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On the Existence of Strong Solutions to a Coupled Fluid-Structure Evolution Problem

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To the memory of my Father

Abstract. We consider here a model of fluid-structure evolution problem which, in particular, has been largely studied from the numerical point of view. We prove the existence of a strong solution to this problem.

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1. Introduction and main results

This work originates essentially from some problems in arteries, where fluid and structure models are coupled. Fluid flows are here described by the Navier–Stokes equations (a good approximation for flows in large vessels). Concerning the structure model we will consider the so-called generalized string model, see Quarteroni, Tuveri and Veneziani [14] Eq. (27), Formaggia, Gerbeau, Nobile, and Quarteroni, [6] Eq. (7), Quarteroni and Formaggia [13], Eq. (4.23). For recent overviews on this problem the reader is refered to [13] and [10]. For some related papers we also refer the reader (without any claim of completeness) to references [2], [3], [4], [9], [10] and to the bibliography of all the above papers. The results and proofs given in this paper have been made available, in particular, in a preprint published in 2001 by the author's Mathematical Department.

It is worth noting that the proofs given below may be simply modified to consider other vessel models. The consideration of a definite model is done here for the sake of clearness. The choice of model (1.3), where in particular it may be $\alpha = 0$, is done since real numerical applications have been having large success in medical communities.

Our main objective will be to establish a rigorous result on the existence of strong solutions to initial-boundary value problems, in which the crucial point is the study of the interaction of fluid and structure. To our knowledge, this is the first rigorous proof of the above result for models like that considered here. Another interesting problem would be to obtain a rigorous proof of the existence of weak solutions. An *a priori* estimate in this direction is well known. See, for instance, [4], [14] Eq. (41), and [6] Lemma 1.1.

Since our main concern is the study of the interaction of fluid and structure, the mathematical obstacles coming out from the artificial consideration of just a segment of vessel are avoided by considering data and solutions which are periodic in the "vessel direction" (with period L, equal to the length of the vessel). The expression "an x-periodic function f" means here the restriction to $[0, L]$ of a periodic function f on the variable $x \in \mathbf{R}$, with period L. More precisely, $f(x+L) = f(x)$, for each (or almost all) $x \in \mathbf{R}$. Clearly, f may depend on other variables (typically y, z, and t). We will consider the 2-D problem. Hence the moving structure consists of two separate boundaries. In order to simplify the presentation we will assume that one of these boundaries is fixed. The consideration of two moving boundaries can be treated in a similar way, by using the change of variables (2.3) instead of (2.1).

We consider a family of x-periodic plane curves Γ_t, $t \in [0, T]$, with equations

$$y = 1 + \eta(t, x), \qquad x \in [0, L]. \tag{1.1}$$

Without loss of generality we assume that

$$\int_0^L \eta^0(x)\, dx = 0, \tag{1.2}$$

where $\eta^0(x) = \eta(0, x)$. If (1.2) were not satisfied we may impose it just by replacing the constant 1 in (1.1) by the mean value of $1 + \eta^0(x)$ on $]0, L[$.

The family of curves Γ_t is just one of the unknowns of our problem. In fact, the x-periodic functions $\eta(t, x)$ should solve the evolution problem

$$\begin{cases} \eta_{tt} - \beta\, \eta_{xx} - \gamma\, \eta_{txx} + \alpha\, \eta_{xxxx} + \sigma\, \eta = \Phi, \text{ in } I_T, \\ \eta(0, x) = \eta^0(x), \\ \eta_t(0, x) = \eta^1(x), \end{cases} \tag{1.3}$$

where $I =]0, L[$, $I_T =]0, T[\times I$, Φ is the forcing term (see below), γ is a strictly positive constant and α, β, and σ are nonnegative constants. Note that α may vanish. In fact, we mostly will assume that $\alpha = 0$. We also remark that the addition of a given external force field to the right-hand side of equation $(1.3)_1$ does not give rise to any difficulty.

The x-periodic functions $\eta^0(x)$ and $\eta^1(x)$ are given, moreover $\eta_0(x)$ satisfies (1.2) and also the assumption

$$1 + \eta^0(x) \geq 2\delta_0, \qquad \forall\, x \in I, \tag{1.4}$$

for some positive constant δ_0. This corresponds to the fact that the vessel Ω_t at time $t = 0$ is connected. Here

$$\Omega_t := \{(x, y) : \ 0 < y < 1 + \eta(t, x)\} . \tag{1.5}$$

The function $\Phi(t, x)$ on the right-hand side of (1.3) is defined as follows.

$$\Phi[\eta, v, p] := \left(\rho_1\, p\, n_t - \rho_2\, \nu\, [\nabla v + \nabla v^T] \cdot n_t\right)_{|\Gamma_t} \sqrt{1 + \eta_x^2} \cdot \vec{e_2}, \tag{1.6}$$

where p and v are the pressure and velocity of the fluid, $\nu > 0$ is the (given) viscosity of the fluid and n_t is the unit normal to Γ_t, namely

$$n_t(t, x, 1 + \eta(t, x)) = \frac{-\eta_x\, \vec{e_1} + \vec{e_2}}{\sqrt{1 + \eta_x^2}}. \tag{1.7}$$

Here $\vec{e_1}$ and $\vec{e_2}$ denote respectively the unit vectors in the x and y (orthogonal) directions . Note that, with an obvious notation,

$$d\Gamma_t = \sqrt{1 + \eta_x^2} dx.$$

The presence of the term $\sqrt{1 + \eta_x^2}$ in the right-hand side of Eq. (1.6) is not necessary to obtain our results. However it takes into account that in equation (1.3) Φ denotes density of external forces with respect to dx and not to $d\Gamma_t$. We remark that the above term seems necessary to obtain the estimate (40) in reference [14].

In equation (1.6) ρ_1 and ρ_2 are positive constants that represents, if $\rho_1 = \rho_2$, the ratio between the density of the fluid and that of the vessel. Note that to assume this ratio "sufficiently small" is a physically reasonable assumption. We do not assume here that (necessarily) $\rho_1 = \rho_2$ due to a "mathematical reason". In fact, we are able to prove the existence of a local (in time) strong solution of our problem for an arbitrary positive constant ρ_2. However, the positive constant ρ_1 should be assumed "sufficiently small". We note that it seems possible to drop the smallness condition on ρ_1 by using an energy estimate inspired by that referred above. However, since we do not use this device, the proofs done here does not depend so much on the particular form of the forcing term Φ. (in particular, we may drop the term ∇v^T on the right-hand side of (1.6)). We consider here the particular form (1.6) just for fixing ideas.

Concerning the fluid flow, we consider here the Navier–Stokes equations in the moving domain Ω_t. We set

$$\widetilde{Q}_T := \{(t, x, y) : \ t \in]0, T[, \ (x, y) \in \Omega_t\} ,$$
$$\widetilde{\Sigma}_T := \{(t, x, y) : \ t \in]0, T[, \ (x, y) \in \Gamma_t\} ,$$
$$\Lambda_T := \{(t, x, 0) : \ t \in]0, T[, \ x \in]0, L[\} ,$$

and we consider the initial-boundary value problem

$$\begin{cases} v_t + (v \cdot \nabla) v - \nu \Delta v + \nabla p = 0, \\ \nabla \cdot v = 0 \qquad \text{in } \widetilde{Q}_T; \\ v(0, x, y) = v^0(x, y) \qquad \text{in } \Omega_0; \\ v(t, x, 1 + \eta(t, x)) = \eta_t(t, x)\, \vec{e_2} \qquad \text{on } \widetilde{\Sigma}_T; \\ v(t, x, 0) = 0 \qquad \text{on } \Lambda_T, \end{cases} \tag{1.8}$$

where η is the solution of problem (1.3). As for equation (1.3), the addition of an external force field to the right-hand side of equation $(1.8)_1$ can be handled by routine calculations.

The problems (1.3) and (1.8) are coupled by means of the dependence of the right-hand side of equation $(1.3)_1$ on p and v and by means of equation $(1.8)_4$. We assume the following (necessary) compatibility conditions on the given initial velocity $v^0(x, y)$.

$$\begin{cases} \nabla \cdot v^0 = 0 \qquad \text{in } \Omega_0; \\ v^0(x, 0) = 0, \\ v^0(x, 1 + \eta^0(x)) = \eta^1(x)\, \vec{e_2} \qquad \text{on } I, \end{cases} \tag{1.9}$$

plus the x-periodicity assumption on v^0.

The last compatibility condition to be assumed is

$$\int_0^L \eta^1(x)\, dx = 0. \tag{1.10}$$

This condition is imposed by the divergence theorem, as shown below. Another consequence of the divergence theorem is the following one. Let us write each "pressure" $p(t, x, y)$ in the form

$$p(t, x, y) = p_0(t, x, y) + \phi(t) \tag{1.11}$$

where $p_0(t, x, y)$ satisfies

$$\frac{1}{2} \int_0^L p_0(t, x, 1 + \eta(t, x))\, dx = 0, \quad \forall\, t \in [0, T[, \tag{1.12}$$

and $\phi(t)$ is an arbitrary function of t. We show that if (η, v, p) is a solution of our problem then it must be

$$\phi(t) = \phi\,[\eta, v] := \frac{\nu\, \rho_2}{L\, \rho_1} \int_0^L \left[-\eta_x \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x} \right) + 2 \frac{\partial v_2}{\partial y} \right]_{|\Gamma_t} dx. \tag{1.13}$$

In fact, from $(1.8)_{2,4,5}$, from the x-periodicity of v and from the divergence theorem it readily follows that

$$\int_{\Gamma_t} v \cdot n_t\, d\Gamma_t = \int_0^L \eta_t\, dx = 0, \tag{1.14}$$

for each $t \in [0, T[$ (note, as claimed above, that $\eta^1(x)$ must satisfy the compatibility condition (1.10)). Moreover, (1.14) together with (1.2) shows that

$$\int_0^L \eta(t,x)\,dx = 0, \qquad \forall\, t \in [0,T[. \tag{1.15}$$

By integrating both sides of $(1.3)_1$ on $[0, L]$ and by taking into account (1.15) and the x-periodicity of η, it readily follows that

$$\int_0^L \Phi\,[\eta, v, p]\,dx = 0, \qquad \forall\, t \in [0,T[\,, \tag{1.16}$$

whenever (η, v, p) is a solution. On the other hand, by (1.6) and (1.7), one gets

$$\Phi\,[\eta, v, p] = \left\{\rho_1\, p + \nu\,\rho_2 \left[\eta_x \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) - 2\frac{\partial v_2}{\partial y}\right]\right\}_{|\Gamma_t}, \tag{1.17}$$

where $v = (v_1, v_2) = v_1\,\vec{e_1} + v_2\,\vec{e_2}$. Consequently, it follows that (1.16) is equivalent to saying that p has the form (1.11) with $\phi(t)$ given by (1.13).

Hence our original problem is equivalent to looking for a solution (η, p_0, v) of the systems (1.8) and

$$\begin{cases} \eta_{tt} - \beta\,\eta_{xx} - \gamma\,\eta_{txx} + \alpha\,\eta_{xxxx} + \sigma\,\eta = \Phi\,[\eta, v, p_0] + \rho_1\,\phi[\eta, v], \\ \eta(0,x) = \eta^0(x), \\ \eta_t(0,x) = \eta^1(x), \end{cases} \tag{1.18}$$

where p_0 satisfies (1.12) and ϕ is given by (1.13). Then the solution (η, v, p) of (1.3), (1.8) is given by $(\eta, v, p_0 + \phi\,[\eta, v])$.

In the sequel we use well-known notations to indicate classical functions spaces, like Sobolev spaces of fractional order $H^s(\Omega)$ or $H^s(I)$, and spaces like $L^2(0,T;X)$ or $C([0,T];X)$, where X is a Banach space. We use the same notations for spaces of scalar and of vector valued functions. The symbol # indicates x-periodicity.

It must be pointed out that strict positivity of some or all of the constants α, β, σ could just help the proofs. However, only the assumption $\gamma > 0$ is used here. Under this assumption, the estimates due to $\beta > 0$ or $\sigma > 0$ are, in fact, useless. On the contrary, if $\alpha > 0$, a stronger estimate on η can be obtained (and used to simplify some proofs). For these reasons we assume in our proofs the (less favorable) case $\beta, \alpha, \sigma = 0$. However we state and prove the main existence result (see the next theorem) also for the case $\alpha > 0$.

Our main result is the following.

Theorem 1.1. *Assume that* $\gamma > 0$, *that*

$$\begin{cases} v^0 \in H^1_{\#}(\Omega_0), \\ \eta_0 \in H^{5/2}_{\#}(I),\ \ \eta_1 \in H^{3/2}_{\#}(I), \end{cases} \tag{1.19}$$

vedi nota ⊛ pag. 23

and that the hypotheses (1.2), (1.4), (1.9), and (1.10) hold. Then if ρ_1, $\|\eta^0\|_{L^\infty(I)}$ *and* $\|\eta^0_x\|_{L^\infty(I)}$ *are sufficiently small the problem (1.3), (1.8) has a solution* (η, v, p) *for a sufficiently small value of* T. *Moreover*

$$\begin{cases} v \in L^2(0,T; H^2_\#(\Omega_t)), \\ v_t \in L^2(0,T; L^2(\Omega_t)), \\ p \in L^2(0,T; H^1_\#(\Omega_t)) \end{cases} \tag{1.20}$$

and

$$\begin{cases} \eta_t \in L^2(0,T; H^{5/2}_\#(I)) \cap L^\infty(0,T; H^{3/2}_\#(I)), \\ \eta_{tt} \in L^2(0,T; H^{-1/2}_\#(I)). \end{cases} \tag{1.21}$$

If $\alpha > 0$, *and by assuming that* $\eta_0 \in H^{7/2}_\#(I)$, *one also has* $\eta \in L^\infty(0,T; H^{7/2}_\#(I))$.

