Until the end of the 1970s, in the mathematical literature on the initial–boundary value problem for compressible Euler equations, the (strong) continuous dependence of the solution on the data was an open problem. Below, we give an introduction to some of our contributions to the resolution of this problem. It is, we hope, a suitable introduction for readers interested in the complete proofs. For convenience, we consider the case of the half-space $\Omega = \mathbb{R}^3_+ \equiv \{x\colon\ x_3 > 0\}$. We start Section 6 by stating the existence Theorem 6.1. Then we illustrate our approach to the proof of this result. The original system is decomposed into four systems, namely (6.4), (6.5), (6.6), and (6.7). Roughly speaking, these decomposition circumvent the lack of (4.3) since the first-order hyperbolic system (6.5) has no boundary constraints, and the second-order hyperbolic system (6.6) satisfies (4.3). As already explained in the introduction (to which we once again refer the reader) our approach to strong continuous dependence will be illustrated by appealing to the very simplified system (8.1).

5.2. Compressible Euler equations. The existence problem

At the end of the 1970s, an existence and uniqueness theorem for solutions to the compressible Euler equations was known only for the Cauchy problem, see Klainerman and Majda [33]. The first existence theorem for solutions to the initial–boundary value problem, for the compressible Euler equations, was proved by D.G. Ebin, see [23], under two restrictions: The initial density ρ_0 must be near constant and the initial velocity $v_0(x)$ must be sub-sonic (slightly compressible fluids). In references [4,6] we succeeded in proving the result in the general case. At more or less the same time, there appeared an independent paper by R. Agemi [1] in which he obtains the result proved in reference [6] (i.e., without size restrictions). We also refer the reader to references [12,38]. In [12] we gave a quite complete and simplified version of our proof, in the half-space case. We strongly refer the interested reader to this quite readable paper. It should be noted, however, that the flat boundary case is much easier to treat.

The set of the above papers is a fundamental step in the mathematical theory of compressible fluids, since they filled a big gap in this theory.

Let us recall some notation: $\mathbb{R}^3_+ = \mathbb{R}^2 \times \mathbb{R}^+$, $\Gamma = \mathbb{R}^2 \times \{0\}$, $\mathcal{N} =$ unit normal to the boundary Γ. Note that x_3 denotes the normal coordinate to the boundary.

We set $L^2 = L^2(\mathbb{R}^3_+)$, $H^k = H^k(\mathbb{R}^3_+)$, k positive integer, and so on. The L^2 norm is denoted by $\|\cdot\|$, that in H^k by $\|\cdot\|_k$. We write $\partial_t = \partial/\partial_t$, $\partial_i = \partial/\partial_{x_i}$, for $i = 1, 2, 3$. Moreover, $(v \cdot \nabla)w = \sum_j v_j \partial_j w$.

For $T > 0$, we set $Q_T = [0, T] \times \mathbb{R}^3_+$, $\Sigma_T = [0, T] \times \Gamma$, $C_T(H^k) = C([0, T]; H^k)$, $L^2_T(H^k) = L^2(0, T; H^k)$. The canonical norms in these two last spaces are denoted by $\|\cdot\|_{k,T}$ and $[\cdot]_{k,T}$, respectively. The above notation will be used without distinction for scalar and vector fields.

If $\rho = \rho(t, x)$ is the density of the fluid, we denote by $\bar{\rho}$ (a given positive constant) the density of the fluid at infinity. We assume that the pressure law $p(\cdot)$ is of class $C^4(\mathbb{R}^+, \mathbb{R})$, moreover $p'(s) > 0$ for each $s \in \mathbb{R}^+$. The barotropic motion of a non-viscous, compressible, fluid is described by the system

$$\begin{cases} \rho\big[\partial_t v + (v \cdot \nabla)v\big] + \nabla p(\rho) = 0 \quad \text{in } Q_T, \\ \partial_t \rho + \nabla \cdot (\rho v) = 0 \quad \text{in } Q_T, \\ v_3 = 0 \quad \text{on } \Sigma_T, \quad v(0) = v_0, \quad \rho(0) = \rho_0. \end{cases} \tag{5.1}$$

Note that, in the case of a generic domain Ω, boundary conditions like $v_3 = 0$ have the form $v \cdot \mathcal{N} = 0$. Moreover, in Eq. (6.2) below, and similar, derivatives ∂_3 are replaced by $\partial_{\mathcal{N}}$.

By the change of variables $g = \log(\rho/\overline{\rho})$, and by setting $h(s) = p'(\overline{\rho}e^s)$, for each $s \in \mathbb{R}$ (say, $h(g) = p'(\rho)$), one obtains the equivalent system

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + h(g)\nabla g = 0 \quad \text{in } Q_T, \\ \partial_t g + v \cdot \nabla g + \nabla \cdot v = 0 \quad \text{in } Q_T, \\ v_3 = 0 \quad \text{on } \Sigma_T, \quad v(0) = v_0, \quad g(0) = g_0. \end{cases} \tag{5.2}$$

The consideration of an external force field, as well as the proof of the continuous dependence of the solution in terms of it, do not present any particular difficulty, as shown in our original papers. Further, since our aim here is to show some of the main points, we will present our method in the particular framework of H^3 spaces. The proofs apply also in the framework of H^k spaces, for $k \geqslant 3$.

