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On non-Newtonian p-fluids. The pseudo-plastic case

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Abstract

In the following we study a class of stationary Navier–Stokes equations with shear dependent viscosity, under the non-slip (Dirichlet) boundary condition. We consider pseudo-plastic fluids. A fluid is said pseudo-plastic, or shear thinning, if in Eq. (1.1) below one has $p < 2$. We are interested in global (i.e., *up to the boundary*) regularity results, in dimension $n = 3$, for the second order derivatives of the velocity and the first order derivatives of the pressure. We consider a cubic domain Ω and impose the non-slip boundary condition only on two opposite faces. On the other faces we assume periodicity, as a device to avoid effective boundary conditions. This choice is made so that we work in a bounded domain Ω and simultaneously with a flat boundary.

Keywords: Navier–Stokes equations; Non-Newtonian fluids; Boundary value problems; Regularity of solutions

1. Introduction and results

The pioneering papers on the mathematical treatment of the Navier–Stokes equations with shear dependent viscosity are due to O.A. Ladyzhenskaya, see [14] and references therein. See also J.-L. Lions [15] and references therein for related models.

We are interested in proving regularity results up to the boundary for solutions to the Navier–Stokes equations for flows with shear dependent viscosity, under suitable boundary conditions. It is worth noting that vorticity is essentially created at the boundary of the physical domain, and that this boundary is nearly always present in realistic problems.

The cases $p > 2$ and $p < 2$ capture shear thickening and shear thinning phenomena, respectively. See, for instance, [13] and [18]. For regularity results, up to the boundary, in the shear thickening (or dilatant) case we refer to [1,2,4] and [16].

In the following we consider the case $1 < p < 2$. For simplicity, we consider the basic system (1.1). For regularity results, up to the boundary, in $W^{2,q}(\Omega)$-spaces we refer the reader to [3,6,7,9] and [10]. In [3] and [7] the problem is studied in the "cubic domain" framework considered in the sequel. In [6] the problem is studied in regular arbitrary domains (this situation requires very new devices). Finally, in Refs. [9] and [10], the author considers cylindrical domains. See also Remark 1.1 below.

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As in [2] and [3], in order to work with a flat boundary Γ, we are led to consider a cubic domain $\Omega = (]0,1[)^3$ and to impose the boundary condition (1.2) only on the two opposite faces

$$\Gamma_- = \{x\colon\ |x_1|, |x_2| < 1,\ x_3 = 0\}, \qquad \Gamma_+ = \{x\colon\ |x_1|, |x_2| < 1,\ x_3 = 1\}.$$

The problem will be assumed periodic, with period equal to 1, in both the x_1 and the x_2 directions (in this way we avoid artificial singularities due to the corner points). The significant boundary is $\Gamma = \Gamma_- \cup \Gamma_+$, or even $\Gamma = \Gamma_-$. We set $x' = (x_1, x_2)$. By "x'-periodic" we mean periodic of period 1 in both x_1 and x_2 directions.

The above framework enables us to emphasize the very basic ideas.

In the sequel u and π denote, respectively, the velocity and the pressure of a viscous incompressible fluid, $\mathcal{D}u$ denotes the symmetric gradient,

$$\mathcal{D}u = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

and $\mu > 0$ is a fixed constant. Concerning the full Navier–Stokes equations we prove the following result.

Theorem 1.1. *Let u be a solution to the full Navier–Stokes equations*

$$\begin{cases} -\nabla \cdot \left(\left(\mu + |\mathcal{D}u|\right)^{p-2}\mathcal{D}u\right) + (u \cdot \nabla)u + \nabla\pi = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

under the non-slip boundary condition

$$u_{|\Gamma} = 0. \tag{1.2}$$

Set

$$\bar{q} = 4p - 2, \qquad l = \frac{4p-2}{p+1}, \tag{1.3}$$

and

$$p_0 = \frac{20}{11} = 1.8181\ldots. \tag{1.4}$$

Then, under the assumption $p > p_0$

$$u \in W^{1,\bar{q}}(\Omega) \cap W^{2,l}(\Omega) \quad \text{and} \quad \nabla\pi \in L^l(\Omega). \tag{1.5}$$

This result improves that in Ref. [3], where $p_0 = \frac{15}{8} = 1.875$, and that in Ref. [7] where $p_0 = \frac{7+\sqrt{35}}{7} = 1.845\ldots$. See Remark 1.1 below.

