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Regularity of solutions to a non homogeneous boundary value problem for general stokes systems in R^n_+

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Abstract. We give a simple and very complete proof of the existence of a strong (H^2) solution to the non-homogeneous problem (1.1) under the non homogeneous boundary conditions (1.6). Here we consider the half-space case $\Omega = \mathbf{R}^n_+,\ n \geq 3,$ see theorem 1.2. This regularity result was previously obtained by Solonnikov and Ščadilov in reference [33] for the classical Stokes system ($\mu = \lambda = 0,\ g(x) = 0$) in the $3 - D$ homogenous case ($a = 0,\ b = 0$) and Ω a suitable open subset of $\mathbf{R}^3$.

1. Introduction and main results

We are interested on studying systems of Stokes type

$$\begin{cases} -\nu\Delta u - \mu\nabla[\nabla\cdot u] + \nabla p = f(x), \\ \lambda p + \nabla\cdot u = g(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

under the non homogeneous slip boundary condition (1.6). Here Ω is an open set in $\mathbf{R}^n$, Γ denotes its boundary, and $\underline{n}$ the unit external normal to Γ. The constants ν, μ and λ satisfy the assumptions $\nu > 0$, $\lambda \geq 0$ and $\mu + \nu > 0$. When $\mu = \lambda = 0$ and $g(x) = 0$ we obtain the classical Stokes system. System (1.1) may also be used in the study of analogous problems for the Navier-Stokes equations.

We remark that the assumption $\mu \neq 0$ can be easily reduced to $\mu = 0$ by using the second equation (1.1) in order to substitute $\nabla\cdot u$ in the first equation (in this way we see that assumptions on μ are, in fact, not necessary). However, the calculations made under the assumption $\mu \neq 0$ will be useful in studying some problems related to compressible fluids.

On the other hand, the introduction of the parameter λ allows a straightforward extension of the proofs below to the case of bounded domains. In fact, replacing the constraint $\nabla\cdot u = 0$ by $\lambda p + \nabla\cdot u = 0$ allows us to localize the equations (flatten the boundary and prove regularity) in a much simple way than the usual ones. Then, the lack of dependence on λ (a crucial point here) yields the extension

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to the limit case $\lambda = 0$. For other remarks on the role played by the parameter λ, in particular in numerical approximation, we refer the reader to the first Remark in section 1 of reference [5].

In the sequel we denote by

$$T = -p\, I + \nu(\nabla u + \nabla u^T)$$

the stress tensor, and set $\underline{t} = T \cdot \underline{n}$. Hence, with an obvious notation (see [33])

$$T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \tag{1.2}$$

$$t_i = \sum_{k=1}^{n} T_{ik} n_k. \tag{1.3}$$

We also define the linear operator $\underline{\tau}$,

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{1.4}$$

Hence

$$\tau_i(u) = \nu \sum_{k=1}^{n} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k - 2\nu \left[\sum_{k,l=1}^{n} \frac{\partial u_l}{\partial x_k} n_k n_l \right] n_i. \tag{1.5}$$

Note that $\underline{\tau}(u)$ is tangential to the boundary and independent of the pressure p.

In the sequel we consider the slip boundary condition

$$\begin{cases} (u \cdot n)_{|\Gamma} = a(x), \\ \underline{\tau}(u)_{|\Gamma} = b(x), \end{cases} \tag{1.6}$$

where $a(x)$ and $b(x)$ are, respectively, a given scalar field and a given tangential vector field on Γ. This boundary condition is an appropriate model for many important flow problems. Besides the pioneering, main contribution, of Solonnikov and Ščadilov, see [33], this boundary condition has been considered by many authors. See, for instance, [21] and [36], and references therein. For some strongly related boundary conditions see, for instance,[3], [9], [14], [27], [17] and also (free boundaries) [7], [24], [28], [29] and [32]. In reference [18] the authors consider the regularity problem for the Neumann boundary value problem. Concerning the Dirichlet boundary value problem, the bibliography is well known and particularly extensive (see, for instance, [10], [13], [19], [23], [35] and references). Some “nonstandard” boundary value problems are considered in reference [4].

Our main interest is the basic L^2-regularity result, i.e. if $f \in L^2(\Omega)$, $g \in H^1(\Omega)$, $a \in H^{3/2}(\Omega)$, and $b \in H^{1/2}(\Gamma)$ then $u \in H^2(\Omega)$ and $p \in H^1(\Omega)$. From this result we may easily get H^k regularity results, $k > 2$. The existence of weak solutions as well as the justification for their definition will also be studied in detail. The formal calculations that lead to the definition of “weak solution” will be presented for a general open set Ω, for future developments. Here we will study the half-space case. Hence $\Omega = \mathbf{R}^n_+ = \{x : x_n > 0\}$, $\Gamma = \mathbf{R}^{n-1}$. We set $x = (x', x_n)$, $x' = (x_1, ..., x_{n-1})$. The only inconvenience in considering the half space case is that coerciveness holds with respect to the norm $\|\nabla u\|$ and not with respect to $\|\nabla u\| + \|u\|$. Hence, instead of the canonical Sobolev space $H^1(\mathbf{R}^n_+)$ we are led to use $D^1(\mathbf{R}^n_+)$ spaces, defined here as the completion of $C_0^\infty(\overline{\mathbf{R}^n_+})$ with respect to the norm $\|\nabla u\|$.

