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REGULARITY FOR STOKES AND GENERALIZED STOKES SYSTEMS UNDER NONHOMOGENEOUS SLIP-TYPE BOUNDARY CONDITIONS

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Abstract. We give a simple and very complete proof of the existence of strong solutions to the nonhomogeneous problem (1.1) under the nonhomogeneous boundary conditions (1.5). See Theorem 1.2. See also the pioneering paper [42] by V.A. Solonnikov and V.E. Ščadilov, and [6].

1. Introduction and main results

In the sequel we consider the system

$$\begin{cases} -\nu\Delta u - \mu\nabla(\nabla\cdot u) + \nabla p = f(x), \\ \lambda p + \nabla \cdot u = g(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

under the nonhomogeneous slip-boundary condition (1.5), where $\beta \geq 0$. When $\beta > 0$, this condition is called a slip-boundary condition with linear friction. The slip-boundary conditions are an appropriate model for flow problems with free boundaries, for flows past chemically reacting walls, and for many other important flows in the real world. See, for instance [28] and [45]. For a study of the mathematical foundations of fluid mechanics, and for a rigorous deduction of its main equations, we refer the reader to Serrin's classical article [35].

Here Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of (at least) class $C^{1,1}$ (Lipschitz-continuous first derivatives). We denote by $\underline{n}$ the unit-outward normal to Γ. The constants ν, μ, and λ satisfy the assumptions $\nu > 0$, $\mu + \nu > 0$, and $\lambda \geq 0$. When $\mu = \lambda = 0$ and $g(x) = 0$ we obtain the classical Stokes system. The use of our results to study the incompressible Navier-Stokes equations can be easily done by standard manipulations.

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A main concern here is the presentation of a self-contained paper. Details are not skipped, even though this choice increases the length of the exposition.

Our main interest is the basic L^2-regularity result; i.e., if $f \in \mathbb{L}^2(\Omega)$, $g \in H^1(\Omega)$, $a \in H^{3/2}(\Omega)$, and $b \in \mathbb{H}^{1/2}(\Gamma)$, then $u \in \mathbb{H}^2(\Omega)$ and $p \in H^1(\Omega)$. From this result we may easily get $\mathbb{H}^k$ regularity results, $k > 2$. See Theorem 1.2 below. However, the existence of weak solutions (as well as the justification for its definition) will also be studied in great detail.

The existence of weak solutions to problem (1.1), (1.5) with $\beta = 0$ was considered by Solonnikov and Scadilov (see reference [42]) in the case $\lambda = \mu = 0$ and $g = 0$. They also prove H^2 regularity in the case that $a = 0$ and $b = 0$ in (1.5). In reference [6] existence and regularity of the solution are proved in the general case (1.1), (1.5) when $\beta = 0$ and $\Omega = \mathbb{R}^n_+ = \{x : x_n > 0\}, \Gamma = \mathbb{R}^{n-1}$. Here we will assume, just for convenience, that $n = 3$. Actually, this assumption is used only in discussing the particular case in which Ω has axial symmetry.

We remark that the assumption $\mu \neq 0$ is not significant here since it may be eliminated by replacing the term $\nabla \cdot u$ in the first equation (1.1) by its expression obtained from the second equation. In this way we even see that μ can be assumed arbitrary. However, it is useful to consider the case $\mu \neq 0$ without using the above device, in order to be able to apply some of our calculations to problems related to compressible fluids.

Remark. The independence of the main estimates on the nonnegative parameter λ is a central point in our proofs. Some very basic estimates are proved for positive values of λ but then shown to be independent of λ. This easily implies its validity also when $\lambda = 0$.

The introduction of the parameter λ is useful, for instance, in numerical approximation. In fact, it sometimes seems convenient to relax the divergence-free constraint by replacing it by $\lambda p + \nabla \cdot u = 0$, for a "sufficiently small" value of λ (penalization method). Clearly, we have to prove convergence as λ goes to zero and also error estimates. See [21], Chapter II, B2.4 and [44], Chapter I, B6. Sharp estimates of this type were proved in reference [4]. See [4], equations (1.6) and (1.7).

Another reason that leads us to introduce the parameter λ, in this case as an auxiliary tool, is the following: Replacing the constraint $\nabla \cdot u = 0$ by $\lambda p + \nabla \cdot u = 0$ allows us to localize the equations (flatten the boundary and prove regularity) in a much simpler way than the usual ones. Then, the lack of dependence on λ yields the extension to the limit case $\lambda = 0$.

Finally, we point out that an alternative proof of our regularity results could also be done by following the simple, and quite elegant, proof introduced in reference [4]. In this last case the role of the parameter λ is crucial.

It is worth noting that the regularity result has a local character. It is obvious from the proofs that when the boundary condition (1.5) is satisfied only on a piece Γ_0 of Γ then the regularity result holds up to the internal points of Γ_0. This is of interest, in particular, when solutions are subjected to suitable boundary conditions on $\Gamma - \Gamma_0$. In this last case the proof of the existence of the solution could be much simpler. For instance, coerciveness becomes trivial (moreover, the axially symmetric case is no longer distinct from the generic case) if one has a Dirichlet boundary condition on one piece of the boundary.

In the sequel we denote by $T = -pI + \nu(\nabla u + \nabla u^T)$ the stress tensor and by $\underline{t} = T \cdot \underline{n}$ the stress vector. Hence, with obvious notation (see also [42]),

$$T_{ik} = -\delta_{ik} p + \nu\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\Big), \tag{1.2}$$

$$t_i = \sum_{k=1}^{3} T_{ik} n_k. \tag{1.3}$$

We also define the linear operators $u_\tau = u - (u \cdot \underline{n})\underline{n}$ (the tangential component of u) and the tangential component of $\underline{t}$

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{1.4}$$

Note that $\underline{\tau}(u)$ is independent of the pressure p.

In the sequel we consider the slip-boundary condition

$$\begin{cases} (u \cdot n)_{|\Gamma} = a(x), \\ \beta u_\tau + \underline{\tau}(u)_{|\Gamma} = b(x), \end{cases} \tag{1.5}$$

where $\beta \geq 0$ is a given constant, and $a(x)$ and $b(x)$ are, respectively, a given scalar field and a given tangential vector field on Γ. For the study of problems under these or under strongly related boundary conditions see, for $\beta = 0$, [6], [24], [28], [42], and [45]; for $\beta > 0$, [12], [19], [23], [24], and [33]. See also the remark at the end of this section. Related boundary conditions, including free-boundary problems, are studied in references [2], [9], [25], [30], [34], [36], and [39].

Coupled fluid-structure boundary-value problems are another source of very interesting open problems. See, for instance, [5], [22], and references therein. Other kinds of boundary conditions are studied in [3].

We do not refer here to the well known, extremely extensive literature on the nonslip-boundary condition (see, for instance, [26], [18], [44], and references) except for [15], [16], [40], and [41], where the nonslip, nonhomogeneous boundary condition for the evolution problem is studied in a very complete way.

Before stating our main results some remarks related to the geometry of Ω are necessary. This point will be treated here in a quite complete and carefully way.

Assume that Ω can be generated by revolution around a given axis l_1 (or, as a particular case, around two orthogonal axes l_1 and l_2). For convenience, in these cases we say that Ω is *(axially) symmetric*. Assume, without loss of generality, that the origin of the system of coordinates belongs to l_1 (or to l_1 and l_2 if there are two independent axes of symmetry). Further, denote by $\underline{l}_i$ a unit vector in the l_i direction. Set $\underline{\gamma}_i(x) = \underline{l}_i \wedge x$, and define the (one- or two-dimensional) linear space

$$Z = f\{z : z = k\underline{\gamma}_1\} \quad \text{or} \quad Z = \{z : z = k_1\underline{\gamma}_1 + k_2\underline{\gamma}_2\}, \tag{1.6}$$

according to the case under consideration (we could set $Z = \{0\}$ in the generic case).

By definition, we call a *special case* that in which Ω is symmetric and $\beta = 0$. Otherwise we call it the *generic case*. In fact, eight distinct situations occur, according to the fact that Ω is, or is not, symmetric, λ vanishes or does not vanish, and similarly for β. However, in order to simplify the presentation, we distinguish only between the special and the generic case.

In the generic case one shows that there is a unique solution (u, p) to our problem if $\lambda > 0$. If $\lambda = 0$, the solution exists if and only if the compatibility condition

$$\int_\Omega g\, dx = \int_\Gamma a\, d\Gamma \tag{1.7}$$

holds. Moreover, u is unique and p is unique up to a constant. If $\lambda > 0$ and (1.7) holds, we will see that the results are a little stronger than those without the assumption (1.7).

In the "special case" there are nonzero solutions to the homogeneous problem. More precisely, the kernel of the linear problem is just Z. In fact, vector fields $z \in Z$ (together to $p = 0$) solve the homogeneous problem (1.1), (1.5) since $\Delta z = 0, \nabla \cdot z = 0, (z \cdot n)_{|\Gamma} = 0$, and $\underline{\tau}(z)_{|\Gamma} = 0$. Conversely, if (u, p) is a weak solution to the above homogeneous problem, then necessarily $u \in Z$ and $p = 0$ (or an arbitrary constant, if $\lambda = 0$). See Appendix I for the proofs. Hence, in the special case, any solution (u_0, p) to the problem (1.1), (1.5) can be decomposed into the form $u_0 = u + z$, where (u, p) is the particular solution of the nonhomogeneous problem (1.1), (1.5) for which u is orthogonal to Z in L^2, and z is an arbitrary element of Z. Moreover, these solutions

(u_0, p) do exist (and the particular solution u is unique) if and only if the compatibility condition (see also [42])

$$\int_\Omega f \cdot \underline{\gamma}_i dx = -\nu \int_\Gamma b \cdot \underline{\gamma}_i d\Gamma \tag{1.8}$$

is satisfied. As for the generic case, if $\lambda = 0$ we have to assume the compatibility condition (1.7). Moreover, the pressure is unique up to a constant.

Before stating the main results we introduce some notation. The symbol $\|\cdot\|$ denotes the canonical norm in $L^2(\Omega)$. We denote by $H^k(\Omega), k$ a positive integer, the usual Sobolev space of order k, by $H^1_0(\Omega)$ the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$, and by $H^{-1}(\Omega)$ the strong dual of $H^1_0(\Omega)$. The canonical norms in these spaces are denoted by $\|\cdot\|_k$. $L^2_\#$ (respectively $H^1_\#$)denotes the subspace of L^2 (respectively H^1) consisting of functions with mean value equal to 0. We denote the trace spaces of $H^2(\Omega)$ and $H^1(\Omega)$ by $H^{3/2}(\Gamma)$ and $H^{3/2}(\Gamma)$, respectively. See [31].

In notation concerning duality pairings and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domains Ω or Γ, provided that the meaning remains clear. As a rule, integer norms, as well as integer Sobolev spaces, always relate to Ω, and fractional norms and fractional Sobolev spaces always concern the boundary Γ. For instance, $\|\cdot\|_{1/2}= \|\cdot\|_{1/2,\Gamma}$, and $H^{1/2} = H^{1/2}(\Gamma)$.

If X is a Banach space we denote by X' its strong dual space. The symbol $\langle\cdot,\cdot\rangle$ denotes a generic duality pairing, in particular the scalar product in L^2. We set $\mathbb{L}^2 = [L^2(\Omega)]^3$, $\mathbb{H}^s = [H^s(\Omega)]^3$, and $\mathbb{H}^s(\Gamma) = [H^s(\Gamma)]^3$, and define

$$\mathbb{H}^1_z = \left\{v \in \mathbb{H}^1 : \langle v, z\rangle = 0, \quad \forall z \in Z\right\},$$

$$\mathbb{H}^1_\tau = \left\{v \in \mathbb{H}^1 : (v \cdot \underline{n})_{|\Gamma} = 0\right\}, \qquad \mathbb{H}^1_{z,\tau} = \mathbb{H}^1_\tau \cap \mathbb{H}^1_z.$$

Clearly, $\mathbb{H}^1_\tau = \mathbb{H}^1_{z,\tau} \oplus Z$. In the generic case $\mathbb{H}^1_z = \mathbb{H}^1$, since $Z = \{0\}$.

We remark that $\|\nabla v\|$ is a norm in $\mathbb{H}^1_\tau$, which is equivalent to the canonical $\mathbb{H}^1$ norm $\|v\|_1$.

For convenience, we denote by the symbol $\mathbb{V}^1$ the space $\mathbb{H}^1_z$ and by the symbol $\mathbb{V}^1_\tau$ the space $\mathbb{H}^1_{z,\tau}$. Hence, $\mathbb{V}^1_\tau = \mathbb{V}^1 \cap \mathbb{H}^1_\tau$. Note that, in the generic case, $\mathbb{V}^1_\tau = \mathbb{H}^1_\tau$.

We denote by c, $\overline{c}$, c_1, c_2, etc., positive constants that depend, at most, on Ω, ν, μ, and β (but not on λ). After Section 3 these constants may also depend on Γ. The same symbol c may denote different constants, even in the same equation. Finally, given a scalar function p we set

$$\overline{p} = p - |\Omega|^{-1} \int_\Omega p\, dx. \tag{1.9}$$

The main results are the following.

Existence of a weak solution (see also [42]).