Remark. If the data are more regular and the necessary compatibility conditions hold, then the solutions are more regular.

Our problem will be solved by previously transforming it in an equivalent problem on the fixed spatial domain

$$\Omega =]0, L[\times]0, 1[.$$

2. An equivalent problem in a fixed domain

We perform the following change of variables

$$x = x \qquad z = \frac{y}{1+\eta(t,x)} \tag{2.1}$$

that transforms, for each fixed t, Ω_t onto Ω.
Remark. In the case of a domain Ω_t with two moving boundaries

$$\Omega_t := \{(x,y) : \ x \in (0,L), \ 0 + \mu(t,x) < y < 1 + \eta(t,x)\} \tag{2.2}$$

we replace (2.1) by

$$z = \frac{y - \mu(t,x)}{1 + \eta(t,x) - \mu(t,x)} \tag{2.3}$$

which already transforms Ω_t onto Ω. Now $v(t,x,\mu(t,x)) = \mu_t(t,x)\,\vec{e_2}$.

In general, in correspondence to any function $f(x,y)$ defined on Ω_t we associate the function $\widehat{f}(x,z) = f(x,y)$. More precisely,

$$\begin{cases} \widehat{f}(x,z) = f(x, (1+\eta(t,x))\,z), \\ f(x,y) = \widehat{f}\left(x, \dfrac{y}{1+\eta(t,x)}\right). \end{cases} \tag{2.4}$$

Clearly, f and $\widehat{f}$ may depend also on t. In the sequel the differential operators ∇ and Δ, when applied to functions labelled by a *hat*, a *tilde* or a *bar* (like $\widehat{v}, \widetilde{v}$ or $\overline{v}$), act on the variables (x, z).

It readily follows from (2.4) that

$$\begin{cases} f_t = & \widehat{f}_t - z\dfrac{\eta_t}{1+\eta}\widehat{f}_z, \\ f_x = & \widehat{f}_x - z\dfrac{\eta_x}{1+\eta}\widehat{f}_z, \\ f_y = & \dfrac{1}{1+\eta}\widehat{f}_z, \\ f_{xx} = & \widehat{f}_{xx} - 2z\dfrac{\eta_x}{1+\eta}\widehat{f}_{xz} + \left(z\dfrac{\eta_x}{1+\eta}\right)^2 \widehat{f}_{zz} \\ & -z\dfrac{(1+\eta)\eta_{xx} - 2\eta_x^2}{(1+\eta)^2}\widehat{f}_z, \\ f_{yy} = & \dfrac{1}{(1+\eta)^2}\widehat{f}_{zz}, \end{cases} \tag{2.5}$$

where f is taken at point (t, x, y) and $\widehat{f}$ at (t, x, z), $z = y/(1+\eta(t,x))$. In particular

$$\nabla p = (\widehat{p}_x - z\frac{\eta_x}{1+\eta}\widehat{p}_z)\,\vec{e_1} + \frac{1}{1+\eta}\widehat{p}_z\,\vec{e_2}. \tag{2.6}$$

By multiplying equations $(1.8)_{1,2}$ by $1+\eta$ and by performing the above change of variables we obtain

$$\begin{cases} \widehat{v}_t - \nu\Delta\widehat{v} + \nabla\widehat{p} = \widehat{F}\,[\eta, \widehat{v}, \nabla\widehat{p}], \\ \nabla\cdot\widehat{v} = \widehat{g}\,[\eta, \widehat{v}] \qquad \text{in } Q_T; \\ \widehat{v}(0,x,z) = \widehat{v}^0(x,z) \qquad \text{in } \Omega; \\ \widehat{v}(t,x,1) = \eta_t(t,x)\,\vec{e_2} \qquad \text{on } \Sigma_T; \\ \widehat{v}(t,x,0) = 0, \qquad \text{on } \Lambda_T, \end{cases} \tag{2.7}$$

where $\Sigma_T =]0,T[\times\Gamma$, $Q_T =]0,T[\times\Omega$, and $\Gamma =: \{\,(x,1): \; x \in]0,L[\}$. Here

$$\begin{aligned} \widehat{F}(t,x,z) &= \widehat{F}\,[\eta, \widehat{v}, \nabla\widehat{p}] \\ &:= -\,\eta\widehat{v}_t + \left[z\eta_t + \nu z\left(\frac{2\eta_x^2}{1+\eta} - \eta_{xx}\right)\right]\widehat{v}_z \\ &\quad + \nu\left\{-2z\eta_x\widehat{v}_{xz} + \eta\widehat{v}_{xx} + \left[\frac{z^2\eta_x^2 - \eta}{1+\eta}\right]\widehat{v}_{zz}\right\} \\ &\quad + z(\eta_x\widehat{p}_z - \eta\widehat{p}_x)\vec{e_1} - (1+\eta)\widehat{v}_1\widehat{v}_x + (z\eta_x\widehat{v}_1 - \widehat{v}_2)\widehat{v}_z, \end{aligned} \tag{2.8}$$

and

$$\widehat{g}(t,x) = \widehat{g}\,[\eta, \widehat{v}] := -\eta\widehat{v}_{1,x} + z\eta_x\widehat{v}_{1,z}, \tag{2.9}$$

where $\widehat{v}_{1,x} = \partial\widehat{v}_1/\partial x$, and so on. Moreover

$$\widehat{v}^0(x,z) = \widehat{v}^0(x, (1+\eta^0(x))z). \tag{2.10}$$

Clearly $\widehat{v}^0$ is still x-periodic.

The assumptions (compatibility conditions) (1.9) become

$$\begin{cases} \nabla \cdot \widehat{v}^0 = \widehat{g}\,[\eta^0, \widehat{v}^0] = -\eta^0 \widehat{v}^0_{1,x} + z\eta^0_x \widehat{v}^0_{1,z}, \\ \widehat{v}^0(x,0) = 0 \qquad \text{on } I, \\ \widehat{v}^0(x,1) = \eta^1(x)\,\vec{e_2} \qquad \text{on } I. \end{cases} \tag{2.11}$$

The compatibility conditions (1.2) and (1.10) remain unaltered. The same holds with respect to equation (1.18), since this equation is independent of z. However we will denote Φ by $\widehat{\Phi}$ when it is written in terms of $\widehat{p}$ and $\widehat{v}$. From (1.18), by using (2.5) and by taking into account that $z = 1$, one shows that problem (1.18) becomes

$$\begin{cases} \eta_{tt} - \beta\,\eta_{xx} - \gamma\,\eta_{txx} + \alpha\,\eta_{xxxx} + \sigma\,\eta = \widehat{\Phi}\,[\eta, \widehat{v}, \widehat{p}_0] + \rho_1\,\widehat{\phi}\,[\eta, \widehat{v}], \\ \eta(0,x) = \eta^0(x), \\ \eta_t(0,x) = \eta^1(x). \end{cases} \tag{2.12}$$

Here

$$\widehat{\Phi}\,[\eta, \widehat{v}, \widehat{p}] := \rho_1\,\widehat{p} + \nu\rho_2 \left(\frac{\eta_x}{1+\eta} \widehat{v}_{1,z} + \eta_x \widehat{v}_{2,x} - \frac{2+\eta_x^2}{1+\eta} \widehat{v}_{2,z} \right), \tag{2.13}$$

$\widehat{p}_0$ satisfies

$$\frac{1}{L} \int_0^L \widehat{p}_0(t,x,1)\,dx = 0, \qquad \forall t \in [0,T[, \tag{2.14}$$

and

$$\widehat{\phi}(t) = \widehat{\phi}\,[\eta, \widehat{v}] := -\frac{\nu\,\rho_2}{L\,\rho_1} \int_0^L \left(\frac{\eta_x}{1+\eta} \widehat{v}_{1,z} + \eta_x \widehat{v}_{2,z} - \frac{2+\eta_x^2}{1+\eta}\,\widehat{v}_{2,z} \right) dx. \tag{2.15}$$

According to (1.11) $\widehat{p}$ is given by

$$\widehat{p}(t,x,z) = \widehat{p}_0(t,x,z) + \widehat{\phi}\,[\eta, \widehat{v}]. \tag{2.16}$$

This is equivalent to saying that

$$\int_0^L \widehat{\Phi}\,[\eta, \widehat{v}, \widehat{p}]\,dx = 0, \qquad \forall t \in [0,T[, \tag{2.17}$$

which corresponds to (1.14).

For convenience we define

$$\dot{H}^1_{\#}(\Omega) = \left\{ \widehat{p}_0 \in H^1_{\#}(\Omega) : \int_0^L \widehat{p}_0(x,1)\,dx = 0 \right\}. \tag{2.18}$$

Since the integral in (2.18) defines a continuous linear functional on $H^1(\Omega)$ it follows that $\dot{H}^1_{\#}$ is a closed subspace of $H^1_{\#}$ and that there is a positive constant c such that

$$\|\widehat{p}_0\|_{L^2(\Omega)} \le c\|\nabla \widehat{p}_0\|_{L^2(\Omega)}, \qquad \forall \widehat{p}_0 \in \dot{H}^1_{\#}(\Omega). \tag{2.19}$$

Hence $\|\nabla\widehat{p}_0\|_{L^2(\Omega)}$ is a norm in $\dot{H}^1_{\#}$, equivalent to the canonical H^1-norm.

Now, for the reader's convenience, we summarize the approach followed in the sequel in order to solve our problem. One gives x-periodic data $\widehat{v}^0, \eta^0, \eta^1$ satisfying the assumptions (2.11), (1.2), (1.4), and (1.10). These data belong to the functional spaces indicated in equations (1.19). If $\alpha > 0$, we also assume that $\eta_0 \in H^{7/2}_{\#}(I)$. Then we look for an x-periodic solution $(\eta, \widehat{v}, \widehat{p}_0)$ of systems (2.12), (2.7), where $\widehat{F}, \widehat{g}, \widehat{\Phi}$, and $\widehat{\phi}$ are defined respectively by (2.8), (2.9), (2.13), and (2.15). The function $\widehat{p}_0$ should satisfy (2.14). The solution of our problem (in Ω) is then given by $(\eta, \widehat{v}, \widehat{p})$, where $\widehat{p}$ is defined by equation (2.16). Finally, the solution (η, p, v) of the initial problem (1.3),(1.8) is obtained by turning back to the variables (t, x, y), by means of $(2.4)_2$.

In order to solve (2.12), (2.7), we linearize these systems, solve the corresponding linear systems, and look for a fixed point. The linearized systems are obtained by replacing on the right-hand side of (2.12) and (2.7) the unknown functions $\eta, \widehat{v}, \widehat{p}_0$ by given functions $\overline{\eta}, \overline{v}, \overline{p}_0$. Hence we consider the couple of systems

$$\begin{cases} \widetilde{\eta}_{tt} - \gamma\,\widetilde{\eta}_{txx} + \alpha\,\widetilde{\eta}_{xxxx} = \widehat{\Phi}\,[\overline{\eta}, \overline{p}_0, \overline{v}] + \rho_1\,\widehat{\phi}\,[\overline{\eta}, \overline{v}], \\ \widetilde{\eta}(0,x) = \eta^0(x), \\ \widetilde{\eta}_t(0,x) = \eta^1(x), \end{cases} \tag{2.20}$$

(where $\widehat{\Phi}$ and $\widehat{\phi}$ are given respectively by (2.13) and (2.15)) and

$$\begin{cases} \widetilde{v}_t - \nu\,\Delta\widetilde{v} + \nabla\widetilde{p}_0 = \widehat{F}\,[\overline{\eta}, \overline{v}, \nabla\overline{p}], \\ \nabla\cdot\widetilde{v} = \widehat{g}\,[\overline{\eta}, \overline{v}], \qquad \text{in } Q_T; \\ \widetilde{v}(0,x,z) = \widehat{v}^0(x,z), \qquad \text{in } \Omega; \\ \widetilde{v}(t,x,1) = \widetilde{\eta}_t(t,x)\,\vec{e_2}, \qquad \text{on } \Sigma_T; \\ \widetilde{v}(t,x,0) = 0, \qquad \text{on } \Lambda_T, \end{cases} \tag{2.21}$$

where $\widehat{F}$ is defined by (2.8). Note that $\nabla p = \nabla p_0$ if p has the form (2.16), since $\widehat{\phi}\,[\eta, v]$ depends only on t. For convenience we set

$$\overline{\phi}(t) := \widehat{\phi}\,[\overline{\eta}, \overline{v}]. \tag{2.22}$$

Next, given $(\overline{\eta}, \overline{p}_0, \overline{v})$ in a suitable set $\mathbb{K}$, we solve the linear systems (2.20), (2.21). Let $(\widetilde{\eta}, \widetilde{p}_0, \widetilde{v})$ be the solution of this problem. Then we look for a fixed point $(\widetilde{\eta}, \widetilde{p}_0, \widetilde{v}) = (\overline{\eta}, \overline{p}_0, \overline{v})$ in $\mathbb{K}$. This fixed point is just the solution $(\eta, \widehat{p}, \widehat{v})$ of (2.7), (2.12), where $\widehat{p}$ is given by (2.16).

