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Time-Periodic Solutions of the Navier-Stokes Equations in Unbounded Cylindrical Domains – Leray's Problem for Periodic Flows

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Abstract

Poiseuille flows in infinite cylindrical pipes, in spite of their enormous simplicity, have a main role in many theoretical and applied problems. As is well known, the Poiseuille flow is a stationary solution of the Stokes and the Navier-Stokes equations with a given constant flux. Time-periodic flows in channels and pipes have a comparable importance. However, the problem of the existence of time-periodic flows in correspondence to any given time-periodic total flux, is still an open problem. A solution is known only in some very particular cases, for instance, the Womersley flows. Our aim is to solve this problem in the general case.

The above existence result opens the way to further investigations. As an example of this possibility we consider the extension of the classical Leray's problem for Poiseuille flows to arbitrary time-periodic flows.

1. Introduction

We start by discussing the motivation that led us to consider the problem below. Let Ω be a bounded, regular, connected open set in $\mathbb{R}^n$, $n \geqq 1$, and consider a cylindrical $(n+1)$-dimensional pipe $\Lambda_+ = \Omega \times \mathbb{R}_+$, where $\mathbb{R}_+$ denotes the positive real line. We denote by Γ the boundary of Ω. We set $x = (x_1, ..., x_n)$ and denote by z the longitudinal coordinate along the axis of the pipe, say $z = x_{n+1}$. We denote by χ the component of the velocity v in the axial direction z. Note that the physical dimension is $N = n + 1$. By assumption, the fluid adheres to the lateral boundary of the cylinder.

Assume that a viscous incompressible fluid is pumped into the pipe Λ_+ with a given inflow velocity $v_0(x,t) = v(x,z,t)_{|z=0}$. The pointwise values of the inflow velocity are unknown, and not necessarily time periodic, but the total flux $g(t)$ is a time-periodic function, i.e., $\int_\Omega \chi_0(x,t)\,dx = g(t)$, where $\chi_0 = \chi_{|z=0}$. Note that the inflow velocity can be pointwisely quite "chaotic", but the total amount of pumped fluid by unit of time is not. Note that this is a very natural situation in many physical problems (the blood pumped by the heart, for instance). Clearly, the incompressibility of the fluid implies that

$$\int_{\Omega} \chi(x, z, t)\, dx = g(t)\,, \tag{1}$$

$z \geqq 0$, for each cross section $\Omega(z) = \{(x, z) : x \in \Omega\}$ and at any time $t \geqq 0$. We call the flux $g(t)$, in the cross sections of the pipe, the *total flux*. It may be possible that after a long time, in a very long pipe, the outflow velocity "forgets" the particular point wise distribution of the inflow velocity v_0, and merely "remembers" the total flux $g(t)$. If we assume that a unique limit solution exists, in correspondence to a given g, than the solution must be independent of z. In spite of the recognized, theoretical and applied, significance of this very basic problem, a positive answer is known only in a very few cases: for instance, the classical Poiseuille steady flow, when the flux is constant; and the Womersley flow, which corresponds to a quite particular but important class of periodic sinusoidal fluxes in circular pipes, see [26]. The central position occupied by periodic flows in pipes leads us to consider the possibility of replacing Poiseuille and Womersley flows by flows with an arbitrary time-periodic total flux $g(t)$. Now, the basic open problem is to prove the existence of a time-periodic flow with a given time-periodic flux $g(t)$. As we will see, this leads to a non-standard variational problem. Contrary to the stationary case, the *main open problem* is now whether there exists, in an infinite pipe $\Lambda = \Omega \times \mathbb{R}$, a periodic flow with a given time-periodic flux $g(t)$.

As in the Womersley paper, we also have in mind flows in large arteries. Here, the heart beat gives rise to a periodic variation, the pulsatility, and hence to a time periodic total flux $g(t)$. However, this flux is far from being of sinusoidal type. Nevertheless, in many blood flow simulations, the Womersley model is used; this may be due to the lack of information on more general periodic solutions. Concerning blood flow problems see, for instance, [22].

Another motivation for our study is the extension of the famous Leray problem to periodic flows. In the classical formulation, two cylindrical semi-infinite pipes, Λ_1 and Λ_2, are connected by a reservoir Λ_0. We consider the problem of the existence of a viscous, incompressible fluid flow, subjected to convergence to Poiseuille flows, in both pipes, as the distance goes to infinity. A constant flux g is assigned. A fundamental contribution to Leray's problem is that given by AMICK in [1], dedicated to Leray himself, and in [2], to which we refer the interested reader. Leray's problem seems to have been proposed, see [1], by Leray himself to LADYZHENSKAYA, who in [13] attempted an existence proof under no restrictions on the viscosity. As referenced in [1], this problem is also mentioned by FINN in the review paper [7]. For the Leray's and related problems we refer, in particular, to [9], Vol.I, Chap. VI, Sections 1 and 2, and Vol. II, Chap. XI, Sections 1, 2, 3 and 4. Other main references are [3, 6, 8, 11, 12, 14–16, 18–20, 24]. For the Leray problem concerning non-Newtonian fluids we refer the reader to [21] and the references therein.

Note that, due to the arbitrariness of the connection reservoir Λ_0, the "inflow" velocity $v_0(x, t)$ at the second pipe is essentially arbitrary, except for the given constant total flux. Consequently, an intimately related problem, in semi-infinite pipes, is that of the convergence, as the distance goes to infinity, to a Poiseuille flow when a constant (total) inflow flux is given.

In this paper we shall prove that each periodic $g(t)$ corresponds to one and only one periodic flow, parallel to the axis, with a given periodic flux. Once the existence of these basic periodic flows is proved, further developments can be done by adapting to the periodic case the known proofs done for the stationary case (see, for instance, [1] and [9]). For this reason, and also to avoid more technical proofs, we will merely take into account the above application to Leray's problem, and leave to the reader further developments, in particular more stringent results on the asymptotic behavior of the solutions at infinity distance. Other interesting extensions concern problems with more than two exits to infinity and applications to more general fluids.

Summarizing: The main problem is the following: Consider an infinite pipe $\Lambda = \Omega \times \mathbb{R}$ with boundary $\Sigma = \Gamma \times \mathbb{R}$. Let a T-periodic function $g(t)$ be given. We look for T-periodic solutions $v(x, z, t)$ in $\Lambda \times \mathbb{R}$ of the Navier-Stokes equations which are parallel to the z-axis, independent of z, vanish on the boundary Σ and satisfy the flux constraint (1). We give a positive answer to this question in Theorem 1 below.

After this first result, we consider the Stokes equations (54) and prove the existence and uniqueness of the solution of Leray's problem for an arbitrary given time-periodic flow $g(t)$, see Theorem 4. Finally we assume that $n \leqq 4$ and prove the existence of the solution of Leray's problem for the Navier-Stokes equations if the viscosity ν is sufficiently large, see Theorem 5.

Without loss of generality, we assume below that

$$|\Omega| = 1\,, \quad \text{and that} \quad T = 2\pi\,, \tag{2}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

In order to avoid misunderstandings between z and t, we denote by $\mathbb{R}_t$ the real line $\mathbb{R}$ when referring to the time variable t.

It is worth noting that if we replace the adherence boundary condition by a Neumann-type boundary condition (for instance, a slip type boundary condition; see, for instance, [4]) then the above problem becomes trivial.

2. The existence theorem in infinite pipes

Let Ω be as above and consider the Navier-Stokes equations in the cylindrical domain Λ under the non-slip boundary condition on the lateral boundary, namely

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\,\Delta\, v + (v\cdot\nabla)\,v + \nabla p = 0, \\ \qquad\qquad \nabla\cdot v = 0 \quad \text{in } \Lambda\times\mathbb{R}_t\,, \\ \qquad\qquad v = 0 \quad \text{on } \Sigma\times\mathbb{R}_t\,. \end{cases} \tag{3}$$

Here the differential operators Δ and ∇ act on all the variables $(x_1, ..., x_n,\ z)$.

Since we are looking for solutions parallel to the axis of the pipe, we merely consider the longitudinal component χ of the velocity. Moreover, independence of the velocity on z easily implies that the Navier-Stokes equations reduce to

$$\begin{cases} \frac{\partial \chi}{\partial t} - \nu \Delta \chi + \frac{\partial p}{\partial z} = 0 , \\ \frac{\partial p}{\partial x_k} = 0 \quad \text{for } k = 1, ..., n , \\ \chi = 0 \quad \text{on } \Sigma \times \mathbb{R}_t . \end{cases} \tag{4}$$

Hence we are looking for solutions χ of the problem (4) satisfying (10) below and also $\chi(t+T) = \chi(t)$.