Theorem 1.1. *Assume that*

$$f \in (\mathbb{H}^1_\tau)',\ g \in L^2,\ a \in H^{1/2},\ b \in \mathbb{H}^{-1/2}, \tag{1.10}$$

where b is tangential to Γ. In the special case (i.e., if Ω is symmetric and if $\beta = 0$) assume, in addition, that the (necessary) compatibility condition (1.8) *holds. One has the following results:*

(*a*) *If $\lambda > 0$ the problem* (1.1), (1.5) *has a unique weak solution (u, p) in $\mathbb{V}^1 \times L^2$. Moreover,*

$$\|u\|_1^2 + \lambda\|p\|^2 + \|\overline{p}\|^2 \le c([f]_{-1}^2 + \|a\|_{1/2}^2 + \|b\|_{-1/2}^2) + c\lambda^{-1}(\|g\|^2 + \|a\|_{1/2}^2). \tag{1.11}$$

(*b*) *If $\lambda \ge 0$ and* (1.7) *holds, the problem* (1.1), (1.5) *has a unique weak solution $(u, p) \in \mathbb{V}^1 \times L^2_\#$. If $\lambda = 0$ the pressure p is unique up to a constant. Moreover,*

$$\|u\|_1^2 + \lambda\|p\|^2 + \|p\|^2 \le c([f]_{-1}^2 + \|g\|^2 + \|a\|_{1/2}^2 + \|b\|_{-1/2}^2). \tag{1.12}$$

(*c*) *In the special case (hence $Z \neq \{0\}$) the general solution is given by $(u + z, p)$, where (u, p) is the particular solution described in points* (*a*) *or* (*b*) *(hence u is orthogonal to Z) and z is an arbitrary element of Z.*

For the proof see Section 2. In equations (1.11) and (1.12) the symbol $[f]_{-1}$ denotes the norm of f as an element of $(\mathbb{H}^1_\tau)'$.

Remark. It is worth noting that if $\lambda > 0$, as well as if $\lambda = 0$, $g = 0$, and $a = 0$, the existence of the weak solution (without $\|p\|^2$ on the left-hand side of (1.11) or (1.12)) can be proved without resorting to Proposition 1.1 below, as shown in Section 6.

Concerning the existence of a strong solution we prove the following regularity result.

Theorem 1.2. *Assume that Γ is of class $C^{2,1}$. Let λ and the data f, g, a, and b satisfy the conditions assumed in one of the cases considered in Theorem* 1.1, *and let (u, p) be the corresponding weak solution. Assume moreover that*

$$f \in \mathbb{L}^2(\Omega), \quad g \in \mathbb{H}^1(\Omega), \quad a \in H^{3/2}(\Gamma), \quad b \in \mathbb{H}^{1/2}(\Gamma). \tag{1.13}$$

Then (u, p) belong to $\mathbb{H}^2 \times H^1$. Moreover, in case (*b*),

$$\|u\|_2^2 + (1 + \lambda)\|p\|_1^2 \le c(\|f\|^2 + \|g\|_1^2 + \|a\|_{3/2}^2 + \|b\|_{1/2}^2), \tag{1.14}$$

where c is independent of λ. In case (a), *the above estimate is satisfied by replacing c by $c(\lambda)$, where $c(\lambda)$ tends to infinity as λ goes to zero.*

In the special case, the above estimates are satisfied by the particular solution $u \in \mathbb{V}^1_\tau$, i.e., by the solution u for which $\langle u, z\rangle = 0$. Clearly, the solutions $u_0 = u + z$ are regular, as well.

It is not difficult to obtain (from our calculations) a precise expression for the dependence of c and $c(\lambda)$ on ν and μ and also of $c(\lambda)$ on λ.

Concerning the proof, we do not use here potential theory results (in this direction see, for instance, [1], [11], [17], [18], [20], [26], [37], and [38]). A main tool will be Nirenberg's translations method; see [32].

As remarked in [6], the current literature on Stokes and Navier-Stokes systems (even in proving the existence of weak solutions to the homogeneous Dirichlet boundary value problem; see, for instance, [13], [29], [44]) is based on a set of special results which are now accepted as tools at one's disposal. Nevertheless, it is worth noting that the proofs of these results are not at all trivial. In this regard we note that in our approach the only such "special result" to be used is Proposition 1.1 below.

Finally, we mention here a different method, introduced in reference [4], to study the nonhomogeneous Dirichlet boundary-value problem for system (1.1) in an open subset $\Omega \subset \mathbb{R}^n$. In [4] the H^2 a-priori estimate for solutions to the Dirichlet problem is proved in a straightforward way. Then, the effective existence of strong solutions is shown by a very simple (and particularly elegant) new method to which we direct the reader's attention. See [4], Section 4 (for a more classical, and very complete, proof, we refer the reader to [13]).

The following proposition is one of the main tools in this paper.

Proposition 1.1. *Let p be a scalar field in L^2. There is a constant c such that*

$$\|\overline{p}\| \leq c\|\nabla p\|_{-1}, \tag{1.15}$$

where $\overline{p}$ is defined by (1.9).

Classical proofs are given in [14] and [43]. See also the appendix in reference [6]. An alternative, simpler, and very complete proof, is given in Appendix II. We may also obtain the above proposition, for any integrability exponent p, by applying a duality argument to the result stated in [10], Theorem 2′. We refer the reader to the very interesting proof of this last result; see [10].

Remark (added in proof). More recently, we have studied the stationary [7] and the evolution [8] Navier-Stokes equations with shear dependent viscosity, namely

$$\begin{cases} \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot T(u,\pi) = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.16}$$

under slip, or nonslip, boundary conditions. T denotes the stress tensor

$$T = -\pi I + \nu_T(u)\mathcal{D}u, \tag{1.17}$$

where $\mathcal{D}u = \nabla u + \nabla u^T$, $\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}u|^{p-2}$, and ν_0 and ν_1 are strictly positive constants. For convenience, we denote here the pressure by the symbol π. Roughly speaking, for the initial-boundary-value problem in the case $n = 3$, and for sufficiently regular data, we show in particular that $u \in L^2(0,T;W^{2,p'})$, for $p \in (2+\frac{2}{5}, 4)$, and $u \in L^{4-p}(0,T;W^{2,l})$, for $p \in (2+\frac{2}{5}, 3)$, where $l = \frac{3(4-p)}{5-p}$.

2. Proof of Theorem 1.1

We start this section by introducing the formal calculations that led to the definition of weak solution. Let ϕ be any vector field in Ω such that

$$(\phi \cdot \underline{n})_{|\Gamma} = 0. \tag{2.1}$$

In the following, vector fields denoted by ϕ are assumed to verify (2.1). In our functional framework this means that $\phi \in \mathbb{H}^1_\tau$.

From (1.4) it follows that $\underline{\tau}(u) \cdot \phi = \underline{t} \cdot \phi$, for all ϕ satisfying (2.1); hence,

$$\underline{\tau}(u) \cdot \phi = \nu \sum_{i,k=1}^{3} \Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \Big) n_k \phi_i, \tag{2.2}$$

for all ϕ as above. Next we define the bilinear form

$$B(u,\phi) := \int_\Omega \Big[\frac{\nu}{2} (\nabla u + \nabla u^T) \cdot (\nabla \phi + \nabla \phi^T) + (\mu - \nu)(\nabla \cdot u)(\nabla \cdot \phi) \Big]\, dx. \tag{2.3}$$

We remark that

$$B(u,z) = 0, \qquad \forall z \in Z. \tag{2.4}$$

By integrations by parts, and by taking (2.2) into account, one easily shows that

$$B(u,\phi) = -\int_\Omega [\nu \Delta u + \mu \nabla(\nabla \cdot u)] \cdot \phi \, dx + \int_\Gamma \underline{\tau}(u) \cdot \phi \, d\Gamma, \tag{2.5}$$

for all ϕ satisfying (2.1). It readily follows that a sufficiently regular couple (u,p) is a solution of $(1.1)_1$, $(1.5)_2$ if and only if

$$B(u,\phi) - \int_\Omega p \nabla \cdot \phi \, dx = \int_\Omega f \cdot \phi \, dx + \int_\Gamma (b - \beta u) \cdot \phi d\Gamma, \tag{2.6}$$

for all ϕ satisfying (2.1).

Next we introduce the constraint $(1.5)_1$. It will be convenient to reduce this nonhomogeneous boundary condition to the homogeneous one,

$$(v \cdot \underline{n})_{|\Gamma} = 0. \tag{2.7}$$

To accomplish this we consider a vector field w such that

$$(w \cdot \underline{n})_{|\Gamma} = a(x), \qquad \|w\|_1 \le c\|a\|_{1/2}. \tag{2.8}$$

More precisely, since $a \in H^{1/2}$, we fix a linear, continuous map $a \mapsto w$, from $H^{1/2}(\Gamma)$ into $H^1(\Omega)$, such that (2.8) holds. Note that at this point there are no assumptions on $\nabla \cdot w$. However, when we assume that (1.7) holds (a necessary condition if $\lambda = 0$), w must satisfy the compatibility condition $\nabla \cdot w = g$. Hence, when we assume that (1.7) holds, we replace (2.8) by

$$\begin{cases} \nabla \cdot w = g, \\ (w \cdot \underline{n})_{|\Gamma} = a(x), \qquad \|w\|_1 \le c(\|a\|_{1/2} + \|g\|). \end{cases} \tag{2.9}$$

A more complete result is stated and proved in Section 8, Corollary 8.2. Moreover, in the special case we assume, in addition, that $w \in \mathbb{H}^1_z$. This is done merely by replacing the above w by the projection $w - \sum_i \langle w, \underline{\gamma}_i \rangle \underline{\gamma}_i$ (note that (2.9) is also satisfied by the new w since $(z \cdot \underline{n})_{|\Gamma} = 0$ and $\nabla \cdot z = 0$, for all $z \in Z$). We set

$$u = w + v, \tag{2.10}$$

where w is fixed as above and the new unknown v is subject to the constraint (2.7).

Under the change of variables (2.10), the equation (2.6) for u is equivalent to the following equation for v:

$$\begin{aligned} &B(v, \phi) - \langle p, \nabla \cdot \phi \rangle + \beta \int_\Gamma v \cdot \phi \, d\Gamma \\ &= -B(w, \phi) + \langle f, \phi \rangle - \beta \int_\Gamma w \cdot \phi \, d\Gamma + \langle b, \phi \rangle_\Gamma, \end{aligned} \tag{2.11}$$

for all ϕ satisfying (2.1). Hence (u, p) is a solution of $(1.1)_1$, (1.5) if and only if v satisfies (2.7) and, moreover, (v, p) is a solution of (2.11). It follows that (u, p) is a solution of the complete problem (1.1), (1.5) if and only if (v, p) is a solution of

$$\begin{aligned} &B(v, \phi) - \langle p, \nabla \cdot \phi \rangle + \beta \langle v, \phi \rangle_\Gamma + \lambda \langle p, \psi \rangle + \langle \nabla \cdot v, \psi \rangle \\ &= -B(w, \phi) + \langle f, \phi \rangle - \beta \langle w, \phi \rangle_\Gamma + \langle b, \phi \rangle_\Gamma + \langle g, \psi \rangle - \langle \nabla \cdot w, \psi \rangle, \end{aligned} \tag{2.12}$$

for all ϕ satisfying (2.1) and all scalar fields ψ. This is the definition of a weak solution.

In terms of our functional framework, this means that we look for solutions (v,p) of problem (2.12) in the space $\mathbb{H}^1_\tau \times L^2$ for all test functions (ϕ,ψ) in this same space. In fact, this is our definition of a weak solution.

In the special case, by setting $\phi = z$ and $\psi = 0$ in (2.12), and by using (2.4), one shows that (1.8) is a necessary condition for the existence of solutions.

In the sequel we write (2.12) in the abbreviated form

$$a_\lambda(V,\Phi) = L(\Phi), \quad \forall \Phi, \tag{2.13}$$

where, by definition,

$$\begin{cases} a_\lambda(V,\Phi) = B(v,\phi) - \langle p, \nabla\cdot\phi\rangle + \beta\langle v,\phi\rangle_\Gamma + \lambda\langle p,\psi\rangle + \langle \nabla\cdot v,\psi\rangle \\ L(\Phi) = -B(w,\phi) + \langle f,\phi\rangle - \beta\langle w,\phi\rangle_\Gamma + \langle b,\phi\rangle_\Gamma + \langle g,\psi\rangle - \langle \nabla\cdot w,\psi\rangle. \end{cases} \tag{2.14}$$

Here, $V = (v,p)$ and $\Phi = (\phi,\psi)$. Note that

$$L(\Phi) = -B(w,\phi) + \langle f,\phi\rangle - \beta\langle w,\phi\rangle_\Gamma + \langle b,\phi\rangle_\Gamma \tag{2.15}$$

when (1.7) holds, since $\nabla\cdot w = g$.

Clearly a_λ is a bilinear form and L is a linear form. Recall that the test functions ϕ satisfy (2.1) and the solution v should satisfy (2.7).

The above argument shows that (2.10), (2.13) is a natural weak formulation of problem (1.1), (1.5). Hence we state the following definition.

Definition. Assume that (1.10) holds, and let w satisfy (2.8). We say that a pair (u,p) is a *weak solution* of problem (1.1), (1.5) if it belongs to $\mathbb{H}^1\times L^2$, and if $u = w + v$, where $(v,p) \in \mathbb{H}^1_\tau \times L^2$ satisfies (2.13) for each $\Phi \in \mathbb{H}^1_\tau \times L^2$.

It is not difficult to show that the above definition does not depend on the choice of the particular w in equations (2.8) or (2.9).

In the sequel we denote by $\|\cdot\|^2_\Gamma$ the $L^2(\Gamma)$ norm. Moreover, we set $\overline{\nu} = \nu$ if $\mu \geq 0$ and set $\overline{\nu} = \nu + \mu$ if $-\nu < \mu < 0$.

Next we establish two useful estimates for $|L(\Phi)|$.