The assumptions on the functions $\overline{\eta}, \overline{p}_0$, and $\overline{v}$ are the following (see (5.21)).

$$\begin{cases} \overline{\eta}_t \in L^\infty(0,T; H^{3/2}_{\#}(I)) \cap L^2(0,T; H^{5/2}_{\#}(I)), \\ \overline{\eta}_{tt} \in L^2(0,T; H^{-1/2}_{\#}(I)), \end{cases} \tag{2.23}$$

$$\overline{\eta} \in L^{\infty}(0,T; H^{7/2}_{\#}(I)), \qquad \text{only if } \alpha > 0. \tag{2.24}$$

$$\begin{cases} \overline{\eta}(0,x) = \eta^0(x), \\ \overline{\eta}_t(0,x) = \eta^1(x), \end{cases} \tag{2.25}$$

$$1 + \overline{\eta}(t,x) \geq \delta_0 \qquad \text{on } I_T, \tag{2.26}$$

and

$$\int_0^L \overline{\eta}(t,x)\, dx = 0, \qquad \forall\, t \in [0,T[. \tag{2.27}$$

Furthermore

$$\begin{cases} \overline{v} \in L^2(0,T; H^2_{\#}(\Omega)), \\ \overline{v}_t \in L^2(0,T; L^2(\Omega)), \\ \overline{p}_0 \in L^2(0,T; H^1_{\#}(\Omega)), \end{cases} \tag{2.28}$$

$$\int_0^L \overline{p}_0(t,x,1)\, dx = 0, \qquad \text{a.e. in } [0,T[, \tag{2.29}$$

$$\overline{v}(0,x,z) = \widehat{v}^0(x,z) \qquad \text{in } \Omega, \tag{2.30}$$

and

$$\overline{v}_1(t,x,z) = 0 \qquad \text{for } z = 0 \text{ and } z = 1, \tag{2.31}$$

where $\overline{v}_1$ is the first component of $\overline{v}$.

We recall that $(2.28)_{1,2}$ implies that (see [11], Chap. 1)

$$\overline{v} \in C([0,T]; H^1_{\#}(\Omega)). \tag{2.32}$$

Let us end this section by introducing some notations used in the sequel. It will be not difficult to consider the explicit dependence of all the constants that appear in the sequel in terms of the constants L, δ_0, ν, α, β, γ, σ (note that constants concerning Sobolev embedding theorems and similar, on I or on the rectangle Ω, depend only on L). However, in order to simplify notations, we denote simply by c positive constants that depend at most on the above constants. It is worth noting that the same symbol c may denote distinct constants, even in the same equation. A constant c is denoted by c_0, c_1, etc., if its specific value will be taken into account in some calculation.

In order to simplify notations we often drop the symbols I, Ω, and $[0,T]$. For instance, both $L^2(0,T; H^1_{\#}(\Omega))$ and $L^2(0,T; H^1_{\#}(I))$ may be denoted by the symbols $L^2(H^1_{\#})$ or $L^2(0,T; H^1_{\#})$. Or by, respectively, $L^2(H^1_{\#}(\Omega))$ and $L^2(H^1_{\#}(I))$. Analogous simplifications may be used in the sequel for other functional spaces.

For convenience, we define

$$
\begin{aligned}
|||v|||^2 &= \|v\|^2_{L^2(H^2_\#(\Omega))} + \|v_t\|^2_{L^2(L^2(\Omega))},\\
|||v,p_0|||^2 &= |||v|||^2 + \|\nabla p_0\|^2_{L^2(L^2(\Omega))},\\
|||\eta|||^2 &= \|\eta\|^2_{L^\infty(H^{3/2}_\#)} + \|\eta_t\|^2_{L^\infty(H^{3/2}_\#)} + \|\eta_t\|^2_{L^2(H^{5/2}_\#)},\\
[\eta_{tt}] &= \|\eta_{tt}\|_{L^2(H^{-1/2}_\#(I))}.
\end{aligned}
\tag{2.33}
$$

Note that we use the same symbol $|||\ \ |||$ to denote distinct norms.

3. The linearized structure model

Here we consider the linear problem

$$
\begin{cases}
\eta_{tt} - \beta\,\eta_{xx} - \gamma\,\eta_{txx} + \alpha\,\eta_{xxxx} + \sigma\,\eta = \psi(t,x),\\
\eta(0,x) = \eta^0(x),\\
\eta_t(0,x) = \eta^1(x),
\end{cases}
\tag{3.1}
$$

where the data ψ, η^0, η^1 are x-periodic. We also assume, for convenience, that (1.2), (1.10) hold and that

$$
\int_0^L \psi(t,x)\,dx = 0, \qquad \forall\, t \geq 0. \tag{3.2}
$$

We will not prove the (well-known) existence of the solution η but just show the particular *a-priori* estimates that will be used in the sequel.

We start by noting that x-periodic solutions to equation (3.1) must satisfy (1.15). In fact, by denoting the left-hand side of (1.15) by $y(t)$ and by integrating $(3.1)_1$ on I, one gets (due to the x-periodicity property and (3.2)) $y''(t)+\sigma\,y(t) = 0$ on I, $y(0) = y'(0) = 0$. Hence $y(t) = 0$.

The choice of the functional spaces done in the sequel follows from the claim of strong solutions $\widehat{v} \in L^2(0,T;H^2)$, $\widehat{v}_t \in L^2(0,T;L^2)$, and $\nabla\widehat{p} \in L^2(0,T;L^2)$. Hence, well known trace theorems show that the function $\widehat{\Phi}$ given by (2.13) belongs to $L^2(0,T;H^{1/2}(I))$ (provided that η is sufficiently regular; in turn, this regularity, depends just on $\widehat{\Phi}$ since it follows *via* the equation (2.12)).

We start by looking for estimates of η in "low" norms. We multiply both sides of equation $(3.1)_1$ by η_t and integrate on $(0,L)$. We denote here the $L^2(I)$ norm simply by $\|\,.\,\|$. One gets

$$
\frac{1}{2}\frac{d}{dt}\left(\|\eta_t\|^2 + \beta\,\|\eta_x\|^2 + \alpha\,\|\eta_{xx}\|^2 + \sigma\,\|\eta\|^2\right) + \gamma\,\|\eta_{tx}\|^2 \leq c\,\|\psi\|\|\eta_{tx}\| \tag{3.3}
$$

since, by (1.15), $\|\eta_t\| \leq c\|\eta_{tx}\|$. Hence, by using the Cauchy–Schwartz inequality,

$$\frac{d}{dt}\left(\|\eta_t\|^2 + \beta\,\|\eta_x\|^2 + \alpha\,\|\eta_{xx}\|^2 + \sigma\,\|\eta\|^2\right) + \gamma\,\|\eta_{tx}\|^2 \leq \frac{c}{\gamma}\|\psi\|^2. \tag{3.4}$$

From (3.4) it follows, by well-known devices, that

$$\begin{aligned} &\|\eta_t\|^2_{L^\infty(L^2)} + \beta\|\eta_x\|^2_{L^\infty(L^2)} + \alpha\|\eta_{xx}\|^2_{L^\infty(L^2)} + \sigma\|\eta\|^2_{L^\infty(L^2)} + \gamma\|\eta_{tx}\|^2_{L^2(L^2)} \\ &\leq \|\eta^1\|^2 + \beta\|\eta^0_x\|^2 + \alpha\|\eta^0_{xx}\|^2 + \sigma\|\eta\|^2 + \frac{c}{\gamma}\|\psi\|^2_{L^2(L^2)}. \end{aligned} \tag{3.5}$$

Note that the proof of this estimate on η in terms of the data relies just on the fact that $\gamma > 0$, as suggested by the coefficient $\frac{c}{\gamma}$. On the contrary, the constants α, β, σ may vanish. Strict positivity of each of these constants gives additional estimates, as seen from (3.5). Estimates due to the fact that β or σ are strictly positive are essentially included in that due to $\gamma > 0$. Without loosing generality, we will assume in the sequel that

$$\beta = \sigma = 0. \tag{3.6}$$

Concerning α the situation is different. If $\alpha > 0$ we obtain an independent estimate on η_{xx} (which could be used to simplify some points in our proofs). However we will assume that $\alpha = 0$ and made just some remarks concerning the case $\alpha > 0$.

Since we are looking for local solutions in time we will assume from now on that

$$0 < T \leq 1. \tag{3.7}$$

This allows us to simplify some equations by replacing, on the "right-hand sides" of our estimates, distinct powers of T by that with the smaller exponent.

We start by looking for an estimate of $\|\eta\|_{L^\infty(L^2)}$ (note that if $\sigma > 0$ such an estimate would be already included in (3.5)). As

$$\eta(t) = \eta(0) + \int_0^t \eta_t(\tau)d\tau$$

one easily gets

$$\|\eta\|_{L^\infty(L^2)} \leq \|\eta^0\| + \|\eta_t\|_{L^\infty(L^2)}, \tag{3.8}$$

where (3.7) have been used.

By addition of (3.5) and (3.8), and by taking into account (3.6), one shows that

$$\begin{aligned} &\|\eta\|^2_{L^\infty(L^2)} + \|\eta_t\|^2_{L^\infty(L^2)} + \alpha\,\|\eta_{xx}\|^2_{L^\infty(L^\infty)} + \gamma\,\|\eta_t\|^2_{L^2(H^1_\#)} \\ &\leq c\left(\|\eta^0\|^2 + \|\eta^1\|^2 + \alpha\,\|\eta^0_{xx}\|^2 + \|\psi\|^2_{L^2(L^2)}\right). \end{aligned} \tag{3.9}$$

Next we want to estimate higher norms of η. For convenience we set $\lambda = \eta_x$. One has, by differentiation of (3.1) with respect to x

$$\begin{cases} \lambda_{tt} - \gamma\,\lambda_{txx} + \alpha\,\lambda_{xxxx} = \psi_x, \\ \lambda(0,x) = \eta^0_x(x), \\ \lambda_t(0,x) = \eta^1_x(x). \end{cases} \tag{3.10}$$

By applying (3.9) to λ and by adding side by side the equation obtained in that way to (3.9) one gets

$$\begin{aligned}&\|\eta\|^2_{L^\infty(H^1_\#)} + \|\eta_t\|^2_{L^\infty(H^1_\#)} + \alpha\,\|\eta_{xx}\|^2_{L^\infty(H^1_\#)} + \gamma\,\|\eta_t\|^2_{L^2(H^2_\#)} \\ &\leq c\left(\|\eta^0\|^2_{H^1_\#} +, \|\eta^1\|^2_{H^1_\#} + \alpha\,\|\eta^0_{xx}\|^2_{H^1_\#} + \|\psi\|^2_{L^2(L^2)}\right).\end{aligned} \tag{3.11}$$

Note that here we have used the estimate

$$\left|\int_0^L \psi_x\,\lambda_t\,dx\right| \leq \left|\int_0^L \psi\,\lambda_{tx}\,dx\right| \leq \frac{\gamma}{2}\|\lambda_{tx}\|^2 + \frac{1}{2\gamma}\|\psi\|^2.$$

Finally we multiply (3.10) by $-\lambda_{txx}$, integrate over $(0,L)$ and argue as above. By adding the estimate obtained in that way to (3.11) it follows that

$$\begin{aligned}&\|\eta\|^2_{L^\infty(H^2_\#)} + \|\eta_t\|^2_{L^\infty(H^2_\#)} + \alpha\,\|\eta_{xx}\|^2_{L^\infty(H^2_\#)} + \gamma\,\|\eta_t\|^2_{L^2(H^3_\#)} \\ &\leq c\left(\|\eta^0\|^2_{H^2_\#} +, \|\eta^1\|^2_{H^2_\#} + \alpha\,\|\eta^0_{xx}\|^2_{H^2_\#} + \|\psi\|^2_{L^2(H^1_\#)}\right).\end{aligned} \tag{3.12}$$

Remark. We may replace everywhere $L^\infty(0,T)$ by $C([0,T])$. This result is not used in the sequel.

Finally, by interpolation between (3.11) and (3.12), we show that

$$\begin{aligned}&\|\eta\|^2_{L^\infty(H^{3/2}_\#)} + \|\eta_t\|^2_{L^\infty(H^{3/2}_\#)} + \widehat{\alpha}\,\|\eta\|^2_{L^\infty(H^{7/2}_\#)} + \gamma\,\|\eta_t\|^2_{L^2(H^{5/2}_\#)} \\ &\leq c\left(\|\eta^0\|^2_{H^{3/2}_\#} + \|\eta^1\|^2_{H^{3/2}_\#} + \alpha\,\|\eta^0\|^2_{H^{7/2}_\#} + \|\psi\|^2_{L^2(H^{1/2}_\#)}\right),\end{aligned} \tag{3.13}$$

where $\widehat{\alpha} = \min\{\alpha, 1\}$.

In the sequel we drop γ from the left-hand side of (3.13). Constants c may depend on γ (and on α, if $\alpha > 0$).

