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Vorticity and Regularity for Viscous Incompressible Flows under the Dirichlet Boundary Condition. Results and Related Open Problems

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Abstract. In reference [7] it is proved that the solution of the evolution Navier–Stokes equations in the whole of $\mathbf{R}^3$ must be smooth if the direction of the vorticity is Lipschitz continuous with respect to the space variables. In reference [5] the authors improve the above result by showing that Lipschitz continuity may be replaced by 1/2-Hölder continuity. A central point in the proofs is to estimate the integral of the term $(\omega \cdot \nabla)u \cdot \omega$, where u is the velocity and $\omega = \nabla \times u$ is the vorticity. In reference [4] we extend the main estimates on the above integral term to solutions under the slip boundary condition in the half-space $\mathbf{R}^3_+$. This allows an immediate extension to this problem of the 1/2-Hölder sufficient condition.

The aim of these notes is to show that under the non-slip boundary condition the above integral term may be estimated as well in a similar, even simpler, way. Nevertheless, without further hypotheses, we are not able now to extend to the non slip (or adherence) boundary condition the 1/2-Hölder sufficient condition. This is not due to the "nonlinear" term $(\omega \cdot \nabla)u \cdot \omega$ but to a boundary integral which is due to the combination of viscosity and adherence to the boundary. On the other hand, by appealing to the properties of Green functions, we are able to consider here a regular, arbitrary open set Ω.

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1. Introduction

In reference [7] P. Constantin and Ch. Fefferman prove that the solution of the evolution Navier–Stokes equations in the whole of $\mathbf{R}^3$ must be smooth if the direction of the vorticity is sufficiently well behaved in regions of large magnitude of the vorticity. In particular they prove that the solution is regular if the direction of the vorticity is Lipschitz continuous with respect to the space variables. In reference [5] the authors improve the above result by showing that Lipschitz continuity may be replaced by 1/2-Hölder continuity, see (2.4).

A main open problem is the extension of the above type of results to boundary value problems. A fundamental steep in this direction is the extension to boundary value problems of some suitable estimates for the right-hand side of equation (3.2).

In reference [4] we prove these estimates for the slip boundary condition in the half-space $\mathbf{R}^3_+$ and prove, as a simple consequence, the above 1/2-Hölder sufficient condition for regularity. The method introduced in reference [4] to obtain suitable estimates for the right-hand side of equation (3.2) is not particularly tied to the slip boundary condition, and may be used to treat other boundary conditions as well. In fact, in the following we succeed in extending to the non-slip boundary condition case all the useful estimates concerning the term $(\omega\cdot\nabla)u\cdot\omega$. See Theorem 2.4. However, in spite of the estimates for $(\omega\cdot\nabla)u\cdot\omega$, we are not able to extend to the non slip boundary condition the 1/2-Hölder sufficient condition for regularity without an additional assumption (see Theorem 2.4). The new obstacle is due to the "additional" boundary integral that appears on the left-hand side of equation (3.2). This term is due to the combination of viscosity and adherence to the boundary, and not to the nonlinear term. This point should be considered in a deeper form, possibly by taking into account suitable physical arguments.

We also replace the half space by a regular open set Ω. This extension is done by appealing to the structure of the Green's function for the Poisson equation under the Dirichlet boundary condition.

We end this section by proposing the following

Open Problem. Consider the problem (2.1), (2.12) with $\Omega = \mathbf{R}^3_+$. Is the assumption (2.4) (or even(2.2)) sufficient to guarantee the regularity of the solution?

2. Known and new results

In the sequel Ω may denote the whole of $\mathbf{R}^3$, the half-space $\mathbf{R}^3_+$ or a bounded, connected, open set in $\mathbf{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{2,\alpha}$. We denote by $\underline{n}$ the unit outward normal to Γ. We do not introduce standard notation or notation whose meaning is clear from the context. We denote by $\|\cdot\|_p$ the canonical norm in the Lebesgue space $L^p := L^p(\Omega)$, $1 \le p \le \infty$. $H^k := H^k(\Omega)$, k positive integer, denotes the classical Sobolev space. Scalar and vector function spaces are indicated by the same symbol.