As a rule, canonical norms in Sobolev spaces H^s, $s \in \mathbf{R}$, are denoted by $\|\cdot\|_s$ and norms in D^s spaces by $[\,\cdot\,]_s$. The symbol $\|\cdot\|$ denotes an L^2 norm.

Our main results are the following (for notation see the opening of section 3).

Existence of a weak solution (see also [33]).

Theorem 1.1. *Assume that f, g, a and b are as in* (3.4). *Then the problem* (1.1), (1.6) *has a unique weak solution (u, p), i.e., a solution that belongs to $[D^1]^n \times L^2$. Moreover*

$$\overline{\nu}^2\|\nabla u\|^2 + \|p\|^2 \le c\left\{\left(1 + \frac{\overline{\nu}}{\nu + |\mu|}\right)[f]_{-1} + (\nu + |\mu|)\|g\| + (1 + \nu + |\mu|)\|a\|_{1/2} + \nu[b]_{-1/2}\right\}^2. \qquad (1.7)$$

For the proof see the section 3.

Existence of a strong solution (regularity theorem).

Theorem 1.2. *Assume that f, g, a and b satisfy* (3.4) *and* (4.1). *Then the above weak solution (u, p) of problem* (1.1), (1.6) *is strong, i.e. it belongs to $[\widetilde{H}^2]^n \times H^1$. Moreover*

$$\nu^2\,(\|\nabla^2 u\|^2 + \|\nabla u\|^2) + \|p\|_1^2 \le \overline{c}\,(\|f\| + [f]_{-1} + \nu\,(\|g\|_1 + \|a\|_{1/2} + \|a\|_{3/2} + \|b\|_{1/2} + [b]_{-1/2}))^2. \qquad (1.8)$$

For the proof see section 4. We set $\widetilde{H}^2 = \left\{g \in D^1 : \nabla^2 g \in L^2\right\}$. Note that, in the case of a bounded domain, one has $\widetilde{H}^2 = H^2$. Hence, in a bounded domain (see also [33])

$$\nu^2\,\|u\|_2^2 + \|p\|_1^2 \le \overline{c}\left(\|f\| + \nu\,(\|g\|_1 + \|a\|_{3/2} + \|b\|_{1/2})\right)^2. \qquad (1.9)$$

A similar result, under slip boundary conditions (with, or without, linear friction), is proved in [5].

The existence of weak and strong solutions to problem (1.1), (1.6) was first stated by Solonnikov and Ščadilov, see [33], in the case of a generic open subset Ω, if $n = 3$, $\lambda = \mu = 0$, $g = 0$ and (to prove the H^2-regularity result) $a = 0$, $b = 0$.

On studying the existence of weak solutions, the current literature on Stokes and Navier-Stokes systems (even for the homogeneous Dirichlet boundary value problem; see, for instance, the well know references [10],[23],[35]), is often based on some "special" results as, for instance, the construction of particular divergence free vector fields, suitable decompositions of functional spaces, theorems of De Rham's type, and so on. In this regard we note that the only "extra tool" used in the sequel is the following well known result (note that we assume $p \in L^2(\mathbf{R}^n_+)$. This allows a simplified proof).

Proposition 1.1. *Let $p \in L^2(\mathbf{R}^n_+)$ be a scalar field in $\mathbf{R}^n_+$. Then*

$$\|p\| \leq C\|\nabla p\|_{-1}. \tag{1.10}$$

We use this result to prove the existence of the strong solution and, in the case of weak solutions, to obtain an L^2 estimate for the pressure. In the classical reference [33], the main auxiliary tools used in proving the existence of weak and strong solutions are different from ours. See lemmas 2 and 3 and problems (10) and (11), in [33]. Also the the L^2 estimate for the pressure is obtained in a different way (see the end of section 3 in the above reference. It is worth noting that, if $\lambda > 0$ as well as if $\lambda = 0$, $g = 0$, $a = 0$, the existence of the weak solution (without $\|p\|^2$ on the left hand side of (1.7)) can be proved without resorting to Proposition 1.1. See Appendix II. We do not use potential theoretical results. For this approach, and applications to Stokes and Navier-Stokes equations, we refer to [1],[8], [12],[13], [15],[19],[30], [31].

This paper is organized as follows. In Section 2 we present some formal calculations that, in fact, justify the definition of a weak solution. This is done for a generic open set Ω. In Section 3 we introduce the functional framework and we prove the Theorem 1.1. In Section 4 we prove the regularity Theorem 1.2.

