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Vorticity and Smoothness in Viscous Flows

Hugo Beirão da Veiga

On the birthday of Olga A. Ladyzhenskaya
with the deepest admiration for her outstanding scientific work

We show a sufficient condition for the regularity of solutions to the evolution Navier–Stokes equations in the three-dimensional case which relates the direction to the amplitude of the vorticity. The proof is done by applying ideas introduced by Constantin and Fefferman [**1**] and recent improvements due to the author and Berselli [**2**]. We follow the last reference in a straightforward way.

1. Introduction

This note is mainly a small tribute to the outstanding scientific work of Olga A. Ladyzhenskaya, in particular, to her fundamental work, pioneering and current, on the Navier–Stokes equations. We present recent results on the regularity of the solutions to the Navier–Stokes equations obtained together with Berselli (cf. [**2**]). Moreover, we adapt the proofs developed in this last reference in order to state a new related result.

Our study takes as a starting point the results of Constantin and Fefferman [**1**]. In [**2**], we relax the assumptions on the direction of vorticity

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|},$$

which are used in [**1**] to ensure the smoothness of the solutions.

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We consider the Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = f & \text{in } \mathbb{R}^3 \times [0,T], \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times [0,T], \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where, for simplicity, we suppose that the external force f is zero and the initial datum is smooth.

We denote by $L^p := L^p(\mathbb{R}^3)$, $1 \leqslant p \leqslant \infty$, the usual Lebesgue spaces equipped with the norm $|\cdot|_p$ and by $H^s := H^s(\mathbb{R}^3)$, $s \geqslant 0$, the classical Sobolev spaces. We indicate with the same symbol both scalar and vector function spaces. $C_w(0,T;L^2)$ denotes the space of weakly continuous functions on $(0,T)$ with the values in L^2.

A classical result (cf. Leray [**3**]) states that for any fixed $T > 0$ there exists at least a *weak solution* of the system (1.1) in (0,T), i.e., a function u such that

$$u \in C_w(0,T;L^2) \cap L^2(0,T;H^1)$$

and

$$\int_0^T \int_{\mathbb{R}^3} \left[u \frac{\partial \varphi}{\partial t} - \nu\, \nabla u \cdot \nabla \varphi - (u \cdot \nabla) u \varphi \right] dx\, dt = \int_{\mathbb{R}^3} u(T)\varphi(T) - u_0 \varphi(0)\, dx$$

for all divergence-free $\varphi \in C^1(0,T;H^1)$. However, it is not known whether weak solutions are unique.

By a *strong solution* we mean a weak solution such that

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2). \tag{1.2}$$

A strong solution exists at least for some $T > 0$. However, it is not known, in general, whether it is or not global in time.

Strong solutions are unique and regular. The main problem is then to prove that for smooth divergence-free initial data there exists a strong (necessarily unique) solution for all time.

It is worth noting that for the problem in $\mathbb{R}^2$ the situation is completely different since it is possible to prove the global existence in time of strong solutions (cf. Ladyzhenskaya [**4**]).

In the two-dimensional case, the vorticity field ω,

$$\omega(x,t) := \nabla \times u(x,t),$$

is always a vector that is perpendicular to the plane of motion (a sufficient condition for the regularity of the solution, even for $n = 3$). On the other hand, taking the curl of the first equation in (1.1), we get

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\, \omega - \nu \Delta \omega = 0.$$

Hence the vorticity satisfies a "linear" evolution equation. In particular, the maximum of the modulus of the vorticity cannot increase.

In the three-dimensional case, taking the curl of the first equation in (1.1), we find

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\, \omega - \nu \Delta \omega = (\omega \cdot \nabla)\, u. \tag{1.3}$$

Now, the modulus of the vorticity can increase and its direction can change.

A simple condition that ensures regularity is that considered in [**5**]. If the vorticity ω satisfies the condition

$$\omega \in L^p(0,T;L^q) \quad \text{for } \frac{2}{p} + \frac{n}{q} \leqslant 2, \quad 1 \leqslant p, \leqslant 2, \tag{1.4}$$

then the solution is regular. We note that the limit case $p = 1$ in (1.4) corresponds to the well-known regularity condition due to Beale, Kato, and Majda [**6**]: $u \in L^1(0,T;W^{1,\infty})$.

In [**1**], the authors prove the regularity of the solutions by assuming hypothesis involving only the direction of the vorticity. As is shown, if $\theta(x, x+y, t)$, the angle between the vorticity ω at points x and $x+y$ at time t, satisfies the inequality

$$|\sin\theta(x,x+y,t)| \leqslant \frac{|y|}{\rho(t)} \quad \text{for} \quad \rho^{-12} \in L^1(0,T),$$

then the solution is necessarily smooth in $(0,T)$. In [**2**], we improve this result by stating a regularity criterion involving the following assumption.

Hypothesis A. *There exist $\beta \in [1/2, 1]$ and $g \in L^a(0,T;L^b)$, where*

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2}, \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right],$$

such that

$$|\sin\theta(x,x+y,t)| \leqslant g(t,x)|y|^\beta \tag{1.5}$$

in the region where the vorticity at both points x and $x+y$ is larger than an arbitrary fixed positive constant K.