Recall that in Eq. (3.1) (see (2.20)) the function $\psi(x,t)$ is given by

$$\psi(x,t) = \widehat{\Phi}\,[\overline{\eta}, \overline{p}_0, \overline{v}] + \rho_1\widehat{\phi}\,[\overline{\eta}, \overline{v}]. \tag{3.14}$$

For clearness of exposition, the proofs of some technical estimates (which are partially straightforward) is postponed to section 6. In section 6 we prove the following estimate.

$$\begin{aligned}&\|\widehat{\Phi}\,[\overline{\eta}, \overline{v}, \overline{p}_0] + \rho_1\widehat{\phi}\,[\overline{\eta}, \overline{v}]\|^2_{L^2(H^{1/2}_\#(I))} \\ &\leq c\rho_1^2\|\nabla\overline{p}_0\|^2_{L^2(Q_T)} + c\rho_2^2\left(1 + \|\eta^0\|^3_{H^{5/2}_\#(I)} + T^{3/2}\|\eta_t\|^3_{L^2(H^{5/2}_\#(I))}\right)^2 |||\overline{v}|||^2.\end{aligned} \tag{3.15}$$

Hence, by applying the estimate (3.13) to the solution $\widetilde{\eta}$ of equation (2.20), we show that

$$\begin{aligned}|||\widetilde{\eta}|||^2 &:= \|\widetilde{\eta}\|^2_{L^\infty(H^{3/2}_\#)} + \|\widetilde{\eta}_t\|^2_{L^\infty(H^{3/2}_\#)} + \widehat{\alpha}\,\|\widetilde{\eta}\|^2_{L^\infty(H^{7/2}_\#)} + \|\widetilde{\eta}_t\|^2_{L^2(H^{5/2}_\#)} \\ &\le c\left(\|\eta^0\|^2_{H^{3/2}_\#} + \|\eta^1\|^2_{H^{3/2}_\#} + \alpha\,\|\eta^0\|^2_{H^{7/2}_\#}\right) + c\,\rho_1^2\,\|\nabla\overline{p}_0\|^2_{L^2(Q_T)} \\ &\quad + c\,\rho_2^2\left(1 + \|\eta^0\|^3_{H^{5/2}_\#(I)} + T^{3/2}\|\eta_t\|^3_{L^2(H^{5/2}_\#(I))}\right)^2 |||\overline{v}|||^2. \end{aligned} \tag{3.16}$$

Note that, if $\alpha > 0$, the definition of $|||\widetilde{\eta}|||$ given here has an additional term with respect to that in (2.33).

We also need a suitable estimate of the $L^2(0,T;H^{-1/2}(I))$ norm of the solution $\widetilde{\eta}$ of (2.20). In section 6 we also show that

$$\begin{aligned}|||\widetilde{\eta}_{tt}|||_{L^2(0,T;H^{-1/2}(\Gamma))} &\le c\rho_1\|\nabla\overline{p}_0\|_{L^2(Q_T)} \\ &\quad + c\rho_2 T^{1/8}\left(1 + \|\eta^0\|^2_{\infty,I} + |||\overline{\eta}|||^2\right)|||\overline{v}||| + cT^{1/2}|||\widetilde{\eta}|||.\end{aligned} \tag{3.17}$$

4. The linearized fluid model equations

Here we consider the linear system (2.21). In section 6 we will show that $\widehat{F}\,[\overline{\eta},\overline{v},\nabla\overline{p}] \in L^2(Q_T)$ and that

$$\begin{aligned}&\|\widehat{F}\,[\overline{\eta},\overline{v},\nabla\overline{p}]\|_{L^2(Q_T)} \\ &\le c\Big[\|\eta^0\|_{L^\infty(I)} + \|\eta^0_x\|_{L^\infty(I)} + \|\eta^0_x\|^2_{L^\infty(I)} + T^{1/3}\|\eta^0\|_{H^{5/2}_\#(I)} \\ &\quad + T^{1/2}\left(\|\overline{\eta}_t\|_{L^\infty(H^{3/2}_\#(I))} + \|\overline{\eta}_t\|_{L^2(H^{5/2}_\#(I))}\right) + T\|\overline{\eta}_t\|^2_{L^2(H^{5/2}_\#(I))}\Big]\,|||\overline{v}||| \\ &\quad + \left(\|\eta^0\|_{L^\infty(I)} + \|\eta^0_x\|_{L^\infty(I)} + c\,T^{1/2}\,\|\overline{\eta}_t\|_{L^2(H^{5/2}_\#(I))}\right)\,\|\nabla\overline{p}\|_{L^2(Q_T)} \\ &\quad + c\,T^{1/4}\left(1 + \|\eta^0\|_{H^{5/2}_\#(I)} + \|\overline{\eta}_t\|_{L^2(H^{5/2}_\#(I))}\right)\,|||\overline{v}|||^2,\end{aligned} \tag{4.1}$$

where the constants c depend at most on ν, δ_0 and L (for notations, see section 6).

On the other hand the reader easily verifies that

$$\widehat{g}\,[\overline{\eta},\overline{v}] = \nabla\cdot\overline{w}, \tag{4.2}$$

where $\overline{w}$ is given by

$$\overline{w}(t,x,z) := \overline{w}\,[\overline{\eta},\overline{v}] = (-\overline{\eta}\,\overline{v}_1)\,\vec{e_1} + (z\overline{\eta}_x\overline{v}_1)\,\vec{e_2}. \tag{4.3}$$

Clearly $\overline{w}$ is x-periodic and satisfies

$$\overline{w}(t,x,0) = \overline{w}(t,x,1) = 0. \tag{4.4}$$

In section 6 we also show that

$$\nabla\cdot\overline{w} \in L^2(0,T;H^1_\#(\Omega)), \quad \overline{w}_t \in L^2(0,T;L^2(\Omega)),$$

and that

$$\begin{cases} \|\nabla\cdot\overline{w}\|_{L^2(H^1_\#)} \le c\Big(\|\eta^0_x\|_{L^\infty(I)} + T^{1/3}\|\eta^0\|_{H^{5/2}_\#} \\ \qquad + T^{1/2}\,\|\overline{\eta}_t\|_{L^2(H^{5/2}_\#)}\Big)\,|||\overline{v}|||, \\ \|\overline{w}_t\|_{L^2(L^2)} \le c\left(\|\eta^0\|_{L^\infty(I)} + \|\eta^0_x\|_{L^\infty(I)}\right)\,\|\overline{v}_t\|_{L^2(Q_T)} \\ \qquad + cT^{1/2}\left(\|\overline{\eta}_t\|_{L^\infty(H^{3/2}_\#)} + \|\overline{\eta}_t\|_{L^2(H^{5/2}_\#)}\right)|||\overline{v}|||, \end{cases} \tag{4.5}$$

where c depends only on L.

In this section we look for solutions $\widetilde{v}$, $\nabla\widetilde{p}_0$ of system (2.21) in the class

$$\widetilde{v}\in H^{2,1}(Q_T), \quad \nabla\widetilde{p}_0\in L^2(Q_T). \tag{4.6}$$

The (necessary) compatibility conditions in order to get solutions in the above class (4.6) are satisfied. In fact the matching, at time $t=0$, between the initial data $\widehat{v}^0$ and the boundary data follows from (2.11) (together with (2.30), $(2.20)_2$ and $(2.20)_3$). Concerning the matching, for $t\ge 0$, between $\widehat{g}$ and the boundary data note that, due to (4.4) and the x-periodicity,

$$\int_\Omega \widehat{g}\,[\overline{\eta},\overline{v}] = \int_\Omega \nabla\cdot\overline{w} = 0. \tag{4.7}$$

This agrees with the fact that the flux of $\widetilde{v}$ through the boundary of Ω vanishes for each $t\ge 0$ since the assumption (2.27) yields

$$\int_0^L \overline{\eta}_t\,\vec{e_2}\cdot\vec{e_2} = 0.$$

We want to show that the solution $\widetilde{v}$, $\nabla\widetilde{p}_0$ of (2.21) satisfies the estimate

$$\begin{aligned} \|\widetilde{v}\|^2_{H^{2,1}_\#(Q_T)} + \|\nabla\widetilde{p}_0\|^2_{L^2(Q_T)} \le c\Big(&\|\widehat{v}^0\|^2_{H^1_\#(\Omega)} + \|\widehat{F}\,[\overline{\eta},\overline{v},\nabla\overline{p}_0]\|^2_{L^2(Q_T)} \\ &+ \|\nabla\cdot\overline{w}\|^2_{L^2(H^1_\#(\Omega))} + \|\overline{w}_t\|^2_{L^2(Q_T)} \\ &+ \|\overline{\eta}_t\|^2_{H^{3/2,3/4}_\#(\Sigma_T)} + \|\overline{\eta}_{tt}\|^2_{L^2(H^{-1/2}_\#(\Gamma))}\Big), \end{aligned} \tag{4.8}$$

where

$$H^{2,1}(Q_T) = L^2(0,T;\,H^2(\Omega))\cap H^1(0,T;\,L^2(\Omega)),$$
$$H^{3/2,3/4}(\Sigma_T) = L^2(0,T;\,H^{3/2}(\Omega)\cap H^{3/4}(0,T;\,L^2(\Omega)).$$

The symbol # has the usual meaning (x-periodicity).

The fact that the boundary data $\eta_t\,\vec{e_2}$ is not tangential to the boundary gives rise to an interesting problem connected to trace spaces of divergence free vector fields. Set

$$\mathcal{H}^s(Q_T) = \left\{ v \in L^2(0,T;H^s(\Omega)), \quad v_t \in L^2(0,T;H^{s-2}(\Omega)) \right\}, \; s \geq \frac{1}{2},$$

and

$$\mathcal{V}^s(Q_T) = \{ v \in \mathcal{H}^s(Q_T) \,:\, \nabla \cdot v = 0 \}.$$

A main problem is to determine the exact functional space consisting of the traces on Σ_T of the vector fields $v \in \mathcal{V}^s(Q_T)$. This problem is studied by Fursikov, Gunzburger and Hou (in the Hilbertian case L^2) in references [7], [8], and by Solonnikov (in the L^p case, $1 < p < \infty$) in references [15], [16]. In reference [7] the authors consider the 2-D case for $s = 1$ and divergence free solutions (see,in particular, Theorem 4.2). In reference [8] the authors consider the 3-D case for an arbitrary real $s \geq 1/2$ and divergence free solutions (see, in particular, the Theorems 2.1 and 2.2). In references [7], [8] many other very interesting related problems are studied. In reference [15] the author (by using a complete different method) considers the 3-D case in a half-space for not necessarily divergence free solutions and $s = 2$. See, in particular, the Theorem 1.1. In reference [16] the author consider the 3-D problem in arbitrary domains. See, in particular, the Theorem 1.2. For the case of tangential boundary data see, for instance, Farwig and Sohr [5] and references therein.

The above authors do not consider the x-periodic case. However this does not change any essential feature in their proofs.

Here we are interested on the 2-D case for vector fields $u \in \mathcal{H}^2(Q_T) = H^{2,1}(Q_T)$, satisfying $\nabla \cdot u = \nabla \cdot \overline{w}$, where $\overline{w}$ is a given vector field such that (4.4) and (4.5) holds (actually, our $\overline{w}$ is given by equation (4.3)). We remark that, if the vector field $\overline{w}$ belong to $L^2(0,T;H^2_\#(\Omega))$, then we may replace in equation (2.21) $\widetilde{v}$ by $\widetilde{v} - \overline{w}$, in order to work with a divergence free unknown $\widetilde{v} - \overline{w}$. This device can be done just if $\alpha > 0$, since in this case we could show that $\overline{w} \in H^{2,1}(Q_T)$. However we are mainly interested in the case $\alpha = 0$. Hence a more careful device is needed.

Since for 2-D problems and $s = 2$ the results we need are not explicitly considered by the above authors, we will contemplate it in the sequel. This will be done by resorting to the method used for $s = 1$ in reference [7] and to some results already proved in reference [8] .

For the sake of clearness let us state, in a compact form, the kind of result that will be used here. For vector fields v defined on Q_T, denote respectively by $\gamma_{0,\tau}$ and $\gamma_{0,n}$ the trace operators on Σ_T of the tangential and the normal components of v. Set

$$\begin{cases} G^2_n(\Sigma_T) := \left\{ \vec{a}_n \in L^2(0,T;H^{3/2}_\#(\Gamma)) \cap H^1(0,T;H^{-1/2}_\#(\Gamma)) : \int_\Gamma \vec{a}_n \, d\Gamma = 0 \right\}, \\ G^2_\tau(\Sigma_T) := L^2(0,T;H^{3/2}_\#(\Gamma)) \cap H^{3/4}(0,T;L^2_\#(\Gamma)). \end{cases}$$

Let now $\overline{w}$ be as above and assume, for the time being, that we can construct a vector field $u \in H^{2,1}_{\#}(Q_T)$ such that

$$\begin{cases} \nabla \cdot u = \nabla \cdot \overline{w}, \\ u(t,x,1) = \overline{\eta}_t \, \vec{e_2}\,, \\ u(t,x,0) = 0, \end{cases} \tag{4.9}$$

and that

$$\|u\|^2_{H^{2,1}_{\#}} \le c\Big(\|\nabla \cdot \overline{w}\|^2_{L^2(H^1_{\#})} + \|\overline{w}_t\|^2_{L^2(L^2)} + \|\eta_t\|^2_{H^{3/2,3/4}_{\#}(\Sigma_T)} + \|\eta_{tt}\|^2_{L^2(H^{-1/2}_{\#}(\Gamma))}\Big). \tag{4.10}$$

Note that, due to

$$\|u\|_{C(0,T;H^1_{\#})} \le c\|u\|_{H^{2,1}_{\#}(Q_T)},$$

one also has

$$\|u_{|t=0}\|_{H^1_{\#}} \le \text{ left-hand side of (4.10)}. \tag{4.11}$$

By assuming the existence of the above u, well known results satisfied by solutions of the linear homogeneous system

$$\begin{cases} (\widetilde{v} - u)_t - \nu\Delta(\widetilde{v} - u) + \nabla\widetilde{p}_0 = \widehat{F} - (u_t - \nu\Delta u), \\ \nabla \cdot (\widetilde{v} - u) = 0 \qquad \text{in } Q_T, \\ (\widetilde{v} - u)_{|t=0} = \widehat{v}^0 - u_{|t=0}, \\ (\widetilde{v} - u)(t,x,z) = 0, \qquad \text{for } z = 0 \text{ and } z = 1, \end{cases} \tag{4.12}$$

show that (4.8) holds. Hence our task becomes proving the existence of the solution u of problem (4.9), (4.10). The solution u will be obtained in the form $u = u^{(1)} + u^{(2)}$. The only purpose of the introduction of the auxiliary vector field $u^{(1)}$ is to reduce the search of u to that of the divergence free vector field $u^{(2)} = u - u^{(1)}$ (see (4.14) and (4.15)). We start by constructing $u^{(1)}$.