From the first equation (4) it follows that $\frac{\partial p}{\partial z}$ is independent of z since χ and its derivatives do not depend on z. Hence $p(t, z) = a(t) - \psi(t)\, z$. Since the term $a(t)$ does not affect the velocity field, we may assume that the pressure has the form

$$p(t, z) = -\psi(t)\, z . \tag{5}$$

Note that the significant quantity is here $\nabla p = -\psi(t)\, \boldsymbol{e}_z$, where $\boldsymbol{e}_z$ denotes the unit vector in the z direction.

The full problem becomes

$$\begin{cases} \frac{\partial \chi}{\partial t} - \nu \Delta \chi = \psi(t) \quad \text{in } \Omega \times \mathbb{R}_t , \\ \chi = 0 \quad \text{on } \Gamma \times \mathbb{R}_t , \\ \chi(t+T) = \chi(t) \quad \forall\, t \in \mathbb{R}_t , \end{cases} \tag{6}$$

together with the constraint (10). The unknowns are $\chi(t, x)$ and $\psi(t)$. Note that problem (6) is independent of z. The function $\chi(t, x, z) = \chi(t, x)$ for each z and the function p given by (5) are a solution of problem (4). Actually, until the end of Section 5, functions and equations will not depend on the variable z.

Integration of equation (6) in Ω shows that we must have

$$\psi(t) = g'(t) - \nu \int_\Omega \Delta \chi \, dx . \tag{7}$$

See the Remark 2 at the end of this section.

Summarizing: Let $g(t)$, $t \in \mathbb{R}_t$, be a given real 2π-periodic function. A 2π-periodic solution v of the Navier-Stokes equations (3) in the infinite cylinder Λ, with v of the form

$$v(t, x, z) = (0, ..., 0, \chi(t, x)) \tag{8}$$

and satisfying the flux condition (1) for each $t \in \mathbb{R}_t$, exists if and only if χ is a solution of the problem

$$\begin{cases} \frac{\partial \chi}{\partial t} - \nu \Delta \chi + \nu \int_\Omega \Delta \chi \, dx = g'(t) \quad \text{in } \Omega \times \mathbb{R}_t , \\ \chi = 0 \quad \text{on } \Gamma \times \mathbb{R}_t , \\ \chi(t+T) = \chi(t) \quad \forall\, t \in \mathbb{R}_t , \end{cases} \tag{9}$$

for which

$$\int_{\Omega} \chi(x,t)\,dx = g(t)\,, \tag{10}$$

for each t.

The following existence and uniqueness result of the above solution v is a corollary of Theorem 2 below. For notation, see the next section. The symbol # denotes 2π-periodicity.

Theorem 1. *Let Ω be an open, bounded and connected set in $\mathbb{R}^n$ and consider the infinite cylinder $\Lambda = \Omega \times \mathbb{R}$. Let $g \in H^1_{\#}(\mathbb{R}_t)$ be given. There is a unique solution v of the Navier-Stokes equations* (3) *in Λ which satisfies the adherence boundary condition $v_{|\Sigma} = 0$ for each t, and such that:*

(i) *v is (2π)-time periodic,*
(ii) *v has the form* (8)*,*
(iii) *The total flux satisfies* (10)*.*

Moreover, χ satisfies the estimates

$$\|\Delta\chi\|^2_{L^2_{\#}(\mathbb{R}_t;H)} \leqq c\,\|g\|^2_{L^2_{\#}(\mathbb{R}_t)} + \frac{c}{\nu^2}\,\|g'\|^2_{L^2_{\#}(\mathbb{R}_t)}\,, \tag{11}$$

$$\|\chi'\|^2_{L^2_{\#}(\mathbb{R}_t;H)} \leqq c\,\nu^2\,\|g\|^2_{L^2_{\#}(\mathbb{R}_t)} + c\,\|g'\|^2_{L^2_{\#}(\mathbb{R}_t)}\,, \tag{12}$$

and

$$\|\chi\|^2_{C_{\#}(\mathbb{R}_t;V)} \leqq c\,(1+\nu)\,\|g\|^2_{L^2_{\#}(\mathbb{R}_t)} + c\left(\frac{1}{\nu} + \frac{1}{\nu^2}\right)\|g'\|^2_{L^2_{\#}(\mathbb{R}_t)}\,. \tag{13}$$

Remark 1. If Ω is locally situated on one side of Γ and if Γ is a differentiable manifold of class $C^{1,1}$, or if Ω is convex, then, by well-known elliptic regularity results,

$$\|\Delta\chi\|^2_{L^2_{\#}(\mathbb{R}_t;H)} \simeq \|\chi\|^2_{L^2_{\#}(\mathbb{R}_t;H^1_0\cap H^2)}\,.$$

Moreover, well-known regularity results for the heat equation, yield regularity results for χ and v, depending on the regularity of Γ and $g(t)$. In particular, if Γ and $g(t)$ are infinitely differentiable, so is v in $\overline{\Lambda} \times \mathbb{R}$.

Clearly, partial derivatives of v of any order vanish if they include differentiation with respect to z. Otherwise, derivatives are not integrable (with any exponent) in the whole of the cylinder Λ.

Remark 2. If we assign the pressure gradient $-\psi(t)$ instead of the total flux $g(t)$, see (6), existence, uniqueness and estimates for the solution are immediate. However, it is worth noting that an estimate of $g(t)$ simply follows from the knowledge of the volume of fluid pumped into the pipe. On the contrary, $\psi(t)$ is a typical "outflow product", that cannot be measured at the inflow, at least in real problems. This simple fact is connected to the difficulty of obtaining an explicit *functional relation* between ψ and g alone; see (7).

Remark 3. By applying the change of variables $\tau = \frac{2\pi}{T}\, t$, it readily follows that all the results proved below still hold if we replace the period 2π by an arbitrary period T. It is worth noting that the estimates (17), (18) and (19) (hence (11), (12) and (13)) hold with the same constants on the right-hand side. In particular these constants do not depend on the period T. We believe that our results can be extended to the almost periodic case.

3. Functional framework – an abstract result

The above problem can be easily seen, and solved, as a particular case of a more general class of problems. In our opinion a more "abstract" presentation, in a wider framework, helps in the understanding of the problem.

Let H and V be real separable Hilbert spaces, with V densely and compactly embedded in H, and denote respectively by $(\ ,\ )$ and $\|\ \|$ the scalar product and the norm in H. We identify H with its dual H'. We have $V \subset H \simeq H' \subset V'$ where V' denotes the dual of V. Define an operator A by means of $(Au, v) = a(u, v)$, where a is any symmetric, continuous, bilinear, V-elliptic form over $V \times V$. We take $a(u, v)$ as the scalar product in V, and set $((u, v)) = a(u, v)$. Hence, $(Au, v) = ((u, v))$; moreover,

$$(Av, v) = \|v\|_V^2$$

for each $v \in V$. Note that A is an isomorphism between $D(A)$ and H, where

$$D(A) = \{v \in V : A\,v \in H\}$$

and the norm of an element $v \in D(A)$ is given by $\|A\,v\|$.

Denote by e a fixed element $e \in H$, such that $e \notin V$. Without any loss of generality, we normalize e by assuming that $\|e\| = 1$. Finally we define $w \in D(A)$ as the unique solution of the equation

$$A\,w = e\,. \tag{14}$$

We set

$$C_1^2 = (Aw, w) = \|w\|_V^2\,,$$

and

$$C_0^2 = \|w\|^2\,.$$

Below we shall solve the following problem.

Problem. Let H, V, A and e be as above and let $g(t)$ be a given real, 2π-time periodic function. We look for solutions χ of the linear problem

$$\begin{cases} \chi' + \nu\, A\, \chi\ - \nu\, (A\,\chi, e)\, e = g'(t)\, e\,, \\ \chi(t + T) = \chi(t)\,, \end{cases} \tag{15}$$

such that

$$(\chi(t), e) = g(t)\,. \tag{16}$$

The problem (9)–(10) is a particular case of the problem (15)–(16) as shown by setting (note that classical and universally accepted notations will be used without definition) $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, $A = -\Delta$ with domain $D(A) = \{v \in V : A\,v \in H\}$, and by denoting by e the constant function $e(x) = 1$ for each $x \in \Omega$. Note that

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega)\,,$$

if Ω is of class $C^{1,1}$, or convex.