Lemma 2.1. *Let L be as in definition* (2.14). *Then*

$$|L(\Phi)| \leq c\left(\|a\|_{1/2} + [f]_{-1} + \|b\|_{-1/2}\right)\|\nabla\phi\| + c\|a\|_{1/2}\|\psi\| + \|g\|\|\psi\| \tag{2.16}$$

for each $\Phi \in \mathbb{H}^1_\tau \times L^2$. If, moreover, (1.7) *holds, then one has*

$$|L(\Phi)| \leq c\left(\|a\|_{1/2} + \|g\| + [f]_{-1} + \|b\|_{-1/2}\right)\|\nabla\phi\|. \tag{2.17}$$

Proof. From (2.3) and (2.8) one gets $|B(w,\phi)| \leq c(\nu + |\mu|)\|a\|_{1/2}\|\nabla\phi\|$. Hence, by definition (2.14), (2.16) follows. If (1.7) holds we may use (2.9) instead of (2.8) to show that $|B(w,\phi)| \leq c(\nu + |\mu|)(\|a\|_{1/2} + \|g\|)\|\nabla\phi\|$. From (2.15) we get (2.17). □

Next we establish a basic result in order to prove the coercivity of the bilinear forms.

Lemma 2.2. *Assume that* $v \in \mathbb{H}^1_\tau$. *Then,*

$$B(v,v) = \nu\|\nabla v\|^2 + \mu\|\nabla \cdot v\|^2 - \nu \int_\Gamma \sum_{i,k=1}^{3} \frac{\partial n_k}{\partial x_i} v_i v_k \, d\Gamma. \tag{2.18}$$

In particular,

$$B(v,v) \geq \overline{\nu}\|\nabla v\|^2 - c_0\nu\|v\|^2_\Gamma, \tag{2.19}$$

where c_0 *depends only on* Γ.

Proof. Since $C_0^\infty(\overline{\Omega}) \cap \mathbb{H}^1_\tau$ is dense in $\mathbb{H}^1_\tau$, we may assume that $v \in C_0^\infty(\overline{\Omega})$. By integration by parts one easily shows that

$$B(v,v) = \nu\|\nabla v\|^2 + \mu\|\nabla \cdot v\|^2 + \nu \int_\Gamma \sum_{i,k=1}^{3} \frac{\partial v_k}{\partial x_i} v_i n_k \, d\Gamma. \tag{2.20}$$

Since the boundary Γ is of class $C^{1,1}$ we may extend the normal vector field $\underline{n}$ to a neighborhood of Γ, as a vector field of class $C^{0,1}$. See [31]. As $(v \cdot \underline{n})_{|\Gamma} = 0$, its derivative with respect to the tangential direction v vanishes on Γ. Hence,

$$\sum_{i,k=1}^{3} \frac{\partial v_k}{\partial x_i} v_i n_k = -\sum_{i,k=1}^{3} \frac{\partial n_k}{\partial x_i} v_i v_k,$$

and (2.18) follows. □

The following result shows, in particular, that in the generic case the bilinear form B is coercive over $\mathbb{H}^1_\tau$. In particular, it is coercive whenever $\beta > 0$.

Lemma 2.3. *There is a positive constant* c_1 (*that depends only on* ν, μ, *and* Ω) *such that*

$$B(v,v) \geq c_1\|\nabla v\|^2, \tag{2.21}$$

for each v *in* $\mathbb{V}^1_\tau$. *Hence the bilinear form* B *is coercive over* $\mathbb{V}^1_\tau$. *In particular, the square root of* $B(v,v)$ *is a norm in* $\mathbb{V}^1_\tau$, *equivalent to the canonical* $\mathbb{H}^1$ *norm.*

Proof. We start by showing (see also [42]) that given a positive ϵ there is an $N = N(\epsilon)$ such that

$$\|v\|^2_\Gamma \leq \epsilon\|\nabla v\|^2 + NB(v,v), \qquad \forall v \in \mathbb{V}^1_\tau. \tag{2.22}$$

If this statement is false, then there would be a positive ϵ such that to each natural number n there corresponds an element $v_n \in \mathbb{V}^1_\tau$ such that $\|v_n\|^2_\Gamma \geq \epsilon \|\nabla v_n\|^2 + nB(v_n, v_n)$.

By setting $w_n = v_n/\|v_n\|_\Gamma$ it follows that $\|w_n\|_\Gamma = 1$; moreover,

$$\epsilon \|\nabla w_n\|^2 + nB(w_n, w_n) \leq 1. \tag{2.23}$$

In particular, there is a subsequence (still denoted by w_n) such that w_n converges weakly in $\mathbb{V}^1_\tau$ (hence strongly in $\mathbb{L}^2$) to some $w \in \mathbb{V}^1_\tau$. In particular, w_n converges to w in $\mathbb{L}^2(\Gamma)$; hence, $\|w\|^2_\tau = 1$.

On the other hand, from (2.19), it follows that

$$B(v, v) + c_0 \nu \|v\|^2_\Gamma \geq \overline{\nu} \|\nabla v\|^2. \tag{2.24}$$

Hence, the square root of the left-hand side of (2.24) is a norm in $\mathbb{V}^1_\tau$. Consequently,

$$B(w, w) + c_0 \nu \|w\|^2_\Gamma \leq \liminf \{B(w_n, w_n) + c_0 \nu \|w_n\|^2_\Gamma\}.$$

By using (2.23), and by taking into account that w_n converges strongly in $\mathbb{L}^2(\Gamma)$ to w, it follows that $B(w, w) = 0$. Since, in general,

$$\frac{\nu}{2} \sum_{i,j} (a_{ij} + a_{ji})^2 + (\mu - \nu) \sum_i (a_{ii})^2 \geq \frac{\nu + \mu}{4} \sum_{i,j} (a_{ij} + a_{ji})^2 \tag{2.25}$$

and $\nu + \mu > 0$ it follows that

$$b_{i,j} := \frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i} = 0. \tag{2.26}$$

Below, we will show that w must vanish. Since this contradicts $\|w\|_\tau = 1$, (2.22) follows.

Let us see that $w = 0$. One easily shows that

$$2 \frac{\partial^2 w_i}{\partial x_j \partial x_k} = \frac{\partial b_{i,k}}{\partial x_j} + \frac{\partial b_{i,j}}{\partial x_k} - \frac{\partial b_{j,k}}{\partial x_i}. \tag{2.27}$$

Hence each w_i is a polynomial of degree less than or equal to one, say

$$w_i(x) = a_i + \sum_{k=1}^{3} c_{i,k} x_k.$$

Since $b_{i,k} = 0$ it follows that $c_{i,k} + c_{k,i} = 0$. It readily follows that w can be written in the form $w(x) = \underline{a} + \underline{l} \wedge x$, where the components of the vector field $\underline{l}$ are given by $l_1 = c_{3,2} = -c_{2,3}$, $l_2 = c_{1,3} = -c_{3,1}$, and $l_3 = c_{2,1} = -c_{1,2}$.

By solving the 3×3 linear system $\underline{b} - \underline{l} \wedge \underline{b} = \underline{a}$ (the determinant of which is given by $1 + |\underline{b}|^2$), we write w in the form $w(x) = \underline{b} + \underline{l} \wedge (x - \underline{b})$.

If $\underline{l} = 0$, then $w = \underline{b}$. Hence $w = 0$ due to the assumption $(w \cdot \underline{n})_{|\Gamma} = 0$. If $\underline{l} \neq 0$, the assumption $(w \cdot \underline{n})_{|\Gamma} = 0$ imposes that Γ has axial symmetry with respect to the axis l, parallel to the vector $\underline{l}$, and passing through the "point" $\underline{b}$. Hence Ω is axial symmetric, and we are in the special case. We assume (without loss of generality, by a translation $-\underline{b}$) that $\underline{b} = 0$. Hence $w(x) = \underline{l} \wedge x$; moreover, l is an axis of symmetry. This shows that $w \in Z$. Since w belongs to $\mathbb{V}^1_\tau$, it must be that $w = 0$.

Finally, the estimate (2.21) follows easily from (2.24) together with (2.22), by setting $\epsilon = \overline{\nu}/(2c_0\nu)$ in this last inequality. □

Proof of Theorem 1.1. Continuity of the bilinear form a_λ and of the linear functional L over the whole of $\mathbb{H}^1 \times L^2$ are obvious. From definition (2.14) and from (2.21) it follows that

$$a_\lambda(V, V) \geq c_1 \|\nabla v\|^2 + \lambda \|p\|^2. \tag{2.28}$$

This shows the coerciveness of the bilinear form a_λ over $\mathbb{V}^1_\tau \times L^2$, if $\lambda > 0$. Hence, given $\lambda > 0$, the problem (2.13) (i.e., the problem (2.12)) has a unique solution (v, p) in $\mathbb{V}^1_\tau \times L^2$.

Next we want to prove the estimates (2.16) and (2.17). Let (v, p) be the above weak solution, where $\lambda > 0$. From (2.12) with $\psi = 0$ and $\phi \in C^\infty_0(\Omega)$ it follows that

$$(p, \nabla \phi) = B(u, \phi) - (f, \phi). \tag{2.29}$$

Hence, by (2.5),

$$\nabla p = f + \nu \Delta u + \mu \nabla(\nabla \cdot u) \tag{2.30}$$

as elements of $\mathbb{H}^{-1}$. Note that f acts, by restriction to $\mathbb{H}^1_0$, as an element of $\mathbb{H}^{-1}$. Moreover, $\|f\|_{-1} \leq [f]_{-1}$. In particular,

$$\|\nabla p\|_{-1} \leq [f]_{-1} + (\nu + |\mu|)(\|\nabla v\| + \|w\|_1). \tag{2.31}$$

Consequently, by Proposition 1.1 (recall that $p \in L^2$),

$$\|\overline{p}\| \leq c \left[[f]_{-1} + (\nu + |\mu|)(\|\nabla v\| + \|w\|_1) \right]. \tag{2.32}$$

By using (2.13) with $\Phi = V$ together with (2.28) and (2.16), it readily follows that

$$\|\nabla v\|^2 + \lambda \|p\|^2 \leq c \left\{ [f]^2_{-1} + \|a\|^2_{1/2} + \|b\|^2_{-1/2} \right\} + \frac{c}{\lambda} \left\{ \|a\|^2_{1/2} + \|g\|^2 \right\}. \tag{2.33}$$

Finally, by using (2.32), we get the complete estimate (1.11) under the assumption $\lambda > 0$.

The existence and the uniqueness of the weak solution (u, p) when $a \neq 0$, as well as the estimate (1.11), follow immediately from the corresponding results just proved for the solution (v, p), since $u = v + w$ and w satisfies the estimate (2.8).

Next we consider the case in which (1.7) holds and $\lambda \geq 0$, and we prove (1.12).

As in the previous case, we start by assuming that $\lambda > 0$. We argue as above, just replacing (2.16) by (2.17). This immediately shows that

$$\|\nabla v\|^2 + \lambda \|p\|^2 \leq c \left\{ [f]_{-1}^2 + \|g\|^2 + \|a\|_{1/2}^2 + \|b\|_{-1/2}^2 \right\}. \tag{2.34}$$

The estimate (1.12) now follows by using (2.32). It is worth noting that now $p = \overline{p}$. In fact, by setting $\phi = 0$ and $\psi = 1$ in equation (2.12) one gets

$$\lambda \int_\Omega p \, dx = \int_\Omega g \, dx - \int_\Gamma a \, d\Gamma = 0.$$

Next, we consider the case $\lambda = 0$. Since the estimate (1.12) is independent of λ, it follows, with obvious notation, that the solutions $V_\lambda = (v_\lambda, p_\lambda)$, $\lambda > 0$, of problem (2.13) converge weakly in $\mathbb{V}_\tau^1 \times L_{\#}^2$, as $\lambda \to 0$, to some $V = (v, p) \in \mathbb{V}_\tau^1 \times L_{\#}^2$. Obviously (v, p) satisfies (1.12). By passing to the limit, as $\lambda \to 0$, in equation (2.12) it follows that (v, p) satisfies this same equation with $\lambda = 0$; i.e.,

$$a_0(V, \Phi) = \langle L, \Phi \rangle, \quad \forall \Phi.$$

This proves Theorem 1.1 also in the case $\lambda = 0$. Note that the above limit (v, p) is unique since the weak solution to problem (2.12) with $\lambda = 0$ is unique in the class $\mathbb{V}_\tau^1 \times L_{\#}^2$. In fact, let (v, p) be the difference between two such solutions; i.e., let (v, p) solve the homogeneous problem (2.12) with $\lambda = 0$. By setting $(\phi, \psi) = (v, p)$ it follows that $B(v, v) = 0$, and hence $v = 0$. It follows that $\langle p, \nabla \phi \rangle = 0$ for each ϕ in H_0^1. Hence $\nabla p = 0$ in H^{-1}, and Proposition 1.1 shows that $p = 0$.

The results claimed for weak solutions (u, p) of problem (1.1), (1.5) when $a \neq 0$ follow immediately from that proved for the solution (v, p) when $a = 0$ since $u = v + w$ and w satisfy the estimate (2.9). $\square$

Remark. In fact the proof is not complete. We have shown that there is a (unique) (v, p) in $\mathbb{V}_\tau^1 \times L^2$ such that (2.12) holds for each (ϕ, ψ) in the space $\mathbb{V}_\tau^1 \times L^2$. However, in the special case, one has $Z \neq \{0\}$; hence, $\mathbb{V}_\tau^1$ is smaller than $\mathbb{H}_\tau^1$. We have to show that (2.12) holds for each (ϕ, ψ) in the larger space $\mathbb{H}_\tau^1 \times L^2$. It remains to show (2.12) for test functions (ϕ, ψ) of the particular form $(z, 0)$ and this result holds since for these particular test functions the left-hand side of (2.12) vanishes (recall (2.4)) and the right-hand side reduces to $\langle f, \phi \rangle + \nu \langle b, \phi \rangle$, which vanishes, by assumption (1.8). Hence the couple

(v,p) is a weak solution in the sense of our definition. Finally, the fact that $v+z$ is also a solution is obvious.

