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VORTICITY AND REGULARITY FOR FLOWS UNDER THE NAVIER BOUNDARY CONDITION

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Abstract. In reference [13], by Constantin and Fefferman, a quite simple geometrical assumption on the direction of the vorticity is shown to be sufficient to guarantee the regularity of the weak solutions to the evolution Navier–Stokes equations in the whole of $\mathbf{R}^3$. Essentially, the solution is regular if the direction of the vorticity is Lipschitz continuous with respect to the space variables. In reference [8], among other side results, the authors prove that 1/2-Hőlder continuity is sufficient.

A main open problem remains of the possibility of extending the same kind of results to boundary value problems. Here, we succeed in making this extension to the well known Navier (or slip) boundary condition in the half-space $\mathbf{R}^3$. It is worth noting that the extension to the non-slip boundary condition remains open. See [7].

1. Introduction and known results.

Consider the evolution 3-D Navier–Stokes equations in the whole of $\mathbf{R}^3$, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)\,u - \nu\Delta u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times [0,T], \\ \nabla\cdot\, u = 0 \quad \text{in } \mathbf{R}^3\times[0,T], \\ u(x,0) = u_0(x) \quad \text{in } \mathbf{R}^3\,, \end{cases} \tag{1}$$

where u denotes the velocity field and p the pressure.

It is well known (essentially due to Leray [16]) that given any fixed $T > 0$ there exists at least a *weak solution*

$$u \in C_w(0,T;L^2) \cap L^2(0,T;H^1),$$

of the system (1) in (0,T), where C_w indicates weak continuity. Moreover, the energy estimate

$$\frac{1}{2}|u(t)|_2^2 + \nu\int_0^t \int_{\mathbf{R}^3} |\nabla u(x,\sigma)|^2\,dx\,d\sigma \le \frac{1}{2}|u_0|_2^2 \tag{2}$$

holds for each $t \in (0,T)$.

A weak solution such that

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2) \tag{3}$$

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is called a *strong solution* in $[0, T]$. In the following, we say that u is a strong solution in $[0, T)$ if u is a strong solution in $[0, t]$, for each $t < T$. Strong solutions are regular, unique, and exist at least for some $T^* > 0$.

It is not known whether weak solutions are unique and strong solutions are global in time. Hence many efforts have been made to obtain significant conditions that are sufficient to guarantee the regularity of weak solutions.

In the following we are interested in sufficient conditions on the vorticity ω,

$$\omega(x,t) = \nabla \times u(x,t)\,,$$

that guarantee the regularity of the solution. In the field of analytic (not geometric) conditions on ω, the literature is still quite wide. A typical result is the following (see [3]). If

$$\omega \in L^p(0,T;L^q) \quad \text{for } \frac{2}{p} + \frac{n}{q} \leq 2, \quad 1 \leq p \leq 2, \tag{4}$$

then the weak solution is regular (see also [9]). This result is the extension to the values $p \leq 2$ of the classical sufficient condition

$$u \in L^p(0,T;L^s) \quad \text{for } \frac{2}{p} + \frac{n}{s} \leq 1, \quad 2 \leq p < \infty\,. \tag{5}$$

However, this type of assumptions have an analytical character. On the contrary, references [13] and [8] furnish significant geometrical conditions. Let us illustrate this approach.

Define the direction of the vorticity ξ as

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}\,.$$

In general we will use the notation

$$\widehat{z} = \frac{z}{|z|} \tag{6}$$

if $|z| \neq 0$. Hence $\xi = \widehat{\omega}$. Denote by $\theta(x,y,t)$ the angle between the vorticity ω at two distinct points x and y at time t. In reference [13] the authors open the way to the study of global regularity of solutions of the Navier-Stokes equations via a simple geometrical assumptions on the direction of the vorticity, a very significant physical entity. The authors prove the following result.

Theorem 1. *(see* [13]*). Let u be a weak solution of* (1) *in* $(0,T)$ *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$*. If*

$$\sin\theta(x,y,t) \leq c\,|x-y| \tag{7}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K, then the solution u is strong in $[0,T]$ *and, consequently, is regular.*

Actually, the literal statement in [13] is a little different (see in particular the comment after equation (32) in the above reference). Main ingredients in the proof of the above result are Biot-Savart Law and a significant formula introduced in reference [12]. See equation (7) in [13].