6. An existence theorem

6.1. Statement of the existence theorem

For the sake of brevity, we state the results directly for the system (5.2). From these results one immediately obtains corresponding results in terms of the couple (v, ρ) and initial data (v_0, ρ_0).

We assume here that

$$v_0 \in H^3, \qquad g_0 \in H^3. \tag{6.1}$$

Furthermore, we assume that the initial data satisfy the compatibility conditions

$$v_{0,3} = 0, \qquad \partial_3 g_0 = 0, \qquad \partial_3[v_0 \cdot \nabla g_0 + \nabla \cdot v_0] = 0 \quad \text{on } \Gamma. \tag{6.2}$$

Note that these conditions are *necessary* in order to obtain solutions in the $C_T(H^3)$ space. For $k > 3$, other necessary conditions must be imposed.

Theorem 6.1. *(See [1,4,6].) Assume that* (6.1) *and* (6.2) *hold. There are positive constants c and T, which depend only on $\|v_0\|_3$ and $\|g_0\|_3$ (c increasingly and T decreasingly), such that there is a unique solution (v, g) of problem* (5.2) *in the class $C_T(H^3)$. Further,*

$$\sum_{j=0}^{3} \left\| \partial_t^j (v, g) \right\|_{3-j,T} \leqslant c. \tag{6.3}$$

In the above references the result is proved for quite general domains Ω.

Remark 6.1. Our original proof was divided in two parts [5,6]. In reference [5] we study of the Euler “incompressible” equations with a non-vanishing, divergence assumption, say $\nabla \cdot v = \theta$. See (6.4). In this last paper we introduce a very useful device to proving, in a quite trivial way, existence of strong solutions to systems of partial differential equations. Solutions are identified with fixed points of maps in the framework of reflexive Banach spaces. The idea is trivial, but at the time new. Later on, the same idea has been used by many other authors.

It is not difficult to prove L^2 and H^1 energy type estimates for the system (5.2). Further, by differentiating the equations with respect to x_1, x_2 and t, we obtain in the same way estimates for the higher order derivatives which satisfy suitable boundary conditions. On the other hand, by truncation with a suitable function of the normal direction x_3, vanishing near the boundary, we may obtain interior estimates for all the derivatives. At this point, one would like to express, near the boundary, the first-order normal derivatives in terms of derivatives with respect to t, x_1 and x_2. This would allow to estimate, near the boundary, the missing derivatives. Unfortunately, in the system consisting of the two first equations $(5.2)_1$ and $(5.2)_2$, the matrix that multiplies the vector $(\partial_3 v_1, \partial_3 v_2, \partial_3 v_3, \partial_3 g)$ has rank 2 on the boundary Γ. Hence it is not possible to solve the above algebraic system in the desired way.

6.2. The approach to the existence theorem

Here we present the approach introduced in references [5,6] (roughly speaking, we follow [12]) in order to overcome the obstacle shown above. We replace the system (5.2) by an equivalent system, by appealing to the properties of the divergence and the curl operators, under the slip boundary condition. This leads to, apparently, a more complicate problem. However, this last problem can be separated in three elementary problems (plus an "equality"), as shown in the sequel.

For clearness, assume for a while that Ω is an arbitrary (for convenience, simply connected) domain. Then, a vector field V vanishes in Ω, if and only if $\nabla \times V = 0$ and $\nabla \cdot V = 0$ in Ω, and $V \cdot \mathcal{N} = 0$ in Γ. Hence, we may rewrite the first equation in (5.2) as $\nabla \times V = 0$ in Ω, $\nabla \cdot V = 0$ in Ω, plus $V \cdot \mathcal{N} = 0$ in Γ, where $V \equiv \partial_t v + (v \cdot \nabla)v + h(g)\nabla g$. Straightforward manipulations lead to the following systems of equations (for convenience, we associate the resulting equations in four separate systems), where

$$\zeta \equiv \nabla \times v, \qquad \delta \equiv \nabla \cdot v.$$

One has

$$\begin{cases} \nabla \times v = \zeta, \\ \nabla \cdot v = \delta \quad \text{in } Q_T, \\ \nabla \cdot v = 0 \quad \text{on } \Sigma_T, \end{cases} \tag{6.4}$$

where t is treated as a parameter. Moreover,

$$\begin{cases} \partial_t \zeta + (v \cdot \nabla)\zeta - (\zeta \cdot \nabla)v + (\nabla \cdot v)\zeta = 0 \quad \text{in } Q_T, \\ \zeta(0) = \zeta_0, \end{cases} \tag{6.5}$$