It is worth noting that the case $\|e\| = 1$ is the borderline case. In fact, if we assume that $\|e\| > 1$ then our problem has, in general, no solution. On the other hand, if $\|e\| < 1$, the existence and uniqueness of the solution is trivial.

Remark 4. In the two last sections the notation $(\ ,\ )$ and $\|\ \|$ will be used to denote the scalar product and the norm in functional spaces related to $n+1$ dimensional domains.

We set $L^2_\#(\mathbb{R}_t) = L^2_\#(\mathbb{R}_t;\mathbb{R})$ and $H^1_\#(\mathbb{R}_t) = H^1_\#(\mathbb{R}_t;\mathbb{R})$.

The next two sections are dedicated to proving the following theorem. Recall that the symbol # means 2π-periodicity.

Theorem 2. *Let $g \in H^1_\#(\mathbb{R}_t)$ and $e \in H$, $\|e\| = 1$ and $e \notin V$, be given. Then there is a unique solution χ of the problem* (15) *such that* (16) *holds. Also, $\chi \in L^2_\#(\mathbb{R}_t; D(A)) \cap C_\#(\mathbb{R}_t; V)$, $\chi' \in, L^2_\#(\mathbb{R}_t; H)$ and*

$$\|\chi\|^2_{L^2_\#(\mathbb{R}_t;D(A))} \leqq \widetilde{C}_0\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + \frac{\widetilde{C}}{\nu^2}\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{17}$$

Moreover,

$$\|\chi'\|^2_{L^2_\#(\mathbb{R}_t;H)} \leqq 8\,\widetilde{C}_0\,\nu^2\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + \left(2 + 8\,\widetilde{C}\right)\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,, \tag{18}$$

where $\widetilde{C}$ is the constant that appears in equation (29) *and $\widetilde{C}_0 = max\left\{\widetilde{C},\ C_1^{-4}\right\}$. In particular,*

$$\|\chi\|^2_{C_\#(\mathbb{R}_t;V)} \leqq c\,(1+\nu)\,\|g\|^2_{L^2_\#(\mathbb{R}_t)} + c\,\left(\frac{1}{\nu} + \frac{1}{\nu^2}\right)\,\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{19}$$

Note that the map $\chi \to \nu\,A\,\chi - \nu\,(A\,\chi, e)\,e$ is not defined on V, since in this case $A\,\chi \in V'$ and $e \notin V$. However, even if e should belong to V, the canonical variational techniques, in the functional framework of V, are not recommended here. In fact, the usual scalar multiplication in H of the first equation (15) by χ does not yield a useful estimate.

Scalar multiplication by $A\,\chi$ gives

$$\frac{1}{2}\,\frac{d}{dt}\,\|\chi\|_V^2 + \nu\,\|A\,\chi\|^2 - \nu\,(A\,\chi,\ e)^2 = g'(t)\,(e,\ \chi)\,. \tag{20}$$

Note that

$$(A\,\chi,\ e)^2 \leqq \|A\,\chi\|^2\,.$$

However coercivity fails, even in this last context, since $(A\,\chi,\ e)^2 = \|A\,\chi\|^2$ for $\chi = w$. More precisely, $\nu\,A\,\chi - \nu\,(A\,\chi, e)\,e = 0$, since $A\,w = e$.

Also note that $\|f\|^2 = (f, e)^2$ if and only if $f = c\,e$ for some constant c. Consequently,

$$\|A\,\chi\|^2 = (A\,\chi, e)^2 \qquad \Leftrightarrow \qquad \chi = c\,w\,.$$

The uniqueness of solutions is obvious. In fact, let χ be a periodic solution of the homogeneous problem

$$\chi'(t) + \nu\,A\,\chi - \nu\,(A\,\chi, e)\,e = 0\,. \tag{21}$$

Scalar multiplication by $A\,\chi$ followed by integration on $(0, 2\,\pi)$ shows that $\|A\,\chi\|^2 = (A\,\chi, e)^2$ a.e. in $(0, 2\,\pi)$, as follows from

$$(\chi', A\,\chi) = \frac{1}{2}\,\frac{d}{dt}\,\|\chi\|_V^2$$

and from the periodicity of $\|\chi\|_V$. Hence $\chi = c(t)\,w$. If, moreover, χ satisfies (16) with $g = 0$, then $c(t)$ must vanish identically.

We state the following Lemma in the form needed for later on

Lemma 1. *If*

$$v' \in L^2((a, b);\ H) \quad \textit{and} \quad v \in L^2((a, b);\ D(A))\,, \tag{22}$$

then $v \in C([a, b];\ V)$*. Moreover,*

$$\|v\|^2_{C([a,b];\,V)} \leqq 8\left[\frac{2}{b-a}\,\|v\|_{L^2(a,b;\,H)} + \|v'\|_{L^2(a,b;\,H)}\right]\|v\|_{L^2(a,b;\,D(A))}\,. \tag{23}$$

If

$$\int_a^b v(t)\,dt = 0, \tag{24}$$

then

$$\|v\|^2_{C([a,b];\,V)} \leqq 24\,\|v'\|_{L^2(a,b;\,H)}\,\|v\|_{L^2(a,b;\,D(A))}\,. \tag{25}$$

Proof. The fact that $v \in C([a,b];V)$, together with a suitable estimate, is a very particular case of some well-known results, see [17], Chapter 1, Section 3.1 and [5], Chapter XVIII, Section 1.3. Here, we just want to show the estimate (23). It is well known, (see [17], Chapter 1, section 2.4, Proposition 2.1) that $[D(A),\ H]_{1/2} = V$. Moreover,

$$\frac{1}{2}\,\frac{d}{dt}\,\|v\|_V^2 = (v',\,A\,v)\,. \tag{26}$$

If $v(0) = 0$, it readily follows that

$$\|v\|^2_{C([a,b];V)} \leqq 2\,\|v'\|_{L^2(a,b;H)}\,\|A\,v\|_{L^2(a,b;H)}\,.$$

In the general case we apply the above estimate to the functions $\alpha\,v$ and $(1-\alpha)\,v$, where the real function α belongs to $C^\infty([a,\ b])$, vanishes near a and takes values in $[0,\ 1]$. Since $v = \alpha\,v + (1-\alpha)\,v$, the estimate (23) follows. Finally (25) follows from (23) together with

$$\int_a^b \|u(t)\|_H^2\,dt \leqq \frac{2}{b-a}\left\|\int_a^b u(t)\,dt\right\|_H^2 + (b-a)\int_a^b \|u(t)\|_H^2\,dt\,. \tag{27}$$

□

4. An auxiliary problem

This and the next section are dedicated to proving Theorem 2. We look for the solution $\chi \in L^2_\#(\mathbb{R}_t;\,D(A))$ of the problem (15)–(16) into the form (41), where the unknowns a_k and b_k belong to $D(A)$. This leads to studying the stationary systems (44) in H, for each positive integer k (for a better understanding of this point, see the very beginning of the next section) or, equivalently, to studing the basic system

$$\begin{cases} k\,v + \nu\,A\,u - \nu\,(A\,u,e)\,e = k\,q\,e\,, \\ -k\,u + \nu\,A\,v - \nu\,(A\,v,e)\,e = -k\,p\,e\,, \end{cases} \tag{28}$$

where $k \geqq 1$, p and q are given reals. In this section we prove the following result:

Theorem 3. *Problem* (28) *has one and only one solution* $(u,v) \in D(A) \times D(A)$. *Moreover,*

$$\|Au\|^2 + \|Av\|^2 \leqq \widetilde{C}\left(1 + \left(\frac{k}{\nu}\right)^2\right)(p^2+q^2)\,, \tag{29}$$

where $\widetilde{C}$ depends only on C_0 and C_1 (a simple explicit expression is easily obtained).

Proof. Since A^{-1} is compact, the eigenvalues of A form an increasing sequence of strictly positive reals, λ_j , $j = 1, 2, ...,$

$$A\,w_j = \lambda_j\,w_j\,.$$

The eigenfunctions w_j's are a Hilbertian basis in H; moreover, we can assume

$$(w_i, w_j) = \delta_{i\,j}\,.$$

Note that the w_j's are an orthogonal basis in V. More precisely, $((w_i, w_j)) = \delta_{i\,j}\,\lambda_i\,\lambda_j$.