3. Construction of global weak solutions with an arbitrary support

In this section we show that if (v,p) is an arbitrary weak solution and θ is an arbitrary $C^{1,1}(\mathbb{R}^3)$ scalar function, then $(\theta v, \theta p)$ is as well a weak solution in Ω, with modified data. The new data also have support contained in the support of θ. Moreover, we prove suitable estimates for $(\theta v, \theta p)$ directly in terms of f, g, a, and b.

For the time being θ is as above and v and ϕ are arbitrary elements of $\mathbb{H}^1$.

Write the expression of $B(\theta v, \phi)$, just by substitution of v by θv in $B(v,\phi)$. Then differentiate the products θv by the usual rule. On the other hand, write a corresponding expression for $B(v, \theta\phi)$, by using the symmetry of the bilinear form B. Straightforward calculations show that

$$\begin{aligned}
B(\theta v, \phi) &= B(v, \theta\phi) + \nu \int_\Omega u_k \frac{\partial\theta}{\partial x_i}\frac{\partial\phi_k}{\partial x_i} dx + \nu \int_\Omega u_i \frac{\partial\theta}{\partial x_k}\frac{\partial\phi_k}{\partial x_i} dx \\
&- \nu \int_\Omega \frac{\partial u_k}{\partial x_i}\frac{\partial\theta}{\partial x_i}\phi_k\,dx - \nu \int_\Omega \frac{\partial u_i}{\partial x_k}\frac{\partial\theta}{\partial x_i}\phi_k\,dx - (\mu-\nu)\int_\Omega \frac{\partial\theta}{\partial x_i}\frac{\partial v_i}{\partial x_k}\phi_k\,dx \qquad (3.1)\\
&- (\mu-\nu)\int_\Omega \frac{\partial^2\theta}{\partial x_i \partial x_k} v_i \phi_k\,dx - (\mu-\nu)\int_\Omega (\nabla\cdot u)(\nabla\theta\cdot\phi)\,dx,
\end{aligned}$$

for arbitrary $\phi, v \in \mathbb{H}^1$ and $\theta \in C^{1,1}(\mathbb{R}^3)$. Next we rewrite equation (3.1) in order to avoid terms containing partial derivatives of ϕ. Straightforward integrations by parts show that

$$\begin{aligned}
B(\theta v, \phi) &= B(v, \theta\phi) + \int_\Omega F^{(1)}\phi\,dx + \int_\Omega F^{(2)}\phi\,dx \\
&+ \nu \int_\Gamma (v\cdot n)(\nabla\theta\cdot\phi) d\Gamma + \nu \int_\Gamma l(v)\cdot\phi\,d\Gamma,
\end{aligned} \qquad (3.2)$$

for each $\phi, v \in \mathbb{H}^1$, where

$$\begin{cases}
F_k^1[v] = -\nu\left(\frac{\partial\theta}{\partial x_i}\frac{\partial v_i}{\partial x_k} + 2\frac{\partial\theta}{\partial x_i}\frac{\partial v_k}{\partial x_i} + \frac{\partial\theta}{\partial x_k}\frac{\partial v_i}{\partial x_i}\right) - (\mu-\nu)\left(\frac{\partial\theta}{\partial x_i}\frac{\partial v_i}{\partial x_k} + \frac{\partial\theta}{\partial x_k}\frac{\partial v_i}{\partial x_i}\right) \\
F_k^2[v] = -\nu\left(\frac{\partial^2\theta}{\partial x_i \partial x_k} v_i + \frac{\partial^2\theta}{\partial x_i^2} v_k\right) - (\mu-\nu)\frac{\partial^2\theta}{\partial x_i\partial x_k} v_i,
\end{cases} \qquad (3.3)$$

and

$$l(v) = \frac{\partial\theta}{\partial n} v. \qquad (3.4)$$

We wrote the expressions of the F^i's and l just for completeness. Their exact form is not of interest here. The point is that they are linear, continuous operators on $\mathbb{H}^1$, with values in $\mathbb{L}^2$ and $\mathbb{H}^{1/2}(\Gamma)$ respectively. In fact,

$$||F^i[v]|| \leq c\|\theta\|_{C^{1,1}(\overline{\Omega})}\|v\|_1, \qquad \|l(v)\|_{1/2} \leq c\|\theta\|_{C^{1,1}(\overline{\Omega})}\|v\|_1, \tag{3.5}$$

for *arbitrary* vector fields v in $\mathbb{H}^1$. If v satisfies (2.7), it follows from (3.2) that

$$B(\theta v, \phi) = B(v, \theta\phi) + \langle F^1, \phi\rangle + \langle F^2, \phi\rangle + \nu\langle l(v), \phi\rangle_\Gamma, \tag{3.6}$$

for all $v \in \mathbb{H}^1_\tau$ and all $\phi \in \mathbb{H}^1$. We could drop the term with l by using only functions θ for which $\frac{\partial\theta}{\partial n} = 0$ on Γ.

Note that, in equation (3.6), $l(v)$ is tangential to Γ since v is so.

Since the functions θ will be fixed and finite in number, from now on the constants of type c may depend also on θ. On the other hand,

$$\langle \theta p, \nabla \cdot \phi\rangle = \langle p, \nabla \cdot (\theta\phi)\rangle - \langle p\nabla\theta, \phi\rangle$$

and

$$\langle \nabla \cdot (\theta v), \psi\rangle = \langle \theta\nabla \cdot v, \psi\rangle + \langle v \cdot \nabla\theta, \psi\rangle.$$

Hence, from (3.6) we obtain

$$\begin{aligned} &B(\theta v, \phi) - \langle \theta p, \nabla \cdot \phi\rangle + \beta\langle \theta v, \phi\rangle_\Gamma + \lambda\langle \theta p, \psi\rangle + \langle \nabla \cdot (\theta v), \psi\rangle \\ &= B(v, \theta\phi) - \langle p, \nabla \cdot (\theta\phi)\rangle + \beta\langle v, \theta\phi\rangle_\Gamma + \langle \nabla \cdot v, \theta\psi\rangle + \lambda\langle p, \theta\psi\rangle \\ &\quad + \langle p\nabla\theta, \phi\rangle + \langle v \cdot \nabla\theta, \psi\rangle + \langle F^1, \phi\rangle + \langle F^2, \phi\rangle + \nu\langle l(v), \phi\rangle_\Gamma, \end{aligned} \tag{3.7}$$

for all $v \in \mathbb{H}^1_\tau$.

Note that in the previous calculations it is not assumed that v is a solution. From now on we assume that $(v, p) \in \mathbb{H}^1_\tau \times L^2$ is a solution to problem (2.12) and (ϕ, ψ) are test functions in $\mathbb{H}^1_\tau \times L^2$. Hence $(\theta\phi, \theta\psi)$ is, as well, a test function. Consequently, the first five terms on the right-hand side of (3.7) can be replaced, according to (2.12). This yields

$$\begin{aligned} &B(\theta v, \phi) - \langle \theta p, \nabla \cdot \phi\rangle + \beta\langle \theta v, \phi\rangle_\Gamma + \lambda\langle \theta p, \psi\rangle + \langle \nabla \cdot (\theta v), \psi\rangle \\ &= -B(w, \theta\phi) + \langle \theta f, \phi\rangle - \beta\langle \theta w, \phi\rangle_\Gamma + \langle \theta b, \phi\rangle_\Gamma + \langle \theta g, \psi\rangle - \langle \theta\nabla \cdot w, \psi\rangle \\ &+ \langle p\nabla\theta, \phi\rangle + \langle v \cdot \nabla\theta, \psi\rangle + \langle F^1 + F^2, \phi\rangle + \nu\langle l(v), \phi\rangle_\Gamma. \end{aligned} \tag{3.8}$$

On the other hand, from (3.2) it follows that

$$B(w, \theta\phi) = B(\theta w, \phi) - \langle F^1[w] + F^2[w], \phi\rangle + \nu\langle l(w) + (w \cdot n)\nabla\theta, \phi\rangle_\Gamma. \tag{3.9}$$

By substitution of (3.9) in (3.8) one finally obtains

$$\begin{aligned} &B(\theta v, \phi) - \langle \theta p, \nabla \cdot \phi\rangle + \beta\langle \theta v, \phi\rangle_\Gamma + \lambda\langle \theta p, \psi\rangle + \langle \nabla \cdot (\theta v), \psi\rangle \\ &\qquad = -B(\theta w, \phi) + \langle F, \phi\rangle + \langle G, \psi\rangle + \langle \zeta, \phi\rangle_\Gamma, \end{aligned} \tag{3.10}$$

where

$$\begin{cases} F = \theta f + p\nabla\theta + F^1[v] + F^2[v] + F^1[w] + F^2[w], \\ G = \theta g - \theta\nabla\cdot w + (\nabla\theta)\cdot v, \\ \zeta = \theta b - \beta\theta w + \nu(l(v) + l(w) + (w\cdot n)\nabla\theta). \end{cases} \tag{3.11}$$

It is worth noting that the supports of F, G, and ζ are contained in the support of θ, hence in $\overline{\Omega}_{2\rho}$.

Equation (3.10) shows that $(\theta v, \theta p)$ solves, in the whole of Ω, a problem of type (2.12). By using, in particular, (3.5), it follows that

$$\begin{aligned} &\|F\| \le c\,(\|f\| + \|p\| + \|v\|_1 + \|w\|_1) \\ &\|G\|_1 \le c\left(\|g\|_1 + \|v\|_1 + \|a\|_{3/2}\right) \\ &\|\zeta\|_{1/2} \le c\left(\|b\|_{1/2} + \|v\|_1 + \|w\|_1\right). \end{aligned} \tag{3.12}$$

On estimating $\|G\|_1$ we take into account that here the vector field w satisfies (2.8), or (2.9), with $\|a\|_{1/2}$, $\|g\|$, and $\|w\|_1$ replaced by $\|a\|_{3/2}$, $\|g\|_1$, and $\|w\|_2$, respectively. In particular, $\|w\|_2 \le c(\|a\|_{3/2} + \|g\|_1)$. Hence, we have proved the following result:

Theorem 3.1. *Let $(v,p) \in \mathbb{H}^1_\tau \times L^2$ be a solution to problem* (2.13) (*i.e., to problem* (2.12)) *for each $(\phi,\psi) \in \mathbb{H}^1_\tau \times L^2$. Then* (3.10) *holds, where $F \in \mathbb{L}^2$, $G \in H^1$, and $\zeta \in \mathbb{H}^{1/2}$, defined by* (3.11), *satisfy* (3.12).

Note that in equation (3.10) there are no smallness assumptions on the supports of the test functions $(\phi,\psi) \in \mathbb{H}^1_\tau \times L^2$.

4. The change of variables

It is clear that if the above solution $(\theta v, \theta p)$ is regular, then the original solution (v,p) is regular in the interior of the support of the function θ. Hence, to prove the regularity of (v,p) it is sufficient to prove the regularity of the $(\theta v, \theta p)$'s for a family of θ's such that the union of the (interiors of the) supports of the θ's contains $\overline{\Omega}$. In other words, given an arbitrary, but fixed, point $x_0 \in \overline{\Omega}$, we need just to prove that $(\theta v, \theta p)$ is regular for some θ such that $\theta \neq 0$ in a neighborhood of x_0. If $x_0 \in \Omega$, the proof is much easier; hence, one assumes that $x_0 \in \Gamma$. In order to reduce this problem, by a suitable change of variables, to a problem involving a flat boundary, we need to consider functions θ with a sufficiently small support. Since the functions $(\theta v, \theta p)$ are solutions in the whole of Ω, the transformed functions $(\widetilde{\theta v}, \widetilde{\theta p})$ will be global solutions as well (in fact, their extensions by zero are solutions in the whole half space $\mathbb{R}^3_+$). This fact allows us to avoid the introduction of more technical truncations and related manipulations, whose complete treatment becomes quite onerous in the context of systems with nonhomogeneous boundary conditions.

Note that, since $w \in \mathbb{H}^2$, the $\mathbb{H}^2$ regularity of $u = v + w$ follows from that of v. Hence, we concentrate our attention on the solutions $v \in \mathbb{H}^1_\tau$ of problem (2.12).

Let $x_0 \in \Gamma$ be given, and let π be the tangent plane to Γ at x_0. We assume that the axes of x_i, $i = 1, 2, 3$, are such that the origin coincides with x_0 and the x_3 axis has the direction of the inward normal to Γ at x_0. Hence the axes of x_i, $i = 1, 2$, lie in the plane π. We may use this particular system of coordinates since the analytical expressions that appear on the left-hand side of (3.10) are invariant under orthogonal transformations, since the expressions of the divergence and gradient are invariant.

In the sequel we assume that Γ is a manifold of class C^3. It is worth noting that small modifications in the proofs show that $W^{3,3}$ regularity is sufficient to prove Theorem 1.2. This can be shown by a suitable application of Hölder's inequality (with exponents 2, 3, and 6) and of Sobolev's embedding theorem $H^1 \hookrightarrow L^6$. However, we assume C^3 regularity, to avoid further manipulations. We remark that the proofs presented below also apply if Γ is a $C^{2,1}$ manifold.