In [8], Berselli and the author improve the above result by showing that

$$\sin\theta(x,y,t) \leq c|x-y|^{1/2} \tag{8}$$

is sufficient to guarantee the regularity of weak solutions. More precisely, in [8] we prove the following result:

Theorem 2. *(see [8]). Let u be a weak solution of (1) in $(0,T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Assume that for some $\beta \in [1/2, 1]$ and $g \in L^a(0,T; L^b)$, where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2}, \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right], \tag{9}$$

one has

$$\sin\theta(x,y,t) \le g(t,x)|x-y|^\beta \tag{10}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Then the solution u is strong in $[0,T]$ and, consequently, is regular. In particular (8) alone is a sufficient condition for regularity.

In [4] and [5] we consider some cases in which $\beta \in [0, 1/2]$ and give a sufficient condition for the regularity of weak solutions that involves, simultaneously, the magnitude and the direction of the vorticity. More precisely, we prove the following assertion.

Theorem 3. *(see [4] and [5]). Let u be a weak solution of (1) in $(0,T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Let $\beta \in [0, 1/2]$ and assume that (16) holds in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Assume, moreover, (17),(18). Then the solution u is strong in $[0,T]$ and, consequently, is regular. In particular (8) alone is a sufficient condition for regularity.*

It is self evident that in the above theorems the hypotheses (7), (10) and (16) may be relaxed by assuming that they are satisfied merely for $|x-y| < \delta$, with an arbitrary positive constant δ.

We end this section by some remarks.

Remark 1. In the assumptions made in references [13], [8] and [4] the quantity $\sin\theta(x,y,t)$ can be everywhere replaced by

$$|(\widehat{x-y}, \xi(x))\, Det(\widehat{x-y}, \xi(y), \xi(x))|\,. \tag{11}$$

The above claim is obvious, since the quantity that comes out in the proofs is just (11). Since

$$|(\widehat{x-y}, \xi(x))\, Det(\widehat{x-y}, \xi(y), \xi(x))| \le \sin\theta(x,y,t)\,, \tag{12}$$

we opt for replacing the above quantity simply by $\sin\theta(x,y,t)$. Clearly $\sin\theta(x,y,t)$ can be replaced as well by any upper bound of (11) as, for instance,

$$|\cos\psi(x,y,t)|\, \sin\phi(x,y,t)$$

where $\psi(x,t)$ denotes the angle between $\xi(x,t)$ and $x-y$, and $\phi(x,y,t)$ denotes the angle between $\xi(y,t)$ and the plane generated by $\xi(x,t)$ and $x-y$.

Remark 2. By simple manipulations of the known proofs one may write many sufficient conditions for regularity as, for instance, conditions obtained by mixing in a single statement the assumptions (10) and (17). Or by letting the parameter β take values in the range $0 < \beta < 3$ (by exploiting here the fact that (62) holds for each β in this range). We warn that $\beta > 1$ implies that $\sin\theta(x,y,t) = 0$. Hence one falls into the well known case in which ω is parallel to a fixed direction.

Concerning extensions to other systems of equations in $\mathbf{R}^3$ we note that in the proofs of the above theorems the real obstacle consists in proving suitable estimates for the nonlinear term $(\omega \cdot \nabla)\, u$, see (26). If in equation (1) we replace the operator Δ by a linear operator L which commutes with the curl operator, we get equation

(26) with the corresponding substitution. Since the estimates for the nonlinear term are still in hands, the extension of the known results to the "$L-$case" became a classical game between parameters and integrability exponents.

In our opinion the *central open problems* are the determination of the best exponent β for which the assumption (16) guarantees the regularity of the solutions without any other additional hypotheses, and the extension of the basic theory to boundary value problems. Concerning the first problem it is worth noting that the proofs given in references [4] formally lead us to believe that the sharpness of the regularity exponent $\beta = \frac{1}{2}$ corresponds to that of the classical condition (5). Hence, the improvement of the exponent $\frac{1}{2}$ seems a very hard goal.

Below we give a first contribution to the second of the above problems by extending to the Navier (or slip) boundary condition (25) the results proved in [4]. We will consider the half-space case $\Omega = \mathbf{R}^3_+$. A fundamental tool is the use of both the Green and the Neumann functions for $\mathbf{R}^3_+$. The extension to regular bounded domains Ω seems possible but is still an open problem.