For each fixed t consider the problem

$$\begin{cases} \Delta q = \nabla \cdot \overline{w} \qquad \text{in } \Omega, \\ \frac{\partial q}{\partial n}(x,z) = 0 \qquad \text{for } z = 0 \text{ and } z = 1. \end{cases} \tag{4.13}$$

For convenience we impose that

$$\int_\Omega q = 0.$$

Clearly, q should be x-periodic.

Due to (4.5) and (4.7) the above problem admits a unique solution $q \in L^2(0,T;H^3_{\#}(\Omega)$. Set $u^{(1)} := \nabla q$. Then $u^{(1)} \in L^2(0,T;H^2_{\#}(\Omega))$, moreover

$$\begin{cases} \nabla u^{(1)} = \nabla \cdot \overline{w}, \\ u^{(1)} \cdot n = 0, \qquad \text{for } z = 0 \text{ and } z = 1, \end{cases} \tag{4.14}$$

and

$$\|u^{(1)}\|_{L^2(H^2_\#)} \leq c\|\nabla \cdot \overline{w}\|_{L^2(H^1_\#)}. \tag{4.15}$$

Furthermore, by using (4.13) and (4.4), one shows that

$$\int_\Omega \nabla(q(t+h) - q(t)) \cdot \nabla\phi = \int_\Omega (\overline{w}(t+h) - \overline{w}(t)) \cdot \nabla\phi,$$

for each $\phi \in H^1_\#(\Omega)$. Note that, in general, we may replace in (4.4) the assumption $\overline{w} = 0$ simply by $\overline{w} \cdot n = 0$. This fact is not used here.

By setting $\phi = q(t+h) - q(t)$ it follows that

$$\left\| \frac{\nabla q(t+h) - \nabla q(t)}{h} \right\|_{L^2} \leq \left\| \frac{\overline{w}(t+h) - \overline{w}(t)}{h} \right\|_{L^2}.$$

Since $\overline{w}_t \in L^2(0,T;L^2(\Omega))$, it follows that

$$\|u_t^{(1)}\|_{L^2(L^2)} \leq \|\overline{w}_t\|_{L^2(L^2)}.$$

This estimate together with (4.15), shows that

$$\|u^{(1)}\|^2_{H^{1,2}_\#(Q_T)} \leq c\left(\|\nabla \cdot \overline{w}\|^2_{L^2(H^1_\#)} + \|\overline{w}_t\|^2_{L^2(L^2)}\right). \tag{4.16}$$

□

In the sequel we denote by Σ the union of Σ_T and Λ_T and by $u_\tau^{(1)}$ the tangential component of $u^{(1)}$ on Σ (*i.e.* $u_\tau^{(1)} = u_1^{(1)}$). Since $H^{3/2,3/4}(\Sigma)$ is just the trace space of $H^{2,1}(Q_T)$ on Σ (see Lions–Magenes [12], vol II, Chap. 4,Theorem 2.3) it follows that

$$\|u_\tau^{(1)}\|_{H^{3/2,3/4}_\#(Q_T)} \leq c\|u^{(1)}\|_{H^{2,1}_\#(Q_T)}. \tag{4.17}$$

Next, we look for a solution $u^{(2)}$ of the problem

$$\begin{cases} \nabla \cdot u^{(2)} = 0 & \text{on } Q_T; \\ u^{(2)} \cdot n = \overline{\eta}_t, & \\ u_\tau^{(2)} = -u_\tau^{(1)} & \text{on } \Sigma_T; \\ u^{(2)} \cdot n = 0, & \\ u_\tau^{(2)} = -u_\tau^{(1)} & \text{on } \Lambda_T, \end{cases} \tag{4.18}$$

such that

$$\|u^{(2)}\|^2_{H^{2,1}_\#(Q_T)} \leq c\left(\|\overline{\eta}_t\|^2_{L^2(H^{3/2}_\#(\Gamma))} + \|\overline{\eta}_{tt}\|^2_{L^2(H^{-1/2}_\#(\Gamma))} + \|u_\tau^{(1)}\|^2_{H^{3/2,3/4}_\#(\Sigma)}\right). \tag{4.19}$$

If such a solution exists, the vector field $u = u^{(1)} + u^{(2)}$ satisfies (4.9) and (4.10).

Let us show the existence of $u^{(2)}$. As already pointed out, we extend here to solutions in $\mathcal{H}^2(Q_T)$ the argument developed in reference [7] for solutions in $\mathcal{H}^1(Q_T)$.

The construction has a local character, hence we can work separately for the two boundaries $z = 0$ and $z = 1$. We start by considering the problem concerning the boundary $z = 1$, *i.e.*, Σ_T. In the sequel we denote $u^{(2)}$, $\overline{\eta}_t$ and $-u_\tau^{(1)}$ by respectively w, a_n and a_τ. Hence we want to solve the problem

$$\begin{cases} \nabla \cdot w = 0 & \text{in } Q_T; \\ w_2 = a_n, & \\ w_1 = a_\tau & \text{on } \Sigma_T, \end{cases} \tag{4.20}$$

where $\vec{a} = a_\tau \, \vec{e_1} + a_n \, \vec{e_2}$,

$$\int_0^L a_n \, dx = 0, \qquad \forall t \in [0,T], \tag{4.21}$$

and

$$a_n \in B^0, \qquad a_\tau \in B_1. \tag{4.22}$$

For convenience, we set

$$\begin{cases} B^0 := L^2(0,T; H_\#^{3/2}(\Gamma)) \cap H^1(0,T; H_\#^{-1/2}(\Gamma)), \\ B_1 := L^2(0,T; H_\#^{3/2}(\Gamma)) \cap H^{3/4}(0,T; L_\#^2(\Gamma)). \end{cases}$$

Following [7], we look for the solution w in the form

$$w = \operatorname{Rot} F,$$

where F is a scalar field. Hence

$$w_1 = \partial_z F, \quad w_2 = -\partial_x F.$$

We impose to F that

$$F(t,0,0) = 0. \tag{4.23}$$

Equations $(4.20)_{2,3}$ become respectively

$$\begin{cases} (\partial_x F)_{|\Gamma} = -a_n, \\ (\partial_z F)_{|\Gamma} = a_\tau, \end{cases} \tag{4.24}$$

for each $t \in [0,T]$. Due to (4.21) and (4.23), equation $(4.24)_1$ is equivalent to

$$F(t,x,0) = \lambda(t,x) := \int_0^x -a_n(t,\xi) \, d\xi. \tag{4.25}$$

Set

$$\mathcal{H}_\#^3(Q_T) = L^2(0,T; H_\#^3(\Omega)) \cap H^1(0,T; H_\#^1(\Omega)) \tag{4.26}$$

and denote by γ_0 the trace operator on Σ_T and by γ_1 the trace operator of the normal derivative ∂_z on Σ_T.

Assume, for the time being, that

$$\begin{cases} \gamma_0(\mathcal{H}^3_\#(Q_T)) = B_0, \\ \gamma_1(\mathcal{H}^3_\#(Q_T)) = B_1, \end{cases} \tag{4.27}$$

where

$$B_0 := L^2(0,T;H^{5/2}_\#(\Gamma)) \cap H^1(0,T;H^{1/2}_\#(\Gamma)),$$

and set $\gamma = (\gamma_0, \gamma_1)$. Also assume that there exists a linear continuous lifting γ^{-1},

$$\gamma^{-1}: \; B_0 \times B_1 \to \mathcal{H}^3_\#(Q_T),$$

which is a right inverse of (γ_0, γ_1), i.e., $\gamma \circ \gamma^{-1} = I$. The verification of these two properties is given below.

Let now a_τ and a_n, satisfying (4.21) and (4.22), be given. By using (4.25) we construct the function $\lambda \in B_0$. Next, by applying γ^{-1} to the pair (λ, a_τ), we get a function $F \in \mathcal{H}^3_\#(Q_T)$ such that $\gamma_0 F = \lambda$ and $\gamma_1 F = a_\tau$. Hence $w = \mathrm{Rot}\, F$ satisfies (4.20). As $F \in \mathcal{H}^3_\#(Q_T)$, it follows that w belongs to $L^2(0,T;H^2_\#(\Omega)) \cap H^1(0,T;L^2(\Omega))$, moreover

$$\|w\|^2_{L^2(0,T;H^2_\#)} + \|w_t\|^2_{L^2(0,T;L^2)} \le c\left(\|a_n\|^2_{B^0} + \|a_\tau\|^2_{B_1}\right) . \tag{4.28}$$

Next we replace $w = \mathrm{Rot}\, F$ by $w = \mathrm{Rot}\,(\phi\, F)$, where $\phi = \phi(z)$ is a C^∞ function such that $\phi(z) = 1$ near $z = 1$ and $\phi(z) = 0$ near $z = 0$. The new vector field w still satisfies the above properties $(4.18)_{1,2,3}$ and (4.28), moreover it vanishes near Λ_T.

Finally, by setting $a_n = 0$, $a_\tau = -u^{(1)}_\tau$ on Λ_T, we construct, in a similar way, a function $\overline{F}$ such that the vector field $w = \mathrm{Rot}\, \overline{F}$ satisfies $(4.18)_{1,4,5}$. By considering a C^∞ function $\overline{\phi}(z)$ such that $\overline{\phi}(z) = 1$ near $z = 0$ and $\overline{\phi}(z) = 0$ near $z = 1$, we see that the vector field $\mathrm{Rot}\,(\overline{\phi}\, \overline{F})$ satisfies $(4.18)_{1,4,5}$ and vanishes near $z = 1$.

The vector field $u^{(2)} = \mathrm{Rot}\,(\phi\, F + \overline{\phi}\, oF)$ satisfies (4.18) and (4.19). □

It remains to show both (4.27) and the existence of γ^{-1}. We will see that these properties follow from results already proved in reference [8]. The existence of the restriction operator $\gamma = (\gamma_0, \gamma_1)$ follows from the theorem 3.1 in the above reference, where $s = 3$. This is shown by using the equation (3.26) in reference [8] for $k = 0$ and the equation (3.27) for $k = 1$. The existence of the lifting operator γ^{-1} follows from the theorem 3.2 in the above reference. Here we use (3.44) for $k = 0$ and (3.45) for $k = 1$. Concerning the case $k = 1$, note that $1/2 < s - k < 5/2$. Hence, for $k = 1$ and $s = 3$, the space on the left-hand side of equation (3.45) in the above reference is just our space B_1. See the Remark 3.2 in reference [8]. Consequently, as already shown, there exists a solution u of system (4.9) satisfying (4.10). As explained above, this guarantees the existence of a (unique) solution u of system (2.21) such that (4.8) holds.

From (4.8), (4.1) and (4.5) we finally get the estimate

$$
\begin{aligned}
&\|\widetilde{v}\|^2_{H^{2,1}_{\#}(Q_T)} + \|\nabla \widetilde{p}_0\|^2_{L^2(Q_T)} \\
&\le c\|\widetilde{v}^0\|^2_{H^1_{\#}(\Omega)} + c\Big(\|\eta^o\|^2_{L^\infty(I)} + \|\eta^o_x\|^2_{L^\infty(I)} \\
&\quad + \|\eta^o_x\|^4_{L^\infty(I)} + T^{2/3}\|\eta^o\|^2_{H^{5/2}_{\#}(I)} \\
&\quad + T\|\overline{\eta}_t\|^2_{L^\infty(H^{3/2}_{\#})} + T\|\overline{\eta}_t\|^2_{L^2(H^{5/2}_{\#})} + T^2\,\|\overline{\eta}_t\|^4_{L^2(H^{5/2}_{\#})}\Big)|||\overline{v}|||^2 \\
&\quad + c\Big(\|\eta^o\|^2_{L^\infty(I)} + \|\eta^o_x\|^2_{L^\infty(I)}\Big)\|\overline{v}_t\|^2_{L^2(Q_T)} \\
&\quad + cT^{1/2}\Big(1 + \|\eta^o\|^2_{H^{5/2}_{\#}(I)} + \|\overline{\eta}_t\|^2_{L^2(H^{5/2}_{\#})}\Big)|||\overline{v}|||^4 \\
&\quad + c\Big(\|\eta^o\|^2_{L^\infty(I)} + \|\eta^o_x\|^2_{L^\infty(I)} + T\,\|\overline{\eta}_t\|^2_{L^2(H^{5/2}_{\#})}\Big)\|\nabla\overline{p}_0\|^2_{L^2(Q_T)} \\
&\quad + c\|\overline{\eta}_t\|^2_{H^{3/2,3/4}_{\#}(\Sigma_T)} + c\|\overline{\eta}_{tt}\|^2_{L^2(H^{-1/2}_{\#}(\Gamma))}.
\end{aligned}
\tag{4.29}
$$

5. Existence of the fixed point

For convenience we consider in the sequel the case $\alpha = 0$. The slight modifications to be done if $\alpha > 0$ are left to the interested reader.