2. Some formal calculations

We start by showing the particular form of some of the operators and equations, introduced in the previous section, when $\Omega = \mathbf{R}^n_+$. In this case $n_i = \delta_{in}$ and

$$\tau_i(u) = \nu\left(\frac{\partial u_i}{\partial x_n} + \frac{\partial u_n}{\partial x_i}\right) - 2\nu\frac{\partial u_n}{\partial x_n}\delta_{in}.$$

Hence

$$\begin{cases} \tau_j(u) = \nu\left(\dfrac{\partial u_j}{\partial x_n} + \dfrac{\partial u_n}{\partial x_j}\right), \quad 1 \leq j \leq n-1, \\ \tau_n(u) = 0. \end{cases} \tag{2.1}$$

Moreover, if $a(x) \equiv 0$, the condition (1.6) reads

$$\begin{cases} u_n = 0, \\ \nu\dfrac{\partial u_j}{\partial x_n} = b_j(x), \quad 1 \leq j \leq n-1. \end{cases} \tag{2.2}$$

Now let ϕ be any vector field in Ω such that

$$(\phi \cdot \underline{n})_{|\Gamma} = 0. \tag{2.3}$$

Remark. In the following, vector fields denoted by ϕ are always assumed to verify (2.3).

From (1.5) it follows that

$$\underline{\tau}(u) \cdot \phi = \nu \sum_{i,k=1}^{n} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k \phi_i \tag{2.4}$$

and, if $\Omega = \mathbf{R}^n_+$,

$$\underline{\tau}(u) \cdot \phi = \nu \sum_{j=1}^{n-1} \left(\frac{\partial u_j}{\partial x_n} + \frac{\partial u_n}{\partial x_j} \right) \phi_j, \tag{2.5}$$

since $\phi_n = 0$.

Next we define the bilinear form

$$B(u, \phi) := \int_\Omega \left[\frac{\nu}{2} (\nabla u + \nabla u^T) \cdot (\nabla \phi + \nabla \phi^T) + (\mu - \nu)(\nabla \cdot u)(\nabla \cdot \phi) \right] dx. \tag{2.6}$$

By integrations by parts, and by taking (2.4) into account, one easily shows that

$$B(u, \phi) = - \int_\Omega [\nu \Delta u + \mu \nabla (\nabla \cdot u)] \cdot \phi \, dx + \int_\Gamma \underline{\tau}(u) \cdot \phi \, d\Gamma. \tag{2.7}$$

It readily follows that a (sufficiently regular) couple (u, p) is a solution of (1.1), $(1.6)_2$ if and only if

$$B(u, \phi) - (p, \nabla \cdot \phi) = (f, \phi) + (b, \phi)_\Gamma, \quad \forall \phi, \tag{2.8}$$

ϕ satisfying (2.3).

Until the functional framework is stated in a precise way (see section 3) the duality pairings $(\cdot, \cdot)$ and $(\cdot, \cdot)_\Gamma$ should be seen as integrals over Ω and Γ respectively.

Next we introduce the constraint $(1.6)_1$. It will be convenient to reduce this non homogeneous boundary condition to the homogeneous one

$$(v \cdot \underline{n})_{|\Gamma} = 0. \tag{2.9}$$

To accomplish this we consider a vector field w such that

$$(w \cdot \underline{n})_{|\Gamma} = a(x), \tag{2.10}$$

and set

$$u = w + v, \tag{2.11}$$

where the new unknown v is subject to the constraint (2.9).

Equation (2.8) becomes

$$B(v, \phi) - (p, \nabla \cdot \phi) = -B(w, \phi) + (f, \phi) + (b, \phi)_\Gamma, \quad \forall \phi, \tag{2.12}$$

ϕ satisfying (2.3). Hence (u, p) is a solution of $(1.1)_1$, (1.6) if and only if (v, p) is a solution of (2.12). It readily follows that (u, p) is a solution to the complete problem (1.1), (1.6) if and only if (v, p) is a solution of

$$\begin{aligned} &B(v, \phi) - (p, \nabla \cdot \phi) + \lambda(p, \psi) + (\nabla \cdot v, \psi) \\ &\quad = -B(w, \phi) + (f, \phi) + (b, \phi)_\Gamma + (g, \psi) - (\nabla \cdot w, \psi), \end{aligned} \tag{2.13}$$

for each ϕ satisfying (2.3) and each scalar field ψ. In the sequel we write (2.13) in the abbreviate form

$$a_\lambda(V, \Phi) =< L, \Phi >, \quad \forall \Phi, \tag{2.14}$$

where, by definition,

$$\begin{aligned} a_\lambda(V, \Phi) &= B(v, \phi) - (p, \nabla \cdot \phi) + \lambda(p, \psi) + (\nabla \cdot v, \psi), \\ < L, \Phi > &= -B(w, \phi) + (f, \phi) + (b, \phi)_\Gamma + (g, \psi) - (\nabla \cdot w, \psi), \end{aligned}$$

and

$$V = (v, p), \ \Phi = (\phi, \psi).$$

Clearly a_λ is a bilinear form and L is a linear form. Recall that the test functions ϕ satisfy (2.3) and the solution v should verify (2.9). The above argument shows that (2.11), (2.14) is a natural weak formulation of problem (1.1), (1.6). In the next section we fix the functional framework.