We set $V_m = \text{span}\,\{w_1, w_2, ..., w_m\}$ and look for $u_m,\ v_m \in V_m$ such that

$$\begin{cases} (k\,v_m + \nu\,A\,u_m - \nu\,(A\,u_m, e)\,e, \phi) = k\,q\,(e, \phi)\,, \\ (-k\,u_m + \nu\,A\,v_m - \nu\,(A\,v_m, e)\,e, \phi) = -k\,p\,(e, \phi) \end{cases} \tag{30}$$

for each $\phi \in V_m$. Since the $\lambda_l\,w_l, l = 1, ..., m$ form a basis of V_m, the problem (30) is equivalent to the system of $2m$ equations obtained by replacing the ϕ's by the above $\lambda_l\,w_l$, $l = 1, ..., m$. Note that, formally, this corresponds to multiplication of the equations by $A\,w_l$ (and not by w_l, as usual). Clearly, we look for u_m and v_m of the form

$$u_m = \sum_1^m \alpha_j\,w_j\,, \quad v_m = \sum_1^m \beta_j\,w_j\,. \tag{31}$$

Straightforward calculations show that (30) is equivalent to the $2m$ dimensional system

$$\begin{cases} k\,\lambda_l\,\beta_l + \nu\,\sum_{j=1}^m \left[\delta_{j\,l} - (w_j, e)\,(e, w_l)\right]\lambda_j\,\alpha_j = k\,q\,(e, w_l)\,, \\ -k\,\lambda_l\,\alpha_l + \nu\,\sum_{j=1}^m \left[\delta_{j\,l} - (w_j, e)\,(e, w_l)\right]\lambda_j\,\beta_j = -k\,p\,(e, w_l)\,, \end{cases} \tag{32}$$

where l runs from 1 to m. Recall that $(w_j,\ w_l) = \delta_{j\,l}$. It is convenient to interpret (32) as a system on the unknown $2m$-dimensional column vector

$$X = (\lambda_1\,\alpha_1, ..., \lambda_m\,\alpha_m, \lambda_1\,\beta_1, ..., \lambda_m\,\beta_m) = (X_1,\ X_2).$$

Set $\gamma_{ij} = \delta_{ij} - (w_j, e)\,(e, w_l)$, $j,\ l = 1, ..., m$, and denote by M the corresponding $m \times m$ matrix. The $2\,m \times 2\,m$ matrix of the system (32) has the form

$$\mathcal{M} = \begin{bmatrix} M & kI \\ -kI & M \end{bmatrix}.$$

Since $X^T\,\mathcal{M}\,X = X_1^T\,M\,X_1 + X_2^T\,M\,X_2$ it follows that $\mathcal{M}$ is positive definite if and only if M is positive definite. Let's prove that this last property holds. Denote by $\overline{e}$ the orthogonal projection in H of e onto V_m. Then, with clear notation,

$$\sum \gamma_{jl}\,\xi_j\,\xi_l = |\xi|^2 - (\xi, \overline{e})\,(\overline{e},\ \xi) \geqq (1 - \|\overline{e}\|^2)\,|\xi|^2$$

for each $\xi \in \mathbb{R}^m$. Since $e \notin V_m$ it follows that $\|\overline{e}\| < 1$. We have proved that problem (30) admits one and only one solution in $V_m \times V_m$.

It is worth noting that the strict positivity of M follows here from the fact that $e \notin V_m$. Since $\|\overline{e}\|$ converges to 1 as m goes to infinity, the behaviour of the system as m goes to infinity is not obvious.

To obtain a suitable estimate, we multiply the first m equations (32) by $\lambda_l\, \alpha_l$, the last m equations by $\lambda_l\, \beta_l$, and sum up for $l = 1, ..., m$. This is equivalent to multiplying the system (32), on the left, by the transpose of X. We obtain

$$\begin{aligned} &\nu \sum_{j,l=1}^{m} \left[\delta_{j,\,l} - (w_j, e)\,(e, w_l)\right] \big((\lambda_j\, \alpha_j)\,(\lambda_l\, \alpha_l) + (\lambda_j\, \beta_j)\,(\lambda_l\, \beta_l)\big) \\ &= k \sum_{l=1}^{m} \lambda_l\,(e, w_l)\,(q\, \alpha_l - p\, \beta_l)\,. \end{aligned} \tag{33}$$

Equation (33) can be written in the equivalent form

$$\nu\, \|A\, u_m\|^2 + \nu\, \|A\, v_m\|^2 - \nu \left[(A\, u_m, e)^2 + (A\, v_m, e)^2\right] \tag{34}$$

$$= k\, q\, (A\, u_m, e) - k\, p\, (A\, v_m, e)\,. \tag{35}$$

Hence,

$$\|A\, u_m\|^2 + \|A\, v_m\|^2 \leqq \tfrac{k^2}{4\,\nu^2}\,(p^2 + q^2) + 2\,\left[(A\, u_m, e)^2 + (A\, v_m, e)^2\right]\,. \tag{36}$$

On the other hand, for each $\phi \in V_m$,

$$(A\, \phi - (A\, \phi, e)\, e,\; w) = (\phi, e) - C_1^2\, (A\, \phi, e)\,,$$

and also

$$\|A\, \phi - (A\, \phi, e)\, e\|^2 = \|A\, \phi\|^2 - (A\, \phi, e)^2\,.$$

Consequently,

$$C_1^4\, (A\, \phi, e)^2 \leqq 2(\phi, e)^2 + 2\, C_0^2 \left[\|A\, \phi\|^2 - (A\, \phi, e)^2\right]\,. \tag{37}$$

Hence, from (37), and by appealing to (35), one proves that

$$\begin{aligned} &C_1^4 \left[(A\, u_m, e)^2 + (A\, v_m, e)^2\right] \\ &\leqq 2\left[(u_m, e)^2 + (v_m, e)^2\right] + 2\, C_0^2\, \frac{k}{\nu}\, \left[q\, (A\, u_m, e) - p\, (A\, v_m, e)\right]\,. \end{aligned} \tag{38}$$

Now, we turn back to system (30). By setting $\phi = \overline{e}$ in both equations, straightforward calculations show that

$$\begin{cases} (v_m, e) = q\, \|\overline{e}\|^2 - \nu\, \frac{1 - \|\overline{e}\|^2}{k}\, (A\, u_m, e)\,, \\ (u_m, e) = p\, \|\overline{e}\|^2 + \nu\, \frac{1 - \|\overline{e}\|^2}{k}\, (A\, v_m, e)\,. \end{cases} \tag{39}$$

By appealing to (39) (note that $(q\,\|\overline{e}\|^2 + B)^2 \leqq 2\,q^2 + 2\,B^2)$) we easily show from (38) that

$$\left[C_1^4 - 4\,\nu^2\left(\frac{1-\|\overline{e}\|^2}{k}\right)^2\right]\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right]$$
$$\leqq 4\,(p^2+q^2) + C_0^2\left\{\frac{k^2}{\varepsilon\,\nu^2}\,(p^2+q^2) + \varepsilon\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right]\right\}$$

for each positive real ε. Note that $\|\overline{e}\|$ converges to 1 as m goes to ∞ and that $k \geqq 1$. Hence, on the left-hand side of the above inequality, we may replace k by 1 and assume that m is sufficiently large so that the coefficient under square brackets is larger than $\frac{C_1^4}{2}$. Hence, by setting $\varepsilon = \frac{C_1^4}{4\,C_0^2}$, we show that

$$C_1^4\left[(A\,u_m, e)^2 + (A\,v_m, e)^2\right] \leqq 16\left[1 + \left(\frac{C_0}{C_1}\right)^4\left(\frac{k}{\nu}\right)^2\right](p^2+q^2)\,. \quad (40)$$

From this last estimate, together with (36), we easily obtain (29) with u and v replaced by u_m and v_m, respectively. From this estimate follows the weak convergence in $D(A) \times D(A)$ of the pair (u_m, v_m) to a solution (u, v) of (28). Clearly, (29) holds. □

5. Proof of Theorem 2: the existence of the periodic solution in an infinite cylinder

In the following we look for the solution $\chi \in L^2_{\#}(\mathbb{R}_t; D(A))$ of the problem (15)–(16) in the form

$$\chi(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos k\,t + \sum_{k=1}^{\infty} b_k \sin k\,t\,, \quad (41)$$

where the unknowns a_k and b_k belong to $D(A)$.

The data $g \in L^2_{\#}(\mathbb{R}_t)$ is written in the form

$$g(t) = p_0 + \sum_{k=1}^{\infty} p_k \cos k\,t + \sum_{k=1}^{\infty} q_k \sin k\,t\,, \quad (42)$$

where the p's and q's are constants.