The construction of the fixed point is based on the estimates (4.29), (3.16), and (3.17). We use here a fruitful method, introduced in reference [1], which allows a straightforward application of Schauder's fixed point theorem to proving existence of sufficiently strong solutions to nonlinear PDE problems in reflexive Banach spaces (more generally, in duals of B-spaces). This device has been applied to many specific problems by several authors (without a possible, but necessarily restrictive, formalization of the method).

We start by proving (5.3) below. One has

$$
\|\overline{\eta}_t\|_{L^2(H^{3/2}_{\#}(I))} \le c T^{1/2}\,\|\overline{\eta}_t\|_{L^\infty(H^{3/2}_{\#}(I))} \tag{5.1}
$$

and also

$$
\|\overline{\eta}_t\|_{H^{3/4}(L^2(I))} \le c\|\overline{\eta}_t\|^{1/4}_{L^2(H^{3/2}_{\#}(I))}\|\overline{\eta}_t\|^{3/4}_{H^1(H^{-1/2}_{\#}(I))}.
$$

It readily follows that

$$
\|\overline{\eta}_t\|_{H^{3/4}(L^2(I))} \le c T^{1/8}\left(\|\overline{\eta}\|_{L^\infty(H^{3/2}_{\#}(I))} + \|\overline{\eta}_t\|^{1/4}_{L^\infty(H^{3/2}_{\#})}\|\overline{\eta}_{tt}\|^{3/4}_{L^2(H^{-1/2}_{\#})}\right). \tag{5.2}
$$

In particular, by (5.1) (5.2), one easily gets

$$
\|\overline{\eta}_t\|^2_{H^{3/2,3/4}_{\#}(\Sigma_T)} \le c T^{1/4}(|||\overline{\eta}|||^2 + \|\overline{\eta}_{tt}\|^2_{L^2(H^{-1/2}_{\#})}). \tag{5.3}
$$

□

We set

$$
\begin{aligned}
A_0 &= \|\eta^0\|^2_{L^\infty(I)} + \|\eta^0_x\|^2_{L^\infty(I)} + \|\eta^0_x\|^4_{L^\infty(I)}\,,\\
B_0 &= \|\eta^0\|^2_{H^{5/2}_\#(I)}\,,\\
C_0 &= \|\eta^0\|^2_{H^{3/2}_\#(I)} + \|\eta^1\|^2_{H^{3/2}_\#(I)}\,,\\
D_0 &= \rho_1^2 + \rho_2^2(1 + \|\eta^0\|^3_{H^{5/2}_\#(I)})\,,\\
E_0 &= \|\widehat{v}_0\|^2_{H^1_\#(\Omega)}\,.
\end{aligned}
$$

Note that, in the proofs below, these quantities are fixed constants since the initial data are given. From (4.29) and (5.3) it follows that

$$
\begin{aligned}
|||\widetilde{v},\widetilde{p}_0|||^2 \leq c_1\,E_0 + c_2\,A_0|||\overline{v},\overline{p}_0|||^2 + c_3\,[\widetilde{\eta}_{tt}]^2 \\
+ c\,T^{2/3}\,B_0|||\overline{v},\overline{p}_0|||^2 + c\,T(|||\overline{\eta}|||^2 + |||\overline{\eta}|||^4)|||\overline{v},\overline{p}_0|||^2 \qquad (5.4)\\
+ c\,T^{1/2}(1 + B_0 + |||\overline{\eta}|||^2)|||\overline{v},\overline{p}_0|||^4 + cT^{1/4}|||\overline{\eta}|||^2 + c\,T^{1/4}\,[\overline{\eta}]^2\,.
\end{aligned}
$$

On the other hand (3.16) and (3.17) show that

$$
|||\widetilde{\eta}|||^2 \leq c_4(C_0 + D_0 + \rho_2^2\,T^{3/2}|||\overline{\eta}|||^{3/2})|||\overline{v},\overline{p}_0|||^2 \tag{5.5}
$$

and that

$$
[\widetilde{\eta}_{tt}]^2 \leq c_0\,\rho_1^2|||\overline{v},\overline{p}_0|||^2 + c\,\rho_2 T^{1/4}(1 + A_0^2 + |||\overline{\eta}|||^2)|||\overline{v},\overline{p}_0|||^2 + c_8 T^{1/2}|||\overline{\eta}|||. \tag{5.6}
$$

We impose to the initial data η_0 that the norms $\|\eta^0\|_{L^\infty(I)}$ and $\|\eta^0_x\|_{L^\infty(I)}$ are sufficiently small, in such a way that

$$
c_2\,A_0 \leq \frac{1}{2}. \tag{5.7}
$$

Now we introduce, in a constructive way, the bounds to be imposed on the norms of the functions $(\overline{\eta},\overline{v},\overline{p}_0)$ in order to get the desired fixed point $(\eta,\widetilde{v},\widetilde{p}_0) = (\overline{\eta},\overline{v},\overline{p}_0)$ by means of Schauder's fixed point theorem.

We assume that the functions $\overline{v}$ and $\overline{p}_0$ satisfy

$$
|||\overline{v},\overline{p}_0|||^2 \leq 4(c_1 E_0 + K) := K_0^2. \tag{5.8}
$$

The value of the positive constant K is fixed (sufficiently large) in such as way that the set $\mathbb{K}$, see (5.21) below) is not empty. Note that if K_0 is too small than assumption (2.30) cannot be satisfied by any $\overline{v}$.

Concerning the function $\overline{\eta}$, we assume that

$$
|||\overline{\eta}|||^2 \leq c_4(C_0 + D_0)(4c_1\,E_0 + 4K) + K := K_1^2, \tag{5.9}
$$

and that

$$
[\overline{\eta}_{tt}]^2 \leq \frac{1}{4c_3}(4c_1\,E_0 + 4K) + \frac{K}{2c_3} := K_2^2. \tag{5.10}
$$

The positive constant K is fixed in such a way that the set $\mathbb{K}$ below (see (5.15)) is not empty, *i.e.*, in such a way that (2.25)–(2.27) can be satisfied for some $\overline{\eta}$. We increase the previous value of K, if necessary.

Finally we impose to ρ_1 to be sufficiently small in such a way that

$$c_3\, c_0\, \rho_1^2 \leq \frac{1}{4}. \tag{5.11}$$

□

By using (5.7)–(5.11) if follows from (5.4) that

$$|||\widetilde{v}, \widetilde{p}_0|||^2 \leq c_1\, E_0 + 2(c_1\, E_0 + K) + \frac{K}{2} + (c_1\, E_0 + K) + c_5\, T^{1/4} \Lambda_0, \tag{5.12}$$

where Λ_0 is a well determined constant.

On the other hand (5.5) yields

$$|||\widetilde{\eta}|||^2 \leq c_4(C_0 + D_0) \cdot 4(c_1\, E_0 + K) + c_6\, T^{3/2} \Lambda_1, \tag{5.13}$$

where Λ_1 is a fixed constant.

Finally, (5.6) yields

$$[\widetilde{\eta}_{tt}]^2 \leq \frac{1}{4c_3}(4c_1\, E_0\, + 4K) + c_7\, T^{1/4}\, \Lambda_2 + c_8\, T^{1/2} |||\widetilde{\eta}|||^2. \tag{5.14}$$

Next we impose smallness conditions on T.

We assume that

$$c_5\, T^{1/4} \Lambda_0 \leq \frac{K}{2}. \tag{5.15}$$

Hence (5.12) gives

$$|||\widetilde{v}, \widetilde{p}_0|||^2 \leq 4(c_1\, E_0 + K) = K_0^2, \tag{5.16}$$

that corresponds to (5.8). We also assume that

$$c_6\, T^{3/2} \Lambda_1 \leq K \tag{5.17}$$

in such a way that (5.13) gives

$$|||\widetilde{\eta}|||^2 \leq c_4(C_0 + D_0)(4c_1\, E_0 + 4K) + K = K_1^2, \tag{5.18}$$

which corresponds to (5.9). Finally we impose that

$$c_7\, T^{1/4} \Lambda_2 + c_8\, T^{1/2} K_1 \leq \frac{K}{2c_3}, \tag{5.19}$$

in such a way that (5.14), together with (5.18), yields

$$[\widetilde{\eta}_{tt}]^2 \leq \frac{1}{4c_3}(4c_1\, E_0 + 4\, K) + \frac{K}{2c_3}, \tag{5.20}$$

which corresponds to (5.10).

□

The above calculations lead to defining the set $\mathbb{K}$ as follows.

$$\mathbb{K} := \{(\overline{\eta}, \overline{v}, \overline{p}_0) : \ (2.23)–(2.31) \text{ and } (5.8)–(5.10) \text{ hold}\}, \tag{5.21}$$

where T is fixed in such a way that (5.15), (5.17), (5.19) are satisfied.

Next, denote by $\mathcal{T}$ the map

$$\mathcal{T}(\overline{\eta}, \overline{v}, \overline{p}_0) = (\widetilde{\eta}, \widetilde{v}, \widetilde{p}_0)$$

defined by (2.20), (2.21). As shown by equations (5.16), (5.18), (5.20) one has

$$\mathcal{T}(\mathbb{K}) \subset \mathbb{K}.$$

The set $\mathbb{K}$ is clearly a convex, bounded subset of the Hilbert space

$$\mathcal{H} = \left\{(\overline{\eta}, \overline{v}, \overline{p}_0) \in L^2(\Sigma_T) \times L^2(Q_T) \times H^{-1}(0,T; L^2(\Omega))\right\}.$$

Note that $(2.28)_3$ together with (2.29) are equivalent to

$$\overline{p}_0 \in L^2(0,T; \dot{H}^1_{\#}(\Omega)).$$

It is easily seen that the immersion

$$L^2(0,T; \dot{H}^1_{\#}(\Omega)) \hookrightarrow H^{-1}(0,T; L^2(\Omega)) \tag{5.22}$$

is compact.

Let us show that $\mathbb{K}$ is a compact subset of $\mathcal{H}$.

$\mathbb{K}$ is closed: in fact, assume that a sequence $(\overline{\eta}^n, \overline{v}^n, \overline{p}_0^n) \in \mathbb{K}$ and that $(\overline{\eta}^n, \overline{v}^n, \overline{p}_0^n) \to (\overline{\eta}, \overline{v}, \overline{p}_0)$ in $\mathcal{H}$. As $\overline{\eta}^n \to \overline{\eta}$ in $L^2(0,T; L^2(I))$ it follows that $\overline{\eta}_t^n \to \overline{\eta}_t$ and $\overline{\eta}_{tt}^n \to \overline{\eta}_{tt}$ in $\mathcal{D}'(0,T; L^2(I))$, where this last symbol denotes the functional space of distributions on $]0,T[$ with values in $L^2(I)$.

Since $|||\overline{\eta}^n||| \le K_1$, well-known results on compact embeddings show that $\overline{\eta}^n \rightharpoonup \overline{\eta}$ weakly-* in $L^\infty(0,T; H^{3/2}(I))$, that $\overline{\eta}_t^n \rightharpoonup \overline{\eta}_t$ weakly in $L^2(0,T; H^{5/2}(I))$ and weakly-* in $L^\infty(0,T; H^{3/2}(I))$, and that $\overline{\eta}_{tt}^n \rightharpoonup \overline{\eta}_{tt}$ weakly in $L^2(0,T; H^{-1/2}(I))$.

By the lower semi-continuity of the norms with respect to the above weak convergences it follows that

$$|||\overline{\eta}||| \le \liminf_{n\to\infty} |||\overline{\eta}^n||| \le K_1,$$

$$[\overline{\eta}_{tt}] \le \liminf_{n\to\infty} [\overline{\eta}_{tt}^n] \le K_2.$$

Hence η satisfies (2.23) and (5.9), (5.10). Finally it is easily seen that the above weak (and weak-*) convergences are sufficiently strong to "pass to the limit" in (2.25), (2.26), (2.27), in order to show that the limit $\overline{\eta}$ also satisfies these last three properties. For instance, it is easily shown that $\overline{\eta}^n \to \overline{\eta}$ uniformly in Σ_T. Hence $(2.25)_1$, (2.26), (2.27) hold for the limit $\overline{\eta}$. On the other hand

$$H^{5/8}(H^{5/8}(I)) = \left[H^0(H^{5/2}), H^1(H^{-1/2})\right]_{5/8}.$$

Since $H^{5/8}(0,T) \hookrightarrow C^{0,1/8}([0,T])$ and the embedding $H^{5/8}(I) \hookrightarrow H^{1/2}(I)$ is compact, it follows (by Ascoli–Arzelà's theorem) that $\overline{\eta}_t^n \to \overline{\eta}_t$ in $C([0,T]; H^{1/2}(I))$. Hence $\overline{\eta}_t(0) = \eta_1$, *i.e.* $\overline{\eta}$ satisfies $(2.25)_2$.