Similar ideas have been applied in reference [7] for the non-slip boundary condition

$$u = 0 \quad on \quad \Gamma\,, \tag{13}$$

by appealing to the Green function for Ω. In this last situation all the suitable estimates concerning the non linear term $(\omega \cdot \nabla)\, u \cdot \omega$ are proved, however a new obstacle (due to the specific boundary condition (13)) appears and regularity under the *sole* assumption $\sin\theta(x,y,t) \le c|x-y|^{1/2}$ (or even $\sin\theta(x,y,t) \le c|x-y|$) remains an open problem. See the remarks after (29).

2. **New results.** The main result in this paper is the following. Definitions concerning the slip boundary condition and the functional space V are given afterwards.

Theorem 4. *Let $u_0 \in V$ and let u be a weak solution of the Navier-Stokes equations in $[0,\,T) \times \mathbf{R}^3_+$, namely,*

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)\,u - \nu\Delta u + \nabla p = 0 \quad in\ \mathbf{R}^3_+ \times [0,T), \\ \nabla\cdot u = 0 \quad in\ \mathbf{R}^3_+ \times [0,T), \\ u(x,0) = u_0(x) \quad in\ \mathbf{R}^3_+, \end{cases} \tag{14}$$

endowed with the slip boundary condition

$$\begin{cases} u_3 = 0, \\ \nu \dfrac{\partial u_j}{\partial x_3} = 0, \quad 1 \le j \le 2. \end{cases} \tag{15}$$

Let $\beta \in [0, 1/2]$ and assume that, for almost all $t \in]0,\,T[$,

$$\sin\theta(x,y,t) \le c|x-y|^\beta \tag{16}$$

for almost all (x,y). Moreover, suppose that

$$\omega \in L^2(0,T;L^r), \tag{17}$$

where

$$r = \frac{3}{\beta+1}. \tag{18}$$

Then the solution u is strong in $[0,T]$ and, consequently, is regular. If $\beta = 1/2$ the assumption (17) is superfluous.

The last claim follows from the fact that weak solutions satisfy (17) for $r = 2$.

A main point in the sequel is that the boundary conditions (15) yield

$$\begin{cases} \omega_1 = \omega_2 = 0, \\ \nu \dfrac{\partial \omega_3}{\partial x_3} = 0 \end{cases} \tag{19}$$

on Γ.

Remark 3. In Theorem 4 it is sufficient that the assumption (16) holds in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. This extension is proved by introducing a suitable decomposition of ω, see [13], according to the regions where $|\omega(x)|$ is larger or smaller than K. Concerning this point we refer the reader to the Remark 3.1 in reference [7].

Remark 4. The results proved here holds as well under the boundary conditions (see [2])

$$\begin{cases} u \cdot n = 0, \\ \omega \times n = 0\,, \end{cases} \tag{20}$$

since for $\Omega = \mathbf{R}^3_+$ these boundary conditions are still given by (15).

Next we recall some definitions concerning the *slip boundary condition*. It is superfluous to give here the well known variational formulation of the problem considered in Theorem 4. We merely remark that the standard functional framework in studying the boundary condition (15) is

$$V = \left\{ v \in [H^1(\mathbf{R}^3_+)]^2 \times H^1_0(\mathbf{R}^3_+) : \nabla \cdot v = 0 \right\} .$$

See [6].

Even though we consider here the Navier-Stokes equations in the half-space $\mathbf{R}^3_+ = \left\{ x \in \mathbf{R}^3 : x_3 > 0 \right\}$ it is suitable to describe the slip boundary condition (25) in the general case of an open set Ω in $\mathbf{R}^3$. Γ denotes the boundary of Ω and $\underline{n}$ the unit external normal to Γ. We denote by

$$T = -p\, I + \nu(\nabla u + \nabla u^T)$$

the stress tensor, and set $\underline{t} = T \cdot \underline{n}$. Hence, with an obvious notation

$$T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \tag{21}$$

$$t_i = \sum_{k=1}^{n} T_{ik} n_k. \tag{22}$$

We also define the linear operator $\underline{\tau}$,

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{23}$$

Hence

$$\tau_i(u) = \nu \sum_{k=1}^{n} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k - 2\nu \left[\sum_{k,l=1}^{n} \frac{\partial u_l}{\partial x_k} n_k n_l \right] n_i. \tag{24}$$

Note that $\underline{\tau}(u)$ is tangential to the boundary and independent of the pressure p.