Substitution in equation (15) yields

$$A\,a_0 - (A\,a_0, e)\,e = 0\,, \quad (43)$$

together with

$$\begin{cases} k\,b_k + \nu\,A\,a_k - \nu\,(A\,a_k, e)\,e = k\,q_k\,e\,, \\ -k\,a_k + \nu\,A\,b_k - \nu\,(A\,b_k, e)\,e = -k\,p_k\,e\,, \end{cases} \quad (44)$$

where k runs from 1 to ∞. Equation (43) is equivalent to

$$a_0 = \widetilde{c}\, w\,, \tag{45}$$

where $\widetilde{c}$ is an arbitrary constant. We anticipate that the value of the constant $\widetilde{c}$ will be uniquely determined by the constraint (16).

On the other hand, each of the infinite systems (44), $k \in \mathbb{N}$, has the form (28). Theorem 3 shows that the coefficients a_k and b_k are uniquely determined. Moreover, the estimate (29) shows that

$$\|A\, a_k\|^2 + \|A\, b_k\|^2 \leqq \widetilde{C} \left(1 + \left(\frac{k}{\nu}\right)^2\right) (p_k^2 + q_k^2) \tag{46}$$

for each $k \in \mathbb{N}$. On the other hand,

$$A\,\chi(t) = \widetilde{c}\, e + \sum_{k=1}^{\infty} (A\, a_k)\, \cos k\, t + \sum_{k=1}^{\infty} (A\, b_k) \sin k\, t\,. \tag{47}$$

It readily follows from (47) that

$$\begin{aligned} \|\chi\|^2_{L^2_\#(\mathbb{R}_t;\, D(A))} &= \int_0^{2\pi} (A\,\chi(t), A\,\chi(t))_H\, dt \\ &= 2\pi\, \widetilde{c}^2 + \pi \sum_{k=1}^{\infty} \left(\|A\, a_k\|^2 + \|A\, b_k\|^2\right)\,. \end{aligned}$$

Finally, by appealing to (46),

$$\|\chi\|^2_{L^2_\#(\mathbb{R}_t;\, D(A))} \leqq 2\pi\, \widetilde{c}^2 + \widetilde{C}\, \pi \sum_{k=1}^{\infty} (p_k^2 + q_k^2) + \widetilde{C}\, \pi\, \nu^{-2} \sum_{k=1}^{\infty} k^2\, (p_k^2 + q_k^2)\,. \tag{48}$$

Now we impose (16) by choosing the value of the arbitrary constant $\widetilde{c}$. By scalar multiplication of both sides of equation (15) by e we show that

$$\frac{d}{dt}[(\chi, e) - g(t)] = 0\,. \tag{49}$$

On the other hand, by (41) and (45) it follows that

$$(\chi(t), e) = \widetilde{c}\,(w, e) + \sum_{k=1}^{\infty} (a_k, e)\, \cos\, kt + \sum_{k=1}^{\infty} (b_k, e)\, \sin\, kt\,. \tag{50}$$

Differentiation with respect to t of this last equation and of equation (42), and appealing to equation (49) show that we must have

$$(a_k, e) = p_k \quad \text{and} \quad (b_k, e) = q_k\,.$$

Hence, from (50) and (42), it follows that

$$(\chi(t), e) = \widetilde{c}\,(w, e) - p_0 + g(t)\,. \tag{51}$$

Consequently, to get (16), we have to impose that $\widetilde{c} = \frac{p_0}{C_1^2}$. This shows, see (45), that (16) holds if and only if in (41) we set

$$a_0 = \frac{p_0}{C_1^2}\, w\,.$$

Finally, from (48) we get

$$\|\chi\|^2_{L^2_\#(\mathbb{R}_t;\,D(A))} \leqq 2\pi\, \frac{p_0^2}{C_1^4} + \widetilde{C}\,\pi \sum_{k=1}^{\infty}(p_k^2 + q_k^2) + \frac{\widetilde{C}}{\nu^2}\|g'\|^2_{L^2_\#(\mathbb{R}_t)}\,. \tag{52}$$

This proves (17). The estimate (18) follows from (17) together with the first equation (15). Finally, the estimate (19) follows from (17), (18), (23) and (25). We obtain (19) by decomposing χ into the time-independent component a_0 and the time-dependent component. This last one, see (41), has vanishing mean-value in each period.

6. The Leray's problem–Stokes case

The aim of this and of the next section is to show that the results proved in the previous sections can be applied to extend and solve Leray's problem in the periodic case. More sophisticated results, as well as extensions to more general cases, can possibly be done by adapting the known proofs in the stationary case. In particular, this may include more stringent results on the asymptotic behavior as z goes to infinity, the extension to more then two exit pipes, and the consideration of non-Newtonian fluids.

Here Θ is an unbounded, connected open subset of $\mathbb{R}^{n+1}$, locally situated on one side of its boundary, consisting of a "reservoir" Λ_0 with two cylindrical exits to infinity, namely Λ_1 and Λ_2. We denote by $\overline{x} = (x_1, ..., x_n, x_{n+1})$ the system of space coordinates in $\mathbb{R}^{n+1}$. The two semi-infinite pipes can be described, possibly in two different systems of coordinates, in the form $\Lambda_i = \Omega_i \times \mathbb{R}_+$, where the sections Ω_i may have different shape and measure. In this framework, we denote by $z \in \mathbb{R}_+$ the axial coordinate in both cylinders and set $x = (x', z)$. Obviously, in this last case $x' = (x_1, ..., x_n)$ does not denote the same $(x_1, ..., x_n)$ that appears in the above definition of $\overline{x}$.

We set

$$\Lambda_i^r = \{(x, z) \in \Lambda_i : z < r\}\,,$$

and

$$\Theta_r = \Lambda_0 \cup \Lambda_1^r \cup \Lambda_2^r\,.$$

We define

$$\mathcal{V} = \left\{ v \in C_0^\infty(\Theta) : \nabla \cdot v = 0 \right\}\,,$$

and denote respectively by $\mathbb{H}$ and $\mathbb{V}$ the closure of $\mathcal{V}$ in $L^2(\Theta)$ and $H^1(\Theta)$. The scalar products in the spaces $\mathbb{H}$ and $\mathbb{V}$ are denoted respectively by

$$(u, v) = \int_\Theta u \cdot v \, d\overline{x} \qquad \text{and} \qquad ((u, v)) = (\nabla u, \nabla v) = \int_\Theta \nabla u \cdot \nabla v \, d\overline{x}\,.$$

In particular,

$$\mathbb{V} = \left\{ v \in H_0^1(\Theta) : \nabla \cdot v = 0 \right\}.$$

Due to the structure of the unbounded set Θ, Poincaré's inequality $\|v\| \leqq \widetilde{c}\, \|\nabla v\|$ holds for each $v \in H_0^1(\Theta)$. In particular,

$$\|v\|_{\mathbb{H}} \leqq \widetilde{c}\, \|v\|_{\mathbb{V}} \tag{53}$$

for each $v \in \mathbb{V}$. Hence, in $\mathbb{V}$, the Dirichlet norm $\|v\|_{\mathbb{V}} = \sqrt{((v, v))}$ is equivalent to the the canonical $H^1(\Theta)$ norm.

Denote by $\chi_i(x, t)$, for $i = 1$ and $i = 2$, the basic time-periodic flows described in Theorem 2 in connection with the sections $\Omega = \Omega_i$ and with a given, arbitrary, periodic flux $g(t)$. Set $\chi_i(x, z, t) = (0, ..., 0, \chi_i(x, t))$. For convenience we denote $\chi_i(x, z, t)$ simply by $\chi_i(x, t)$.

We look for solutions to the following problem:

Problem PL. Given a real (2π)-time-periodic function $g(t)$ find a (2π)-time-periodic function $v(t, x, z)$ of the Stokes evolution problem

$$\begin{cases} \frac{\partial v}{\partial t} - \nu \Delta v + \nabla p = 0\,, \\ \qquad \nabla \cdot v = 0 \qquad \text{in} \quad \Theta \times \mathbb{R}_t\,; \\ \qquad v = 0 \qquad \text{on} \quad (\partial \Theta) \times \mathbb{R}_t\,, \\ v(t + 2\pi) = v(t) \quad \forall\, t \in \mathbb{R}_t\,, \end{cases} \tag{54}$$

such that

$$\sup_{t \in \mathbb{R}_t} \|v(t) - \chi_i(t)\|_{H^1(\Lambda_i)} \leqq constant\,, \quad i = 1, 2\,. \tag{55}$$

Remark 5. We will show that (69) holds, where $u(t) = v(t) - \chi_i(t),\ i = 1,\ 2$, in Λ_i.