plus

$$\begin{cases} (\partial_t + v \cdot \nabla)^2 g - \nabla \cdot \big(h(g)\nabla g\big) = \sum (\partial_i v_j)(\partial_j v_i) \quad \text{in } Q_T, \\ \partial_3 g = 0 \quad \text{on } \Sigma_T, \\ g(0) = g_0, \quad (\partial_t g)(0) = g_1, \end{cases} \tag{6.6}$$

and, finally,

$$\delta = -(\partial_t g + v \cdot \nabla g). \tag{6.7}$$

It is worth noting that in the general case of a non-flat boundary, the boundary condition $\partial_3 g = 0$ is replaced by

$$\partial_{\mathcal{N}} g = \frac{1}{h(g)} \sum (\partial_i \mathcal{N}_j) v_i v_j \quad \text{on } \Sigma_T.$$

We get the expressions of the new initial data $\zeta_0(x)$ and $g_1(x)$ in terms of the given initial data, $\zeta_0(x) = \nabla \times v_0$ and $g_1(x) = -(v_0 \cdot \nabla g_0 + \nabla \cdot v_0)$. From (6.2) it follows that $\partial_3 g_0 = \partial_3 g_1 = 0$ on Γ, which are precisely the necessary compatibility conditions to solve (6.6).

Roughly speaking (see the introduction of reference [10]; for a simplified version see [12, Section 1]), we solve the above sequence of problems as follows. We start by giving a triad (ϕ, θ, q) in a suitable non-empty, convex, closed set $\mathbf{K}$, and by looking for a fixed point of the map

$$S : (\phi, \theta, q) \to (\zeta, \delta, g), \tag{6.8}$$

as follows (roughly speaking, both θ and δ should be seen here as $\nabla \cdot v$, and ϕ and ζ as $\nabla \times v$): For each given couple (ϕ, θ) we solve the elliptic system (6.4) with (ζ, δ) replaced by (ϕ, θ). This gives v. Note that this v depends on the particular couple (ϕ, θ). Clearly, v will represent the true velocity field only after the fixed point (similar remarks applies below). As a second step, we solve the Euler type equation (6.5), which gives ζ. Note that this transport equation does not require compatibility conditions. Hence the normal direction may be treated as the other directions. Next, we solve the linear hyperbolic initial–boundary value problem (6.6) obtained by using the previous vector field v and by replacing $h(g)$ by $h(q)$. This gives g. Note that, in (6.6), the derivative $\partial_3^2 g$ appears multiplied by the coefficient

$v_3^2 - h(g)$. This coefficient does not vanish on the boundary, since is given by $-h(g)$. This allows us to express $\partial_3^2 g$ in terms of tangential derivatives, in a neighborhood of Γ. Finally, δ is simply obtained from (6.7).

The above construction defines the map (6.8). Obviously, the main problem is to be able to solve the above sequence of problems with very sharp estimates, in order to prove the existence of the fixed point (uniqueness is easily shown). The more complex system is (6.6), in particular due to the non-homogeneous boundary condition for $\partial_{\mathcal{N}} g$, when the boundary is not flat.

7. Hadamard's well-posedness theorem

Let now (v_0^ν, g_0^ν) be a sequence of initial data satisfying the assumptions (6.1) and (6.2). One has the following result, which shows that the problem (5.2) is well posed in the classical Hadamard's sense. For $k = 3$ and $\Omega = \mathbb{R}^3_+$ the result was proved in reference [13].

Theorem 7.1. *Let* (v_0, g_0) *and* (v_0^ν, g_0^ν), $\nu \in \mathbb{N}$, *be couples of initial data satisfying the conditions* (6.1) *and* (6.2). *Further, assume that*

$$\lim_{\nu\to\infty} \big\| \big(v_0^\nu - v_0, g_0^\nu - g_0\big) \big\|_3 = 0. \tag{7.1}$$

Let $(v, g) \in C_{T_0}(H^3)$ *be the solution of problem* (5.2) *in* Q_{T_0}, *for some* $T_0 > 0$ (*see Theorem* 6.1). *Then*

$$\lim_{\nu\to\infty} \sum_{j=0}^{3} \big\| \partial_t^j (v_\nu - v, g_\nu - g) \big\|_{3-j,T_0} = 0, \tag{7.2}$$

where (v_ν, g_ν) *is the solution to the problem*

$$\begin{cases} \partial_t v_\nu + (v_\nu \cdot \nabla) v_\nu + h(g_\nu)\nabla g_\nu = 0 & \text{in } Q_{T_0}, \\ \partial_t g_\nu + v_\nu \cdot \nabla g_\nu + \nabla \cdot v_\nu = 0 & \text{in } Q_{T_0}, \\ v_{\nu,3} = 0 & \text{on } \Sigma_{T_0}, \\ v_\nu(0) = v_0^\nu, \quad g_\nu(0) = g_0^\nu. \end{cases} \tag{7.3}$$

Such a solution exists in Q_{T_0} *for sufficiently large values of* n.