The regularity results for solutions to the system (1.1) follow from corresponding results for solutions to the simplified system

$$\begin{cases} -\nabla \cdot \left(\left(\mu + |\mathcal{D}u|\right)^{p-2}\mathcal{D}u\right) + \nabla\pi = f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.6}$$

For the above reason, in the following we start by proving results for solutions to this last system. As a second step, we show that the regularity results stated in Theorem 1.2 below still hold in the presence of the convective term $(u \cdot \nabla)u$, provided that $p > p_0$. Just the same argument show that the extensions of Theorems 1.3 and 1.4 hold as well, under the same hypothesis on p.

We denote by the symbol c positive constants, independent of u, and by

$$C = C\left(\|\mathcal{D}u\|_p\right),$$

positive constants that may depend on $\|\mathcal{D}u\|_p$. Constants of type c and C may change from equation to equation.

Further, we denote by $q' = \frac{q}{q-1}$ the dual exponent of q. For $1 < r < 3$, we define the Sobolev embedding exponent r^* by the equation

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3}. \tag{1.7}$$

In the sequel we prove the following results. For notation see the next section, in particular (2.8) and (2.9).

Theorem 1.2. *Assume that $f \in L^{p'}(\Omega)$ and let $u \in V_p$ be a solution to the problem* (1.6), (1.2)*, where $\mu > 0$ and $1 < p < 2$. Then $D_*^2 u$ belongs to $L^p(\Omega)$ and $\nabla_* \pi$ belongs to $L^2(\Omega)$. Moreover,*

$$\left\| D_*^2 u \right\|_p \leqslant C \| f \|_{p'} \tag{1.8}$$

and

$$\| \nabla_* \pi \|_2 \leqslant C \| f \|_{p'}. \tag{1.9}$$

Next, we define exponents

$$r(q) = \frac{2q}{2(2-p)+q}, \qquad \lambda(q) = \frac{2q}{2-p+q}, \qquad \mathcal{Q}(q) = \frac{6q}{8-4p+q}. \tag{1.10}$$

One has the following (conditional) result.

Theorem 1.3. *Assume, in addition to the hypothesis of Theorem* 1.2*, that $p > \frac{3}{2}$ and that*

$$\mathcal{D}u \in L^q(\Omega), \tag{1.11}$$

for some q satisfying $p \leqslant q \leqslant 6$. Then $D_*^2 u \in L^{\lambda(q)}$, and

$$\left\| D_*^2 u \right\|_{\lambda(q)} \leqslant C \| f \|_{p'} \left\| \mu + |\mathcal{D}u| \right\|_q^{\frac{2-p}{2}}. \tag{1.12}$$

Further,

$$u \in W^{1,\mathcal{Q}(q)}(\Omega) \cap W^{2,r(q)}(\Omega). \tag{1.13}$$

Moreover,

$$\| u \|_{2,r(q)} \leqslant C \left(1 + \| \mathcal{D}u \|_q^{2-p} \right) \| f \|_{p'} \tag{1.14}$$

and

$$\| u \|_{1,\mathcal{Q}(q)} \leqslant C \left(1 + \| \mathcal{D}u \|_q^{\frac{2(2-p)}{3}} \right) \| f \|_{p'}. \tag{1.15}$$

Note that the exponent $\mathcal{Q}(q)$ is strictly greater then the Sobolev embedding exponent $r^*(q)$, given by (1.7). On the other hand, $l = r(\bar{q})$.

Since (1.11) holds for $q = p$, Theorem 1.3 allows a bootstrap argument, that leads to the following result.

Theorem 1.4. *Under the assumptions of Theorem* 1.2*, and $p > \frac{3}{2}$, one has*

$$u \in W^{1,\bar{q}}(\Omega) \cap W^{2,l}(\Omega), \quad \text{and} \quad \nabla \pi \in L^l(\Omega). \tag{1.16}$$

Moreover,

$$\| u \|_{1,\bar{q}} \leqslant C \left(1 + \| f \|_{p'}^{\frac{3}{2p-1}} \right) \tag{1.17}$$

and

$$\| u \|_{2,l} \leqslant C \left(\| f \|_{p'} + \| f \|_{p'}^{\frac{5-p}{2p-1}} \right). \tag{1.18}$$

It is worth noting that $\bar{q} > l^*$. On the other hand, from (1.12) and (1.17) it follows that

$$D_*^2 u \in L^{\lambda(\bar{q})},$$

where $\lambda(\bar{q}) = \frac{4(2p-1)}{3p} > l$.