We note that Hypothesis A is satisfied, in particular, if

$$|\sin\theta(x,x+y,t)| \leqslant c|y|^{1/2}. \tag{1.6}$$

Theorem 1.1. *Suppose that u is a weak solution of* (1.1) *in* $(0,T)$ *with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$, and that Hypothesis* A *is satisfied. Then the solution u is strong in* $(0,T)$ *and, consequently, is regular.*

In this paper, we consider the case in which $\beta \in [0, 1/2]$ by stating a sufficient condition for the regularity of solutions that involves, simultaneously, the modulus and the direction of the vorticity. The proof follows that presented in [**2**]. Hypothesis A is replaced with the following assumption.

Hypothesis B. *Let $\beta \in [0, 1/2]$. Suppose that*

$$|\sin\theta(x, x+y, t)| \leqslant c|y|^{\beta} \tag{1.7}$$

in the region where the vorticity at both points x and $x+y$ is larger than an arbitrary fixed positive constant K. Moreover, suppose that

$$\omega \in L^2(0,T; L^r), \tag{1.8}$$

where

$$r = \frac{3}{\beta+1}. \tag{1.9}$$

We will prove the following assertion (cf. Sec. 3).

Theorem 1.2. *Suppose that u is a weak solution of (1.1) in $(0,T)$ with $u_0 \in H^1$ and $\nabla \cdot u_0 = 0$. Let Hypothesis B be satisfied. Then the solution u is strong in $(0,T)$ and, consequently, is regular.*

Remark 1.3. It is clear that it is possible to relax Hypotheses A and B by assuming that (1.5) and (1.7) are satisfied only for $|y| \leqslant \delta$ for an arbitrary positive constant δ.

Remark 1.4. It is worth noting that in the two extreme cases, $\beta = 1/2$ and $\beta = 0$, the above result coincides with two already known results. For $\beta = 1/2$ the assumptions of Theorem 1.2 coincide with those of Theorem 1.1 since, in both cases, they reduce to (1.6). In this regard, we note that if $\beta = 1/2$, then $r = 2$. Hence the assumption (1.8) in Theorem 1.2 is necessarily satisfied (due to the energy estimate (2.6)).

On the other hand, if $\beta = 0$, the proof given below fails. However, the statement in Theorem 1.2 still holds as a consequence of the results proved by us in [5]. In fact, since $r = 3$, the assumption (1.4) is satisfied for $n = q = 3$ and $p = 2$ (since $r = 3$).

2. Some Known Results

We recall some results proved in [7].

The Biot–Savart law reads

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \left(\nabla \frac{1}{|y|}\right) \times \omega(x+y)\, dy. \tag{2.1}$$

Differentiating (2.1), we obtain the following expression for the *strain matrix*:

$$S[\omega](x) = \frac{1}{2}[\nabla u(x) + (\nabla u(x))^*] = \frac{3}{4\pi}\,\text{P.V.}\int_{\mathbb{R}^3} M(\widehat{y}, \omega(x+y))\frac{dy}{|y|^3}. \tag{2.2}$$

Here, $\widehat{y}$ is the unit vector in the direction of y, while

$$M(\widehat{y}, \omega) := \frac{1}{2}\left[\widehat{y} \otimes (\widehat{y} \times \omega) + (\widehat{y} \times \omega) \otimes \widehat{y}\right]$$

is a symmetric traceless matrix that defines a proper singular operator since, for each fixed ω, its mean value on the unit sphere vanishes.

Using this formula, we can express α as follows:

$$\alpha(x) := S[\omega](x)\xi(x) \cdot \xi(x) \tag{2.3}$$

on the set $\{x \in \mathbb{R}^3 : \ |\omega(x)| > 0\}$.

From the representation formula for $S[\omega](x)$ one can deduce the formula

$$\alpha(x) := \frac{3}{4\pi}\,\text{P.V.}\int_{\mathbb{R}^3} D(\widehat{y}, \xi(x+y), \xi(x))|\omega(x+y)|\frac{dy}{|y|^3}, \tag{2.4}$$

where

$$D(a,b,c) := (a \cdot c)\,\text{Determinant}(a,b,c). \tag{2.5}$$

We note that, in (2.4), the determinant $D(\widehat{y}, \xi(x+y), \xi(x))$ vanishes if $\xi(x+y) = \pm\xi(x)$. Alignment or anti-alignment of the vorticity depletes the nonlinearity.

Finally, we recall the well-known energy estimate for weak solutions

$$\frac{1}{2}|u(t)|_2^2 + \nu\int_0^t\int_{\mathbb{R}^3} |\nabla u(x,\sigma)|^2\, dx\, d\sigma \leqslant \frac{1}{2}|u_0|_2^2 \tag{2.6}$$

stated in Leray's paper [**3**].

3. Proof of Theorem 1.2

Using the continuation principle for strong solutions, it is easy to see that, in order to prove the above theorems, it suffices to prove the following lemma.