For the extension of the above result from $H^3(\Omega)$ to $H^k(\Omega)$ spaces, where Ω is an open, bounded, connected subset of $\mathbb{R}^n$, $n \geqslant 2$, of class C^{k+2}, and k is any integer satisfying $k > \frac{n}{2} + 1$, see Theorem 1.4 in reference [14] (in this last reference we give a more effective statement, by appealing to neighborhoods instead of sequences).

We note that under the hypotheses of the above theorem one easily shows that

$$\lim_{\nu\to\infty} \sum_{j=0}^{2} \big\| \partial_t^j (v_\nu - v, g_\nu - g) \big\|_{2-j,T_0} = 0, \tag{7.4}$$

and, as an immediate consequence,

$$\lim_{\nu\to\infty} \sum_{j=0}^{2} \big\| \partial_t^j \big(h(g) - h(g_\nu)\big) \big\|_{2-j,T_0} = 0. \tag{7.5}$$

In fact, by energy type estimates, one trivially shows that $\|v_\nu - v\|_{0,T}$ and $\|g_\nu - g\|_{0,T}$ go to zero as $\nu \to \infty$. Since $\|\cdot\|_2 \leqslant c \|\cdot\|_3^{\frac{2}{3}} \|\cdot\|_0^{\frac{1}{3}}$, (7.4) follows.

The proof of Theorem 7.1 is quite complex and well constructed. In the following we concentrate just on one of the more characteristic obstacles, that appears when trying to prove the theorem. In order to isolate this point, we will operate drastic simplifications in the above systems of equations. In this way we hope to be able to briefly illustrate our approach.

8. The perturbation theory

In the sequel we restrict our considerations to the following drastic simplification of the system (6.6)

$$\begin{cases} \partial_t^2 g - \nabla \cdot (l \nabla g) = 0 \quad \text{in } Q_T, \\ \partial_3 g = 0 \quad \text{on } \Sigma_T, \\ g(0) = g_0, \quad (\partial_t g)(0) = g_1, \end{cases} \tag{8.1}$$

together with

$$\begin{cases} \partial_t^2 g_\nu - \nabla \cdot (l_\nu \nabla g_\nu) = 0 \quad \text{in } Q_T, \\ \partial_3 g_\nu = 0 \quad \text{on } \Sigma_T, \\ g_\nu(0) = g_0^\nu, \quad (\partial_t g_\nu)(0) = g_1^\nu, \end{cases} \tag{8.2}$$

where $l = h(g)$ and $l_\nu = h(g_\nu)$. Note that from (6.3) it follows that

$$\sum_{j=0}^{3} \left\| \partial_t^j l_\nu \right\|_{3-j,T} \leqslant c. \tag{8.3}$$

Further, from (7.5),

$$\lim_{\nu \to \infty} \sum_{j=0}^{2} \left\| \partial_t^j (l - l_\nu) \right\|_{2-j,T_0} = 0. \tag{8.4}$$

In the above systems (8.1) and (8.2), the initial data satisfy the (necessary) compatibility conditions $\partial_3 g_0 = \partial_3 g_1 = \partial_3 g_0^\nu = \partial_3 g_1^\nu = 0$ on Γ. We want to prove that if

$$\lim_{\nu \to \infty} \left(\left\| g_0^\nu - g_0 \right\|_3 + \left\| g_1^\nu - g_1 \right\|_2 \right) = 0, \tag{8.5}$$

then

$$\sum_{j=0}^{3} \left\| \partial_t^j (g_\nu - g) \right\|_{3-j,T} = 0. \tag{8.6}$$

The more natural way to prove (8.6) would be to appeal to the energy estimates for the difference $g_\nu - g$, solution of the initial–boundary value problem

$$\begin{cases} \partial_t^2 (g - g_\nu) - \nabla \cdot \left[l \nabla (g - g_\nu) \right] = \nabla \cdot \left[(l - l_\nu) \nabla g_\nu \right] \quad \text{in } Q_T, \\ \partial_3 (g - g_\nu) = 0 \quad \text{on } \Sigma_T, \\ (g - g_\nu)(0) = g_0 - g_0^\nu, \quad \partial_t (g - g_\nu)(0) = g_1 - g_1^\nu. \end{cases} \tag{8.7}$$

However this approach fails, as explained in the next section. A new device is needed.

8.1. The ϵ-pivot system

As the function $\nabla \cdot [(l - l_\nu) \nabla g_\nu]$ does not belong to H^2, but merely to H^1, we can not obtain from (8.7) an H^3 estimate for $(g - g_\nu)(t)$.

It is also worth noting that if the right-hand side of the first equation in (8.7) belongs to $L_T^p(H^k)$, the solution would merely belongs to $C_T(H^{k+1})$, although we are dealing with a second-order equation. This loss of regularity, which does not occur for elliptic and parabolic equations, still subsists in the hyperbolic case, even when the coefficients l and $l - l_\nu$ are as regular as we want.