Remark 1.1. The thread of the proofs follow that one introduced in Ref. [1] for the case $p > 2$ and developed in some subsequent papers. We start by proving that weak solutions in the energy space $W^{1,p}(\Omega)$ actually belong to a suitable $W^{2,s}(\Omega)$ space. By Sobolev classical embedding theorems this last space is continuously embedded in $W^{1,s^*}(\Omega)$, where $s^* > p$. This situation leads to a quite natural bootstrap argument. On the other hand, as in previous papers, we obtain better regularity for derivatives in the tangential directions than in the normal direction. Hence, in the bootstrap argument, we may use anisotropic embedding theorems, see [20], instead of the classical Sobolev theorems. This fruitful idea is due to Berselli, see [7]. By introducing this new device in the proof given in [3], this author succeed in improving the results stated in this last reference. In particular, he shows that the solutions to problem (1.6) belong to $W^{1,\bar{q}-\epsilon}(\Omega) \cap W^{2,l-\epsilon}(\Omega)$, for any arbitrarily small $\epsilon > 0$, where $\bar{q}$ and l are the ones defined in (1.3). Since these exponents are greater than those in [3], better regularity results follow.

The above regularity results for solutions to the system (1.6) are the starting point for proving similar results for solutions to the full Navier–Stokes system (1.1). Actually, we assume that the convective term is part of the right-hand side. So, a crucial point is to prove sharp estimates for the solutions u of (1.6) in terms of the right-hand side f. Below, we prove the estimates (1.8), (1.14), (1.15) and (1.18) with better powers of $\|f\|_{p'}$ than the ones in the previous papers. This improvement, which is the very responsible for a better value for p_0 in Theorem 1.1, is a consequence of a better estimate for the quantities $I_s(u)$, see below. We prove that $I_s(u) \leqslant \|f\|_{p'}^2$, see (3.14), instead of $I_s(u) \leqslant \|f\|_{p'}^{p'}$, as in [3] and [7]. See Remark 3.1 below.

Remark 1.2. We cannot overlook that in applications to real fluids we often meet in the left-hand side of Eq. (1.1) an additional term $-\nu_0 \Delta u$, say

$$\begin{cases} -\nu_0 \Delta u - \nabla \cdot \big((\mu + |\mathcal{D}u|)^{p-2} \mathcal{D}u \big) + (u \cdot \nabla) u + \nabla \pi = f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.19}$$

However, from a mathematical point of view, the case $\nu_0 = 0$ looks more challenging since (as the reader may easily realize by inserting this term in the proofs below) the presence of the above term makes the proofs easier. See Remark 5.1 in Ref. [2] (further, in (1.19) we may assume that $\mu = 0$). In particular, see [5] and [11], solutions u to the problem (1.19) under (for instance) the boundary condition (1.2) satisfy

$$u \in W^{2,2}(\Omega) \cap W^{1,q}(\Omega), \quad \forall q < \infty. \tag{1.20}$$

2. Some auxiliary results

For the reader's convenience we repeat here some well-known results (not due to us).

We set $\partial_i f = \frac{\partial f}{\partial x_i}$, $\partial^2_{ij} f = \frac{\partial^2 f}{\partial x_i \partial x_j}$. We use the same notation for functional spaces and norms for both scalar and vector fields. The symbol $\|\cdot\|_p$ denotes the canonical norm in $L^p(\Omega)$, moreover $\|\cdot\| = \|\cdot\|_2$. Further, $W^{1,p}(\Omega)$ denotes the usual Sobolev space.

We set

$$V_p = \big\{ v \in W^{1,p}(\Omega)\colon\ (\nabla \cdot v)_{|\Omega} = 0;\ v_{|\Gamma} = 0;\ v \text{ is } x'\text{-periodic} \big\}. \tag{2.1}$$

By appealing to inequalities of Korn's type, one shows that there is a positive constant c such that

$$\|\nabla v\|_p + \|v\|_p \leqslant c \|\mathcal{D}v\|_p, \tag{2.2}$$

for each $v \in V_p$. Hence the two quantities above are equivalent norms in V_p.

Assume that $f \in (V_p)'$. We say that u is a *weak solution* to problem (1.6), (1.2) if $u \in V_p$ satisfies

$$\frac{1}{2} \int_\Omega \nu_T(u) \mathcal{D}u \cdot \mathcal{D}v \, dx = \int_\Omega f \cdot v \, dx \tag{2.3}$$

for all $v \in V_p$. Concerning existence of weak solutions under the assumptions $p < 2$ and $n \geqslant 3$, we refer the reader to [12,19] and references therein.