Lemma 3.1. *Let u be a weak solution of (1.1) on $(0,\tau)$, $0 < \tau \leqslant T$ and, at the same time, a strong solution in $(0,\tau')$ for each $\tau' < \tau$. Let Hypothesis B be satisfied. Then u is a strong solution in $(0,\tau)$.*

Since the solutions can be assumed to be regular in $(0,\tau')$, we can prove the estimates in $(0,\tau')$ without introducing smoothing processes to justify our calculations.

Next we give the proof of Theorem 1.2. The proof is a straightforward adaptation of that given in [**2**], to which the reader is referred for details. Multiplying Eq. (1.3) by ω and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 = \int_{\mathbb{R}^3} S[\omega](x)\,\omega(x)\cdot\omega(x)\, dx. \tag{3.1}$$

In the sequel, the symbol c denotes any positive constant that depends on neither the solution u nor k and ν. The same symbol c will denote distinct constants.

Let K be the positive constant in Hypothesis B. We split $\omega(x)$ as $\omega(x) = \omega_1(x) + \omega_2(x)$, where

$$\omega_2(x) = \begin{cases} \omega(x), & |\omega(x)| > k, \\ 0, & |\omega(x)| \leqslant k. \end{cases}$$

This gives rise, in the natural way, to a decomposition $S[\omega] = S[\omega_1]+S[\omega_2]$. The integrand $S[\omega]\omega \cdot \omega$ splits into a sum of eight terms of the form $S[\omega_i](x)\omega_j \cdot \omega_k$, $i, j, k = 1, 2$. Clearly, the more difficult term to estimate is that where $i, j, k = 2$. It is just the one that requires the use of Hypothesis B. In the sequel, we will focus on this term. Actually, this is (essentially) equivalent to the assumption $K = 0$. Hence we will assume that $K = 0$ and refer the reader to **[2]** for the treatment of the remaining terms.

By (2.3), we have

$$S[\omega]\omega \cdot \omega = |\omega|^2 \alpha\, \xi \cdot \xi.$$

Consequently, using (2.4), one gets

$$S[\omega]\omega \cdot \omega = \frac{3}{4\pi}|\omega(x)|^2 \,\mathrm{P.V.} \int\limits_{\mathbb{R}^3} D(\widehat{y}, \xi(x+y), \xi(x))|\omega(x+y)| \frac{dy}{|y|^3}.$$

Moreover, the assumption (1.7) shows that

$$|D(\widehat{y}, \xi(x+y), \xi(x))| \leqslant c|y|^\beta.$$

It readily follows that

$$|S[\omega](x)\omega(x) \cdot \omega(x)| \leqslant \frac{3}{4\pi}|\omega(x)|^2 I(x),$$

where

$$I(x) = \int\limits_{\mathbb{R}^3} |\omega(x+y)| \frac{dy}{|y|^{3-\beta}}.$$

Recall that here $\beta > 0$. The case $\beta = 0$ follows from **[5]**.

The Hardy–Littlewood–Sobolev inequality in R^n (cf., for example, Stein **[8**, Chap. V, Sec. 1.2]) states that if $f \in L^r$ for $1 < r < n$, then

$$I(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+y)}{|y|^{n-\beta}}\, dy$$

belongs to L^q, $1 < q < \infty$, for $1/q = 1/r - \beta/n$. Furthermore, the mapping $f \mapsto I$ is linear and continuous from L^r into L^q. Using this inequality with $n = 3$ for β and r as in Hypothesis B (hence $q = 3$), we get

$$|I(x)|_3 \leqslant c|\omega|_r \leqslant c|\omega|_r.$$

Using Hölder's inequality with exponents 3, 2, and 6, one can show that

$$\left| \int\limits_{\mathbb{R}^3} S[\omega](x)\omega(x) \cdot \omega(x)\, dx \right| \leqslant c|\omega|_r|\omega|_6|\omega|_2. \tag{3.2}$$

Since $|\omega|_6 \leqslant c|\nabla\omega|_2$, from (3.2) it follows that

$$\left| \int\limits_{\mathbb{R}^3} S[\omega](x)\omega(x) \cdot \omega(x)\, dx \right| \leqslant \frac{\nu}{4}|\nabla\omega|_2^2 + c\,\nu^{-1}|\omega|_r^2|\omega|_r^2. \tag{3.3}$$

From (3.1) and (3.3) we find

$$\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 \leqslant c\nu^{-1}|\omega|_r^2\, |\omega|_2^2. \tag{3.4}$$

The energy inequality (2.6) shows that $|\omega|_2^2$ is integrable on $(0, \tau)$ since $|\omega|_2 = |\nabla u|_2$. On the other hand, by the assumption (1.8), $|\omega|_r^2$ is integrable on $(0, \tau)$. Hence the right-hand side of (3.4) is integrable on $(0, \tau)$. By a standard argument, we have $\omega \in L^\infty(0, \tau; L^2) \cap L^2(0, \tau; H^1)$. It follows that u satisfies (1.2) on $(0, \tau)$. Hence u is a strong solution in that interval.

Now, we can extend u, as a strong (regular) solution, to some interval $[\tau, \tau + \varepsilon)$, $\varepsilon > 0$, by starting from the "initial" data $u(\tau)$ which belongs to H^1.

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