By tacking into account that the crucial obstacle is the lack of regularity of the "pivot" function g_ν on the right-hand side of (8.7) (and hopping to be able to estimate the term $l - l_\nu$ in a suitable way) we have trying to apply the following.

Alternative strategy. To single out, in correspondence to any $\epsilon > 0$, a "pivot" function

$$g_\epsilon \in L^\infty_T\big(H^4\big)$$

and a positive integer $N(\epsilon)$ such that

$$\|g - g_\epsilon\|_{3,T} < c\epsilon \quad \text{and} \quad \|g_\nu - g_\epsilon\|_{3,T} < c\epsilon \quad \text{for each } \nu > N(\epsilon).$$

Here the positive constant c should be independent of the other quantities involved in the above equations. Note that the $L^\infty_T(H^4)$-norm of g_ϵ necessarily blows up as ϵ goes to zero.

Our first attempt to construct the g_ϵ functions was to consider the auxiliary system

$$\begin{cases} \partial_t^2 g_\epsilon - \nabla\cdot(l\nabla g_\epsilon) = 0 \quad \text{in } Q_T, \\ \partial_3 g_\epsilon = 0 \quad \text{on } \Sigma_T, \\ g_\epsilon(0) = g_0^\epsilon, \quad (\partial_t g_\epsilon)(0) = g_1^\epsilon, \end{cases} \tag{8.8}$$

where $g_0^\epsilon \in H^4$, $g_1^\epsilon \in H^3$, and

$$\lim_{\epsilon\to 0}\big(\big\|g_0^\epsilon - g_0\big\|_3 + \big\|g_1^\epsilon - g_1\big\|_2\big) = 0, \tag{8.9}$$

where g_0 and g_1 are the initial data in Eq. (8.1). Unfortunately, since $l(t) \notin H^4$, we cannot expect $g^\epsilon(t) \in H^4$, as desired. As a first attempt to overcome this obstacle we may try to approximate, in Eq. (8.8), the term $l(t)$ with a family $l_\epsilon(t) \in H^4$ (in the linear case this means $l_\epsilon = h(g_\epsilon)$). However, technical obstacles lead us to avoiding this way. Hence, we must look for an additional idea to improve our strategy. This is the subject of the next two subsections.

8.2. *The H^k via H^{k-1} device*

We start from the following two observations:

(i) The last obstacle described in the above section does not subsist if we look for a perturbation theorem in H^2 instead of H^3, provided that the coefficient l (on the left-hand side) still stays in H^3.

(ii) By differentiation of Eq. (8.1) with respect to the tangential coordinates x_1 or x_2, we get

$$\begin{cases} \partial_t^2 \partial_* g - \nabla\cdot(l\nabla \partial_* g) = \nabla\cdot\big[(\partial_* l)\nabla g\big] \quad \text{in } Q_T, \\ \partial_3\partial_* g = 0 \quad \text{on } \Sigma_T, \\ \partial_* g(0) = \partial_* g_0, \quad (\partial_t\partial_* g)(0) = \partial_* g_1, \end{cases} \tag{8.10}$$

where the symbol ∂_* indicates here differentiation with respect to x_1 and x_2. Since the coefficient $l \in H^3$ remains unchanged, we may appeal to the system (8.10) to try obtaining a perturbation theorem in H^2 for the first-order derivatives $\partial^* g$. For Cauchy problems, we also differentiate with respect to x_3. For initial–boundary value problems this is not useful, due to the lack of a suitable boundary condition. However (8.10) also holds if we differentiate with respect to t, provided that we replace $\partial_* g_0$ and $\partial_* g_1$ by g_1 and $\nabla\cdot(l(0)\nabla g_0)$, respectively.

By applying the above argument to Eqs. (8.2) we get

$$\begin{cases} \partial_t^2 \partial_* g_\nu - \nabla\cdot(l_\nu\nabla \partial_* g_\nu) = \nabla\cdot\big[(\partial_* l_n)\nabla g_n\big] \quad \text{in } Q_T, \\ \partial_3\partial_* g_\nu = 0 \quad \text{on } \Sigma_T, \\ \partial_* g_\nu(0) = \partial_* g_{0,\nu}, \quad (\partial_t\partial_* g_\nu)(0) = \partial_* g_{1,n}. \end{cases} \tag{8.11}$$

As for (8.10), in the initial–boundary value problem case we also consider differentiation with respect to t.

Estimate $g - g_\nu$ in H^3 is equivalent to estimating the derivatives $\partial_*(g - g_\nu)$ in H^2, if $\partial_* \equiv \{\partial_1, \partial_2, \partial_3\}$. However, in the case of the initial–boundary value problem we are constrained to consider

$$\partial_* \equiv \{\partial_1, \partial_2, \partial_t\}.$$

Summarizing, the lack of H^4-regularity for the coefficients $l(t)$ and $l_\nu(t)$ is not an insuperable barrier, since the H^3 regularity enjoyed by these "coefficients" is sufficient to obtain H^2 estimates for an appropriate set of first-order derivatives $\partial_* g$. So, we need to establish a sufficiently sharp perturbation theorem in H^2. We consider this problem in the next subsection.