One has (see, for instance [3])

$$\|\mathcal{D}u\|_p^p \leqslant c \|f\|_{-1,p'} \|u\|_{1,p} + |\Omega| \mu^p, \tag{2.4}$$

$$\|\nabla u\|_p^{p-1} \leqslant c \big(\|f\|_{-1,p'} + \mu^{p-1} \big), \tag{2.5}$$

and

$$\|\pi\|_{L^{p'}_{\#}} \leqslant c\big(\|f\|_{-1,p'} + \mu^{p-1}\big), \tag{2.6}$$

where $L^{\alpha}_{\#} = L^{\alpha}/\mathbb{R}$.

The following well-known result (see Nečas [17]) is of capital importance in the sequel. If a distribution g is such that $\nabla g \in W^{-1,\alpha}(\Omega)$, then $g \in L^{\alpha}(\Omega)$ and

$$\|g\|_{L^{\alpha}_{\#}} \leqslant c\|\nabla g\|_{W^{-1,\alpha}}. \tag{2.7}$$

We end this section by introducing some more notation.

We denote by D^2u the set of all the second derivatives of u. The meaning of expressions like $\|D^2u\|$ is clear. The symbol D^2_*u denotes any of the second order derivatives $\partial^2_{ik}u_j$ except for the derivatives $\partial^2_{33}u_j$, if $j=1$ or $j=2$. Moreover,

$$\big|D^2_*u\big|^2 := \big|\partial^2_{33}u_3\big|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(3,3)}}^{3} \big|\partial^2_{ik}u_j\big|^2. \tag{2.8}$$

Similarly, ∇_* may denote any first order partial derivative, except for $\partial/\partial x_3$. In particular,

$$|\nabla_*\pi|^2 := \sum_{j=1}^{2}|\partial_j\pi|^2. \tag{2.9}$$

3. Regularity of the tangential derivatives. Proof of Theorem 1.2

In this section we prove the estimates (1.8) and (1.9).

In order to avoid arguments already developed in [1] and [2], we replace the use of differential quotients simply by differentiation.

We start by the following useful result.

Lemma 3.1. *Assume that $u \in V_p$. Then*

$$\big\|D^2_*u\big\|_p \leqslant c\|\nabla_*\mathcal{D}u\|_p. \tag{3.1}$$

The above estimate holds for any $p > 1$.

Proof. Since $\partial_s u$, $s=1,2$, vanishes for $x_3=0$ and $x_3=1$, we may apply (2.2) with $v=\partial_s u$, if $s=1,2$. This device yields

$$\big\|\nabla_*(\nabla u)\big\|_p + \|\nabla_* u\|_p \leqslant c\big\|\nabla_*(\mathcal{D}u)\big\|_p \leqslant c\|\nabla_*\mathcal{D}u\|_p. \tag{3.2}$$

On the other hand, by appealing to the constraint $\nabla\cdot u = 0$, we show that

$$\big\|D^2_*u\big\|_p \leqslant c\|\nabla_*\nabla u\|_p. \qquad \square \tag{3.3}$$

In the sequel $s=1,2$. Hence x_s denotes each of the two tangential directions to Γ. Consequently translations in these two directions are admissible (in the usual sense).

In order to fix ideas we state some well-known results in the particular situation considered here. We define the tensor S as

$$S = \big(\mu + |D|\big)^{p-2}D, \tag{3.4}$$

where D is an arbitrary tensor. One has

$$\frac{\partial S_{ij}}{\partial D_{kl}}C_{ij}C_{kl} \geqslant (p-1)\big(\mu+|D|\big)^{p-2}|C|^2, \tag{3.5}$$

for all tensors C. Summation on repeated indexes is assumed except for the index s below.

Define, for $s = 1, 2$,

$$J_s(u) =: \int_\Omega \nabla \cdot \left[\left(\mu + |\mathcal{D}| \right)^{p-2} \mathcal{D} \right] \cdot \partial_{ss}^2 u \, dx. \tag{3.6}$$

For convenience, here and in the sequel, we set

$$\mathcal{D} = \mathcal{D}u.$$

By two integrations by parts, and by taking into account that $\mathcal{D}u$ is symmetric one shows, after some manipulations, that

$$J_s(u) = \int_\Omega \partial_s \left[\left(\mu + |\mathcal{D}| \right)^{p-2} \mathcal{D} \right] : \partial_s \mathcal{D} \, dx. \tag{3.7}$$

Consequently,

$$J_s(u) = \int_\Omega \frac{\partial}{\partial D_{kl}} \left[\left(\mu + |D| \right)^{p-2} D_{ij} \right] (\partial_s \mathcal{D}_{kl})(\partial_s \mathcal{D}_{ij}) \, dx, \tag{3.8}$$

where derivatives with respect to D_{kl} are evaluated at the point $D = \mathcal{D}$. Hence, by (3.5), the following result follows.