Consider the evolution 3-D Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0 \quad \text{in } \Omega\times[0,+\infty), \\ \nabla\cdot u = 0 \quad \text{in } \Omega\times[0,+\infty), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

Roughly speaking, it is well known that under suitable boundary conditions there exists at least one *weak solution* in $[0,+\infty)$ and, for a suitable $\tau > 0$, a strong solution in $[0,\tau)$. We assume the reader is well acquainted with this material. It may be superfluous to recall that it is not known whether weak solutions are unique and whether strong solutions are global in time. These are among the most important and challenging open problems in mathematics. Many efforts have been made to obtain significant conditions that are sufficient to guarantee the regularity of weak solutions. In reference [7] the authors open the way to the study of global regularity of solutions of the Navier–Stokes equations via a simple geometrical assumption on the direction of the vorticity, a very significant physical entity. Denote by ω the vorticity of the velocity field u,

$$\omega(x,t) = \nabla \times u(x,t),$$

define the direction of the vorticity as

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}$$

and denote by $\theta(x,y,t)$ the angle between the vorticity ω at two distinct points x and y at time t. In reference [7] the authors prove, in particular, the following result.

Theorem 2.1 (see [7]). *Let be $\Omega = \mathbf{R}^3$ and let u be a weak solution of* (2.1) *in* $(0,T)$ *with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. If, for a.a. $t \in (0,T)$,*

$$\sin\theta(x,y,t) \leq c|x-y| \tag{2.2}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K, then the solution u is strong in $[0,T]$ *and, consequently, is regular.*

The main ingredients in the proof of the above result are the Biot–Savart Law

$$u(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3}\left(\nabla\frac{1}{|y|}\right) \times \omega(x+y)\,dy, \tag{2.3}$$

and a particularly significant formula introduced by Constantin in reference [6]. See equation (7) in [7].

In [5] Berselli and the author improve the above result by showing that

$$\sin\theta(x,y,t) \leq c|x-y|^{1/2} \tag{2.4}$$

is sufficient to guarantee the regularity of weak solutions. More precisely

Theorem 2.2 (see [5]). *Let $\Omega = \mathbf{R}^3$ and let u be a weak solution of* (2.1) *in* $(0,T)$ *with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Assume that for some $\beta \in [1/2, 1]$ and $g \in L^a(0,T;L^b)$, where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2}, \quad a \in \left[\frac{4}{2\beta-1}, \infty\right], \tag{2.5}$$

one has

$$\sin\theta(x,y,t) \leq g(t,x)|x-y|^{\beta} \tag{2.6}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Then the solution u is strong in $[0,T]$ and, consequently, is regular. In particular, (2.4) *alone is a sufficient condition for regularity.*

The central and more challenging open problem is the improvement of the best exponent β for which the assumption (2.9) guarantees the regularity of the solutions *without* any other additional hypotheses. It is worth noting that the proof given in reference [3] formally leads us to believe that the sharpness of the regularity exponent $\beta = 1/2$ corresponds to that of the classical sufficient condition for regularity

$$u \in L^p(0,T;L^s) \quad \text{for } \frac{2}{p} + \frac{n}{s} \leq 1, \quad 2 \leq p < \infty. \tag{2.7}$$

Consequently, the above improvement appears quite difficult to obtain.

Another central problem is the extension of the basic theory to boundary value problems. This is proved in reference [4] in the half-space case $\Omega = \mathbf{R}^3_+$ for the slip boundary condition (see, for instance, [13], [2], [4] for the definition), which is an appropriate model for many important flow problems. We prove, among other side results, that the above 1/2-Hölder assumption still remains a sufficient condition for regularity under the slip boundary condition. More precisely, we prove the following result.

Theorem 2.3 (see [4]). *Let be $\Omega = \mathbf{R}^3_+$ and let the initial data u_0 belong to $V =: H^1 \times H^1 \times H^1_0$ and be divergence free. Let u be a weak solution in $[0,T) \times \mathbf{R}^3_+$ of the Navier–Stokes equations* (2.1) *under the slip boundary condition*

$$u_3 = 0; \qquad \frac{\partial u_j}{\partial x_3} = 0, \quad 1 \leq j \leq 2. \tag{2.8}$$

Let $\beta \in [0,1/2]$ and assume that, for a.a. $t \in (0,T)$,

$$\sin\theta(x,y,t) \leq c|x-y|^\beta \tag{2.9}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Moreover, suppose that

$$\omega \in L^2(0,T;L^r), \tag{2.10}$$

where

$$r = \frac{3}{\beta+1}. \tag{2.11}$$

Then the solution u is strong in $[0,T]$ and, consequently, is regular. Note that if $\beta = \frac{1}{2}$ the assumption (2.11) *is superfluous.*

We were not able to prove a similar result under the non-slip boundary condition. The aim of the present paper is essentially to give a contribution to the treatment of the crucial term $(\omega \cdot \nabla)u \cdot \omega$ in the presence of the non-slip boundary condition, which is the first steep in trying to extend (at least partially) to this boundary condition the result already proved for the slip boundary condition. In this more difficult case a regularity coefficient $\beta = 1$ would already be a very interesting result.