3. Existence of the weak solution

In general, in notation concerning duality pairings and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domains $\mathbf{R}^n_+$ or $\mathbf{R}^{n-1} \equiv \Gamma$, provided that the meaning remains clear. For instance, the symbol $\| \, . \, \|$ denotes an L^2-norm, either in $\mathbf{R}^n_+$ and in $\mathbf{R}^{n-1}$.

If X is a Banach space we denote by X' its (strong) dual space. We will not define some canonical notation as, for instance, the functional spaces $L^2(\mathbf{R}^n_+)$, $H^k(\mathbf{R}^n_+)$, $H^1_0(\mathbf{R}^n_+)$, $H^{-1}(\mathbf{R}^n_+)$, $H^{1/2}(\mathbf{R}^{n-1})$ and the corresponding norms $\| \, . \, \|$, $\| \, . \, \|_k$, $\| \, . \, \|_1$, $\| \, . \, \|_{-1}$, $\| \, . \, \|_{1/2,\Gamma}$. By convention, “integer norms”, as well as “integer

Sobolev spaces", always relate to $\mathbf{R}^n_+$, and "fractional norms" always concern the boundary $\Gamma = \mathbf{R}^{n-1}$. For instance, $\| \, . \, \|_{1/2} = \| \, . \, \|_{1/2,\Gamma}$, and $H^{1/2} = H^{1/2}(\mathbf{R}^{n-1})$. For the study of fractional Sobolev spaces see, for instance, [25].

We define $D^1 := D^{1,2}(\mathbf{R}^n_+)$ as the completion of $C_0^\infty(\overline{\mathbf{R}^n_+})$ (or $C^k(\overline{\mathbf{R}^n_+})$, $k \geq 1$) with respect to the norm $\|\nabla v\|$. Moreover, D_0^1 is the completion of $C_0^\infty(\mathbf{R}^n_+)$ with respect to $\|\nabla v\|$. It is well-known (by Sobolev embedding theorems) that ($n > 2$)

$$D^1 = \left\{ v : \; v \in L^r, \; \nabla v \in L^2 \right\}, \tag{3.1}$$

where $1/r = 1/2 - 1/n$. In particular, the norms $\|\nabla v\|$ and $\|\nabla v\| + \|v\|_{L^r}$ are equivalent in D^1 and in D_0^1. This can be shown by extending $C^k(\overline{\mathbf{R}^n_+})$ to $C^k(\mathbf{R}^n)$ (by the well known reflection method) and then by applying the corresponding result in the whole space (see [22]). See also [16], Theorems I.2 and I.4, and Remark 1 on page 234.

Clearly, the usual Sobolev spaces H_0^1 and H^1 are dense and strictly contained in D_0^1 and D^1, respectively. In particular, it follows that $L^{r'} \hookrightarrow (D^1)' \hookrightarrow (H^1)'$ and $L^{r'} \hookrightarrow (D_0^1)' \hookrightarrow H^{-1}$, where $r' = r/(r-1)$.

Since restrictions to bounded sets of functions in D^1 belong to Sobolev spaces $W^{1,2}$, it follows that their trace on the boundary $\mathbf{R}^{n-1}$ is well defined as an element of $W^{\frac{1}{2}}_{loc}(\mathbf{R}^{n-1})$. Obviously, functions in D_0^1 have vanishing trace on $\mathbf{R}^{n-1}$. Trace spaces in $\mathbf{R}^{n-1}$ may be studied, in a convenient way, by resorting to the Fourier transform. However, for convenience, we also apply to reference [16]. The trace space of D^1 is denoted here by $D^{1/2} = D^{1/2}(\mathbf{R}^{n-1})$,. Actually, it is the completion of $C_0^\infty(\mathbf{R}^{n-1})$ with respect to the norm induced in this space by the norm $\|\nabla v\|$ in $C_0^\infty(\overline{\mathbf{R}^n_+})$. Hence, it consists of functions (distributions) ψ that have "half derivative" in $L^2(\mathbf{R}^{n-1})$ (in the usual Fourier-transform sense) and that belong to $L^s(\mathbf{R}^{n-1})$, where s is given by the Sobolev embedding exponent

$$\frac{1}{s} = \frac{1}{2} - \frac{1/2}{n-1}. \tag{3.2}$$

See [16], Theorem II.3 and Definition II.1.

We set $D^{-1/2} = (D^{1/2})'$. Norms in $D^{1/2}$ and $D^{-1/2}$ are denoted respectively by $[\, . \,]_{1/2}$ and $[\, . \,]_{-1/2}$. Note that, by (3.2), one has $L^{s'} \hookrightarrow D^{-1/2}$ where $s' = 2(n-1)/n$.

It is worth noting that our main interest here is the local regularity up to the boundary. This leads us to avoid the use of heavier functional frameworks specially related to behavior at infinity. This extension can be easily done by the interested reader. For this kind of functional framework (without any claim of completeness) we refer to [16], [2], [13], and bibliography.