The slip boundary condition reads

$$\begin{cases} (u \cdot \underline{n})_{|\Gamma} = 0, \\ \underline{\tau}(u)_{|\Gamma} = 0\,. \end{cases} \tag{25}$$

We consider here homogeneous boundary conditions. When $\Omega = \mathbf{R}^3_+$, the equations (25) have the form (15). See [6], Equation (2.2).

The slip boundary condition (25) was proposed by Navier, see [18]. We point out that this condition, and similar ones, are an appropriate model for many important flow problems. Besides the pioneering mathematical contribution [21] by Solonnikov and Ščadilov, this boundary condition has been considered by many authors. See, for instance, [1], [6], [11], [14], [15], [17], [19], [20], [23] and references therein.

3. **Proof of Theorem 4.** We denote by $|\cdot|_p$ the canonical norm in the Lebesgue space $L^p := L^p(\mathbf{R}^3)$, $1 \leq p \leq \infty$. $H^s := H^s(\mathbf{R}^3)$, $0 \leq s$, denotes the classical Sobolev spaces. Scalar and vector function spaces are indicated by the same symbol.

From now on we set

$$\Omega = \mathbf{R}^3_+ \quad \text{and} \quad \Gamma = \left\{ x \in \mathbf{R}^3 : \, x_3 \, = 0 \right\} .$$

For convenience, we mostly will use the Ω, Γ notation.

Since $u_0 \in H^1$, the solution is strong, hence regular, in $[0,\, \tau)$, for some $\tau > 0$. Let $\tau \leq T$ be the maximum of these values. We will show that, under this hypothesis, u is strong in $[0,\, \tau]$. Hence, by a continuation principle, u is strong in $[\tau,\, \tau + \varepsilon)$. This shows that $\tau = T$. Without loss of generality we assume that the solution u is regular in $[0,\, T)$ and we prove that this implies regularity in $[0,\, T]$.

By taking the curl of both sides of the first equation (14) we find, for each $t < T$,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\, \omega - \nu \Delta \omega = \, (\omega \cdot \nabla)\, u \,, \tag{26}$$

in $\mathbf{R}^3_+$. Moreover, by taking the scalar product in L^2 of both sides of (26) with ω, we get

$$\frac{1}{2} \frac{d}{dt} |\omega|_2^2 + \nu |\nabla \omega|_2^2 = \int_\Omega (\omega \cdot \nabla)\, u \, \cdot \omega(x)\, dx. \tag{27}$$

Note that

$$-\nu \int_\Omega \Delta \omega \cdot \, \omega \, dx = \, \nu |\nabla \omega|_2^2 + \, \nu \int_\Gamma \frac{\partial\, \omega}{\partial\, x_3} \cdot \, \omega \, d\Gamma \tag{28}$$

since $\underline{n} = \, (0, 0, -1)$. Under the boundary condition (15) it readily follows from (19) that

$$\int_\Gamma \frac{\partial\, \omega}{\partial\, x_3} \cdot \, \omega \, d\Gamma = \, 0 \,.$$

However, under the non-slip boundary condition (13) one gets

$$\nu \int_\Gamma \frac{\partial\, \omega}{\partial\, x_3} \cdot \, \omega \, d\Gamma = \frac{\nu}{2} \frac{d}{d\, x_3} \int_{\Gamma(x_3)} (\omega_1^2 + \, \omega_2^2)\, d\Gamma \,. \tag{29}$$

If we are able to control this quantity in a suitable way, then the Theorem 4 applies to the non-slip boundary condition as well, as easily shown by a simple adaptation of the proofs given here.

Set, for each tern (j, k, l), $j, k, l \in \, \{1, 2, 3\}$,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation}\,, \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation}\,, \\ 0 & \text{if two indexes are equal}\,. \end{cases} \tag{30}$$

One has

$$(a \times b)_j = \, \epsilon_{jkl} \, a_k \, b_l \,, \tag{31}$$

and

$$(\nabla \times v)_j = \epsilon_{jkl} \frac{\partial v_l}{\partial x_k}, \tag{32}$$

where here, and in the sequel, the usual convention about summation of repeated indexes is assumed.