Lemma 3.2. *Let be $s = 1, 2$. Then*

$$J_s(u) \geqslant (p-1) I_s(u), \tag{3.9}$$

where

$$I_s(u) = \int_\Omega \left(\mu + |\mathcal{D}| \right)^{p-2} |\partial_s \mathcal{D}u|^2 \, dx. \tag{3.10}$$

Next multiply both sides of the first equation (1.6) by $\partial_{ss}^2 u$ and integrate over Ω. By appealing to (3.6) and (3.9), it readily follows that

$$I_s(u) \leqslant \frac{1}{p-1} \|f\|_{p'} \left\| \partial_{ss}^2 u \right\|_p \leqslant \frac{1}{p-1} \|f\|_{p'} \left\| \partial_s (\nabla u) \right\|_p, \tag{3.11}$$

for $s = 1, 2$. Hence, from (3.1), one gets

$$I_s(u) \leqslant \frac{1}{p-1} \|f\|_{p'} \|\partial_s \mathcal{D}u\|_p, \tag{3.12}$$

for $s = 1, 2$.

Finally we appeal to the conditional assumption (1.11). In the following lemma λ is not necessarily given by $\lambda(q)$.

Lemma 3.3. *Let be $1 < p < 2$, $p \leqslant q$, and $1 < \lambda \leqslant \lambda(q)$. Assume that $\mathcal{D}u \in L^q$. Then $D_*^2 u \in L^\lambda$, moreover*

$$\left\| D_*^2 u \right\|_\lambda \leqslant c I_s(u)^{\frac{1}{2}} \left\| \mu + |\mathcal{D}u| \right\|_{\frac{(2-p)\lambda}{2-\lambda}}^{\frac{2-p}{2}}. \tag{3.13}$$

Proof. Note that $\lambda \leqslant \lambda(q)$ is equivalent to $\frac{(2-p)\lambda}{2-\lambda} \leqslant q$.

One has

$$\int |\partial_s \mathcal{D}u|^\lambda \, dx = \int \left(\mu + |\mathcal{D}u| \right)^{\frac{(p-2)\lambda}{2}} |\partial_s \mathcal{D}u|^\lambda \left(\mu + |\mathcal{D}u| \right)^{\frac{(2-p)\lambda}{2}} \, dx.$$

By Hölder's inequality with exponents $\frac{2}{\lambda}$ and $\frac{2}{2-\lambda}$ it readily follows (3.13). We also appeal to (3.1). $\square$

Corollary 3.1. *One has*

$$I_s(u) \leqslant c\|f\|_{p'}^2 \|\mu + |\mathcal{D}u|\|_p^{2-p} \leqslant C\|f\|_{p'}^2 \tag{3.14}$$

and

$$\|D_*^2 u\|_p \leqslant c\|f\|_{p'} \|\mu + |\mathcal{D}u|\|_p^{2-p} \leqslant C\|f\|_{p'}. \tag{3.15}$$

Proof. From (3.13), with $\lambda = p$, it follows that

$$\|D_*^2 u\|_p \leqslant c I_s(u)^{\frac{1}{2}} \|\mu + |\mathcal{D}u|\|_p^{\frac{2-p}{2}},$$

which, together with (3.12) leads to the desired estimates. □

Remark 3.1. In Ref. [3], by appealing to the inequality $a^p \leqslant b^{p-2}a^2 + b^p$, see [8], we prove (3.14) with $\|f\|_{p'}^2$ replaced by $\|f\|_{p'}^{p'}$. Here we do not appeal to the above kind of inequalities (for details, and for a comparison between the two distinct proofs, see Lemma 3.2 in Ref. [3]). The present improvement will display its weight in Section 6, where we apply our results to the full Navier–Stokes equations, and f is replaced by $F = f - (u \cdot \nabla)u$.

The following corollary follows immediately by setting in (3.13) $\lambda = \lambda(q)$ and $\lambda = r(q)$, respectively.