8.3. The perturbation theorem in H^2

For simplicity we restrict ourselves to the Cauchy problem. For a few remarks on the initial–boundary value problem see the next subsection. The next step is to prove a *perturbation theorem* in H^2 for the linear system

$$\begin{cases} \partial_t^2\phi - \nabla\cdot(l\nabla\phi) = f \quad \text{in } Q_T, \\ \partial_3\phi = 0 \quad \text{on } \Sigma_T, \\ \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1. \end{cases} \tag{8.12}$$

A sequence of perturbed systems must be considered

$$\begin{cases} \partial_t^2\phi_\nu - \nabla\cdot(l_\nu\nabla\phi_\nu) = f_\nu \quad \text{in } Q_T, \\ \partial_3\phi_\nu = 0 \quad \text{on } \Sigma_T, \\ \phi_\nu(0) = \phi_0^\nu, \quad \partial_t\phi_\nu(0) = \phi_1^\nu. \end{cases} \tag{8.13}$$

We want a perturbation theorem in H^2, to be applied to the three systems satisfied by the first-order derivatives $\partial_* g$. Further, in order to prove the perturbation theorem in H^3, we have to apply to the ϵ-device. The proof that the solutions $g_\epsilon \in L_T^\infty(H^3)$ of the ϵ-systems (8.8) satisfy, in particular, the properties (8.9) (where the index 3 should be replaced by 2) is out of the aims of these notes. We merely state here the following result, which is a particular case of Theorem 6.3 in reference [13] (an H^k perturbation result is proved in [15, Theorem 4.1]).

Theorem 8.1. *Given $\epsilon > 0$ there exists $C = C(\epsilon)$ such that*

$$\|\phi - \phi_\nu\|_{2,T}^2 \leqslant c\epsilon + c\big(\|\phi_0 - \phi_0^\nu\|_2^2 + \|\phi_1 - \phi_1^\nu\|_1^2\big) + c[f - f_\nu]_{1,T}^2 + C(\epsilon)[l - l_\nu]_{2,T}^2, \tag{8.14}$$

where the constants c are independent of ϵ and of all the functions involved in the problem.

By applying Theorem 8.1 to the solutions $\phi = \partial_* g$ and $\phi_\nu = \partial_* g_\nu$ of Eqs. (8.10) and (8.11), and by tacking into account (8.4) and (8.5), one gets

$$\|\partial_* g - \partial_* g_\nu\|_{2,T}^2 \leqslant c\epsilon + c\big([g - g_\nu]_{3,T}^2 + c[l - l_\nu]_{3,T}^2\big), \tag{8.15}$$

for $\nu > N(\epsilon)$. Note that here $f = \nabla\cdot[(\partial_* l)\nabla g]$, $f_\nu = \nabla\cdot[(\partial_* l_\nu)\nabla g_\nu]$, $\phi_0 = \partial_* g_0$, $\phi_0^\nu = \partial_* g_0^\nu$, etc.

At this point we distinguish between the well-posedness problem for non-linear systems and the (strictly related) perturbation theorem for linear systems. In the first case we want to prove that the non-linear problem (8.1) is well posed, where $l = h(g)$ and $l_\nu = l(g_\nu)$. Hence we use the estimate

$$[l - l_\nu]_{3,T} \leqslant c[g - g_\nu]_{3,T}.$$

In the second case we look for a perturbation theorem. Here, the coefficients l and l_ν are given, and we assume that

$$\lim_{\nu\to\infty}[l - l_\nu]_{3,T} = 0.$$

In both cases we arrive to an estimate like

$$\|\partial_* g - \partial_* g_\nu\|_{2,T}^2 \leqslant c\epsilon + c[g - g_\nu]_{3,T}^2, \tag{8.16}$$

for $\nu > N(\epsilon)$ for a suitable $N(\epsilon)$ (that could be larger than the above one). Now, by adding the estimates (8.16) written for each of the first-order derivatives ∂_* of g (see, however, Section 8.4), we obtain an estimate like

$$\|g - g_\nu\|_{3,T}^2 \leqslant c\epsilon + cT\|g - g_\nu\|_{3,T}^2, \tag{8.17}$$

for $n > N(\epsilon)$. Hence, for sufficiently small positive values of T, which are independent of ϵ,

$$\|g - g_\nu\|_{3,T}^2 \leqslant c\epsilon \quad \text{for } \nu > N(\epsilon). \tag{8.18}$$

This shows the convergence of $g_\nu(t)$ to $g(t)$ in H^3, uniformly with respect to t in $[0, T]$. Actually, our proofs show that $\partial_t^j g_\nu(t)$ converges to $\partial_t^j g(t)$ in $C([0, T]; H^{3-j})$, for $j = 0, 1, 2, 3$. Finally, we have shown that the above results hold in the whole of the existence interval $[0, T_0]$ of a strong solution g.