Since

$$-\Delta u = \nabla \times (\nabla \times u) - \nabla (\nabla \cdot u), \tag{33}$$

it follows that

$$\begin{cases} -\Delta u = \nabla \times \omega \quad \text{in} \quad \Omega\,; \\ \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0 \quad \text{in} \quad \Gamma\,, \\ u_3 = 0 \quad \text{in} \quad \Gamma\,, \end{cases} \tag{34}$$

for each t.

In the sequel

$$G(x,y) = \frac{1}{4\pi} \left(\frac{1}{|x-y|} - \frac{1}{|x-\overline{y}|} \right) \tag{35}$$

denotes the Green's function for the Dirichlet boundary value problem in the half space, where

$$\overline{y} = (y_1, y_2, -y_3)\,,$$

and

$$N(x,y) = \frac{1}{4\pi} \left(\frac{1}{|x-y|} + \frac{1}{|x-\overline{y}|} \right) \tag{36}$$

denotes the classical Neumann's function for the half space $\mathbf{R}^3_+$.

For convenience we set

$$\epsilon_1 = \epsilon_2 = 1\,, \epsilon_3 = -1\,.$$

Note that $\overline{y}_k = \epsilon_k\, y_k$. Analogously, $\overline{\omega}_k = \epsilon_k\, \omega_k$, and so on.

By appealing to (32) and (34) it follows that

$$\begin{cases} -\Delta u_j = \epsilon_{jkl} \dfrac{\partial \omega_l}{\partial x_k} \quad \text{in} \quad \Omega\,, \\ \dfrac{\partial u_j}{\partial x_3} = 0 \quad \text{in} \quad \Gamma\,, \end{cases} \tag{37}$$

for $j = 1$ and $j = 2$. From (37) one gets

$$u_j(x) = \int_\Omega N(x,y)\, \epsilon_{jkl} \frac{\partial \omega_l(y)}{\partial y_k}\, dy\,. \tag{38}$$

Hence, by an integration by parts,

$$u_j(x) = -\int_\Omega \epsilon_{jkl} \frac{\partial N(x,y)}{\partial y_k} \omega_l(y)\, dy + \int_\Gamma N(x,y)\, \epsilon_{jkl}\, \omega_l(y)\, n_k\, dy\,, \tag{39}$$

for $j = 1, 2$.

For $j = 3$ it follows from (34) that

$$u_3(x) = \int_\Omega G(x,y)\, (\nabla \times \omega(y))_3\, dy\,. \tag{40}$$

By appealing to (32) and by taking into account that $G(x,y) = 0$ if $y \in \Gamma$, an integration by parts yields

$$u_3(x) = -\int_\Omega \epsilon_{3kl} \frac{\partial G(x,y)}{\partial y_k} \omega_l(y)\, dy\,. \tag{41}$$

Remark. Note that for the boundary value problem (13) the equation (41) holds for $j = 1, 2, 3$. This easily would lead, just by simplifying the proofs presented in the sequel, to the extension of Theorem 4 to solutions of the boundary value problem (13) *provided that* one is able to control the boundary integral (29).

Next we replace in equations (39) and (40) the terms $\frac{\partial N(x,y)}{\partial y_k}$ and $\frac{\partial G(x,y)}{\partial y_k}$ by its explicit expressions obtained from (36) and (35). One easily shows that

$$u_j(x) = a_j(x) - b_j(x) + \gamma_j(x) , \quad j = 1, 2, 3, \tag{42}$$

where

$$\begin{cases} a_j(x) = -\dfrac{1}{4\pi} \displaystyle\int_\Omega \epsilon_{jkl} \frac{x_k - y_k}{|x - y|^3} \omega_l(y)\, dy , \\ b_j(x) = \dfrac{1}{4\pi} \epsilon_j \displaystyle\int_\Omega \epsilon_{jkl}\, \epsilon_k \frac{x_k - \overline{y}_k}{|x - \overline{y}|^3} \omega_l(y)\, dy \end{cases} \tag{43}$$

and

$$\gamma_j(x) = \frac{1}{2\pi} \int_\Gamma \epsilon_{jkl}\, \omega_l(y)\, n_k \frac{dy}{|x - y|} . \tag{44}$$

Actually, for $j = 3$, there are no boundary integral in the expression (40) of u_3. However (42) is correct, since $\gamma_3(x) = 0$. In fact $\epsilon_{3kl}\, \omega_l(y)\, n_k = \epsilon_{33l}\, \omega_l(y) = 0$ on Γ.