The constraint (55) implies convergence of v to the χ_i's, as the coordinate z go to infinity, uniformly with respect to t. In fact, a straightforward argument (see, for instance, [9]) shows that

$$\lim_{z \to +\infty} \|v(t) - \chi_i(t)\|_{H^{\frac{1}{2}}(\Omega_i)} = 0 \tag{56}$$

uniformly with respect to t.

The solution v of problem (54)–(55) has the form $v = v_0 + u$, where v_0 is an auxiliary flow which coincides on the exit pipes Λ_i with the basic periodic flows χ_i, and u is a perturbation of v_0 that goes to zero when the distance z along the exit pipes goes to infinity. More precisely, we have the following result:

Theorem 4. *Let* $g \in H^1_\#(\mathbb{R}_t)$ *be given. There is a unique solution* v *of problem* (54)–(55). *The solution* v *can be written in the form* $v = v_0 + u$, *where* v_0 *is a solution of the problem* (58) *and* u *solves* (61). *The flows* v_0 *and* u *satisfy, respectively, the estimates* (59) *and* (69). *Moreover,* $u = v - \chi_i \quad in \quad \Lambda_i\,, \quad i = 1, 2,$ *satisfies the asymptotic estimate* (56).

Remark 6. Further regularity results are easily proved. In particular, if Ω is *regular* (say, of class $C^{1,1}$, or convex), then

$$D(\mathcal{A}) = H^2(\Theta) \cap \mathbb{V}(\Theta)\,, \tag{57}$$

and if Θ and g are of class C^∞ so is v. For the definition of $\mathcal{A}$ see the end of this section.

It is worth noting that the convergence of the solution v to the limit functions χ_i, as z goes to ∞, is stronger than that implied by (55) alone. If the data are regular, exponential decay should occur, as for the case of the Poiseuille flow. See [1] and [9], Sections VI.1 and VI.2.

For convenience, when writing some of the main estimates, we consider the explicit case in which (57) holds. However, the results below hold in the general case, by merely replacing $H^2(\Theta)$ by $D(\mathcal{A})$.

The first step of the proof of Theorem 4 consists of constructing a time-periodic vector field v_0 in Θ such that

$$\begin{cases} v_0 \in L^2_\#(\mathbb{R}_t; H^2(\Theta_1))\,, \\ v_0' \in L^2_\#(\mathbb{R}_t; L^2(\Theta_1))\,; \\ \qquad \nabla \cdot v_0 = 0 \quad \text{in} \quad \Theta \times \mathbb{R}_t\,, \\ \qquad v_0 = 0 \quad \text{on} \quad (\partial\,\Theta) \times \mathbb{R}_t\,, \\ \qquad v_0(t + 2\pi) = v_0(t)\ \forall\, t \in \mathbb{R}_t, \\ \qquad v_0 = \chi_i \quad \text{in} \quad \Lambda_i \times \mathbb{R}_t\,, \quad i = 1, 2\,. \end{cases} \tag{58}$$

The construction of the vector field v_0 is done by freezing the variable t. This construction is done by following that of the extended Poiseuille vector field q in [1]. See also [24] and [9], Chapter VI, section 1 for details. Following the notation in [9], we chose the truncation functions $\zeta_i(z) \in C_0^\infty(\mathbb{R}^{n+1})$, $i = 1,\ 2$, equal to 1 in $\Lambda_i - \Lambda_i^1$ and vanishing on $\Theta - \Lambda_i$. The map $(\chi_1, \chi_2) \to v_0$ is linear, moreover

$$\|v_0\|_{H^2(\Theta_1)} \leqq c\,(\|\chi_1\|_{H^2(\Lambda_1^1)} + \|\chi_2\|_{H^2(\Lambda_2^1)})\,.$$

Similar estimates hold by replacing H^2 by H^1_0 or by L^2. These facts show that

$$\|v_0\|^2_{L^2_\#(\mathbb{R}_t; H^2(\Theta_1))} \leqq c \sum_{i=1}^{2} \|\chi_i\|^2_{L^2_\#(\mathbb{R}_t; H^2(\Lambda_i^1))}\,.$$

The linearity of the map $(\chi_1, \chi_2) \to v_0$ yields

$$\|v_0'\|^2_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \leqq c \sum_{i=1}^{2} \|\chi_i'\|^2_{L^2_\#(\mathbb{R}_t;L^2(\Lambda_i^1))} .$$

By appealing to (11), (12) and (13), it follows that

$$\begin{cases} \|v_0\|^2_{L^2_\#(\mathbb{R}_t;H^2(\Theta_1))} \leqq c\,(1 + \nu^{-2})\,\|g\|^2_{H^1_\#(\mathbb{R}_t)} , \\ \|v_0\|^2_{C_\#(\mathbb{R}_t;H^1(\Theta_1))} \leqq c\,(\nu + \nu^{-2})\,\|g\|^2_{H^1_\#(\mathbb{R}_t)} , \\ \|v_0'\|^2_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \leqq c\,(1 + \nu^2)\,\|g\|^2_{H^1_\#(\mathbb{R}_t)} . \end{cases} \tag{59}$$

Next, we look for solutions of problem (54) in the form

$$v = v_0 + u . \tag{60}$$

By setting $f(t) = -\left(\frac{\partial v_0}{\partial t} - \nu\,\Delta\,v_0\right)$, the problem (54) becomes

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\,\Delta\,u + \nabla\,p = f(t) , \\ \nabla \cdot u = 0 \qquad \text{in} \quad \Theta \times \mathbb{R}_t ; \\ u = 0 \qquad \text{on} \quad (\partial\,\Theta) \times \mathbb{R}_t , \\ u(t + 2\pi) = u(t) \quad \forall\, t \in \mathbb{R}_t . \end{cases} \tag{61}$$

Next we exploit the fact that $v_0 = \chi_i$ in Λ_i, where the functions $\chi_i(x, z, t) = \chi_i(x, t)$ satisfy (6) with $\psi(t) = \psi_i(t)$ and $\psi_i(t)$ satisfies (7) with $\chi(t) = \chi_i(t)$.

By adding $\nabla \sum_i (z\,\zeta_i(z)\,\psi_i(t))$ to the pressure term that appears in the left-hand side of the first equation (61) we show that we can replace $f(t)$ by

$$f(t) = -\left(\frac{\partial\,v_0}{\partial\,t} - \nu\,\Delta\,v_0\right) + \sum_i \frac{\partial}{\partial\,z}\,(z\,\zeta_i(z)\,\psi_i(t)) .$$

Since $v_0 = \chi_i$ in Λ_i, the first equation (6) shows that $f(t)$ vanishes on $\Lambda_1^1 \cup \Lambda_2^1$, i.e.,

$$supp\ f \subset \Theta_1 . \tag{62}$$

Furthermore, the estimate (59), together with (7) and (11), shows that

$$\|f\|_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \leqq c\,(1 + \nu)\,\|g\|_{H^1_\#(\mathbb{R}_t)} . \tag{63}$$

Let us set the problem (61) in a variational form.

We look for $u \in L^2_\#(\mathbb{R}_t;\ \mathbb{V})$ such that

$$\frac{d}{d\,t}(u,\ v) + \nu\,((u,\ v)) = (f(t),\ v) \qquad \forall\, v \in \mathbb{V} \tag{64}$$

in the distributional sense. We denote by $f(t)$ the orthogonal projection of the above $f(t)$ over $\mathbb{H}$. Note that the orthogonal projection of $f(t)$ over $\mathbb{H}^{\perp}$ is a gradient, which does not affect the solution of equation (64). We have

$$\|f\|_{L^2_{\#}(\mathbb{R}_t;\mathbb{H})} \leqq c\,(1+\nu)\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,. \tag{65}$$

Under the estimate (65) it is known that problem (64) admits a unique solution $u \in L^2_{\#}(\mathbb{R}_t;\mathbb{V})$. In fact, due to (53), the existence and uniqueness of a solution $u \in L^2(0,\,2\pi;\mathbb{V})$ of the Cauchy problem $u(0) \in \mathbb{H}$ is well known. In particular, $u' \in L^2(0,\,2\pi;\mathbb{H})$. Straightforward calculations show that the map $S:\,u(0) \to u(2\pi)$ is a strict contraction in $\mathbb{H}$. Moreover $S(B) \subset B$ for a sufficiently large ball $B \subset \mathbb{H}$. This proves the existence of a unique fixed point $u(0) = u(2\pi)$. The proof of the above result is a simplification of the one given in the next section for solutions of the system (73). It is worth noting that the assumption (84) is due to the presence of the nonlinear terms in the left hand side of (73). The simplified system (61) does not require this assumption.