We prove here the following result.

Theorem 2.4. *Let Ω be a bounded, connected, open set in $\mathbf{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{2,\alpha}$ and let $u_0 \in H^1_0(\Omega)$ satisfy $\nabla \cdot u_0 = 0$. Let u be a weak solution of the Navier–Stokes equations* (2.1) *in $[0,T) \times \Omega$ endowed with the non-slip boundary condition*

$$u_{|\Gamma} = 0. \tag{2.12}$$

Let $\beta \in [0, 1/2]$ and assume that, for a.a. $t \in (0,T)$, (2.9) *holds in the region where the vorticity at both points x and $x+y$ is larger than an arbitrary fixed positive constant K. Moreover, suppose that* (2.10) *is satisfied, where r is given by* (2.11). *Then the estimate*

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + (\nu - \epsilon)\|\nabla\omega\|_2^2 - \frac{\nu}{2}\int_\Gamma \frac{\partial|\omega|^2}{\partial n} d\Gamma \le c h(t)\|\omega\|_2^2 \tag{2.13}$$

holds, where $\epsilon > 0$ is arbitrary and $h(t)$, given by (3.31), *satisfies*

$$h \in L^1(0,T). \tag{2.14}$$

If, in addition, an upper bound of the form

$$\frac{1}{2}\int_\Gamma \frac{\partial|\omega|^2}{\partial n} d\Gamma \le C_0 \int_\Omega |\nabla\omega|^2\, dx + B(t)\int_\Omega |\omega|^2\, dx \tag{2.15}$$

holds for some positive constant $C_0 < 1$ and for some $B(t) \in L^1(0,T)$, then u is regular in $]0,T]$. Note that if $\beta = \frac{1}{2}$ the assumption (2.10) *is superfluous.*

The last claim follows from the fact that weak solutions satisfy (2.10) for $r = 2$.

3. Proof of Theorem 2.4

A weak solution such that, for each $\epsilon > 0$, $u \in L^\infty(0,\tau;H^1) \cap L^2(\epsilon,\tau;H^2)$ is called here a *strong solution* in $[0,\tau]$. In the following, we say that u is a strong solution in $[0,\tau^*)$ if u is a strong solution in $[0,\tau]$, for each $\tau < \tau^*$.

Since $u_0 \in H^1$, our solution is strong, hence regular, in $[0,\tau)$, for some $\tau > 0$. Let τ be the maximum of these values in the interval $[0,T]$. If one is able to show that, under our hypotheses, u is necessarily strong in $[0,\tau]$ then, by a continuation principle, u is strong in $[\tau, \tau+\varepsilon)$. This shows that $\tau = T$. Without loss of generality, we assume in the sequel that the solution u is regular in $(0,T)$ and prove that this implies regularity in $(0,T]$.

By taking the curl of both sides of the first equation (2.1) we find, for each $t < T$,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega - \nu \Delta \omega = (\omega \cdot \nabla) u, \tag{3.1}$$

in Ω. Moreover, by taking the scalar product in L^2 of both sides of (3.1) with ω, we get

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_2^2 + \nu \|\nabla \omega\|_2^2 - \nu \int_\Gamma \frac{\partial \omega}{\partial n} \cdot \omega d\Gamma = \int_\Omega (\omega \cdot \nabla) u \cdot \omega \, dx. \tag{3.2}$$

Set, for each triad (j, k, l), $j, k, l \in \{1, 2, 3\}$,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation,} \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation,} \\ 0 & \text{if two indexes are equal.} \end{cases} \tag{3.3}$$

One has

$$(a \times b)_j = \epsilon_{jkl} a_k b_l, \tag{3.4}$$

and

$$(\nabla \times v)_j = \epsilon_{jkl} \frac{\partial v_l}{\partial x_k}, \tag{3.5}$$

where here, and in the sequel, the usual convention about summation of repeated indexes is assumed.