Note that the decomposition (42) corresponds to separate the $(x - y)-$terms from the $(x - \overline{y})-$terms, and not G from N.

Clearly

$$\begin{aligned} ((\omega \cdot \nabla)\, u \cdot \omega)\,(x) &\equiv \frac{\partial u_j(x)}{\partial x_i} \omega_i(x)\omega_j(x) \\ = \frac{\partial a_j(x)}{\partial x_i} \omega_i(x)\omega_j(x) &- \frac{\partial b_j(x)}{\partial x_i} \omega_i(x)\omega_j(x) + \frac{\partial \gamma_j(x)}{\partial x_i} \omega_i(x)\omega_j(x) . \end{aligned} \tag{45}$$

We start by proving that the last term in the right hand side of (45) vanishes.

Lemma 1. *One has*

$$\frac{\partial \gamma_j(x)}{\partial x_i} \omega_i(x)\omega_j(x) = 0 , \quad \forall x \in \Omega . \tag{46}$$

Proof.

From equation (44) it follows that

$$\frac{\partial \gamma_j(x)}{\partial x_i} = \frac{1}{2\pi} P.V. \int_\Gamma \epsilon_{kjl}\, (x_i - y_i)\, \omega_l(y)\, n_k \frac{dy}{|x - y|^3} . \tag{47}$$

Hence

$$\frac{\partial \gamma_j(x)}{\partial x_i} \omega_i(x)\omega_j(x) = \frac{1}{2\pi} P.V.\int_\Gamma (\widehat{x-y}) \cdot \omega(x)\ Det\, (n(y), \omega(x), \omega(y))\ \frac{dy}{|x-y|^2} . \tag{48}$$

Since $n(y)$ and $\omega(y)$ are parallel for $y \in \Gamma$, (46) follows.

Next we prove the following result (recall definition (6)):

Lemma 2. *For each* $x \in \Omega$

$$\frac{\partial a_j(x)}{\partial x_i} \omega_i(x)\, \omega_j(x) = \frac{3}{4\pi} P.V.\int_\Omega \big((\widehat{x-y}) \cdot \omega(x)\big)\ Det\, \big((\widehat{x-y}), \omega(y), \omega(x)\big)\ \frac{dy}{|x-y|^3} \tag{49}$$

and

$$\frac{\partial\, b_j(x)}{\partial\, x_i}\,\omega_i(x)\,\omega_j(x) = -\frac{3}{4\,\pi}\int_\Omega \Big((\widehat{x-\overline{y}})\cdot\,\omega(x)\Big)\; Det\,\Big((\widehat{x-\overline{y}}), \omega(y), \overline{\omega}(x)\Big)\;\frac{dy}{|x-\overline{y}|^3}\,. \tag{50}$$

Proof.
By differentiation of $a_j(x)$ with respect to x_i we show that

$$\frac{\partial\, a_j(x)}{\partial\, x_i} = -\frac{1}{4\,\pi}\,P.V.\int_\Omega \epsilon_{jkl}\,\left[\frac{\delta_{ik}}{|x-\,y|^3} - 3\,\frac{(x_i-\,y_i)(x_k-\,y_k)}{|x-\,y|^5}\right]\omega_l(y)\,dy\,. \tag{51}$$

Straightforward calculations, left to the reader (use the combinatorial ϵ-operators), show that

$$\frac{\partial\, a_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) = -\frac{3}{4\,\pi}\,P.V.\int_\Omega (\widehat{x-y})\cdot\,\omega(x)\;\, Det\,\big((\widehat{x-y}), \omega(x), \omega(y)\big)\;\frac{dy}{|x-y|^3}\,. \tag{52}$$

This proves (49).