8.4. A final remark

Concerning the fact that Eqs. (8.10) and (8.11) do not hold if differentiation is in the normal direction, we note the following. Roughly speaking, the above equations hold "far from the boundary" (by truncation, etc.). On the other hand, as already mentioned, our modified system allows to express the missing normal derivatives, near the boundary, in terms of non-normal derivatives. This is a quite difficult point, which requires new, non trivial devices. Note the following. In Eq. (6.6), the coefficient $v_3^2 - h(g)$ that multiplies the normal derivative $\partial_3^2 g$, does not vanish on the boundary, since is given by $-h(g)$. This allows us to express $\partial_3^2 g$ in terms of tangential derivatives, in a neighborhood of Γ. Above, for simplicity, we have considered the system (8.1), where the coefficient $v_3^2 - h(g)$ is replaced by l. Similarly, $v_{\nu,3}^2 - h(g_\nu)$ is replaced by l_n, and so on. Without these simplifications, the situation is much harder to treat. For simplicity, let us discuss this point just in the framework of the existence theorem (say, no ν-systems, ϵ device, and so on). We may prove uniform estimates for $\partial_3^2 g$ in a δ-neighborhood of the boundary, $\delta > 0$, if in this neighborhood the coefficient $v_3^2 - h(g)$ is strictly positive. However, this coefficient may vanish in interior points, since $v_3^2 = 0$ just on the boundary. Hence, an estimate from below for $\delta(x_3)$ depends on uniform estimates from below for the quantity $v_3^2 - h(g)$. However, this point requires estimates for the unknown $\partial_3^2 g$ near the boundary. We do not treat this delicate question here. However the *very basic idea* is the following. In system (8.10) (as well as in (8.11)) the Neumann boundary condition does not hold if $\partial_* = \partial_3$ since $\partial_3 \partial_* g = 0$ on Σ_T is false in general. However, instead of considering the Neumann boundary condition $\partial_3(\partial_3 g) = 0$ we consider the Dirichlet boundary condition $\partial_3 g = 0$.

Clearly, if we want strong well-posedness results, the above devices must be developed for $g - g_\nu$ and $g_\nu - g_\epsilon$ (recall (8.8)) and not simply for g.

Part III. Mach number and incompressible limits

Here we want just recall some quite general results proved in reference [21] (see also [18]), concerning singular limits (incompressible and inviscid) in the strong norms for solutions to the equations of motion of compressible fluids, depending on the Mach number λ^{-1} and on the viscosity coefficients $\mu \in [0, \mu_0]$ and $\zeta \in [0, \zeta_0]$. The spatial domain is the n-dimensional torus, $n \geqslant 2$. Equations are

$$\begin{cases} \rho_t + v \cdot \nabla\rho + \rho\nabla \cdot v = 0, \\ \rho\big[v_t + (v \cdot \nabla)v\big] + \lambda^2 \nabla p(\rho) = \mu\Delta v + \zeta\nabla(\nabla \cdot v), \\ \rho(0) = \bar{\rho}_0 + \rho_0(x), \quad v(0) = v_0(x), \end{cases} \tag{8.19}$$

where $\bar{\rho}_0$, the mean density of the fluid, is a fixed positive constant. Without loss of generality, we set that $\bar{\rho}_0 = 1$. For convenience, we assume that the pressure law has the form $p(\lambda, \rho) = \lambda^2 p(\rho)$, where $p(\cdot)$ is a fixed $C^{k+2}(\mathbb{R}^+; \mathbb{R})$ function and $p'(s) > 0$ for all $s > 0$. We denote by k_0 the smallest integer larger than $n/2$ and by $k \geqslant k_0 + 1$ a fixed integer. However our proofs hold if $p(\lambda, \rho)$ satisfies the assumptions in reference [9]. The crucial point is to assume that $\lim p'(\lambda, \rho) = \infty$ as $\lambda \to \infty$.

As in the previous sections, we are interested in proving *strong* continuous dependence, and this is the main novelty of our results. Moreover, in the literature, it is in general assumed that, as the Mach number λ^{-1} goes to zero, the viscosity coefficients remain fixed (hence one deals with Navier–Stokes or with Euler equations, separately). See the fundamental, pioneering, paper [33], by Klainerman and Majda, and also [34,35]. Hence one deals with the Navier–Stokes compressible equations if the (fixed) viscosity coefficients does not vanish, and with the Euler compressible equations if $\mu = \zeta = 0$. In [21] we studied the behavior of (ρ, v) as (simultaneously) the Mach number λ^{-1} goes to zero, the viscosity coefficient μ converges to a value $\bar{\mu} \geqslant 0$ and ζ stays bounded. Note that we do not exclude the (more challenging case) in which $\bar{\mu} = 0$. In this case the solution to the Navier–Stokes *compressible* equations converge (strongly) to the solution w of the *incompressible* Euler equations

$$\begin{cases} \nabla \cdot w = 0, \\ w_t + (w \cdot \nabla)w + \nabla\pi = \bar{\mu}\Delta w, \\ w(0) = w_0(x), \end{cases} \tag{8.20}$$

where $\nabla \cdot w_0 = 0$.