Since

$$-\Delta u = \nabla \times (\nabla \times u) - \nabla (\nabla \cdot u) \tag{3.6}$$

it follows that

$$\begin{cases} -\Delta u = \nabla \times \omega & \text{in} \quad \Omega, \\ \\ u = 0 & \text{in} \quad \Gamma, \end{cases} \tag{3.7}$$

for each t. Let now $G(x, y)$ be the Green's function for the Dirichlet boundary value problem in Ω. Since the boundary Γ is regular it is a classical result that

$$G(x, y) = \frac{1}{4\pi |x - y|} + \gamma(x, y), \tag{3.8}$$

where $\gamma(x, y)$ is a regular function in $\Omega \times \Omega$. See [10] and [9]. See also [8], chapter 4, paragraphs 2 and 4. In particular, one has

$$\left| \frac{\partial^2 G(x, y)}{\partial y_k \partial x_i} \right| \le \frac{c}{|x - y|^3}. \tag{3.9}$$

The above results are contained in the general theory developed by V. A. Solonnikov in references [11] and [12]. See in particular Theorem 1.1 in reference [12]. From (3.7) it follows that

$$u(x) = \int_\Omega G(x, y) \nabla \times \omega(y) \, dy, \tag{3.10}$$

for $x \in \Omega$. By considering a single component u_j, $j = 1, 2, 3$, in equation (3.10), by appealing to (3.5), and by taking into account that $G(x, y) = 0$ if $y \in \Gamma$, an integration by parts yields

$$u_j(x) = -\int_\Omega \epsilon_{jkl} \frac{\partial G(x,y)}{\partial y_k} \omega_l(y)\, dy. \tag{3.11}$$

Hence

$$\frac{\partial u_j(x)}{\partial x_i} = -P.V. \int_\Omega \epsilon_{jkl} \frac{\partial^2 G(x,y)}{\partial x_i \partial y_k} \omega_l(y)\, dy. \tag{3.12}$$

It readily follows that

$$\begin{aligned} ((\omega \cdot \nabla) u \cdot \omega)(x) &\equiv \frac{\partial u_j(x)}{\partial x_i} \omega_i(x) \omega_j(x) \\ &= -\int_\Omega \epsilon_{jkl} \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \omega_i(x) \omega_j(x) \omega_l(y)\, dy. \end{aligned} \tag{3.13}$$

Following [5], we split $\omega(x)$ as

$$\omega(x) = \omega^{(1)}(x) + \omega^{(2)}(x), \tag{3.14}$$

where $\omega^{(1)}(x) = \omega(x)$ and $\omega^{(2)}(x) = 0$ if $|\omega(x)| \le K$ and $\omega^{(1)}(x) = 0$ and $\omega^{(2)}(x) = \omega(x)$ if $|\omega(x)| > K$. Next we replace $\omega(x)$ by $\omega^{(1)}(x) + \omega^{(2)}(x)$ in the right-hand side of equation (3.13). In this way we may write the main equation (3.2) in the form

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\omega\|_2^2 + \nu \|\nabla \omega\|_2^2 - \nu \int_\Gamma \frac{\partial \omega}{\partial n} \cdot \omega d\Gamma \\ &= -\sum_{\alpha,\beta,\gamma=1}^{2} \epsilon_{jkl} \int_\Omega \omega_i^{(\alpha)}(x) \omega_j^{(\beta)}(x) \left\{ \int_\Omega \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \omega_l^{(\gamma)}(y)\, dy \right\} dx \\ &= \sum_{\alpha,\beta,\gamma=1}^{2} \int_\Omega \mathcal{K}_{\alpha,\beta,\gamma}(x)\, dx, \end{aligned} \tag{3.15}$$

with obvious notation. See (3.21) below.

We start by estimating the terms on the right-hand side of for (3.15) for which $(\alpha, \beta, \gamma) \neq (2, 2, 2)$. From (3.8) it follows that the convolution kernel $\frac{\partial^2 G(x,y)}{\partial y_k \partial x_i}$ satisfies the assumptions required by the Calderon–Zygmund inequality. It follows in particular that

$$\|I^{(2)}\|_2 \le c \|\omega\|_2, \tag{3.16}$$

and

$$\|I^{(1)}\|_4 \le c \|\omega^{(1)}\|_4, \tag{3.17}$$

where $I^{(\gamma)}$ denotes anyone of the integrals $(1 \le i, k \le 3)$

$$I^{(\gamma)}(x) = \int_\Omega \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \omega_l^{(\gamma)}(y)\, dy.$$

It readily follows by appealing to Hölder's inequality and to (3.16) that

$$\left| \int_\Omega \mathcal{K}_{\alpha,\beta,2}(x)\, dx \right| \le cK \|\omega\|_2^2 \tag{3.18}$$

when $(\alpha, \beta) \ne (2, 2)$.