Next we consider the b term. By differentiation of $b_j(x)$ with respect to x_i one gets

$$\frac{\partial\, b_j(x)}{\partial\, x_i} = \frac{1}{4\,\pi}\int_\Omega \epsilon_{jkl}\,\epsilon_j\,\epsilon_k\,\left[\frac{\delta_{ik}}{|x-\,\overline{y}|^3} - 3\,\frac{(x_i-\,\overline{y}_i)(x_k-\,\overline{y}_k)}{|x-\,\overline{y}|^5}\right]\omega_l(y)\,dy\,. \tag{53}$$

Hence,

$$\begin{aligned}\frac{\partial\, b_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) &= \frac{1}{4\,\pi}\int_\Omega \epsilon_{jil}\,\epsilon_i\,\omega_i(x)\,\epsilon_j\,\omega_j(x)\,\omega_l(y)\,\frac{dy}{|x-\,\overline{y}|^3} \qquad (54)\\ &-\frac{3}{4\,\pi}\int_\Omega \epsilon_{jkl}\,\left[(x_i-\,\overline{y}_i)\,\omega_i(x)\right]\,\epsilon_k\,(x_k-\,\overline{y}_k)\,\epsilon_j\,\omega_j(x)\,\omega_l(y)\,\frac{dy}{|x-\,\overline{y}|^5}\,.\end{aligned}$$

In accordance to previous notation we set

$$\overline{\omega} = (\omega_1,\,\omega_2,\,-\omega_3)\,.$$

It follows that

$$\begin{aligned}\frac{\partial\, b_j(x)}{\partial\, x_i}\,\omega_i(x)\omega_j(x) &= -\frac{1}{4\,\pi}\int_\Omega Det\,(\overline{\omega}(x),\,\overline{\omega}(x),\,\omega(y))\,\frac{dy}{|x-\,\overline{y}|^3} \qquad (55)\\ &+\frac{3}{4\,\pi}\int_\Omega \Big((\widehat{x-\,\overline{y}})\cdot\,\omega(x)\Big)\; Det\,\Big((\widehat{x-\,\overline{y}}), \overline{\omega}(x), \omega(y)\Big)\;\frac{dy}{|x-\,\overline{y}|^3}\,.\end{aligned}$$

This proves (50).

From (45), (46), (49) and (50) we get the following statement.

Lemma 3. *Under the above hypothesis one has the following identity.*

$$\begin{aligned}&((\omega\cdot\nabla)\,u\cdot\,\omega)\,(x) \qquad (56)\\ &= -\frac{3}{4\,\pi}\,P.V.\int_\Omega \Big((\widehat{x-\,y})\cdot\,\omega(x)\Big)\; Det\,\Big((\widehat{x-\,y}), \omega(x), \omega(y)\Big)\;\frac{dy}{|x-\,y|^3}\\ &\quad -\frac{3}{4\,\pi}\int_\Omega \Big((\widehat{x-\,\overline{y}})\cdot\,\omega(x)\Big)\; Det\,\Big((\widehat{x-\,\overline{y}}),\, \overline{\omega}(x),\,\omega(y)\,\Big)\;\frac{dy}{|x-\,\overline{y}|^3}\\ &=: I_1(x) +\, I_2(x)\,. \qquad (57)\end{aligned}$$

In the following two lemmas we estimate the integrals over Ω of the above quantities I_1 and I_2. The proof of the first lemma is by now standard.

Lemma 4. *For each $t \in (0, T)$ the following estimate holds.*

$$\left| \int_\Omega I_1(x)\, dx \right| \leq \frac{\nu}{4} |\nabla \omega|_2^2 + \frac{c}{\nu} |\omega|_r^2 |\omega|_2^2 . \tag{58}$$

Proof. It readily follows from (16) that, for each $t \in [0, T[$,

$$|I_1(x)| \leq \frac{3}{4\pi} P.V. \int_\Omega |\sin\theta(x,y)|\, |\omega(x)|^2 |\omega(y)| \frac{dy}{|x-y|^3} . \tag{59}$$