Canonical devices (formally, scalar multiplication in $\mathbb{H}$ of both sides of the first equation (61) by u, followed by integration in $(0,\,2\pi)$) show that

$$\nu\,\|u\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{V})} \leqq c\,\|f\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{H})}\,,$$

where we have used Poincaré's inequality (53). Finally, by (65),

$$\nu\,\|u\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{V})} \leqq c\,(1+\nu)\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,. \tag{66}$$

This same bound holds as well for $\|u'\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{V}')}$, as easily seen from (64) and (66).

Further regularity: Following a classical technique, we define an unbounded operator $\mathcal{A}$ in $\mathbb{H}$ by setting

$$(\mathcal{A}u,\,v)_{\mathbb{H}} = ((u,v))_{\mathbb{V}}\,.$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ consists of the set of elements $u \in \mathbb{V}$ for which the map $v \to ((u,v))_{\mathbb{V}}$ is an element of $\mathbb{H}'$. Hence $\mathcal{A}u \in \mathbb{H}' \cong \mathbb{H}$. The equation (64) gives rise to the equation

$$u' + \nu\,\mathcal{A}u = f(t)\,. \tag{67}$$

Scalar multiplication by $\mathcal{A}u$ and integration over $(0, 2\pi)$ show that

$$\nu\,\|\mathcal{A}u\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{H})} \leqq c\,(1+\nu)\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,. \tag{68}$$

We have used here the periodicity of the solution and (63). The periodicity of the solution shows that the integral on $(0,\,2\pi)$ of the scalar product $(u',\,\mathcal{A}u)_{\mathbb{H}} = \frac{1}{2}\,\frac{d}{dt}\,\|u\|^2_{\mathbb{V}}$ vanishes. A similar estimate for u' follows directly from the equation (67), by appealing to the above estimate for $\mathcal{A}u$. Finally, from these two estimates it follows that

$$\begin{aligned}&\|u'\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{H})} + (\nu^{-1} + \nu^{1/2})\,\|u\|_{C_{\#}(\mathbb{R}_t;\,\mathbb{V})} + \nu\,\|\mathcal{A}u\|_{L^2_{\#}(\mathbb{R}_t;\,\mathbb{H})}\\ &\quad\leqq c\,(1+\nu)\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,.\end{aligned} \tag{69}$$

Note that the estimate for the second term on the left hand side of (69) follows by appealing to (23), with H replaced by $\mathbb{H}$ and so on. Note that $\mathbb{V} = [D(\mathcal{A}),\ \mathbb{H}]_{1/2}$, independently of (57). See [17], Chap.I, Eq. (2.42).

The uniqueness of the solution in the class $L^2_{\#}(\mathbb{R}_t;\ \mathbb{V})$ follows by setting $f = 0$ in equation (64), and by following standard techniques.

Finally, if Ω is regular, (57) holds. This can be seen by arguing as in [1]. See also [9], VI.1, Lemma 1.2. A main point here is that the sections Ω_i do not depend on z.

Remark 7. We may also start by considering problem (64) in the truncated domains Θ_r by replacing everywhere Θ by Θ_r). This problem admits one and only one solution u_r. For a very elementary proof see, for instance, [23], Chap. 7, problem 7.1–2. We can easily verify that the main estimates do not depend on the parameter r. We denote by u_r the extension by zero of u_r to the whole of Θ. Since the estimates do not depend on r, we may extract an increasing sequence r_n, converging to ∞, and such that the sequence $u_n = u_{r_n}$ converges weakly in $L^2(\mathbb{R}_t; \mathbb{V})$ to some element u. Moreover, u'_n converges weakly to u' in $L^2(\mathbb{R}_t;\ \mathbb{H})$. By choosing test functions $v \in \mathcal{V}$, passing to the limit in the variational equation as n goes to ∞ and, finally, appealing to a density argument, we prove that the limit function u satisfies the variational equation for any test function $v \in \mathbb{V}$.

7. The Leray's problem in the Navier-Stokes case ($N \leqq 4$)

In the following we assume that the dimension $N = n + 1$ satisfies $N \leqq 4$. Note that the physical meaningful situation corresponds to $n = 2$. For brevity, we assume here that the Ω_i's are regular.

In the case of the Navier-Stokes equations, we look for solutions v of the following problem:

Problem PLNS. Given a real (2π)-time-periodic function $g(t)$ find a (2π)-time-periodic function $v(t, x, z)$ of the Navier-Stokes evolution problem

$$\begin{cases} \frac{\partial v}{\partial t} - \nu\,\Delta\,v + (v \cdot \nabla)\,v + \nabla\,p = 0\,, \\ \nabla \cdot v = 0 \quad \text{in} \quad \Theta \times \mathbb{R}_t\,, \\ v = 0 \quad \text{on} \quad (\partial\,\Theta) \times \mathbb{R}_t\,, \\ v(t + 2\pi) = v(t) \ \ \forall\, t \in \mathbb{R}_t\,, \end{cases} \tag{70}$$

such that, for $i = 1,\ 2$,

$$\|v - \chi_i\|_{L^\infty_{\#}(\mathbb{R}_t;L^2(\Lambda_i))} + \|v - \chi_i\|_{L^2_{\#}(\mathbb{R}_t;H^1(\Lambda_i))} \leqq \text{ constant}\,. \tag{71}$$

The constraint (71) implies convergence of v, in a weak sense, to the χ_i's, as the coordinate z go to infinity. See the end of this section.

Theorem 5. *Let $g \in H^1_\#(\mathbb{R}_t)$ be given. There is a positive constant c_0, that depends only on Θ, such that, if* (84) *holds, the problem* (70)–(71) *has at least one solution v. The solution v can be written in the form $v = v_0 + u$, where v_0 satisfies* (58) *and* (59) *and u satisfies* (87) *and* (73). *In particular, $u = v - \chi_i \quad in \quad \Lambda_i\,, \quad i = 1, 2,$ satisfies the asymptotic estimates* (89) *and* (90).

As in the previous section, we look for solutions v in the form

$$v = v_0 + u\,. \tag{72}$$

Now the problem (61) is replaced here by

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\,\Delta\,u + \quad (u\cdot\nabla)\,u + (v_0\cdot\nabla)\,u + \\ \qquad (u\cdot\nabla)\,v_0 + \nabla\,p = f(t)\,, \\ \nabla\cdot u = 0 \quad \text{in} \quad \Theta\times\mathbb{R}_t\,; \\ u = 0 \quad \text{on} \quad (\partial\,\Theta)\times\mathbb{R}_t, \\ u(t+2\,\pi) = u(t) \quad \forall\,t\in\mathbb{R}_t\,, \end{cases} \tag{73}$$

where

$$f(t) = -\left(\frac{\partial\,v_0}{\partial\,t} - \nu\,\Delta\,v_0 + (v_0\cdot\nabla)\,v_0\right) + \sum_i \frac{\partial}{\partial\,z}\,(z\,\zeta_i(z)\,\psi_i(t))$$

satisfies (62). This last property follows from (6), since $(\chi_i\cdot\nabla)\,\chi_i = 0$.

By appealing to the Sobolev embedding theorem $H^1(\Theta_1)\subset L^4(\Theta_1)$ it readily follows that

$$\|(v_0\cdot\nabla)\,v_0\|^2_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \leqq \|v_0\|^2_{L^\infty_\#(\mathbb{R}_t;H^1(\Theta_1))}\,\|v_0\|^2_{L^2_\#(\mathbb{R}_t;H^2(\Theta_1))}\,.$$

Hence, by (59), we have

$$\|(v_0\cdot\nabla)\,v_0\|_{L^2_\#(\mathbb{R}_t;L^2(\Theta_1))} \leqq c\,\sqrt{\nu+\nu^{-4}}\,\|g\|^2_{H^1_\#(\mathbb{R}_t)}\,. \tag{74}$$

Consequently, (65) is replaced here by

$$\|f\|_{L^2_\#(\mathbb{R}_t;\mathbb{H})} \leqq c\,(1+\nu)\,\|g\|_{H^1_\#(\mathbb{R}_t)} + c\,\sqrt{\nu+\nu^{-4}}\,\|g\|_{H^1_\#(\mathbb{R}_t)}\,. \tag{75}$$

We look for $u\in L^2_\#(\mathbb{R}_t;\,\mathbb{V})$ such that

$$\begin{aligned} &\frac{d}{d\,t}(u,\,v) + \nu\,((u,\,v)) + ((u\cdot\nabla)\,u,\,v) + ((v_0\cdot\nabla)\,u,\,v) + ((u\cdot\nabla)\,v_0,\,v) \\ &\quad = (f(t),\,v) \quad \forall\,v\in\mathbb{V}\,, \end{aligned} \tag{76}$$

in the distributional sense.