For vector fields we set

$$\begin{cases} \widetilde{\mathbb{H}}^1 = [D^1]^{n-1} \times D_0^1, \\ \widetilde{\mathbb{H}}^{-1} = (\widetilde{\mathbb{H}}^1)' \\ \widetilde{\mathbb{H}}^{1/2} = [D^{1/2}]^{n-1}, \\ \widetilde{\mathbb{H}}^{-1/2} = (\widetilde{\mathbb{H}}^{1/2})'. \end{cases} \tag{3.3}$$

Obviously, $\widetilde{\mathbb{H}}^{-1} = [(D^1)']^{n-1} \times (D_0^{-1})'$ and $\widetilde{\mathbb{H}}^{-1/2} = [D^{-1/2}]^{n-1}$.

In the sequel f, g, a, and b are given as follows:

$$f \in \widetilde{\mathbb{H}}^{-1},\ g \in L^2,\ a \in H^{1/2},\ b \in \widetilde{\mathbb{H}}^{-1/2}. \tag{3.4}$$

According to above remarks, we may the above assumptions on f and b simply by

$$\begin{cases} f \in [L^{r'}(\mathbf{R}_+^n)]^n, \\ b \in [L^{s'}(\mathbf{R}^{n-1})]^{n-1}, \end{cases} \tag{3.5}$$

since this yields $f \in \widetilde{\mathbb{H}}^{-1},\ b \in \widetilde{\mathbb{H}}^{-1/2}$.

In connection to (2.10), we fix a linear continuous map $a \mapsto w_n$ from $H^{1/2}$ into $H^1(\mathbf{R}_+^n)$ such that the first equation (3.6) holds and $\|w_n\|_1 \le c\|a\|_{1/2}$. In particular, $a \mapsto w = (0, ..., 0, w_n)$ defines a linear continuous map from $H^{1/2}$ into $[H^1(\mathbf{R}_+^n)]^n \hookrightarrow [D^1(\mathbf{R}_+^n)]^n$ such that

$$\begin{cases} w_n = a(x) \quad \text{on } \Gamma, \\ \|\nabla w\| \le c[a]_{1/2}. \end{cases} \tag{3.6}$$

Definition. *Assume that* (3.4) *holds and let* w *satisfy* (3.6). *We say that a pair* (u, p), *belonging to* $[D^1]^n \times L^2$, *is a* <u>*weak solution*</u> *to problem* (1.1), (1.6) *if* $u = w + v$, *where* (v, p) *belongs to* $\widetilde{\mathbb{H}}^1 \times L^2$ *and satisfies* (2.14) *for each* $\Phi \in \widetilde{\mathbb{H}}^1 \times L^2$.

In the sequel we set $\overline{\nu} = \nu$ if $\mu \ge 0$ and set $\overline{\nu} = \nu + \mu$ if $-\nu < \mu < 0$.

Proof of Theorem 1.1. The proof consists in showing that the bilinear form a_λ is continuous and coercive and the linear functional L is continuous. Continuity is trivially verified. Let us show coerciveness. Assume that v is regular. By integration by parts one has

$$\frac{1}{2}(\nabla v + \nabla v^T, \nabla v + \nabla v^T) = \|\nabla v\|^2 + \|\nabla \cdot v\|^2 \tag{3.7}$$

since, on $\Gamma = \mathbf{R}^{n-1}$,

$$\sum_{i,k=1}^{n} \frac{\partial v_k}{\partial x_i} v_i n_k = \sum_{j=1}^{n-1} \frac{\partial v_n}{\partial x_j} v_j = 0.$$

Equation (3.7) holds for any $v \in \widetilde{\mathbb{H}}^1$, as follows by approximation of v by regular vector fields. Hence

$$a_\lambda(V, V) \geq \nu\|\nabla v\|^2 + \mu\|\nabla \cdot v\|^2 + \lambda\|p\|^2$$
$$\geq \overline{\nu}\|\nabla v\|^2 + \lambda\|p\|^2. \tag{3.8}$$

This shows the coerciveness of the bilinear form a_λ over $\widetilde{\mathbb{H}}^1 \times L^2$, if $\lambda > 0$. Hence, if $\lambda > 0$, the problem (1.1), (1.6) has a unique weak solution (u, p). It readily follows that these solutions satisfy the estimate (6.1). However we are interested in proving the stronger estimate (1.7), for each $\lambda \geq 0$. This estimate for $u = v + w$ is equivalent to the estimate (3.14) for v. Since these estimates do not depend on λ, they readily follow also for $\lambda = 0$. Let us start by proving the estimate (3.14) for an arbitrary $\lambda > 0$.

From definition (2.6) and (3.6) one gets $|B(w, v)| \leq c\,(\nu + |\mu|)\|a\|_{1/2}\,\|\nabla v\|$. Hence, by the definition of L it follows that

$$|< L, V >| \leq c[(\nu + |\mu|)\|a\|_{1/2} + [f]_{-1} + [b]_{-1/2}]\,\|\nabla v\|$$
$$+c\,\|a\|_{1/2}\|p\| + \|g\|\,\|p\|. \tag{3.9}$$