Corollary 3.2. *Let be $1 < p < 2$ and $p \leqslant q$, and assume that $\mathcal{D}u \in L^q$. Then $D_*^2 u \in L^{\lambda(q)}$, moreover*

$$\|D_*^2 u\|_{\lambda(q)} \leqslant C\|f\|_{p'} \|\mu + |\mathcal{D}u|\|_q^{\frac{2-p}{2}}. \tag{3.16}$$

In particular, (1.12) *holds. Further,*

$$\|D_*^2 u\|_{r(q)} \leqslant C\|f\|_{p'} \|\mu + |\mathcal{D}u|\|_{\frac{q}{2}}^{\frac{2-p}{2}}. \tag{3.17}$$

Proof of Eq. (1.9). By differentiation of the first equation (1.6) with respect to x_s, one gets

$$\nabla \partial_s \pi = -\nabla \cdot \partial_s\big(\big(\mu + |\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big) + \partial_s f. \tag{3.18}$$

On the other hand one easily shows that

$$\partial_s\big(\big(\mu + |\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big) = \big(\mu + |\mathcal{D}u|\big)^{p-2}\partial_s\mathcal{D}u + (p-2)\big(\mu + |\mathcal{D}u|\big)^{p-3}|\mathcal{D}u|^{-1}(\mathcal{D}u \cdot \partial_s\mathcal{D}u)\mathcal{D}u. \tag{3.19}$$

Hence,

$$\big|\partial_s\big(\big(\mu + |\mathcal{D}u|\big)^{p-2}\mathcal{D}u\big)\big| \leqslant (3-p)\mu^{\frac{p-2}{2}}\big(\mu + |\mathcal{D}u|\big)^{\frac{p-2}{2}}|\partial_s\mathcal{D}u|, \tag{3.20}$$

almost everywhere in Ω. Hence, by (3.18), and by appealing to (2.7), we prove that

$$\|\partial_s\pi\|_2 \leqslant c\mu^{\frac{p-2}{2}} I_s(u)^{\frac{1}{2}} + c\|f\|_2 \tag{3.21}$$

for $s \neq 3$. Hence, by appealing to (3.14), one gets

$$\|\partial_s\pi\|_2 \leqslant C\|f\|_{p'}, \tag{3.22}$$

which is just (1.9). □

4. Proof of Theorem 1.3

For the proof of the following result we refer the reader to [3, Section 4]. See also [1].

Lemma 4.1. *Assume, for convenience, that $p > \frac{3}{2}$. Then the following pointwise estimate*

$$\left(p - \frac{3}{2}\right) \sum_{l=1}^{2} \left|\partial_{33}^2 u_l\right| \leqslant c\left|D_*^2 u(x)\right| + c\big(\mu + |\mathcal{D}u|\big)^{2-p}\big(\left|\nabla^* \pi\right| + |f|\big) \tag{4.1}$$

holds, almost everywhere in Ω.

Since $(\mu + |\mathcal{D}u|)^{2-p} \in L^{\frac{q}{2-p}}$ and $\nabla^* \pi \in L^2$, Hölder's inequality shows that the right-hand side of (4.1) is integrable with power r. More precisely, by appealing to (3.17) and (1.9), it readily follows that ($l = 1, 2$)

$$\left\|\partial_{33}^2 u_l\right\|_{r(q)} \leqslant c\left\|D_*^2 u\right\|_{r(q)} + c\left\|\mu + |\mathcal{D}u|\right\|_q^{2-p}\big(\left\|\nabla^* \pi\right\|_2 + \|f\|_2\big). \tag{4.2}$$

From the above estimate together with (3.17) and (3.22), straightforward manipulations show that

$$\left\|\partial_{33}^2 u_*\right\|_{r(q)} \leqslant C\big(1 + \|\mathcal{D}u\|_q^{2-p}\big)\|f\|_{p'} \tag{4.3}$$

where, following our notation, u_* denotes the two first components of u. Eq. (1.14) follows.

Next we state the following particular case of more general results proved by Troisi in Ref. [20].