Let $x_0 \in \Gamma$ be given, and let $(x', x_3) = (x_1, x_2, x_3)$ be the above system of coordinates. By the definition of a C^3 manifold, there is a positive real a and a real function $x_3 = h(x')$, of class C^3 on the sphere $\{x' : |x'| < a\}$, such that the points x for which $x_3 = h(x')$ belong to Γ, the points x for which $h(x') < x_3 < a + h(x')$ belong to Ω, and the points x for which $-a + h(x') < x_3 < h(x')$ belong to $\mathbb{R}^3 - \Omega$. Without loss of generality, we assume that $a \leq 1$. We define

$$\begin{aligned} &I_r = \{x : |x'| < r, -r + h(x') < x_3 < r + h(x')\}, \\ &\Omega_r = \{x \in I_r : h(x') < x_3\}, \qquad \Gamma_r = \{x \in I_r : x_3 = h(x')\}, \end{aligned} \tag{4.1}$$

where $0 < r < a$. Note that I_r is a neighborhood of x_0 with "size" r, $\Omega_r = \Omega \cap I_r$, and $\Gamma_r = \Gamma \cap I_r$. We set

$$\begin{aligned} &J_r = \{y : |y'| < r, -r < y_3 < r\} \\ &Q_r = \{y \in J_r : 0 < y_3\}, \quad \Lambda_r = \{y \in J_r : y_3 = 0\}. \end{aligned} \tag{4.2}$$

We consider the change of variables $y = Tx$ given by

$$(y_1, y_2, y_3) = (x_1, x_2, x_3 - h(x')), \ (x_1, x_2, x_3) = (y_1, y_2, y_3 + h(y')). \tag{4.3}$$

For each r, T is a C^3 diffeomorphism of I_r onto J_r that maps Ω_r onto Q_r and Γ_r onto Λ_r. If $y = Tx$, then $y' = Tx'$. Hence, $h(x') = h(y')$ and $\frac{\partial h(x')}{\partial x_j} = \frac{\partial h(y')}{\partial y_j}$, for $j = 1, 2$. For this reason, we will not indicate the independent variables when dealing with h functions. Also note that the Jacobian determinant of the above transformation is equal to 1.

Given $x_0 \in \Gamma$ we fix, once and for all, a positive ρ such that $3\rho < r$. For convenience, we set $Q = Q_{3\rho}$, $\Lambda = \Lambda_{3\rho}$. Next we fix a function $\theta \in C_0^2(\mathbb{R}^3)$ (or even in $C_0^\infty(\mathbb{R}^3)$) such that $U := \operatorname{supp}\,\theta \subset I_{2\rho}$ and $\theta \neq 0$ (say, $\theta = 1$) on I_ρ. Our aim is to prove that $(\theta v, \theta p)$ belongs to $\mathbb{H}^2(\Omega_{3\rho}) \times L^2(\Omega_{3\rho})$. This shows that $(v,p) \in \mathbb{H}^2(\Omega_\rho) \times L^2(\Omega_\rho)$. Note that supp $(\theta v, \theta p)$ is contained in $\overline{\Omega}_{2\rho}$, as well as the supports of the corresponding data in equation (3.10). To prove the regularity of (v,p) in $\overline{\Omega}$ we cover Γ by a finite number of neighborhoods I_ρ, as described above. Hence, we use just a finite number of fixed functions h and θ. For that reason it would be superfluous to indicate the dependence of the constants c on these functions.

In the sequel we will use the following notation:

$$\widetilde{f}(y) = f(T^{-1}(y)), \tag{4.4}$$

where here f denotes an arbitrary scalar or vector field. As a rule, $f = f(x)$ and $\widetilde{f} = \widetilde{f}(y)$. Moreover, partial derivatives and differential operators when applied to f functions concern the x variables and when applied to $\widetilde{f}$ functions concern the y variables.

The regularity of $(\theta v, \theta p)$ on $\overline{\Omega}_{3\rho}$ is equivalent to that of the corresponding transformed functions $(\widetilde{\theta\, v}, \widetilde{\theta\, p})$ on $\overline{Q}_{3\rho}$. Note that these functions, as well as the transformed data, have support contained in $\overline{Q}_{2\rho}$. Hence $(\widetilde{\theta\, v}, \widetilde{\theta\, p})$, extended by 0 to all of $\mathbb{R}^3_+$, is a solution on the whole of $\mathbb{R}^3_+$.

The (covariant) transform of a vector field v via the change of coordinates $y = Tx$ is given by

$$\widetilde{v}_j(y) = v_j; \quad \widetilde{v}_3(y) = v_3 - (\partial_1 h)v_1 - (\partial_2 h)v_2, \tag{4.5}$$

where $j = 1, 2$, and the v_i functions are calculated at the point $(y', y_3 + h(y'))$. Conversely,

$$v_j(x) = \widetilde{v}_j; \quad v_3(x) = \widetilde{v}_3 + (\partial_1 h)\widetilde{v}_1 + (\partial_2 h)\widetilde{v}_2, \tag{4.6}$$

where $j = 1, 2$, and the $\widetilde{v}_i$ functions are calculated at the point $(x', x_3 - h(x'))$.

Note that divergences remain invariant:

$$(\nabla_x \cdot v)(x) = (\nabla_y \cdot \widetilde{v})(y), \tag{4.7}$$

where y= T x. Moreover, $\widetilde{v}(y)$ is tangential to the boundary $\Lambda_{3\rho}$; i.e.,

$$\widetilde{v}_3 = 0 \qquad on \qquad \Lambda_{3\rho} \tag{4.8}$$

if (and only if) v is tangential to $\Gamma_{3\rho}$. A main point here is that

$$\partial_j h(0,0) = 0, \qquad j = 1, 2, \tag{4.9}$$

which holds since π is tangential to Γ at x_0. From (4.6) it follows that

$$\begin{cases} \frac{\partial v_j}{\partial x_k} = \frac{\partial \widetilde{v}_j}{\partial y_k} - (\partial_k h)\frac{\partial \widetilde{v}_j}{\partial y_3}; \\ \frac{\partial v_j}{\partial x_3} = \frac{\partial \widetilde{v}_j}{\partial y_3}; \\ \frac{\partial v_3}{\partial x_k} = \frac{\partial \widetilde{v}_3}{\partial y_k} - (\partial_k h)\frac{\partial \widetilde{v}_3}{\partial y_3} \\ + \sum_{j=1}^2 (\partial_j \partial_k h)\, \widetilde{v}_j + \sum_{j=1}^2 (\partial_j h) \left[\frac{\partial \widetilde{v}_j}{\partial y_k} - (\partial_k h)\frac{\partial \widetilde{v}_j}{\partial y_3} \right]; \\ \frac{\partial v_3}{\partial x_3} = \frac{\partial \widetilde{v}_3}{\partial y_3} + \sum_{i=1}^2 (\partial_i h)\frac{\partial \widetilde{v}_i}{\partial y_3}, \end{cases} \tag{4.10}$$

where the partial derivatives of $\widetilde{v}$ are calculated at $y = Tx$. We wrote the above expressions just for completeness. Their exact forms are not of interest here. The useful properties are that the coefficients $\partial_j h$ are regular functions (C^2, in our case) and satisfy (4.9). Hence, instead of (4.10), we simply write

$$\frac{\partial v_i}{\partial x_k} = \frac{\partial \widetilde{v}_i}{\partial y_k} + \sum (\partial h)\widetilde{v}_y + \sum (\partial h)^2 \widetilde{v}_y + \sum (\partial^2 h)\widetilde{v}, \tag{4.11}$$

where ∂h could indicate any first-order partial derivative of h. More generally, $(\partial h)^k$ could indicate any product of k first-order partial derivatives of h. Moreover, $(\partial^k h)$ denotes any partial derivative of order k of h. It now seems clear the meaning of symbols like $(\partial h)^m (\partial^k h)^s$.

Our next aim is to write equation (3.10) in terms of the new variables y. By using (4.11), one verifies that

$$\begin{aligned} &\int_\Omega [\nabla(\theta v) + \nabla(\theta v)^T] \cdot [\nabla\phi + \nabla\phi^T] dx \qquad (4.12) \\ &= \int_Q [\nabla(\widetilde{\theta\, v}) + \nabla(\widetilde{\theta\, v})^T] \cdot [\nabla\widetilde{\phi} + \nabla\widetilde{\phi}^T] dy + \sum_{m=1}^4 \int_Q (\partial h)^m (\widetilde{\theta\, v})_y \widetilde{\phi}_y \, dy \\ &+ \sum_{s=0}^2 \int_Q (\partial h)^s (\partial^2 h)(\widetilde{\theta\, v}\widetilde{\phi}_y + (\widetilde{\theta\, v})_y \widetilde{\phi}) dy + \sum \int_Q (\partial^2 h)^2 \widetilde{\theta\, v}\widetilde{\phi}\, dy. \end{aligned}$$

We set $\widetilde{f}_y = \partial_y \widetilde{f}$, and so on. Note that in the transformation formulae (4.12) v is an arbitrary vector field in $\mathbb{H}^1$ (not necessarily a solution). This situation prevails through equation (4.17). In particular, (4.17) holds for $v = w$.

Let us simplify our notation, by avoiding some useless information. Consider, for instance, the four coefficients $(\partial h)^m$ in equation (4.12). The point is that these coefficients are of class C^2 (since we assume that h is of class C^3) and, moreover, they vanish at the origin $(y_1, y_2) = (0, 0)$, due to (4.9). Hence, we will denote by the symbol $H_0' = H_0'(y_1, y_2)$ functions at least of class C^2 and such that $H_0'(0, 0) = 0$, and simply by $H' = H'(y_1, y_2)$ functions at least of class C^2. More precisely, as in the above example, these functions will be products of derivatives of h of order less than or equal to one. This is the reason we insert a prime on the above symbol. Similarly, the coefficients $(\partial h)^s(\partial^2 h)$ will be denoted by H''. This symbol denotes, in general, C^1 functions of $y' = (y_1, y_2)$. The coefficient $(\partial^2 h)^2$ is, as well, an H'' function. As a rule, the primes on the symbols H denote the order of the higher partial derivatives of h that appear in the particular coefficient H. As a last example, a product of an H'' by an H''' coefficient is an H''' coefficient (which is the highest order that will appear in the sequel). Finally, multiplicative constants c are incorporated in this type of coefficient.

With the above notation, equation (4.12) can be written as

$$\begin{aligned}&\int_\Omega [\nabla(\theta v) + \nabla(\theta v)^T] \cdot [\nabla\phi + \nabla\phi^T]dx \\ &= \int_Q [\nabla(\widetilde{\theta v}) + \nabla(\widetilde{\theta v})^T] \cdot [\nabla\widetilde{\phi} + \nabla\widetilde{\phi}^T]dy + R(\widetilde{\theta v}, \widetilde{\phi}),\end{aligned} \tag{4.13}$$

where the bilinear form $R = R(\widetilde{\theta v}, \widetilde{\phi})$ is given by

$$R(\widetilde{\theta v}, \widetilde{\phi}) = \int_Q H_0'(\widetilde{\theta v})_y \widetilde{\phi}_y\, dy + \int_Q H''(\widetilde{\theta v}\widetilde{\phi}_y + (\widetilde{\theta v})_y\widetilde{\phi})\, dy + \int_Q H''\widetilde{\theta v}\widetilde{\phi}\, dy. \tag{4.14}$$

For later use, note that the first term on the right-hand side of (4.14) (i.e., the second one on the right-hand side of (4.12)) has the form

$$\int_Q H_0'(\widetilde{\theta v})_y \widetilde{\phi}_y\, dy = \int_Q \sum_{i,j,k,l=1}^{3} c_{kl}^{ij}(y') \frac{\partial(\widetilde{\theta v})_i}{\partial y_k} \frac{\partial(\widetilde{\phi})_j}{\partial y_l} dy. \tag{4.15}$$

Next, by using (4.7) it follows that

$$\int_\Omega (\nabla \cdot \theta v)(\nabla \cdot \phi)dx = \int_Q (\nabla \cdot \widetilde{\theta v})(\nabla \cdot \widetilde{\phi})dy. \tag{4.16}$$

Consequently, by (4.13), it follows that

$$B(\theta v, \phi) = \widetilde{B}(\widetilde{\theta v}, \widetilde{\phi}) + R(\widetilde{\theta v}, \widetilde{\phi}), \tag{4.17}$$

where $\widetilde{B}$ is defined as B in (2.3), by replacing Ω by Q. According to the above conventions, the coefficient $\nu/2$ was incorporated in the functions of type H. As already noted, equation (4.17) holds with v replaced by w.

Next, by (4.7), one gets

$$\langle \theta p, \nabla \cdot \phi \rangle_\Omega = \langle \widetilde{\theta p}, \nabla \cdot \widetilde{\phi} \rangle_Q. \tag{4.18}$$

On the other hand, by (4.6),

$$\begin{aligned}
&\sum_{i=1}^{3}\int_\Omega F_i\phi_i\,dx \qquad (4.19)\\
&=\int_Q\Big[\widetilde{F}_1\widetilde{\phi}_1+\widetilde{F}_2\widetilde{\phi}_2+\Big(\widetilde{F}_3+\frac{\partial h}{\partial y_1}\widetilde{F}_1+\frac{\partial h}{\partial y_2}\widetilde{F}_2\Big)\Big(\widetilde{\phi}_3+\frac{\partial h}{\partial y_1}\widetilde{\phi}_1+\frac{\partial h}{\partial y_2}\widetilde{\phi}_2\Big)\Big]dy,
\end{aligned}$$

where $F=(F_1,F_2,F_3)$ is defined in (3.11). Hence, with our simplified notation,

$$\langle F,\phi\rangle_\Omega=\int_Q H'\widetilde{F}\widetilde{\phi}\,dy, \qquad (4.20)$$

for some H' functions (a summation of terms of the form $H'\widetilde{F}\widetilde{\phi}$ is understood). Finally, it readily follows that

$$\langle\zeta,\phi\rangle_\Lambda=\int_\Lambda\widetilde{\zeta}\cdot\widetilde{\phi}H'(y_1,y_2)dy_1\,dy_2,\quad \langle\theta v,\phi\rangle_\Lambda=\int_\Lambda\widetilde{\theta\,v}\cdot\widetilde{\phi}H'(y_1,y_2)dy_1\,dy_2, \qquad (4.21)$$

where, in this particular case,

$$H'(y_1,y_2)=\Big(1+\Big(\frac{\partial h}{\partial y_1}\Big)^2+\Big(\frac{\partial h}{\partial y_2}\Big)^2\Big)^{\frac{1}{2}}.$$

The fact that (v,p) is a solution of the original problem was not used to prove the above transformation formulae for integrals. Now we will use these transformation formulae to write (3.10) in terms of the y coordinates. From now on (v,p) is a solution of problem (2.12).