From (59) and (16) one gets

$$\int_\Omega |I_1(x)|\, dx \leq \int_\Omega c\, |\omega(x)|^2 I(x)\, dx , \tag{60}$$

where $I(x)$ is the Riesz potential

$$I(x) = \int_\Omega |\omega(y)| \frac{dy}{|x-y|^{3-\beta}} . \tag{61}$$

By a well known Hardy-Littlewood-Sobolev inequality (see [22], Chapter V), if $\beta \in (0,3)$ and $\omega \in L^r(\Omega)$, for some $r \in (1, \frac{3}{\beta})$, then

$$|I|_q \leq c\, |\omega|_r , \tag{62}$$

where

$$\frac{1}{q} = \frac{1}{r} - \frac{\beta}{3} .$$

By this inequality with β and r given by (18) it follows that $|I|_3 \leq c\, |\omega|_r$. From equation (60) by appealing to Hőlder's inequality (with exponents $3, 2$ and 6) and by a Sobolev's embedding theorem one shows that (58) holds. To prove the next lemma one has to estimate the angle between $\overline{\omega}(x)$ and $\omega(y)$ instead of $\omega(x)$ and $\omega(y)$. We succeed in obtaining a suitable estimate by appealing to points $P\,x,\ P\,y \in \Gamma$, and to the fact that for $z \in \Gamma$, $\omega(z)$ is orthogonal to Γ.

Lemma 5. *For each $t \in (0, T)$ the following estimate holds.*

$$\left| \int_\Omega I_2(x)\, dx \right| \leq \frac{\nu}{4} |\nabla \omega|_2^2 + \frac{c}{\nu} |\omega|_r^2 |\omega|_2^2 . \tag{63}$$

Proof. From the boundary condition (15) it readily follows that

$$\omega(z) = (0, 0, \omega_3(z)) , \quad \overline{\omega}(z) = (0, 0, -\omega_3(z)), \quad \forall z \in \Gamma . \tag{64}$$

Since the solution u is assumed to be regular for $t \in (0, T)$, in this range the assumption (16) holds *up to the boundary.*

Define P as the orthogonal projection of $\overline{\Omega}$ onto Γ. From (64) one gets

$$\xi(P\,y) = +e_3 , \quad \text{or} \quad \xi(P\,y) = -e_3 , \quad \forall y \in \overline{\Omega} , \tag{65}$$

where e_3 is the unit vector in the positive x_3-direction. It readily follows from (65) and (16) that

$$\sin(\xi(y), \pm e_3) \leq c\, y_3^\beta , \quad \forall y \in \overline{\Omega} , \tag{66}$$

since $|y - Py| = y_3$. The presence of the symbol $\pm$ in an equation means that the equation holds with both signs. The symbol $\sin(a, b)$ denotes the sinus of the angle between the two vectors a and b. Since $\overline{\xi} = (\xi_1, \xi_2, -\xi_3)$, one also has

$$\sin(\overline{\xi}(x), \pm e_3) \leq c x_3^\beta , \quad \forall x \in \overline{\Omega} . \tag{67}$$

Next we consider the three unit vectors $\overline{\xi}(x)$, $\xi(y)$, and e_3. By identifying the angle $\angle(a,b)$ of two unit vectors a and b with the length of a geodesic on a spherical surface of radius equal to one, one shows that

$$\angle(a,b) \leq \angle(a,c) + \angle(c,b)\,.$$

Consequently, by appealing to (67) and to (66) we prove that

$$\sin(\overline{\xi}(x), \xi(y)) \leq 2\,c|x - \overline{y}|^{\beta}\,, \quad \forall x,\, y \in \overline{\Omega}\,. \tag{68}$$

On the other hand the expression of $I_2(x)$ shows, in particular, that

$$|I_2(x)| \leq \frac{3}{4\,\pi} \int_{\Omega} \sin(\overline{\xi}(x), \xi(y))\, |\omega(x)|^2\, |\omega(y)|\, \frac{dy}{|x - \overline{y}|^3}\,.$$

Due to (68) one has

$$|I_2(x)| \leq c \int_{\Omega} |\omega(x)|^2\, |\omega(y)|\, \frac{dy}{|x - \overline{y}|^{3-\beta}}\,. \tag{69}$$

By noting that $|x - \overline{y}| \geq |x - y|$, we may end the proof as in Lemma 4.

End of the proof of Theorem 4.

From (27), (56) and lemmas 4 and 5 it follows that

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \frac{\nu}{2}\,|\nabla\omega|_2^2 \leq \frac{c}{\nu}\,|\omega|_r^2\,|\omega|_2^2\,. \tag{70}$$

Since $|\omega|_r^2$ is integrable in $(0,\,T)$ a well known argument shows that

$$u \in L^{\infty}(0,T;H^1) \cap L^2(0,T;H^2)\,.$$

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