Proposition 4.1. *Let Ω be as above, and let $v \in W^{1,1}(\Omega)$. Assume that*

$$\partial_k v \in L^{p_k}, \quad \textit{for } k = 1, 2, 3, \tag{4.4}$$

where

$$\frac{1}{\bar{p}} := \frac{1}{3} \sum_{k=1}^{3} \frac{1}{p_k} - \frac{1}{3}. \tag{4.5}$$

Then $v \in L^{\bar{p}}(\Omega)$ and

$$\|v\|_{\bar{p}} \leqslant c \prod_{k=1}^{3} \|\partial_k v\|_{p_k}^{\frac{1}{3}} + c\|v\|_p. \tag{4.6}$$

Obviously, we may replace $\|v\|_p$ by any other L^s norm, $s \geqslant 1$.

An essential point in order to get the limit exponent l in the calculation that follows, is that the constant c on the right-hand side of (4.6) does not depend on the values of the exponents p_k used in the sequel. This property holds provided that $\bar{p}$ lies bounded away from 3. This follows essentially from Eq. (1.15) in the above reference (note, however, that all the values p_k that will be used here lie bounded away from 3).

Further, note that the exponent $\mathcal{Q}(q)$, defined by (1.10), satisfies

$$\frac{1}{\mathcal{Q}(q)} = \frac{1}{3}\left(\frac{2}{\lambda(q)} + \frac{1}{r(q)} - 1\right).$$

Next, we apply Troisi's Theorem to (the components of) ∇u. By appealing to (1.12) and to (4.3) we prove (1.15).

5. The boot-strap argument. Proof of Theorem 1.4

The proof follows that in Ref. [1]. See also [2]. The same basic ideas were also applied in Refs. [3] and [7].

Since $\mathcal{D}u \in L^p(\Omega)$, it follows from Theorem 1.3 that $\mathcal{D}u \in L^{\mathcal{Q}(p)}(\Omega)$, where $\mathcal{Q}(p) = \frac{6p}{8-3p}$. Since this last exponent is greater than p, we may start an induction argument. Recall that our constants C are independent on the integrability exponents used by us.

Define the strictly increasing sequence

$$\begin{cases} q_1 = p, \\ q_{n+1} = \mathcal{Q}(q_n). \end{cases} \tag{5.1}$$

Further, note that the exponent $\bar{q}$ given by (1.3) is the limit $\bar{q} = \lim_{n\to\infty} q_n$. In particular, $\bar{q}$ is the fixed point of the map $q \to \mathcal{Q}(q)$,.

On the other hand, from (1.15) it follows that

$$\|u\|_{1,q_{n+1}} \leqslant C\|f\|_{p'}\bigl(1 + \|u\|_{1,q_n}^{\frac{2(2-p)}{3}}\bigr). \tag{5.2}$$

With an obvious notation, we write this equation in the form

$$a_{n+1} \leqslant b\bigl(1 + a_n^{\alpha}\bigr).$$

Note that $0 < \alpha < 1$. By arguing as in [2] we prove that

$$\|u\|_{1,q_n} := a_n < 2\bigl(b + b^{\frac{1}{1-\alpha}}\bigr),$$

at least for sufficiently large values of n. Consequently, $\|u\|_{1,\bar{q}}$ is bounded by the right-hand side of the above equation. This shows that

$$\|u\|_{1,\bar{q}} \leqslant C\bigl(\|f\|_{p'} + \|f\|_{p'}^{\frac{3}{2p-1}}\bigr).$$

The estimate (1.17) follows. Further, the estimate (1.18) follows easily by applying once more the estimate (1.14), now with $q = \bar{q}$, and by taking into account (1.17). Note that $r(\bar{q}) = l$.

Finally, the L^l regularity of $\frac{\partial\pi}{\partial x_3}$, hence the global regularity of $\nabla\pi$ (recall (1.9)), is easily obtained from (1.6). In fact, this equation, written for $j = 3$, furnishes an explicit expression for $\frac{\partial\pi}{\partial x_3}$ in terms of functions already estimated. Actually,

$$|\partial_3\pi| \leqslant c\bigl(\mu + |\mathcal{D}u(x)|\bigr)^{p-2}\bigl|D^2u(x)\bigr| + \bigl|f(x)\bigr|, \tag{5.3}$$

almost everywhere in Ω. Explicit estimates for $\|\partial_3\pi\|_l$ are left to the reader.

6. The Navier–Stokes equations

In this section we prove Theorem 1.1.