Similar arguments show that $|||\overline{v}, \overline{p}_0||| \leq K_0$ and that the limit $(\overline{v}, \overline{p}_0)$ satisfies the properties (2.29), (2.30), (2.31). For instance, the sequence $\overline{v}^n$ is bounded in $C([0,T]; H^1)$ and in $C^{0,1/2}([0,T]; L^2)$. Hence, by Ascoli–Arzelà's theorem, $\overline{v}^n \to \overline{v}$ in $C([0,T]; L^2)$. Consequently $\overline{v}(0) = \widehat{v}_0$, *i.e.* the limit $\overline{v}$ satisfies (2.30). □

Compactness of $\mathbb{K}$ follows easily. For instance, given a sequence $\eta^n \in \mathbb{K}$, well-known compact embedding theorems show that the boundedness of $|||\eta^n|||$ implies the convergence in $L^2(\Sigma_T)$ of (subsequences) η^n. A similar argument applies to velocities and pressures.

Let us show that the map $\mathcal{T} : \mathbb{K} \to \mathbb{K}$ is continuous with respect to the $\mathcal{H}$ topology. Assume that a sequence $(\overline{\eta}^n, \overline{v}^n, \overline{p}_0^n) \in \mathbb{K}$ converges in $\mathcal{H}$ to a point $(\overline{\eta}, \overline{v}, \overline{p}_0)$. Boundedness of the above sequence with respect to the norms used in the definition of $\mathbb{K}$ implies convergence to the same point with respect to the corresponding weak (or weak-*) topologies. For instance, $\overline{v}^n \rightharpoonup \overline{v}$ weakly in $L^2(H^2)$ and weakly-* in $L^\infty(L^2)$, and $\overline{v}_t^n \rightharpoonup \overline{v}_t$ weakly in $L^2(L^2)$.

Since the sequence of solutions $(\widetilde{\eta}^n, \widetilde{v}^n, \widetilde{p}_0^n)$ of problems (2.21), (2.22) (corresponding to the sequence of given functions $(\overline{\eta}^n, \overline{v}^n \overline{p}_0^n))$ also belongs to $\mathbb{K}$, it follows that subsequences of $(\widetilde{\eta}^n, \widetilde{v}^n, \widetilde{p}_0^n)$ converge to elements $(\widetilde{\eta}, \widetilde{v}, \widetilde{p}_0)$ with respect to the above weak (or weak-*) topologies. For instance, $\widetilde{v}^n$ and $\widetilde{v}_t^n$ (subsequences) converge with respect to the weak topologies referred above for the sequences $\overline{v}^n$ and $\overline{v}_t^n$. The convergence of both the sequences in the weak topologies is largely sufficient to pass to the limit in equations (2.20), (2.21) (written now for the approximating sequences) in order to verify that the limit $(\widetilde{\eta}, \widetilde{v}, \widetilde{p}_0)$ is the solution corresponding to $(\overline{\eta}, \overline{v}, \overline{p}_0)$. Since the solution to the linear problems (2.20), (2.21) is unique (for each given $(\overline{\eta}, \overline{v}, \overline{p}_0)$) it also follows that all the sequence $(\widetilde{\eta}^n, \widetilde{v}^n, \widetilde{p}_0^n)$ must converge to a unique limit $(\widetilde{\eta}, \widetilde{v}, \widetilde{p}_0) = \mathcal{T}(\overline{\eta}, \overline{v}, \overline{p}_0)$. Hence $\mathcal{T}$ is continuous.

The above fixed point is clearly a solution $(\eta, \widehat{v}, \widehat{p}_0)$ of (2.7), (2.12).

Finally, we obtain $\widehat{p}$ by using (2.16). Clearly, $(\eta, \widehat{v}, \widehat{p})$ is a solution of (2.7) (since $\widehat{\phi}[\eta, \widehat{v}]$ depends only on t).It is also a solution of (2.12) with right-hand side given by $\widehat{\Phi}[\eta, \widehat{v}, \widehat{p}]$ since this last quantity is equal to the right-hand side of (2.12). □

Finally, we turn back to the variables (t, x, y). By using the transformation formulae (2.5) we obtain from $(\eta, \widehat{v}, \widehat{p})$ a solution (η, v, p) of the initial system (1.8). Moreover, η is a solution of (1.3) since $\Phi[\eta, v, p] = \widehat{\Phi}[\eta, \widehat{v}, \widehat{p}]$, by the construction done in Section 1.

The fact that the couple (v, p) has the regularity claimed in Theorem 1.1 follows easily from the regularity of the transformation formulae (2.5), by doing calculations similar to that done in order to show, in Section 6, that $\widehat{F} \in L^2(Q_T)$. Note that, from the point of view of regularity, the change of variables $y \to z$ and its reciprocal $z \to y$ are similar since $0 < \delta_0 \leq 1 + \eta(t, x) \leq$ *constant*. □

6. Auxiliary estimates. Proofs

In this section we prove the estimates (4.1), (4.5), (3.15), and (6.26). We start by proving (4.1). Since in definition (2.8) p appears only in terms of its gradient, we may indistinctly use p or p_0.

The functions $\eta(t,x)$, defined for $x \in \Gamma$, are extended to Ω simply by setting $\eta(t,x,z) = \eta(t,x)$, $z \in]0,1[$. Consequently, the values of the norms of η as a function defined on Ω coincide with the corresponding norms of η as a function on I.

For notational convenience in this section we mostly denote $\overline{\eta}, \overline{v}, \overline{p}_0, \overline{p}, \overline{w}$ simply by η, v, p_0, p, w.

We also use in this section the following abbreviate notation:

$$A \preceq B$$

means that $A \leq c\,B$, for some positive constant c.

We start by estimating the single terms that appear in the definition of $\widehat{F}$. In general, the estimates obtained below could be rough whenever this choice do not have a substantial effect on the main final result to be proved.

One has

$$\|\eta\, v_t\|_{L^2(Q_T)} \preceq \|\eta\|_{L^\infty(Q_T)} |||\overline{v}|||,$$

and

$$\|\eta(t)\|_{L^\infty(0,T;L^\infty(I))} \leq \|\eta^0\|_{L^\infty(I)} + \int_0^t \|\eta_t(\tau)\|_{L^\infty(0,T;L^\infty(I))}\, d\tau.$$

Consequently

$$\|\eta\|_{L^\infty(Q_T)} \preceq \|\eta^0\|_{L^\infty(I)} + T^{1/2} \|\eta_t\|_{L^2(0,T;H^{3/2}(I))}. \tag{6.1}$$

Hence

$$\|\eta\, v_t\|_{L^2(Q_T)} \preceq \left(\|\eta^0\|_{L^\infty(I)} + T^{1/2} \|\eta_t\|_{L^2(0,T;H^{3/2}(I))} \right) |||v|||. \tag{6.2}$$

□

Next (here $v_z = \partial_z v$, and so on)

$$\|z\, \eta_t\, v_z\|_{L^2(Q_T)} \preceq \|\eta_t\|_{L^2(0,T;L^\infty(\Omega))} |||v|||.$$

On the other hand

$$\|\eta_t\|^2_{L^2(0,T;L^\infty(\Omega))} \leq T \|\eta_t\|^2_{L^\infty(0,T:L^\infty(\Omega))} = T \|\eta_t\|^2_{L^\infty(0,T;L^\infty(I))}.$$

Since $H^{3/2}(I) \hookrightarrow L^\infty(I)$ it follows that

$$\|\eta_t\|_{L^2(0,T;L^\infty(\Omega))} \preceq T^{1/2} \|\eta_t\|_{L^\infty(0,T;H^{3/2}(I))}. \tag{6.3}$$

Hence

$$\|z\, \eta_t\, v_z\|_{L^2(Q_T)} \preceq T^{1/2} \|\eta_t\|_{L^\infty(0,T;H^{3/2}(I))} |||v|||. \tag{6.4}$$

□

Next, by applying (6.1) to η_x, it follows that

$$\|\eta_x\|_{L^\infty(Q_T)} \preceq \|\eta_x^0\|_{L^\infty(I)} + T^{1/2}\|\eta_t x\|_{L^2(0,T;H^{3/2}(I))}. \tag{6.5}$$

On the other hand (2.26) yields

$$\|1/(1+\eta)\|_{L^\infty(Q_T)} \leq \frac{1}{\delta_0}. \tag{6.6}$$

Consequently, by (6.5) and by taking into account that $v_z \in L^\infty(0,T;L^2(\Omega))$, it follows

$$\left\| z\frac{\eta_x^2}{1+\eta}v_z \right\|_{L^2(Q_T)} \preceq \frac{1}{\delta_0}\left(\|\eta_x^0\|_{L^\infty(I)} + T^{1/2}\|\eta_{tx}\|_{L^2(0,T;H^{3/2}(I))}\right)^2 T^{1/2}|||v|||. \tag{6.7}$$

□

Now

$$\eta_{xx}(t) = \eta_{xx}(0) + \int_0^t \eta_{txx}(\tau)\,d\tau$$

yields

$$\|\eta_{xx}\|_{L^\infty(0,T;H^{1/2}(I))} \leq \|\eta_{xx}^0\|_{H^{1/2}(I)} + T^{1/2}\|\eta_{txx}\|_{L^2(0,T;H^{1/2}(I))}. \tag{6.8}$$

In particular

$$\|\eta_{xx}\|_{L^\infty(0,T;L^6(\Omega))} \preceq \|\eta_{xx}^0\|_{H^{1/2}(I)} + T^{1/2}\|\eta_{txx}\|_{L^2(0,T;H^{1/2}(I))},$$

since $H^{1/2}(I) \hookrightarrow L^p(I)$, for each $p < +\infty$. On the other hand

$$H^{1/3}(\Omega) = \left[L^2(\Omega), H^1(\Omega)\right]_{1/3},$$

moreover $H^{1/3}(\Omega) \hookrightarrow L^3(\Omega)$. Hence

$$\|\nabla v\|_{L^3(\Omega)} \preceq \|\nabla v\|^{2/3}\|\nabla v\|^{1/3}_{H^1(\Omega)}.$$

It readily follows that $\nabla v \in L^6(0,T;L^3(\Omega))$ and that

$$\|\nabla v\|_{L^6(0,T;L^3(\Omega))} \preceq \|\nabla v\|^{2/3}_{L^\infty(0,T;L^2(\Omega))}\|\nabla v\|^{1/3}_{L^2(0,T;H^1(\Omega))} \preceq |||v|||.$$

In particular

$$\|\nabla v\|_{L^2(0,T;L^3(\Omega))} \preceq T^{1/3}|||v|||. \tag{6.9}$$

From (6.8) and (6.9) we finally obtain

$$\|z\eta_{xx}v_z\|_{L^2(Q_T)} \preceq T^{1/3}\left(\|\eta_{xx}^0\|_{H^{1/2}(I)} + T^{1/2}\|\eta_{txx}\|_{L^2(0,T;H^{1/2}(I))}\right)|||v|||. \tag{6.10}$$

□

Now we consider the terms containing second order derivatives of v. By using (6.6) one shows that the left-hand side of (6.11) below is bounded by

$$c\left\{\|\eta_x\|_{L^\infty(Q_T)} + \|\eta\|_{L^\infty(Q_T)} + \frac{1}{\delta_0}(\|\eta_x\|^2_{L^\infty(Q_T)} + \|\eta\|_{L^\infty(Q_T)})\right\}|||v|||.$$

Hence, by (6.1) and (6.5) it readily follows that

$$\begin{aligned}&\left\|-2z\,\eta_x\,v_{xz}+\eta\,v_{xx}+\frac{z^2\eta_x^2-\eta}{1+\eta}v_{zz}\right\|_{L^2(Q_T)}\\&\le c\Big(1+\frac{1}{\delta_0}\Big)\Big\{\|\eta^0\|_{L^\infty(I)}+T^{1/2}\|\eta_t\|_{L^2(0,T;H^{3/2}(I))}+\|\eta^0_x\|_{L^\infty(I)}\\&\quad+T^{1/2}\|\eta_t\|_{L^2(0,T;H^{5/2}(I))}+\|\eta^0_x\|^2_{L^\infty(I)}+T\|\eta_t\|^2_{L^2(0,T;H^{5/2}(I))}\Big\}|||v|||.\end{aligned}\tag{6.11}$$

□

Next we consider the term that depends on the pressure. By means of (6.1) and (6.5) one shows that

$$\begin{aligned}&\|z(\eta_x\,p_z-\eta\,p_x)\vec{e_1}\|_{L^2(Q_T)}\\&\le\Big(\|\eta^0\|_{L^\infty(I)}+\|\eta^0_x\|_{L^\infty(I)}+cT^{1/2}\|\eta_t\|_{L^2(0,T;H^{5/2}(I))}\Big)\,|||\nabla p|||_{L^2(Q_T)}.\end{aligned}\tag{6.12}$$

□

Finally, we consider the terms originating from the "nonlinear" term. Since $\|\cdot\|^2_{L^3(\Omega)}\le\|\cdot\|_{L^2(\Omega)}\|\cdot\|_{L^6(\Omega)}$ and $H^1(\Omega)\hookrightarrow L^p(\Omega)$ for each finite p (here $p=6$), it readily follows that

$$\|\nabla v\|_{L^4(0,T;L^3(\Omega))}\le c\|v\|^{1/2}_{L^\infty(0,T;H^1(\Omega))}\|v\|^{1/2}_{L^2(0,T;H^2(\Omega))}.$$

Consequently

$$\|v\cdot\nabla v\|_{L^2(0,T;L^2(\Omega))}\le cT^{1/4}\|v\|^{3/2}_{L^\infty(0,T;H^1(\Omega))}\|v\|^{1/2}_{L^2(0,T;H^2(\Omega))}.$$

By using (6.1) and (6.5) it follows

$$\begin{aligned}&\|-(1+\eta)v_1\,v_x+(z\,\eta_x\,v_1-v_2)v_z\|_{L^2(0,T;L^2(\Omega))}\\&\le cT^{1/4}\left(1+\|\eta^0\|_{H^{5/2}(I)}+T^{1/2}\|\eta_t\|_{L^2(0,T;H^{5/2}(I))}\right)|||v|||^2.\end{aligned}\tag{6.13}$$

In conclusion, from equations (6.2), (6.4), (6.7), (6.10), (6.11), (6.12) and (6.13) one easily shows that (4.1) holds. For convenience, in this last equation we assume that (3.7) holds. □

Next we prove $(4.5)_1$. Recall that $\nabla\cdot w\simeq-\eta Dv+z\,\eta_x\,Dv$, where D means a first order derivative with respect to x or z.