On the other hand, by appealing to Hölder's inequality and to (3.17), it follows that

$$\left| \int_\Omega \mathcal{K}_{2,2,1}(x)\, dx \right| \le c \|\omega\|_4 \|\omega\|_2 \|\omega^{(1)}\|_4. \tag{3.19}$$

Since $\|\omega\|_4 \le \|\omega\|_2^{\frac{1}{4}} \|\nabla\omega\|_2^{\frac{3}{4}}$ it readily follows by Young's inequality that the right-hand side of (3.19) is bounded by

$$\epsilon\nu \|\nabla\omega\|_2^2 + c(\epsilon\nu)^{-\frac{3}{5}} \|\omega^{(1)}\|_4^{\frac{8}{5}} \|\omega\|_2^2.$$

Since $\|\omega^{(1)}\|_4 \le K^{\frac{1}{2}} \|\omega\|_2^{\frac{1}{2}}$ it follows that

$$\int_\Omega \mathcal{K}_{2,2,1}(x)\, dx \le \epsilon\nu \|\nabla\omega\|_2^2 + c(\epsilon\nu)^{-\frac{3}{5}} K^{\frac{4}{5}} \|\omega\|_2^{\frac{4}{5}} \|\omega\|_2^2. \tag{3.20}$$

Note hat we could better exploit the fact that Ω has finite measure together with $|\omega^{(1)}(x)| \le K$.

Next we consider the (2,2,2) term. Here we will exploit the geometrical structure of the trilinear form

$$\mathcal{K}_{\alpha,\beta,\gamma}(x) := -\int_\Omega \epsilon_{jkl} \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \omega_i^{(\alpha)}(x) \omega_j^{(\beta)}(x) \omega_l^{(\gamma)}(y)\, dy \tag{3.21}$$

with respect to (generical) vector fields $\omega^{(\alpha)}, \omega^{(\beta)}$, and $\omega^{(\gamma)}$. In fact,

$$\mathcal{K}_{\alpha,\beta,\gamma}(x) = \int_\Omega \mathcal{F}[\omega^{(\alpha)}](x,y) \cdot \omega^{(\beta)}(x) \times \omega^{(\gamma)}(y)\, dy, \tag{3.22}$$

where by definition $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is the linear operator defined by setting

$$\mathcal{F}_k[\omega](x,y) = \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \omega_i(x) \tag{3.23}$$

for an arbitrary vector field ω. Note that, by (3.9),

$$|\mathcal{F}[\omega](x,y)| \le \frac{c}{|x-y|^3} |\omega(x)|. \tag{3.24}$$

In particular,

$$\mathcal{K}_{2,2,2}(x) = P.V. \int_\Omega \mathcal{F}[\omega^{(2)}](x,y) \cdot \omega^{(2)}(x) \times \omega^{(2)}(y)\, dy. \tag{3.25}$$

By appealing to (3.24), (3.9) and assumption (2.9) we show that

$$|\mathcal{K}_{2,2,2}(x)| \le c(\Omega) \int_\Omega |\omega(x)|^2 |\omega(y)| \frac{dy}{|x-y|^{3-\beta}}. \tag{3.26}$$

Consequently

$$\int_\Omega |\mathcal{K}_{2,2,2}(x)|\, dx \le c \int_\Omega |\omega(x)|^2 I(x)\, dx, \tag{3.27}$$

where $I(x)$ is the Riesz potential

$$I(x) = \int_\Omega |\omega(y)| \frac{dy}{|x-y|^{3-\beta}}. \tag{3.28}$$

On the other hand, by Hardy–Littlewood–Sobolev inequality (see [14], Chapter V)

$$\|I\|_3 \le c\|\omega\|_r, \tag{3.29}$$

since

$$\frac{1}{3} = \frac{1}{r} - \frac{\beta}{3}.$$

Hence, from equation (3.27), by appealing to Hölder's inequality with exponents $2, 6$ and 3, to (3.29) and to a Sobolev embedding theorem, one shows that the absolute value of the integral on the left-hand side of (3.27) is bounded by $c\,(\|\omega\|_2(\|\omega\|_2 + \|\nabla\omega\|_2)\|\omega\|_r)$. Hence

$$\int_\Omega |\mathcal{K}_{2,2,2}(x)|\, dx \le \epsilon\nu \|\nabla\omega\|^2 + c\left(\frac{1}{\epsilon\nu}\|\omega\|_r^2 + \|\omega\|_r\right)\|\omega\|_2^2. \tag{3.30}$$