Remark. Clearly, in equation (2.12) there are no restrictions on the size of the supports of the test functions $(\phi,\psi)\in\mathbb{H}^1_\tau\times L^2$. After the multiplication of the solution (v,p) by θ we obtain equation (3.10), where the test functions remain the same as above. Consequently, after the change of variables $x\mapsto y$, there are no smallness assumptions on the supports of the (transformed) test functions $(\widetilde{\phi},\widetilde{\psi})$. In particular, $\widetilde{\psi}$ may be any element in $\mathbb{H}^1(Q)$ such that $\widetilde{\phi}_3=0$ on Λ and Ψ any element in $L^2(Q)$. Actually, these test functions are completely free outside $\overline{Q}_{2\rho}$ since all the integrals vanish outside this set.

Let us turn back to equation (3.10). We start with the ϕ terms. Set $\psi=0$ in equation (3.10). By using (4.17), (4.18), (4.20), and (4.21) one gets

$$\begin{aligned}
&\widetilde{B}(\widetilde{\theta\,v},\widetilde{\phi})-\langle\widetilde{\theta\,p},\nabla\cdot\widetilde{\phi}\rangle+\beta\langle H'\widetilde{\theta\,v},\widetilde{\phi}\rangle_\Lambda \qquad (4.22)\\
&=-\widetilde{B}(\widetilde{\theta\,w},\widetilde{\phi})+\langle H'\widetilde{\zeta},\widetilde{\phi}\rangle_\Lambda-R(\widetilde{\theta\,v},\widetilde{\phi})+R(\widetilde{\theta\,w},\widetilde{\phi})+\langle H'\widetilde{F},\widetilde{\phi}\rangle.
\end{aligned}$$

Next, we write the "ψ" part of equation (3.10) in terms of the y variables. By setting $\phi = 0$ one gets (in Ω)

$$\lambda\langle \theta p, \psi\rangle + \langle \nabla \cdot (\theta v), \psi\rangle = \langle G, \psi\rangle.$$

Due to (4.7) one shows that

$$\lambda\langle \widetilde{\theta\, p}, \widetilde{\psi}\rangle_Q + \langle \nabla \cdot (\widetilde{\theta\, v}), \widetilde{\psi}\rangle_Q = \langle \widetilde{G}, \widetilde{\psi}\rangle_Q. \tag{4.23}$$

From (3.12) it follows that

$$\begin{cases} \|\widetilde{F}\| \le c\,(\|f\| + \|p\| + \|v\|_1 + \|w\|_1) \\ \|\widetilde{G}\|_1 \le c\left(\|g\|_1 + \|v\|_1 + \|a\|_{3/2}\right) \\ \|\widetilde{\zeta}\|_{1/2} \le c\left(\|b\|_{1/2} + \|v\|_1 + \|w\|_1\right). \end{cases} \tag{4.24}$$

Note that the supports of $\widetilde{F}$, $\widetilde{G}$, and $\widetilde{\zeta}$ are contained in $\overline{Q}_{2\rho}$.

By addition of (4.22) and (4.23) one obtains the complete transformation formulae of (2.12), whose meaning, due to the arbitrary nature of $\widetilde{\phi}$ and $\widetilde{\psi}$, is that $(\widetilde{\theta\, v}, \widetilde{\theta\, p})$ is a weak solution in Q (hence in $\mathbb{R}^3_+$) with perturbed data. The solution and the data have compact support in $\overline{Q}_{2\rho}$.

5. The fundamental estimate

For notational convenience, in this section the symbols v, p, w, ϕ, and ψ will be used with the following meaning:

$$v = \widetilde{\theta\, v}; \quad p = \widetilde{\theta\, p}; \quad w = \widetilde{\theta\, w}; \quad \phi = \widetilde{\phi}; \quad \psi = \widetilde{\psi}. \tag{5.1}$$

We set, for arbitrary scalar or vector fields f,

$$\tau_h f(y_1, y_2, y_3) = f(y_1 + h, y_2, y_3)$$

or

$$\tau_h f(y_1, y_2, y_3) = f(y_1, y_2 + h, y_3),$$

where the index j, $j = 1$ or $j = 2$, denotes the direction y_j of the above translation. In the calculations that follow j is assumed to be fixed. We also set

$$f_h = \tau_h f; \qquad \Delta_h f = \frac{f_h - f}{h},$$

where $h \in \mathbb{R}$. Note that translations are in the tangential directions.

We start from equations (4.22) and (4.23). Note that there are no assumptions on the supports of the test functions (ϕ, ψ). In fact, they could be arbitrary elements of $\mathbb{H}^1_\tau(\mathbb{R}^3_+) \times L^2(\mathbb{R}^3_+)$. However, if $|h| < \rho$ (sufficient for our purposes), they have support on $Q \cup \Lambda$. In particular, $\|\phi_h\| \le c\|\phi_y\|$, and similarly for ψ. Just for notational convenience we will assume this property.

We start from equation (4.22). By replacing ϕ by $-\Delta_{-h}\phi$, it follows that

$$\widetilde{B}(\Delta_h v, \phi) - \langle \Delta_h p, \nabla \cdot \phi \rangle + \beta \langle H' v, -\Delta_{-h}\phi \rangle_\Lambda = -\widetilde{B}(\Delta_h w, \phi) \qquad (5.2)$$
$$+ \langle H' \widetilde{\zeta}, -\Delta_{-h}\phi \rangle_\Lambda - R(v, -\Delta_{-h}\phi) + R(w, -\Delta_{-h}\phi) + \langle H' \widetilde{F}, -\Delta_{-h}\phi \rangle.$$

Note that (recall definition (4.14))

$$R(v, -\Delta_{-h}\phi) = \langle \Delta_h (H'_0 v_y), \phi_y \rangle_Q + \langle \Delta_h (H'' v), \phi_y \rangle_Q$$
$$+ \langle \Delta_h (H'' v_y), \phi \rangle_Q + \langle \Delta_h (H'' v), \phi \rangle_Q. \qquad (5.3)$$

Next we estimate the first term on the right-hand side of (5.3). One has (we drop the symbols Q)

$$|\langle \Delta_h (H'_0 v_y), \phi_y \rangle| \leq |\langle H'_0 (\Delta_h v_y), \phi_y \rangle| + |\langle (\Delta_h H'_0) v_y, \phi_y \rangle|. \qquad (5.4)$$

Let $\epsilon > 0$ be given. Since $H'_0(0,0) = 0$, we may fix $\rho > 0$ in such a way that

$$|H'_0(y_1, y_2)| \leq \epsilon \qquad \text{in} \quad Q_{3\rho}. \qquad (5.5)$$

Hence, the first term on the right-hand side of (5.4) is bounded by

$$\epsilon \|\Delta_h v_y\| \| \phi_y\|.$$

On the other hand, it is easily shown that the second term on the right-hand side of (5.4), as well as all the remaining terms on the right-hand side of (5.3), are bounded by $c(\|v\| + \|v_y\|)(\|\phi\| + \|\phi_y\|)$. It follows that

$$|R(v, -\Delta_{-h}\phi)| \leq \epsilon \|\Delta_h v_y\| \|\phi_y\| + c\|v_y\| \|\phi_y\|. \qquad (5.6)$$

Above, we have used the C^3 (or $C^{2,1}$) regularity of Γ, since it gives rise to the regularity of the H-type functions. In this context, the reader may verify that Γ of class $W^{3,3}$ would be sufficient. In this case, the H' functions will be of class $W^{2,3}$, and the H'' functions of class $W^{1,3}$.

Clearly, (5.6) holds by setting $v = w$. However, a much rougher estimate is sufficient here, namely

$$|R(w, -\Delta_{-h}\phi)| \leq c\|w\|_2 \|\phi_y\|. \qquad (5.7)$$

Note that $\|\Delta_h w_y\| \leq \|w\|_2$. Next,

$$|\widetilde{B}(\Delta_h w, \phi)| \leq c\|w\|_2 \|\phi_y\|, \qquad (5.8)$$

and

$$|\langle H' \widetilde{F}, -\Delta_{-h}\phi \rangle| \leq c\|\widetilde{F}\| \|\phi_y\|. \qquad (5.9)$$

Note that $\|\Delta_{-h}\phi\| \leq \|\partial\phi / \partial y_j\| \leq \|\phi_y\|$, where $\phi_y = \nabla \phi$.

On the other hand,

$$|\langle H' \widetilde{\zeta}, -\Delta_{-h}\phi \rangle_\Lambda| \leq c\|H' \widetilde{\zeta}\|_{1/2} \|\phi_{y_j}\|_{-1/2},$$

where the fractional Sobolev norms concern the flat boundary Λ (or, equivalently, the whole plane $\mathbb{R}^2$) and $j = 1$ or 2. Since

$$\|\phi_{y_j}\|_{-1/2} \leq \|\phi\|_{1/2} \leq \|\phi\|_{H^1(Q)},$$

one gets

$$|\langle H'\widetilde{\zeta}, -\Delta_{-h}\phi\rangle_\Lambda| \leq c\|H'\widetilde{\zeta}\|_{1/2}\|\phi_y\|. \tag{5.10}$$

Similarly, one easily shows that

$$|\langle H'v, -\Delta_{-h}\phi\rangle_\Lambda| \leq c\|v_y\|\|\phi_y\|. \tag{5.11}$$

From (5.2), taking into account the equations (5.6), (5.7), (5.8), (5.9), (5.10), and (5.11), one gets

$$|\widetilde{B}(\Delta_h v, \phi) - \langle \Delta_h p, \nabla\cdot\phi\rangle| \leq \epsilon\|\Delta_h v_y\|\|\phi_y\| + c(\|v_y\| + \|w\|_2 + \|\widetilde{F}\| + \|\widetilde{\zeta}\|_{1/2})\|\phi_y\|. \tag{5.12}$$

By setting $\phi = \Delta_h v$ one gets

$$\begin{aligned} &|\widetilde{B}(\Delta_h v, \Delta_h v) - \langle \Delta_h p, \nabla\cdot\Delta_h v\rangle| \\ &\leq \epsilon\|\Delta_h v_y\|^2 + c(\|v_y\| + \|w\|_2 + \|\widetilde{F}\| + \|\widetilde{\zeta}\|_{1/2})\|\Delta_h v\|. \end{aligned} \tag{5.13}$$

Next we use the equation (4.23). Replacing ψ by $-\Delta_{-h}\psi$, one gets, by standard devices,

$$\lambda\langle \Delta_h p, \psi\rangle_Q + \langle \nabla\cdot(\Delta_h v), \psi\rangle_Q = \langle \Delta_h\widetilde{G}, \psi\rangle_Q,$$

and by replacing ψ by $\Delta_h p$

$$\lambda\|\Delta_h p\|^2 + \langle \nabla\cdot(\Delta_h v), \Delta_h p\rangle_Q = \langle \Delta_h\widetilde{G}, \Delta_h p\rangle_Q. \tag{5.14}$$

Finally, from (5.13) and (5.14) it follows that

$$\begin{aligned} \widetilde{B}(\Delta_h v, \Delta_h v) + \lambda\|\Delta_h p\|^2 \leq \epsilon\|\Delta_h v_y\|^2 + c(\|v_y\| + \|w\|_2 + \|\widetilde{F}\| \\ + \|\widetilde{\zeta}\|_{1/2})\|\Delta_h v\| + c(\|\widetilde{G}\|_1 + \|w\|_2)\|\Delta_h p\|. \end{aligned} \tag{5.15}$$

Note that all the functions that appear in equation (5.15) have compact support contained in $\overline{Q}$ (assume $|h| \leq \rho$). Hence norms in Q, in terms of the variable y, are bounded by the corresponding norms in $\Omega_{3\rho}$, in terms of the variable x. Actually (see (5.17) below), some of these last norms will be bounded by the corresponding norms in the whole of Ω.

Remark that

$$|\widetilde{B}(\Delta_h v, \Delta_h v)| \geq \overline{\nu}\|\nabla\Delta_h v\|^2. \tag{5.16}$$

This is shown by applying (2.18) with respect to the variable y (with Ω replaced by Q), by taking into account that $\Delta_h v$ has compact support contained in $\overline{Q}$, and that the boundary integral vanishes since $v_3 = 0$ on Λ and $\underline{n} = (0, 0, 1)$.