On the other hand, from (2.14) (i.e.,(2.13)) with $\Psi = 0$ and $\phi \in C_0^\infty(\mathbf{R}^n_+)$ it follows that

$$(p, \nabla\phi) = B(u, \phi) - (f, \phi), \tag{3.10}$$

hence, by (2.7) (see the first remark below),

$$\nabla p = f + \nu\Delta v + \mu\nabla[\nabla \cdot v] \tag{3.11}$$

as elements of $[H^{-1}]^n$. In particular $\nabla p \in [H^{-1}]^n$ (in fact this follows directly from $p \in L^2$. See the second remark below), moreover

$$\|\nabla p\|_{-1} \leq [f]_{-1} + (\nu + |\mu|)\|\nabla v\|. \tag{3.12}$$

Consequently, by Proposition 1.1,

$$\|p\| \leq c[\,[f]_{-1} + (\nu + |\mu|)\|\nabla v\|\,]. \tag{3.13}$$

By using (3.8) (the nonnegative term $\lambda\|p\|^2$ will be not taken into account here since it is not uniform on λ) together to (3.9) and (3.13), it readily follows that $\overline{\nu}^2\|\nabla v\|^2$ is bounded by the right hand side of the inequality

$$\overline{\nu}^2\|\nabla v\|^2 + \overline{\nu}(\nu + |\mu|)^{-2}\|p\|^2 \leq C\left\{\left[1 + \frac{\overline{\nu}}{\nu + |\mu|}\right][f]_{-1}\right.$$
$$\left. +(\nu + |\mu|)(\|g\| + \|a\|_{1/2}) + [b]_{-1/2}\right\}^2, \tag{3.14}$$

where (at this point) the coefficient $\overline{\nu}/(\nu + |\mu|)$ is superfluous. Finally, by using (3.13), we get the complete estimate (3.14).

Since the estimate (3.14) is independent of λ, it follows, with obvious notation, that the solutions $V_\lambda = (v_\lambda, p_\lambda)$, $\lambda > 0$, of problem (2.14) converge weakly in $\widetilde{\mathbb{H}}^1 \times L^2$ to some $V = (v, p) \in \widetilde{\mathbb{H}}^1 \times L^2$, as $\lambda \to 0$. Obviously (v, p) satisfies (3.14). By passing to the limit in equation (2.14) as $\lambda \to 0$, it follows that

$$a_0(V, \Phi) = < L, \Phi >, \quad \forall \Phi.$$

This proves the Theorem 1.1 also in the case $\lambda = 0$. Finally (1.7) follows from (3.14) and (3.6), since $u = w + v$. □

Remark. Note that, in Equation (3.10), f acts on ϕ, by restriction to $[H_0^1(\mathbf{R}^n_+)]^n$, a closed subspace of $[D^1(\mathbf{R}^n_+)]^n$. The norm of f, as an element of $[H^{-1}]^n$, is bounded by $[f]_{-1}$ (alternatively, we could assume that $f \in [H^{-1}]^n$).

Remark. It is worth noting that, from the very beginning of the proof of theorem 1.1, it is already well known that $p \in L^2$ (since $\lambda > 0$). This shows that we need the result established in Proposition 1.1 only in the particular case in which p is a priori given in L^2 (instead of H^{-1}). This is another favorable aspect of the general formulation followed here, since under the above hypothesis on p the proof of Proposition 1.1 could be substantially simplified.

4. Proof of Theorem 1.2

In the sequel $\lambda \geq 0$ is fixed. Hence we denote a_λ simply by a. Here we assume that (3.4) holds and, moreover, that

$$\begin{cases} f \in [L^2(\mathbf{R}^n_+)]^n, \\ \nabla g \in [L^2(\mathbf{R}^n_+)]^n, \\ a \in H^{3/2}(\mathbf{R}^{n-1}), \\ b \in [\widetilde{H}^{1/2}(\mathbf{R}^{n-1})]^{n-1}. \end{cases} \tag{4.1}$$

In a first step, for technical reasons due to the fact that the domain $\mathbf{R}^n_+$ is unbounded, we assume (in addition to (4.1)) that (3.5) holds.

In addition to (3.6), the linear continuous map $a \mapsto w$ is now fixed in such a way that

$$\|\nabla^2 w\| \leq c\|a\|_{3/2}. \tag{4.2}$$

Next we apply the translation method (see[26]). Let j, $1 \leq j \leq n-1$, be fixed, let $h \neq 0$, and define the translation operator $\tau_h z(x) = z(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n)$. In equation (2.14) set $\phi = \tau_{-h}\phi$, $\psi = \tau_{-h}\psi$ and denote the equation obtained in that way by $(2.14)_{-h}$. Take the difference between the equations $(2.14)_{-h}$ and (2.14). For convenience, denote $\tau_h z$ simply by z_h. Due to the equality $< v, z_{-h} > = < v_h, z >$ it follows that

$$a\left(\frac{V_h - V}{h}, \Phi\right) = \left\langle \frac{L_h - L}{h}, \Phi \right\rangle, \qquad \forall \Phi \in \widetilde{\mathbb{H}}^1 \times L^2. \tag{4.3}$$

Hence, by (3.14)

$$\overline{\nu}^2\|\nabla(v_h - v)/h\|^2 + \frac{\overline{\nu}^2}{(\nu+|\mu|)^2}\|(p_h - p)/h\|^2$$
$$\leq c\Bigg[(\nu+|\mu|)(\|\partial g/\partial x_j\| + \|\partial a/\partial x_j\|_{1/2}) +$$
$$+(1+\frac{\overline{\nu}}{\nu+|\mu|})\|f\| + [b]_{1/2}\Bigg]^2, \tag{4.4}$$

since $\|(g_h - g)/h\| \leq \|\partial g/\partial x_j\|$, $\|(a_h - a)/h\|_{1/2} \leq \|\partial a/\partial x_j\|_{1/2}$, $[(f_h - f)/h]_{-1} \leq \|f\|$, and $[(b_h - b)/h]_{-1/2} \leq c[b]_{1/2}$. For the readers convenience, the last two inequalities will be proved in Appendix I.