By (3.15), (3.18), (3.20) and (3.30), one proves (2.13) where

$$h(t) = K + (\epsilon\nu)^{-\frac{3}{5}} K^{\frac{4}{5}} |\omega|_2^{\frac{4}{5}} + (\epsilon\nu)^{-1}|\omega|_r^2 + |\omega|_r. \tag{3.31}$$

Clearly (2.14) holds. Note that $|\omega|_2^{\frac{4}{5}} \in L^1(0,T)$. The first part of Theorem 2.4 is proved.

Furthermore, if (2.15) holds then

$$\frac{d}{dt}|\omega|_2^2 \le C\,(h(t) + B(t))\,|\omega|_2^2. \tag{3.32}$$

This shows that

$$\omega \in L^\infty(0,T; L^2(\Omega)),$$

consequently

$$u \in L^\infty(0,T; H^1(\Omega)).$$

Well know results lead to further regularity for the solution u.

Remark 3.1. This remark concerns the effect of the decomposition (3.14) on the singular integral that appears in the right-hand side of equation (3.13). We consider only the main term $\frac{1}{4\pi|x-y|}$ of $G(x,y)$, since $\gamma(x,y)$ is not significant here. In this case equation (3.12) has the form

$$\frac{\partial u_j(x)}{\partial x_i} = -\frac{1}{4\pi}\epsilon_{jkl} P.V. \int_\Omega \left[\delta_{ik} - 3\frac{(x_i - y_i)(x_k - y_k)}{|x-y|^2}\right] \omega_l(y) \frac{dy}{|x-y|^3}. \tag{3.33}$$

Note that $\delta_{ik} - 3\frac{z_i z_k}{|z|^2}$ is homogeneous of degree 0 and has vanishing integral over the unit spherical surface $|z| = 1$ (cancelation property). Equation (3.13) can be written in the form

$$\begin{aligned}\frac{\partial u_j(x)}{\partial x_i}\omega_i(x)\omega_j(x) = -\frac{1}{4\pi} P.V. \int_\Omega \Big[&\mathrm{Det}\,(\omega(x), \omega(x), \omega(y)) \\ &+ 3(\widehat{x-y}) \cdot \omega(x) \mathrm{Det}\left((\widehat{x-y}), \omega(x), \omega(y)\right)\Big] \frac{dy}{|x-y|^3},\end{aligned} \tag{3.34}$$

where $\widehat{x-y} = \frac{x-y}{|x-y|}$. Since $\mathrm{Det}\,(\omega(x), \omega(x), \omega(y)) = 0$ it follows that

$$\begin{aligned}&\frac{\partial u_j(x)}{\partial x_i}\omega_i(x)\omega_j(x) \\ &= -\frac{3}{4\pi} P.V. \int_\Omega (\widehat{x-y}) \cdot \omega(x) \mathrm{Det}\left((\widehat{x-y}), \omega(x), \omega(y)\right) \frac{dy}{|x-y|^3}.\end{aligned} \tag{3.35}$$

For each fixed x the homogeneous kernel on the right-hand side of (3.35) satisfies the cancelation property. However this property gets lost if the first vector field $\omega(x)$ does not coincide with the second one. Hence, in making use of the decomposition (3.14), we appeal to (3.34) and not to the simplified formula (3.35).

Finally, for the convenience of readers interested in improving our results, we note that in the particular case in which Ω is replaced by the half space $\mathbf{R}^3_+$ the boundary integral on the left-hand side of equation (2.15) has the form

$$-\int_\Gamma \frac{\partial(\omega_1^2 + \omega_2^2)}{\partial x_3} d\Gamma. \tag{3.36}$$

Hence, by appealing to (2.12) and to $\nabla \cdot u = 0$ it readily follows that this integral may also be written in the form

$$\int_\Gamma \frac{\partial}{\partial x_3} \left[\left(\frac{\partial u_1}{\partial x_3}\right)^2 + \left(\frac{\partial u_2}{\partial x_3}\right)^2\right] d\Gamma. \tag{3.37}$$

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