For clearness, let us turn back to the original notation (recall (5.1)). By using (5.15) with $\epsilon = \overline{\nu}/2$, (5.14), and (5.16) it follows that

$$\begin{aligned} \|\Delta_h(\widetilde{\theta\, v})_y\|_Q^2 + \lambda\|\Delta_h\widetilde{\theta\, p}\|_Q^2 \leq c(\|v_y\|_\Omega + \|p\|_\Omega + \|w\|_{2,\Omega} + \|f\|_\Omega \\ + \|b\|_{1/2,\Gamma})\|\Delta_h\widetilde{\theta\, v}\| + c(\|g\|_{1,\Omega} + \|a\|_{3/2,\Gamma} + \|w\|_{2,\Omega} + \|v_y\|_\Omega)\|\Delta_h\widetilde{\theta\, p}\|, \end{aligned} \tag{5.17}$$

where we have taken into account the estimates (4.24) and also that $\|(\widetilde{\theta\, v})_y\|_Q \leq c\|v_y\|_\Omega$ and $\|\widetilde{\theta\, w}\|_{2,Q} \leq \|w\|_{2,\Omega}$. Note that $\|w\|_{2,\Omega} \leq c(\|a\|_{3/2,\Gamma} + \|g\|_{1,\Omega})$, as already remarked after equation (3.12). The estimate (5.17) is not sufficient for proving the regularity Theorem 1.2 for weak solutions under the hypotheses (b) in Theorem 1.1. We will need the following device. Turn back to equation (5.2) and assume that $\phi \in C_0^\infty(Q)$. Note that the two boundary integrals vanish. Obviously we obtain, just as above, the equation (5.12) without the term involving $\widetilde{\zeta}$ (we remark that the more accurate estimate (5.5) is not necessary here since we simply replace ϵ by c). Taking into account that

$$|\widetilde{B}(\Delta_h v, \phi)| \leq c\|\Delta_h v_y\|\|\phi_y\|,$$

it follows that

$$|\langle\Delta_h p, \nabla\cdot\phi\rangle| \leq c(\|\Delta_h v_y\| + \|v_y\| + \|w\|_2 + \|\widetilde{F}\|)\|\phi_y\|, \qquad \forall\phi \in C_0^\infty(Q).$$

Hence,

$$\|\nabla\Delta_h p\|_{-1} \leq c(\|\Delta_h v_y\| + \|v_y\| + \|w\|_2 + \|\widetilde{F}\|).$$

Turning back to our more complete notation, and using estimates already shown, we prove that the left-hand side $\|\nabla\Delta_h p\|_{-1}$ of the above equation is bounded by the right-hand side of equation (5.18) below. Equation (5.18) follows by Proposition 1.1. In fact, the mean value of $\Delta_h p$ in Q vanishes (for $|h| \leq \rho$) since

$$\int_Q [\tau_h(\widetilde{\theta\, p}) - (\widetilde{\theta\, p})]dy = 0.$$

Note that translations are in the tangential directions and $\widetilde{\theta\, p}$ has compact support in $\overline{Q}_{2\rho}$. This shows that

$$\|\Delta_h p\| \leq c(\|\Delta_h(\widetilde{\theta\, v})_y\| + \|v_y\|_\Omega + \|w\|_{2,\Omega} + \|f\|_\Omega + \|p\|_\Omega). \tag{5.18}$$

6. Proof of Theorem 1.2

In this section we will use Nirenberg's translation method (see [32]) to prove the H^2 regularity.

From (5.17), by using Cauchy-Schwartz inequality, and by taking into account that the weak solution (u,p) satisfies (1.11), one easily gets

$$\begin{aligned}&\|\Delta_h(\widetilde{\theta\, v})_y\|_Q^2 + \lambda\|\Delta_h\widetilde{\theta\, p}\|_Q^2 \le \\ &c\tfrac{1+\lambda}{\lambda}(\|f\|^2 + \|g\|_1^2 + \|a\|_{3/2}^2 + \|b\|_{1/2}^2) + \tfrac{c}{\lambda^2}(\|g\|^2 + \|a\|_{1/2}^2),\end{aligned} \tag{6.1}$$

where the norms on the right-hand side concern Ω or Γ. This shows that

$$\begin{aligned}&\|D_*^2(\widetilde{\theta\, v})\|_Q^2 + \lambda\|D_*(\widetilde{\theta\, p})\|_Q^2 \le \\ &c(\lambda)(\|f\|^2 + \|g\|_1^2 + \|a\|_{3/2}^2 + \|b\|_{1/2}^2 + \|g\|^2 + \|a\|_{1/2}^2),\end{aligned} \tag{6.2}$$

where, in general, $c(\lambda)$ goes to infinity as λ tends to zero. For convenience, D_* indicates any first derivative with respect to the y variables except for D_{y_3}, and D_*^2 indicates any second derivative with respect to the y variables except for $D_{y_3}^2$.

The estimate (6.2) is sufficient to prove the regularity of the weak solutions described in part (a) of Theorem 1.1. For case (b), in particular when $\lambda = 0$, we have to appeal to Proposition 1.1.

Assume that (v,p) is the weak solution in Theorem 1.2, part (b). From (1.12) it follows, in particular, that

$$\|v_y\|_\Omega + \|p\|_\Omega \le cM, \tag{6.3}$$

where $M = \|f\|_\Omega + \|g\|_{1,\Omega} + \|a\|_{3/2,\Gamma} + \|b\|_{1/2,\Gamma}^2$. From equations (5.17), (5.18), and (6.3), straightforward manipulations yield

$$\|\Delta_h(\widetilde{\theta\, v})_y\|_Q^2 + (1+\lambda)\|\Delta_h\widetilde{\theta\, p}\|_Q^2 \le cM^2. \tag{6.4}$$

By taking into account that the translations τ_h can be done in the two tangential directions y_1 and y_2, it follows that

$$\|D_*^2(\widetilde{\theta\, v})\|_Q + (1+\lambda)\|D_*\widetilde{\theta\, p}\|_Q \le cM. \tag{6.5}$$

Now we turn back to equation (4.22). Note that (as for (2.5))

$$\widetilde{B}(\widetilde{\theta\, v},\widetilde{\phi}) = -\int_Q \Big[\nu\Delta\widetilde{\theta\, v} + \mu\nabla(\nabla\cdot\widetilde{\theta\, v})\Big]\cdot\widetilde{\phi}\,dy, \tag{6.6}$$

for all $\widetilde{\phi} \in C_0^\infty(Q)$. Hence, by using in equation (4.22) test functions $\widetilde{\phi} \in C_0^\infty(Q)$ we show that

$$\begin{aligned}&-\nu\Delta\widetilde{\theta\, v} - \mu\nabla(\nabla\cdot\widetilde{\theta\, v}) + \nabla(\widetilde{\theta\, p}) \\ &= -\widetilde{B}(\widetilde{\theta\, w},\widetilde{\phi}) - R(\widetilde{\theta\, v},\widetilde{\phi}) + R(\widetilde{\theta\, w},\widetilde{\phi}) + \langle H'\widetilde{F},\widetilde{\phi}\rangle, \qquad \forall\widetilde{\phi}\in C_0^\infty(Q).\end{aligned} \tag{6.7}$$

Straightforward calculations together with estimates already shown prove that

$$|\widetilde{B}(\widetilde{\theta\, w},\widetilde{\phi})| \le c\|\widetilde{\theta\, w}\|_2\|\widetilde{\phi}\| \le cM\|\widetilde{\phi}\|$$

and

$$|\langle H'\widetilde{F}, \widetilde{\phi}\rangle| \leq c\|\widetilde{F}\|\|\widetilde{\phi}\| \leq cM|\widetilde{\phi}\|.$$

On the other hand, from (4.14) and integrations by parts, it follows that

$$\begin{aligned} R(\widetilde{\theta v}, \widetilde{\phi}) = &-\langle H_0'\nabla(\widetilde{\theta v})_y, \widetilde{\phi}\rangle - \langle(\nabla H_0')(\widetilde{\theta v})_y, \widetilde{\phi}\rangle \\ &- \langle\nabla(H''\widetilde{\theta v}), \widetilde{\phi}\rangle + \langle H''(\widetilde{\theta v})_y, \widetilde{\phi}\rangle + \langle H''\widetilde{\theta v}, \widetilde{\phi}\rangle, \end{aligned} \tag{6.8}$$

for all $\widetilde{\phi} \in C_0^\infty(Q)$. It is easily shown that the last four terms on the right-hand side of (6.8) are bounded by $c\|(\widetilde{\theta v})_y\|\|\widetilde{\phi}\|$, hence by $cM\|\widetilde{\phi}\|$. By writing (6.8) with v replaced by w, one easily shows that

$$|R(\widetilde{\theta w}, \widetilde{\phi})| \leq c\|w\|_2\|\widetilde{\phi}\| \leq cM\|\widetilde{\phi}\|.$$

From equations (6.7) and (6.8), by taking into account the above estimates, and by writing the first term on the right-hand side of (6.8) in a more detailed form (see (4.15)) one gets (we write the three scalar equations, $j = 1, 2, 3$)

$$\{-\nu\Delta\widetilde{\theta v} - \mu\nabla(\nabla\cdot\widetilde{\theta v}) + \nabla(\widetilde{\theta p})\}_j = \sum_{i,k,l=1}^{3} c_{kl}^{ij}(y')\frac{\partial^2(\widetilde{\theta v})_i}{\partial y_k \partial y_l} + T_j, \tag{6.9}$$

where $T \in L^2(Q)$ and $\|T\| \leq cM$. Since the L^2 norms of the second partial derivatives of $\widetilde{\theta v}$, which are of type $D_*^2(\widetilde{\theta v})$, have already been estimated (see (6.5)), we include all the terms in the above summation for which $(k,l) \neq (3,3)$ in the term T. We do the same with the single terms on the left-hand side of (6.9) that concern second derivatives $D_*^2(\widetilde{\theta v})$, as well as $\nabla_* p$. It follows that (6.9) can be written in the form (we write $c^{ij} = c_{33}^{ij}$)

$$\begin{cases} (\nu + c^{11})D_3^2(\widetilde{\theta v})_1 + c^{21}D_3^2(\widetilde{\theta v})_2 + c^{31}D_3^2(\widetilde{\theta v})_3 = T_1, \\ c^{12}D_3^2(\widetilde{\theta v})_1 + (\nu + c^{22})D_3^2(\widetilde{\theta v})_2 + c^{32}D_3^2(\widetilde{\theta v})_3 = T_2, \\ c^{13}D_3^2(\widetilde{\theta v})_1 + c^{23}D_3^2(\widetilde{\theta v})_2 + (\mu + \nu + c^{33})D_3^2(\widetilde{\theta v})_3 - D_3 p = T_3, \end{cases} \tag{6.10}$$

where $D_3 = \partial/\partial y_3$ and $D_3^2 = D_3 D_3$.

On the other hand, equation (4.23) shows that

$$\lambda\widetilde{\theta p} + D_3(\widetilde{\theta v})_3 = -D_1(\widetilde{\theta v})_1 - D_2(\widetilde{\theta v})_2 + \widetilde{G}.$$

By differentiating with respect to y_3 and by taking into account the second equation (4.24), it follows that

$$D_3^2(\widetilde{\theta v})_3 + \lambda D_3\widetilde{\theta p} = T_4, \tag{6.11}$$

where $T_4 \in L^2(Q)$ satisfies $\|T_4\| \leq cM$.

Finally, we consider the 4×4 linear system in the four unknowns $D_3^2(\widetilde{\theta\, v})_j$, $j = 1, 2, 3$, and $D_3\widetilde{\theta\, p}$, consisting of equations (6.10) and (6.11). Now recall that the coefficients $c_{kl}^{ij}(y')$ are functions of type $H_0'(y')$, i.e., functions of class $C^2(Q)$ that vanish for $y' = 0$. Hence, by fixing a sufficiently small ρ or, equivalently, a function θ with a sufficiently small support U, the absolute value of the functions c_{kl}^{ij} is bounded in Q by an arbitrarily small positive constant δ. By choosing δ sufficiently small (in terms of ν and $\nu+\mu$) one easily shows that the determinant of the above system is larger than $c\overline{\nu}$, for some constant c, uniformly on Q (and independently of λ). For the reader's convenience, we note that the above determinant is given by $(\nu+c)^2-c^2+\lambda[(\nu+c)^2(\nu+\mu+c)+2c^3-c^2(3\nu+\mu+3c)]$, where c represents a generic coefficient $c_{kl}^{ij}(y')$. This shows that the second derivatives $D_3^2(\widetilde{\theta\, v})$, as well as $D_3 p$, belong to $L^2(Q)$ and have L^2 norm bounded by a constant times M. Consequently,

$$\|\widetilde{\theta\, v}\|_2 + (1+\lambda)\|\widetilde{\theta\, p}\|_1 \le cM. \tag{6.12}$$

Hence, the estimate (1.14) holds.

In case (b), we work with the estimate (6.2) instead of (6.5). Arguing as above, we prove that

$$\|\widetilde{\theta\, v}\|_2 + \lambda\|\widetilde{\theta\, p}\|_1 \le c(\lambda)M, \tag{6.13}$$

where $c(\lambda)$ is as in Theorem 1.2 (it is not difficult to obtain a more precise form for $c(\lambda)$).

7. Appendix I

Here we consider the homogeneous system (1.1), (1.5) (i.e., f, g, a, and b vanish) and show that the solutions are just the elements of Z. We start by showing the following result.

Proposition 7.1. *Let $\underline{a}$ and $\underline{l}$ be two given vectors, $\underline{l} \neq 0$, and define*

$$z(x) = \underline{a} + \underline{l} \wedge x. \tag{7.1}$$

Then $z \neq 0$ is a solution to the homogeneous problem (1.1), (1.5) *if and only if $\beta = 0$ and Ω is axially symmetric with respect an axis l parallel to $\underline{l}$.*

Proof. The axis l is constructed as in the proof of Lemma 2.3.