We find it convenient to make the change of variables (for details see [20,21])

$$g = \log(\rho/\rho_0).$$

Eqs. (8.21) are then equivalent to (for details see [20,21])

$$\begin{cases} g_t + v \cdot \nabla g + \nabla \cdot v = 0, \\ v_t + \lambda^2 \phi'(g) \nabla g + \nabla(\nabla \cdot v) = \exp(-g)\big[\mu \Delta v + \zeta \nabla(\nabla \cdot v)\big], \\ g(0) = g_0(x), \quad v(0) = v_0(x), \end{cases} \tag{8.21}$$

where $\phi'(s) \equiv p'(s)$ for all $s \in \mathbb{R}$. Our results are stated in terms of g. To obtain the results in terms of ρ is trivial. See the remarks in [21, p. 314].

Let us briefly give an idea of the results proved in reference [21]. For more details, variants of the result below, and significant remarks we refer to this last reference and also to [20], where some useful preliminaries are proved.

Assume the following conditions on the initial data:

$$\|v_0\|_{k_0+1} \leqslant c_1, \qquad \lambda \|g_0\|_{k_0+1} \leqslant c_1, \tag{8.22}$$

$$\|v_0\|_{k_0+1} \leqslant c_2, \qquad \lambda \|g_0\|_{k_0+1} \leqslant c_2, \tag{8.23}$$

$$\lambda \|\nabla \cdot v_0\| \leqslant c_3, \qquad \lambda^2 \|\nabla g_0\| \leqslant c_3. \tag{8.24}$$

Under these hypotheses, there is a positive constant T, depending only in c_1 (decreasingly), such that the problem (8.21) has a unique strong solution in $[0, T]$. Further,

$$\lambda^2 \|g\|_{k,T}^2 + \|v\|_{k,T}^2 + \mu[\nabla v]_{k,T}^2 + \zeta[\nabla \cdot v]_{k,T}^2 \leqslant C_1\big(\lambda^2 \|g_0\|_{k,T}^2 + \|v_0\|_{k,T}^2\big). \tag{8.25}$$

For other estimates, and more details, see [21, Lemma 1.1, remarks (i), (ii), (iii), (iv)] and [20, Lemmas 1.1, 1.2, 1.3, Corollary 1.4]. These existence results are minor improvements of results stated in [33]. Further, a careful use of standard techniques lead to Theorem 1.5 in reference [20], concerning singular limits. This theorem improves, in some aspects, Theorem 2 in reference [33]. However, the above results do not show strong convergence, i.e., convergence in spaces like $C([0, T]; H^k)$. This is proved in [21, Theorem A]. Let us illustrate this last result. For details and other related results see the original work.

In order to state the result in a clear form, we introduce the following notation. The above constants $k_0, k, \mu_0, \zeta_0, c_1, c_2, c_3$ are fixed. Then we define the corresponding set of admissible data (initial data and parameters)

$$\mathfrak{X} = \big\{(v_0, g_0, \lambda, \mu, \zeta) \in H^k \times H^k \times [1, \infty[\times [0, \mu_0] \times [0, \zeta_0]\text{: (8.22), (8.23), (8.24) hold}\big\}, \tag{8.26}$$

endowed with canonical product norm. Finally we define the map S on $\mathfrak{X}$,

$$S(v_0, g_0, \lambda, \mu, \zeta) = (v, g),$$

where (v, g) is the solution of (8.21) with data and parameters $(v_0, g_0, \lambda, \mu, \zeta) = (v, g)$. One has the following result (see [21, Theorem A]).

Theorem 8.2. *Let $S : \mathfrak{X} \to C([0, T]; H^k \times H^k)$ be defined as above. Then*

$$\lim_{(v_0, \lambda g_0, \lambda, \mu) \to (w_0, 0, \infty, \overline{\mu})} \big\|(v, g) - (w, 0)\big\|_{k,T}^2 + \lambda^2 \|\nabla g\|_{k-1,T}^2 + \|g_t\|_{k-1,T}^2 + \overline{\mu}[v - w]_{k+1,T}^2 = 0, \tag{8.27}$$

where w is the solution to problem (8.20).

Note that $g = 0$ means that $\rho = 1$, which is just the mean density ρ_0. Also note that if $\overline{\mu}$ is positive, hence w is a solution of a Navier–Stokes incompressible equation, then one has an "additional" strong converge of v to w in $L^2([0, T]; H^k)$.

The above kind of results has been extended to magneto-fluid dynamics in reference [37]. Let us report some references, previous to [21], treating the incompressible limit for compressible fluids. In [2,3,18,22,24,33,34,36,38–41] the authors consider inviscid fluids. Viscous stationary fluids where considered in [8,9]. Viscous non-stationary fluids were studied in [33–35].

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