For $N \leqq 4$, the proof of the existence of, *at least*, one periodic solution of the problem (76) follows well-known techniques. The problem can be treated by adapting the classical variational approach, followed in the case of bounded domains, to the domain Θ . The situation is very similar, since Poincaré's inequality holds in Θ . We construct Faedo-Galerkin approximate solutions and show the existence of the limit solution (possibly not unique). Due to the "extra terms" containing the vector field v_0, we have to assume that the viscosity ν is sufficiently large with respect to the $H^1_{\#}(\mathbb{R}_t)$ norm of the periodic flux $g(t)$.

Since the technical aspects are well known, we merely present the main formal calculations.

By setting $v = u$ in equation (76) we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\,\|\nabla u\|^2 - ((u\cdot\nabla)\,u,\ v_0) = (f(t),\ u)\,. \tag{77}$$

By appealing to Hölder's inequality, the Sobolev embedding theorem $H^1(\Theta_1) \subset L^4(\Theta_1)$ and Poincare's inequality we show that

$$\left|\int_{\Theta_1}(u\cdot\nabla)\,u\cdot v_0\,d\overline{x}\right| \leqq c\,\|v_0\|_{H^1(\Theta_1)}\,\|\nabla u\|^2_{L^2(\Theta_1)}\,. \tag{78}$$

Hence, by (59),

$$\left|\int_{\Theta_1}(u\cdot\nabla)\,u\cdot v_0\,d\overline{x}\right| \leqq c\,\sqrt{\nu+\nu^{-2}}\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,\|\nabla u\|^2_{L^2(\Theta_1)}\,. \tag{79}$$

On the other hand,

$$\left|\int_{\Lambda^1}(u\cdot\nabla)\,u\cdot v_0\,d\overline{x}\right| \leqq \int_{z=1}^{+\infty}dz\int_{\Omega}|(u\cdot\nabla)\,u\cdot v_0|\,dx\,, \tag{80}$$

where Λ^1 represents Λ^1_1 or Λ^1_2, and Ω represents Ω_1 or Ω_2. Recall that the sections Ω_i do not depend on z.

Furthermore,

$$\int_{\Omega}|(u\cdot\nabla)\,u\cdot v_0|\,dx \leqq c\,\|v_0\|_{L^4(\Omega)}\,\|\nabla u\|^2_{L^2(\Omega)}\,.$$

By taking into account that $H^1(\Omega_i) \subset L^4(\Omega_i)$ and $v_0 = \chi_i$ in Λ_i, and also by appealing to (13), it follows that

$$\int_{\Omega}|(u\cdot\nabla)\,u\cdot v_0|\,dx \leqq c\,\sqrt{\nu+\nu^{-2}}\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,\|\nabla u\|^2_{L^2(\Omega)}$$

for each t.

Finally, by integration with respect to t, we show that

$$\left|\int_{\Lambda^1}(u\cdot\nabla)\,u\cdot v_0\,dx\right| \leqq c\,\sqrt{\nu+\nu^{-2}}\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,\|\nabla u\|^2_{L^2(\Lambda^1)}\,. \tag{81}$$

From (79) and (81) we obtain

$$|((u\cdot\nabla)\,u,\ v_0)| \leqq c_0\,\sqrt{\nu+\nu^{-2}}\,\|g\|_{H^1_{\#}(\mathbb{R}_t)}\,\|\nabla u\|^2_{L^2(\Theta)}\,. \tag{82}$$

From (77), (75) and (82) it follows that

$$\frac{1}{2}\,\frac{d}{d\,t}\|u\|^2 + \nu\,\|\nabla u\|^2 \leqq c_0\sqrt{\nu + \nu^{-2}}\;\|g\|_{H^1_\#(\mathbb{R}_t)}\,\|\nabla u\|^2 + \|f\|\,\|u\|\,. \tag{83}$$

Hence, if

$$c_0\sqrt{\nu + \nu^{-2}}\,\|g\|_{H^1_\#(\mathbb{R}_t)} \leqq \frac{\nu}{2}\,, \tag{84}$$

then

$$\frac{d}{d\,t}\|u\|^2 + \nu\,\|\nabla u\|^2 \leqq 2\,\|f\|\,\|u\|\,. \tag{85}$$

In particular,

$$\frac{d}{d\,t}\|u\|^2 + c\,\nu\,\|u\|^2 \leqq \frac{c_1}{\nu}\,\|f\|^2\,. \tag{86}$$

Consequently,

$$\|u(t)\|^2 \leqq e^{-c\nu t}\,\|u(0)\|^2 + \frac{c_1}{\nu}\int_0^t e^{-c\nu(t-s)}\,\|f(s)\|^2\,ds\,.$$

It readily follows that the the map $u(0) \to u(2\pi)$ has a fixed point in the ball $B \subset \mathbb{H}$ centered at the origin with radius

$$\rho = \frac{c_1}{\nu}\,\frac{\|f\|^2}{1 - \exp\{-2\pi c\,\nu\}}\,.$$

For details see, for instance, [25] page 60 and [23] page 180. If, as in these references, we use the classical Faedo-Galerkin method, the approximate (periodic) solutions remain inside the ball B. This fact leads to a convergence of a subsequence to a weak solution u of problem (73),

$$u \in L^\infty_\#(\mathbb{R}_t; L^2(\Theta)) \cap L^2_\#(\mathbb{R}_t; H^1(\Theta))\,. \tag{87}$$

Since

$$v = \chi_i + u \quad \text{in} \quad \Lambda_i\,,$$

the solution v of problem (70) satisfies (71).

The convergence of v to χ_i, in Λ_i, $i = 1, 2$, as z goes to ∞, is equivalent to the convergence of u to zero. As shown below, this last property follows from

$$u \in L^\infty_\#(\mathbb{R}_t; L^2(\Lambda)) \cap L^2_\#(\mathbb{R}_t; H^1(\Lambda))\,. \tag{88}$$

Asymptotic behavior. For convenience, we drop in the sequel the index i, $i = 1, 2$, from notations.

We set

$$\Omega(z) = \{(x, z) : x \in \Omega\}\,,$$

and

$$\Lambda_r = \{(x, z) \in \Lambda : z > r\}\,.$$

We have the following result:

Proposition 1. *Set* $p = 2/s$ *and let* u *satisfy* (88). *Then,*

$$\lim_{z\to\infty} \|u\|_{L^p_\#(\mathbb{R}_t;H^{s-\frac{1}{2}}(\Omega(z)))} = 0 \tag{89}$$

for each $s \in [1/2, 1]$. *In particular*

$$\lim_{z\to\infty} \|u\|_{L^p_\#(\mathbb{R}_t;L^q(\Omega(z)))} = 0\,, \tag{90}$$

where $q = 2n/(n+1-2s)$.

Proof. From (88) it easily follows, by interpolation, that $u \in L^p_\#(\mathbb{R}_t; H^s(\Lambda))$ for each $s \in [0, 1]$. If $s > 0$ then

$$\lim_{z\to\infty} \|u\|_{L^p_\#(\mathbb{R}_t;H^s(\Lambda_z))} = 0\,. \tag{91}$$

Since

$$\|u\|_{H^{s-\frac{1}{2}}(\Omega(z))} \leqq c\,\|u\|_{H^s(\Lambda_z)}\,,$$

the conclusion follows. □

We assume now that $s = 0$ and prove that (88) yields

$$\lim_{z\to\infty} \|u\|_{L^4_\#(\mathbb{R}_t;L^2(\Lambda_z))} = 0\,.$$

This estimate can be proved for regular functions and then extended to u by a density argument. Starting from

$$|u(z,x)|^2 \leqq \int_z^{+\infty} |u(s,x)|\,|\frac{\partial u}{\partial z}(s,x)|\,ds\,,$$

we easily show that

$$\|u\|^2_{L^2(\Omega(z))} \leqq \|u\|^2_{L^2(\Lambda_z)}\,\|\nabla u\|^2_{L^2(\Lambda_z)}\,,$$

a.e. in $\mathbb{R}_t$. The conclusion follows by a straightforward argument.

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