It is immediate to verify that any vector field of the form (7.1) satisfies $\nabla\cdot z = 0$, $\Delta z = 0$, and also

$$\frac{\partial z_i}{\partial x_k} + \frac{\partial z_k}{\partial x_i} = 0. \tag{7.2}$$

Note that, in particular, $B(u,z)=0$, for any vector field u. From (7.2) it follows that $T(z,p)=-pI$, $\underline{t}=-p\underline{n}$, and $\underline{\tau}(z)=0$. Hence, z is a solution to the homogeneous problem (1.1), (1.5) if and only if $\beta z_\tau=0$ and $z\cdot\underline{n}=0$ on Γ, hence if and only if $\beta=0$ and $z\cdot\underline{n}=0$ on Γ. This last condition is equivalent to saying that Ω is axially symmetric with respect to the above axis l. Clearly, if z is as above, (z,p) is a solution to our problem if and only if $p=0$ (p a constant if $\lambda\neq 0$).

Next we prove that if $\beta=0$ and Ω is axially symmetric, then necessarily any solution v of the homogeneous problem belongs to Z.

Theorem 7.1. *Consider the special case and assume that (v,p) is a solution to the homogeneous problem. Then $v\in Z$ and $p=0$ (constant, if $\lambda=0$).*

Proof. From (2.12) one gets

$$\begin{cases} B(v,\phi)-\langle p,\nabla\cdot\phi\rangle=0 & \forall\phi\in\mathbb{H}^1_\tau,\\ \lambda\langle p,\psi\rangle+\langle\nabla\cdot v,\psi\rangle=0, & \forall\psi\in L^2. \end{cases} \tag{7.3}$$

Next decompose v, $v\in\mathbb{H}^1_\tau$, as $v=v_0+z$, where $v_0\in\mathbb{V}^1_\tau$ and $z\in Z$. Since $B(v,v)=B(v_0,v_0)$, as remarked in proving the above proposition, and since also $\nabla\cdot v=\nabla\cdot v_0$, it readily follows from (7.3) that $B(v_0,v_0)+\lambda\|p\|^2=0$. Since $v_0\in\mathbb{V}^1_\tau$, it follows from Lemma 2.3 that $v_0=0$. On the other hand, $p=0$ if $\lambda\neq 0$. If $\lambda=0$, it follows from the first equation (2.12) that $\langle p,\nabla\cdot\phi\rangle=0$; hence, $\nabla p=0$.

8. Appendix II

In this section we prove the Proposition 1.1.

Proposition 8.1. *Let $b=(b_1,b_2,0)\in\mathbb{H}^{1/2}(\mathbb{R}^2)$ denote a generic tangential vector field on the boundary $\mathbb{R}^2=\{x:x_3=0\}$. There is a linear map $b\to w$, continuous from $\mathbb{H}^{1/2}(\mathbb{R}^2)$ into $\mathbb{H}^2(\mathbb{R}^3)$, such that*

$$w_{|\mathbb{R}^2}=0,\quad (\nabla\times w)_{|\mathbb{R}^2}=b. \tag{8.1}$$

Moreover, $w_3=0$ on $\mathbb{R}^3$.

Proof. We use the Fourier transform in $\mathbb{R}^3$

$$(\mathcal{F}f)(\xi)=\widehat{f}(\xi)=\int e^{-2\pi i x\cdot\xi}f(x)dx$$

and the inverse Fourier transform

$$(\mathcal{F}^{-1}f)(x)=\int e^{2\pi i x\cdot\xi}\widehat{f}(\xi)d\xi.$$

Moreover, we denote by F the Fourier transform in $\mathbb{R}^2$ and by F^{-1} its inverse. We set $x' = (x_1, x_2)$ and $\xi' = (\xi_1, \xi_2)$.

Let f be a function defined in $\mathbb{R}^3$ and $\gamma_0 f$ be its trace on $\mathbb{R}^2$. Then, the Fourier transform F of the trace is given by

$$F(\gamma_0 f)(\xi') = \int \widehat{f}(\xi) d\xi_3. \tag{8.2}$$

In fact, one easily shows that the inverse Fourier transform F^{-1} of the right-hand side of (8.2) is just $\mathcal{F}^{-1}(\widehat{f})_{|x_3=0}$. In particular, (8.2) shows that $\gamma_0 f = 0$ on $\mathbb{R}^2$ whenever $\widehat{f}(\xi)$ is odd with respect to ξ_3.

We look for $\widehat{w}$ of the form $(\widehat{w}_1, \widehat{w}_2, 0)$, in order that $w_3 = 0$. From (8.2), and by taking into account that differentiation with respect to x_j is transformed in multiplication by $2\pi i \xi_j$, it follows that the second equation (8.1) becomes

$$-\int \widehat{w}_2 \xi_3 d\xi_3 = \frac{1}{2\pi i} \widehat{b}_1, \quad \int \widehat{w}_1 \xi_3 d\xi_3 = \frac{1}{2\pi i} \widehat{b}_2, \tag{8.3}$$

for functions $\widehat{w}_j(\xi)$, $j = 1, 2$, which are odd with respect to ξ_3. In the sequel we seek such functions.

We immediately see that equations (8.3) are satisfied by functions of the following type:

$$\widehat{w}_1(\xi) = \frac{1}{2\pi i} \xi_3 \mu^3(\xi') \theta(\xi_3 \mu(\xi')) \widehat{b}_2(\xi'),$$
$$\widehat{w}_2(\xi) = -\frac{1}{2\pi i} \xi_3 \mu^3(\xi') \theta(\xi_3 \mu(\xi')) \widehat{b}_1(\xi'),$$

where $\theta(z) = \nu e^{-\frac{z^2}{2}}$, $\forall z \in \mathbb{R}$, and ν is such that

$$\int z^2 \theta(z) dz = 1.$$

Note that θ is an even function, infinitesimal at infinity, of infinite order. Since the $\widehat{w}_j$'s are odd with respect to ξ_3, it follows that the w's have zero trace on $\mathbb{R}^2$.

A trivial choice would be $\mu = 1$. However, this choice does not guarantee that $w \in \mathbb{H}^2(\mathbb{R}^3)$. We set

$$\mu(\xi') = \frac{1}{(1 + |\xi'|^2)^{\frac{1}{2}}}.$$

With this choice one has

$$\int_{\mathbb{R}^3} |\widehat{w}_1(\xi)|^2 (1 + |\xi|^2)^2 d\xi \leq \frac{2}{4\pi^2} \int_{\mathbb{R}^3} [(1 + |\xi'|^2)^2 + \xi_3^4] \mu^6 \xi_3^2 \theta^2(\xi_3 \mu) |\widehat{b}_2(\xi')|^2 d\xi.$$

The above integral is bounded by

$$\frac{1}{2\pi^2}\int_{\mathbb{R}^3}\mu^2\xi_3^2\theta^2(\xi_3\mu)|\widehat{b}_2(\xi')|^2 d\xi + \frac{1}{2\pi^2}\int_{\mathbb{R}^3}\xi_3^6\mu^6\theta^2(\xi_3\mu)|\widehat{b}_2(\xi')|^2 d\xi =$$

$$\frac{1}{2\pi^2}\int_{\mathbb{R}^2}(1+|\xi'|^2)^{\frac{1}{2}}|\widehat{b}_2(\xi')|^2 d\xi' + \frac{1}{2\pi^2}\left(\frac{\nu}{8\sqrt{2}}+\frac{\nu^2}{2}\right)\int_{\mathbb{R}^2}(1+|\xi'|^2)^{\frac{1}{2}}|\widehat{b}_2(\xi')|^2 d\xi',$$

since

$$\int_{\mathbb{R}^2}\xi_3^6\mu^6\theta^2(\xi_3\mu)d(\mu\xi_3) = \frac{\nu^2}{8\sqrt{2}}\int z^6 e^{-\frac{z^2}{2}}dz.$$

Hence,

$$\int_{\mathbb{R}^3}|\widehat{w}_1(\xi)|^2\left(1+|\xi|^2\right)^2 d\xi \le C\int_{\mathbb{R}^2}\left(1+|\xi'|^2\right)^{\frac{1}{2}}|\widehat{b}_2(\xi')|^2 d\xi';$$

i.e.,

$$\|w_1\|^2_{\mathbb{H}^2(\mathbb{R}^3)} \le c\|b_2\|^2_{\mathbb{H}^{\frac{1}{2}}(\mathbb{R}^2)},$$

and similarly for w_2.

Corollary 8.1. *Let K be a compact set in $\mathbb{R}^2$, and let $\mathbb{H}^{\frac{1}{2}}(K)$ denote the linear subspace of $\mathbb{H}^{\frac{1}{2}}(\mathbb{R}^2)$ consisting of tangential vector fields $b=(b_1,b_2)$ with compact support in K. Let U be an open subset of $\mathbb{R}^3$ such that $K\subset U$. There is a linear map $b\to v$, continuous from $\mathbb{H}^{\frac{1}{2}}(K)$ into $\mathbb{H}^1(\mathbb{R}^3)$, such that*

$$\nabla\cdot v = 0, \quad v_{|\mathbb{R}_2} = b, \quad supp\ v \in U. \tag{8.4}$$

Proof. Let w denote the vector field constructed in Theorem 8.1, and let χ be a (fixed) smooth function with compact support in U and equal to 1 on an open subset U_0 of $\mathbb{R}^3$, that contains K. The vector field $v=\nabla\times(\chi w)$ satisfies all the desired properties. Let's prove the second equation of (8.4). Since

$$\nabla\times(\chi w) = \chi(\nabla\times w) + (\nabla\chi)\times w,$$

and since $w_{|\mathbb{R}^2}=0$, it follows that

$$\nabla\times(\chi w) = \chi(\nabla\times w) = \chi b$$

on $\mathbb{R}^2$. By the construction, it immediately follows that $\chi b = b$ on $\mathbb{R}^2$.

Corollary 8.2. *Let Ω be a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class $C^{1,1}$, and let*

$$(\overline{b},g)\in\mathbb{H}^{\frac{1}{2}}(\Gamma)\times L^2(\Omega)$$

be such that

$$\int_\Omega g\,dx = \int_\Gamma \overline{b}\cdot n\,d\Gamma.$$

There is a linear continuous map $(\overline{b}, g) \to \overline{v}$, *from* $\mathbb{H}^{\frac{1}{2}}(\Gamma) \times L^2(\Omega)$ *into* $\mathbb{H}^1(\Omega)$, *such that*

$$\begin{cases} \nabla \cdot \overline{v} = \overline{g}, & in \quad \Omega, \\ \overline{v} = \overline{b}, & on \quad \Gamma. \end{cases} \tag{8.5}$$

Proof. First we consider the solution $\Phi \in H^2(\Omega)$ of the Neumann problem $\Delta\Phi = g$ in Ω, $\nabla\Phi \cdot \underline{n} = \overline{b} \cdot \underline{n}$ on Γ. Choose, for instance, the solution Φ with vanishing mean value in Ω. Next, define b on Γ as $b = \overline{b} - \nabla\Phi$. Note that b belongs to $\mathbb{H}^{\frac{1}{2}}(\Gamma)$ and is tangential to Γ.

Finally, we construct a vector field $v \in \mathbb{H}^1(\Omega)$ such that

$$\begin{cases} \nabla \cdot v = 0, & \text{in } \Omega, \\ v = b, & \text{on} \quad \Gamma, \end{cases} \tag{8.6}$$

and set $\overline{v} = v + \nabla\Phi$.

The construction of v follows easily from Corollary 8.2, by using a suitable partition of unity and the change of coordinates $y = Tx$; see (4.3). Vector fields are transformed according to (4.5) (for notation and details, see Section 4).

Hence, to each $x_0 \in \Gamma$ we associate a sufficiently small neighborhood I_r (see (4.1)) in such a way that $\Gamma \cap I_r$ is described in the Cartesian coordinates introduced in Section 4. Next, we cover Γ by a finite number N of these neighborhoods I_{r_j}, $j = 1, \ldots, N$, and consider a partition of unity θ_j, subordinate to this covering. Let j be fixed (below, we drop the symbol j from the notation).

Using the notation introduced in Section 4, we consider in Λ_r the tangential vector field $\widetilde{\theta b}$ (the transform of the vector field θb by means of (4.5)). Corollary 8.1 shows that there is a vector field $\widetilde{v}$ such that

$$\begin{cases} \nabla\widetilde{v} = 0, & \text{in} \quad I_r, \\ \widetilde{v}_{|\Lambda_r} = \widetilde{\theta b}, & supp\, \widetilde{\theta b} \in I_r. \end{cases}$$

Turning back to the x variables, we obtain from $\widetilde{v}(y)$ a vector field $v(x)$, with compact support in Ω_r, such that $\nabla v = 0$ and $v_{|\Gamma_r} = \theta b$. Recall that, due to (4.7), (4.6) transforms divergence-free vector fields into divergence-free vector fields, and tangent vector fields to Λ_r into tangent vector fields to Γ_r.

By addition of all the above vector fields, as j runs from 1 to N, we get a vector field v that satisfies (8.6). The proof of Corollary 8.2 is accomplished.

We learned the above argument from a manuscript of G. Prodi.

Proof of Proposition 1.1. See also [27]. Let p and $\overline{p}$ be as in Proposition 1.1. Further, endow $\mathbb{H}^1_0(\Omega)$ with the norm $\|\nabla v\|$. Clearly, ∇p is an element of $\mathbb{H}^{-1}(\Omega)$. By using the definition of the strong dual norm, together with $(\nabla p, v) = (\overline{p}, \nabla \cdot v)$, it immediately follows that

$$\|\nabla p\|_1 \geq \frac{|(\overline{p}, \nabla \cdot v)|}{\|\nabla v\|},$$

for each v in $\mathbb{H}^1_0(\Omega)$. By using as v the solution $\overline{v}$ of problem (8.5), where $\overline{g} = \overline{p}$ and $\overline{b} = 0$, (1.15) follows. $\square$

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