The main quantities to be estimated in order to bound the left-hand side of $(4.5)_1$ are $\|z\,\eta_{xx}Dv\|_{L^2(Q_T)}$ and $\|z\,\eta_x\,D^2v\|_{L^2(Q_T)}$. The first of these two quantities is bounded by the right-hand side of (6.10). The second one is bounded by $\|\eta_x\|_{L^\infty(Q_T)}|||v|||$, which, in turn, is estimated by resorting to (6.5). We left details to the reader. □

In order to show $(4.5)_2$ we estimate the $L^2(Q_T)$ norm of $\eta_x\, v_t$ just as done above for $\eta_x\, D^2 v$. On the other hand the $L^2(Q_T)$ norm of $\eta_{tx}\, v$ is estimated as follows:

$$\begin{aligned}\|\eta_{tx}\, v\|_{L^2(0,T;L^2(\Omega))} &\le c\,\|\eta_{tx}\|_{L^2(0,T;L^4(\Omega))}\|v\|_{L^\infty(0,T;L^4(\Omega))}\\ &\le cT^{1/2}\|\eta_t\|_{L^\infty(0,T;H^{3/2}(I))}|||v|||.\end{aligned}$$

Details are left to the reader. □

Next we prove (3.15). From $(2.28)_3$ and (2.29) it readily follows that

$$\|p_0\|_{H^1(\Omega)} \le c\|\nabla p_0\|.$$

Note, in this regard, that the left-hand side of (2.29) is a linear continuous functional on $H^1_\#(\Omega)$. Hence, in particular

$$\|p_0\|_{L^2(0,T;H^{1/2}(I))} \le c\|\nabla p_0\|_{L^2(0,T;L^2(\Omega))}. \tag{6.14}$$

Next, we consider the expression of $\widehat{\Phi}$ given by (2.13). One has

$$\begin{aligned}\left\|\frac{2+\eta_x^2}{1+\eta}\, v_{2,z}\right\|_{H^{1/2}(I)} &\le c\left\|\frac{2+\eta_x^2}{1+\eta}\right\|_{H^1(I)}\|v_{2,z}\|_{H^{1/2}(I)}\\ &\le c\left\|\frac{1}{1+\eta}\right\|_{H^1(I)}\|2+\eta_x^2\|_{H^1(I)}\|v\|_{H^2(\Omega)}\\ &\le c(1+\|\eta_x\|^3_{H^1(I)})\|v\|_{H^2(\Omega)}.\end{aligned}$$

In a similar way we estimate the other terms that compose $\widehat{\Phi}-\rho_1\overline{p}_0$. After all, we deduce that

$$\left\|\frac{\eta_x}{1+\eta}v_{1,z}+\eta_x\, v_{2,x}-\frac{2+\eta_x^2}{1+\eta}v_{2,z}\right\|_{H^{1/2}(I)} \le c(1+\|\eta_x\|^2_{H^1(I)})\|v\|_{H^2(\Omega)}, \tag{6.15}$$

as $a+a^2 \le c(1+a^3)$. Since $H^1(I) \hookrightarrow H^{5/2}(I)$, it readily follows that the $L^2(0,T)$ norm of the left-hand side of (6.15) is bounded by

$$c\,(1+\|\eta_x\|^2_{L^\infty(0,T;H^{5/2}(I))})|||v|||.$$

Finally we estimate the $L^\infty(0,T;H^{5/2})$ norm of η_x by arguing as for (6.8). All that yields

$$\|\widehat{\Phi}-\rho_1 p_0\|_{L^2(0,T;H^{1/2}(I))} \le c\,\rho_2\,(1+\|\eta^0\|^3_{H^{5/2}(I)}+T^{3/2}\|\eta_t\|^3_{L^2(0,T;H^{5/2}(I))})|||v|||. \tag{6.16}$$

From (6.16) and (6.14) one shows that (here we use the "bar" notation)

$$\begin{aligned}\|\widehat{\Phi}(\overline{\eta},\overline{v},\overline{p}_0)\|_{L^2(0,T;H^{1/2}(I))} &\le c\rho_1\|\nabla\overline{p}_0\|_{L^2(Q_T)}\\ &\quad + c\rho_2(1+\|\eta^0\|^3_{H^{5/2}(I)}+T^{3/2}\|\overline{\eta}_t\|^3_{L^2(0,T;H^{5/2}(I))})|||\overline{v}|||.\end{aligned} \tag{6.17}$$

□

Next we estimate the $L^2(0,T;H^{1/2}_{\#}(I))$ norm of $\widehat{\phi}(t)=\widehat{\phi}[\eta,v]$. Since $\widehat{\phi}$ depends only on t this norm coincides with the $L^2(0,T;L^2(I))$ norm.

From (2.15) one has, for each t,

$$|\widehat{\phi}(t)| \leq c\frac{\rho_2}{\rho_1}(1+\|\eta_x\|^2_{L^\infty(0,T;L^\infty(I))})\|\nabla v\|_{L^1(I)}. \tag{6.18}$$

Hence, by using (6.5), it follows from (6.18) that

$$\rho_1\|\widehat{\phi}\|_{L^2(0,T)} \leq c\,\rho_2\,(1+\|\eta^0\|^2_{L^\infty(I)}+T\|\eta_t\|^2_{L^2(0,T;H^{5/2}(I))})\|\nabla v\|_{L^2(0,T;L^2(\Gamma))}.$$

Next, by Theorem 2.1, Chap. 4, in reference [12], one has

$$\|\nabla v\|_{H^{1/4}(0,T;L^2(\Gamma))} \leq c\,|||v|||.$$

Since $H^{1/4}(0,T)\hookrightarrow L^4(0,T)$ it readily follows that

$$\|\nabla v\|_{L^2(0,T;L^2(\Gamma))} \leq c\,T^{1/4}\,|||v|||.$$

Hence

$$\rho_1\|\widehat{\phi}\|_{L^2(0,T)} \leq c\,\rho_2\,T^{1/4}(1+\|\eta^0\|^2_{L^\infty(I)}+T\|\eta_t\|^2_{L^2(0,T;H^{5/2}(I))})|||\nabla v|||. \tag{6.19}$$

Note that under the assumption $T\leq 1$ the right-hand side of (6.19) is bounded by that of (6.17). Consequently (3.15) follows from (6.17) and (6.19). □

Finally we estimate the $L^2(0,T;H^{-1/2}(I))$ norm of the solution $\widetilde{\eta}_{tt}$ of equation (2.20). From (2.20) it follows that

$$\begin{aligned}\|\widetilde{\eta}_{tt}\|_{L^2(0,T;H^{-1/2}(I))} \leq{}& \gamma\|\widetilde{\eta}_t\|_{L^2(0,T;H^{3/2}(I))}+c\rho_1\|\widehat{\phi}\|_{L^2(0,T)}\\ &+\|\widehat{\Phi}[\overline{\eta},\overline{v},\overline{p}_0]\|_{L^2(0,T;H^{-1,2}(\Gamma))}.\end{aligned}$$

From (3.16) and (6.19) it readily follows that

$$\begin{aligned}\|\widetilde{\eta}_{tt}\|_{L^2(0,T;H^{-1/2}(I))} \leq{}& c\,T^{1/2}|||\overline{\eta}|||+c\,\rho_2 T^{1/4}(1+\|\eta^0\|^2_{L^\infty(I)}+T\,|||\overline{\eta}|||^2)|||\overline{v}|||\\ &+\|\widehat{\Phi}[\eta,\overline{v},\overline{p}_0]\|_{L^2(0,T;H^{-1,2}(I))}.\end{aligned} \tag{6.20}$$

On the other hand

$$\begin{aligned}\|\widehat{\Phi}[\overline{\eta},\overline{v},\overline{p}_0]\|_{L^2(0,T;L^2(\Sigma_T))} \leq{}& c_1\rho_1\|\nabla\overline{p}_0\|_{L^2(0,T;L^2(\Omega))}\\ &+c\rho_2\left\|\frac{\overline{\eta}_x}{1+\overline{\eta}}\overline{v}_{1,z}+\overline{\eta}_x\,\overline{v}_{2,z}-\frac{2+\overline{\eta}_x^2}{1+\overline{\eta}}\,\overline{v}_{2,z}\right\|_{L^2(0,T;L^2(\Gamma))}.\end{aligned} \tag{6.21}$$

Note that the $H^{-1/2}(I)$ norm was replaced by the stronger $L^2(\Gamma)$ norm. Next

$$\left\|\frac{2+\overline{\eta}_x^2}{1+\overline{\eta}}\,\overline{v}_{2,z}\right\|_{L^2(0,T;L^2(I))} \leq \left\|\frac{2+\overline{\eta}_x^2}{1+\overline{\eta}}\right\|_{L^\infty(0,T;L^4(I))}\|\overline{v}_{2,z}\|_{L^2(0,T;L^4(I))}. \tag{6.22}$$

Since

$$\left\|\frac{2+\overline{\eta}_x^2}{1+\overline{\eta}}\right\|_{L^\infty(0,T;L^4(I))} \leq \frac{1}{\delta_0}\|2+\eta_x^2\|_{L^\infty(0,T;L^4(I))},$$

it readily follows that

$$\left\| \frac{2+\overline{\eta}_x^2}{1+\overline{\eta}} \right\|_{L^\infty(0,T;L^4(I))} \leq c\left(1+\|\overline{\eta}_x\|^2_{L^\infty(0,T;H^{1/2}(I))}\right), \tag{6.23}$$

as $H^{1/2}(I) \hookrightarrow L^\infty(I)$.

On the other hand, by interpolation, one shows that

$$\|\overline{v}\|_{H^{1/8}(0,T;H^{7/4}_{\#}(\Gamma))} \leq c\|\overline{v}\|^{7/8}_{L^2(0,T;H^2_{\#}(\Omega))}\|\overline{v}\|^{1/8}_{H^1(0,T;L^2(\Omega))}.$$

Hence

$$\|\nabla\overline{v}\|_{H^{1/8}(0,T;H^{1/4}(\Gamma))} \leq c|\|\overline{v}\||,$$

and, by well-known embedding theorems, it follows

$$\|\nabla\overline{v}\|_{L^{8/3}(0,T;L^4(\Gamma))} \leq c|\|\overline{v}\||.$$

Consequently

$$\|\nabla\overline{v}\|_{L^2(0,T;L^4(\Gamma))} \leq cT^{1/8}|\|\overline{v}\||. \tag{6.24}$$

From (6.22), (6.23), (6.24), it follows

$$\left\| \frac{2+\overline{\eta}_x^2}{1+\overline{\eta}}\,\overline{v}_{2,z} \right\|_{L^2(0,T;L^2(I))} \leq cT^{1/8}(1+\|\overline{\eta}\|^2_{L^\infty(0,T;H^{3/2}(I))})|\|\overline{v}\||.$$

The other two terms that compose the last term on the right-hand side of (6.21) are treated in a similar way. Hence, from (6.21) it follows that

$$\begin{aligned} \|\widehat{\Phi}[\overline{\eta},\overline{v},\overline{p}_0]\|_{L^2(0,T;L^2(\Gamma))} \leq\ & c\,\rho_1\|\nabla\overline{p}_0\|_{L^2(Q_T)} \\ & + c\rho_2\,T^{1/8}(1+\|\overline{\eta}\|^2_{L^\infty(0,T;H^{3/2}(I))})|\|\overline{v}\||. \end{aligned} \tag{6.25}$$

Finally, turning back to (6.20), one gets

$$\begin{aligned} \|\widetilde{\eta}_{tt}\|_{L^2(0,T;L^2(I))} \leq\ & c\rho_1\|\nabla\overline{p}_0\|_{L^2(Q_T)} \\ & + c\rho_2\,T^{1/8}(1+\|\eta^0\|^2_{L^\infty(I)}+|\|\overline{\eta}\||^2)|\|\overline{v}\|| \\ & + c\,T^{1/2}|\|\widetilde{\eta}\||. \end{aligned} \tag{6.26}$$

□

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