Since

$$\int_\Omega (u\cdot\nabla)u\cdot u\,dx = 0,$$

it readily follows that all the estimates stated in Section 2 for weak solutions hold for solutions u to the complete Navier–Stokes equations

$$\begin{cases} -\nabla\cdot\bigl((\mu + |\mathcal{D}u|)^{p-2}\mathcal{D}u\bigr) + \nabla\pi = F, \\ \nabla\cdot u = 0, \end{cases} \tag{6.1}$$

where

$$F = f - (u\cdot\nabla)u.$$

From (3.15) one gets

$$\bigl\|D_*^2 u\bigr\|_p \leqslant C\bigl(\|f\|_{p'} + \bigl\|(u\cdot\nabla)u\bigr\|_{p'}\bigr). \tag{6.2}$$

Next, by Hölder's inequality,

$$\bigl\|(u\cdot\nabla)u\bigr\|_{p'}^{p'} = \int |u|^{p'}|\nabla u|^{(1-\gamma)p'}|\nabla u|^{\gamma p'}\,dx \leqslant \left(\int |u|^{p'\alpha'}|\nabla u|^{(1-\gamma)p'\alpha'}\,dx\right)^{\frac{1}{\alpha'}}\left(\int |\nabla u|^{\gamma p'\alpha}\,dx\right)^{\frac{1}{\alpha}}, \tag{6.3}$$

where $0 \leqslant \gamma \leqslant 1$ and $1 < \alpha < +\infty$. We impose to γ the restriction

$$\gamma < \frac{2p-1}{3},$$

which is written in the form

$$\gamma = \frac{2p-1-\epsilon}{3}, \tag{6.4}$$

for some (small) $\epsilon > 0$. Further, we define α by the equation

$$\gamma p' \alpha = \bar{q}.$$

It follows

$$\big\|(u \cdot \nabla)u\big\|_{p'} \leqslant \big(K(u)\big)^{\frac{1}{p'\alpha'}} \|\nabla u\|_{\bar{q}}^{\gamma}, \tag{6.5}$$

where (see also [3, Eq. (5.7)])

$$K(u) = \int |u|^{p'\alpha'} |\nabla u|^{(1-\gamma)p'\alpha'}\, dx. \tag{6.6}$$

So

$$K(u) \leqslant \left(\int |u|^{p'\alpha' r}\, dx\right)^{\frac{1}{r}} \left(\int |\nabla u|^{(1-\gamma)p'\alpha' r'}\, dx\right)^{\frac{1}{r'}},$$

where $r > 1$. If there is an r such that

$$p'\alpha' r \leqslant p^* \quad \text{and} \quad (1-\gamma)p'\alpha' r' \leqslant p, \tag{6.7}$$

then $K(u) \leqslant C$, since $\|\nabla u\|_p$ is bounded (independently of $(u \cdot \nabla)u$). Note that

$$\|u\|_{p^*} \leqslant c\|\nabla u\|_p,$$

where $p^* = \frac{3p}{3-p}$.

Conditions (6.7) can be written in the form

$$\frac{p}{p-(1-\gamma)p'\alpha'} \leqslant r \leqslant \frac{p^*}{p'\alpha'}$$

or, equivalently, as

$$\frac{1}{(p-1)-(1-\gamma)\alpha'} \leqslant r \leqslant \frac{p^*}{p\alpha'}. \tag{6.8}$$

Clearly, a suitable value r exists if and only if the left-hand side of the above equation is less or equal to the right-hand side. Straightforward calculations lead to $\alpha \geqslant \frac{3(p-1)}{6p-10-\epsilon}$, hence to

$$\frac{3(4p-2)(p-1)}{(2p-1)p - p\epsilon} \geqslant \frac{3(p-1)}{(6p-10)-\epsilon}.$$

This holds for some $\epsilon > 0$ if (and only if)

$$\frac{3(4p-2)(p-1)}{(2p-1)p} > \frac{3(p-1)}{(6p-10)},$$

which is equivalent to the assumption $p > p_0$. Finally, from the boundedness of $K(u)$ together with (1.17) and (6.5), one gets

$$\|\nabla u\|_{\bar{q}} \leqslant C\big(1 + \|f\|_{p'}^{\frac{3}{2p-1}} + \|\nabla u\|_{\bar{q}}^{\frac{3\gamma}{2p-1}}\big).$$

Since $\frac{3\gamma}{2p-1} < 1$, $\|\nabla u\|_{\bar{q}}$ is bounded.

Further, (6.5) shows that $\|(u \cdot \nabla)u\|_{p'}$ is bounded. Hence $\|F\|_{p'}$ is bounded. Finally, by appealing to Theorem 1.3 with $q = \bar{q}$ and f replaced by F, we prove that $u \in W^{2,l}(\Omega)$.

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