From (4.4) it follows that the estimate

$$\overline{\nu}^2\|\nabla_*^2 u\|^2 + \frac{\overline{\nu}^2}{(\nu+|\mu|)^2}\|\nabla_* p\|^2$$
$$\leq c\Bigg[(1+\nu+|\mu|)(\|\nabla g\| + \|a\|_{3/2}) + (1+\frac{\overline{\nu}}{\nu+|\mu|})\|f\| + [b]_{1/2}\Bigg]^2 \tag{4.5}$$

holds for $\nabla_*^2 v$ (i.e., holds if on the left hand side we replace $\nabla_*^2 u$ by $\nabla_*^2 v$). Here ∇_*^2 denotes second order derivatives except for $\partial^2/\partial x_n^2$ and ∇_* denotes first order derivatives except for $\partial/\partial x_n$.

By taking now into account that $u = w + v$ and (4.2), (4.5) follows.

By (2.14) or by (2.13), written with $\psi = 0$ and $\phi \in C_0^\infty$ and also with $\psi \in C_0^\infty$ and $\phi = 0$, and by taking into account that $u = v + w$, it follows that equation (1.1) holds, for instance, in the sense of distributions.

Next we consider the linear 2×2 system consisting of the n^{th} scalar equation in (1.1) together with the equation obtained by differentiation of the $(n+1)^{st}$ scalar equation (1.1) with respect to x_n

$$-(\nu+\mu)\frac{\partial^2 u_n}{\partial x_n^2} + \frac{\partial p}{\partial x_n} = f_n + \nu\Delta_* u_n + \mu\frac{\partial}{\partial x_n}(\nabla_*\cdot u_n) \equiv F$$
$$\frac{\partial^2 u_n}{\partial x_n^2} + \lambda\frac{\partial p}{\partial x_n} = \frac{\partial g}{\partial x_n} - \frac{\partial}{\partial x_n}(\nabla_*\cdot u_*) \equiv G, \tag{4.6}$$

where Δ_* is the the Laplacian with respect to the variables $x_1, \dots, x_{n-1}$, and $u_* = (u_1, \dots, u_{n-1})$. Note that the absolute value of the determinant of the above system is $1 + \lambda(\nu+\mu)$, which is larger or equal than 1, independently of the values of λ, ν, and μ. By solving this linear system with respect to the unknowns $\partial^2 u_n/\partial x_n^2$ and $\partial p/\partial x_n$ it follows that

$$\begin{cases} |\partial p/\partial x_n| \leq |F + (\nu+\mu)G| \\ |\partial^2 u_n/\partial x_n^2| \leq |G - \lambda F|. \end{cases} \tag{4.7}$$

Note that the individual summands in the definitions of F and G in equation (4.6) are already bounded by the right hand side of (4.5). Hence,

$$\begin{cases} \|F\|^2 \le \|f_n\|^2 + (\nu^2+\mu^2)\|\nabla_*^2 u\|^2, \\ \|G\|^2 \le \|\partial g/\partial x_n\|^2 + \|\nabla_*^2 u\|^2. \end{cases} \tag{4.8}$$

For convenience, from now on, we avoid indicating the explicit dependence of the positive constants in terms of ν, μ, and λ. We denote positive constants, that may depend on these three parameters, by $\overline{c}$. However, it is immediate to verify that the "constants" $\overline{c} = \overline{c}(\nu, \mu, \lambda)$ are uniformly bounded from above if (for instance) $\nu + \mu$ is bounded from below by a positive constant and ν, $|\mu|$ and λ are bounded from above (this is not used in the sequel). From (4.7), (4.8), and (4.5) one gets

$$\left\| \frac{\partial^2 u_n}{\partial x_n^2} \right\|^2 + \left\| \frac{\partial p}{\partial x_n} \right\|^2 \le \overline{c}\left(\|f\|^2 + \|\nabla g\|^2 + \|a\|_{3/2}^2 + [b]_{1/2}^2 \right). \tag{4.9}$$

Finally, the first $n-1$ equations (1.1) show that

$$\nu \frac{\partial^2 u_j}{\partial x_n^2} = -\nu \sum_{l=1}^{n-1} \frac{\partial^2 u_j}{\partial x_l^2} + \mu \frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial p}{\partial x_j} - f, \tag{4.10}$$

for each $j \ne n$. The estimates (4.5) show that also the left hand sides of (4.10) are bounded in the L^2-norm by the right hand side of (4.5). Hence

$$\|\nabla^2 u\|^2 + \|\nabla p\|^2 \le \overline{c}\left(\|f\|^2 + \|\nabla g\|^2 + \|a\|_{3/2}^2 + [b]_{1/2}^2 \right). \tag{4.11}$$

By adding this estimate to (1.7) we finally obtain (1.8) where $\overline{c}$ is as above.

Since (1.8) does not depend on the $L^{r'}$ and $L^{s'}$ norms of f and b (see (3.5)) a well-known argument shows that the assumption (3.5) can be dropped. Alternatively, we may replace the assumption (3.4) on f and b by (3.5) and replace on the right hand side of (1.8) $[f]_{-1}$ and $[b]_{-1/2}$ by, respectively, $\|f\|_{L^{r'}}$ and $\|b\|_{L^{s'}}$.

The proof of Theorem 1.2 is complete.

5. Appendix I

For the reader's convenience we prove here the following inequalities, which were used in section 4 in order to prove (4.4).

$$\begin{cases} [(f_h - f)/h]_{-1} \le \|f\|, \\ [(b_h - b)/h]_{-1/2} \le 2\pi\, [b]_{1/2}. \end{cases} \tag{5.1}$$

Since $\widetilde{\mathbb{H}}^1 \hookrightarrow [L^r]^n$ and f belongs to $[L^{r'}]^n$, it follows that $f \in (\widetilde{\mathbb{H}}^1)'$ and $[f]_{-1} \leq c\,\|f\|_{r'}$. Moreover, f acts as follows: $< f, \phi >= \int f\,\phi\,dx$. Hence,

$$\left|\left\langle \frac{f_h - f}{h}, \phi \right\rangle\right| \leq \left| \int f\, \frac{\phi - \phi_{-h}}{h}\, dx \right| \leq \|f\|\,\|\nabla\phi\|,$$

for each $\phi \in \widetilde{\mathbb{H}}^1$. This shows (5.1).

Next we prove the second equation (5.1). By assumption $b \in \widetilde{\mathbb{H}}^{1/2} \cap [L^{s'}]^{n-1}$. Let $\phi \in \widetilde{\mathbb{H}}^{1/2}$. Since $\widetilde{\mathbb{H}}^{1/2} \hookrightarrow [L^s]^{n-1}$ one has

$$\left\langle \frac{b_h - b}{h}, \phi \right\rangle = \int b\, \frac{\phi - \phi_{-h}}{-h}\, dx' = \int \widehat{b}\, \frac{\widehat{\phi} - \widehat{\phi}_{-h}}{-h}\, d\xi,$$

where $\widehat{b}(\xi)$ is the Fourier transform of b in $\mathbf{R}^{n-1}$. By recalling that

$$\widehat{\tau_{-h}\,\phi}(\xi) = e^{-2\pi\, i\, \xi_j\, h}\, \widehat{\phi}(\xi),$$

straightforward manipulations yield

$$\left|\left\langle \frac{b_h - b}{h}, \phi \right\rangle\right| \leq\, [b]_{1/2} \left(\int |\widehat{\phi}(\xi)|^2\, \frac{|exp(-2\pi\, i\, \xi_j\, h) - 1|^2}{h^2\, |\xi|}\, d\xi \right)^{1/2}.$$

Since $|(e^{i\,\theta} - 1)/\,\theta| \leq 1$, it readily follows that

$$\left|\left\langle \frac{b_h - b}{h}, \phi \right\rangle\right| \leq 2\,\pi\, [b]_{1/2}\, \|\phi\|_{1/2}.$$

6. Appendix II

In this Appendix (see the Remark after theorem 1.1) we show the existence of the weak solution without resort to Proposition 1.1, in the following two cases. Either $\lambda > 0$ or $\lambda = 0$, $a = 0$ and $g = 0$. Note that the second case corresponds to the usual Stokes system.

From inequalities (3.8) and (3.9) it easily follows, by a standard argument that

$$\begin{aligned} \overline{\nu}^2\, \|\nabla u\|^2 + \lambda\, \|p\|^2 \; &\leq c\, \overline{\nu}^{-1}\big[\, [f]_{-1} + (\nu + |\mu|)\, [a]_{1/2} + [b]_{-1/2} \big]^2 \\ &\quad + c\, \lambda^{-1} ([a]_{1/2} + \|g\|)^2. \end{aligned} \tag{6.1}$$

In particular, if $\lambda > 0$, we get a weak solution (u, p) satisfying (6.1).

Next we consider the case $\lambda = 0$. Let us denote the above solutions by (u_λ, p_λ). If $\lambda = 0$, $a = 0$ and $g = 0$, we obtain the solution $(v, p) = (u, p)$ to our problem as being the limit of the above solutions (u_λ, p_λ) as λ goes to 0. This is trivially verified since in this case ($a = 0$, $g = 0$) the estimate (6.1) is uniform in λ. Clearly, in following this simplified method, we lose the estimate of the norm $\|p\|$ (which was obtained in section 3 by appeal to Proposition 1.1). However, this situation is the current one in the literature, even in dealing with the simplest boundary value problems.

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