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Singular Limits in Compressible Fluid Dynamics

H. BEIRÃO DA VEIGA

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Main Notation

We start by introducing the main notation. As usual, $\mathbb{R}^+$ is the set of positive reals and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. We denote by $|\cdot|_p$ and $\|\cdot\|_m$ the canonical norms in the space $L^p = L^p(\Omega)$, $1 \leqq p \leqq \infty$, and in the L^2-Sobolev space $H^m = H^m(\Omega)$, respectively. We set $\|\cdot\| = \|\cdot\|_0$. The point x belongs to the n-dimensional torus, $n \geqq 2$, identified here with the set $\Omega = [0, 1]^n$.

We denote by $\|\cdot\|_{m,T}$ and $[\cdot]_{m,T}$ the canonical norms in $L^\infty(0, T; H^m)$ and $L^2(0, T; H^m)$, respectively. The function $x \to f(t, x)$, for a fixed t, is sometimes denoted by $f(t)$.

We denote by k_0 the smallest integer larger than $n/2$ and by k a fixed integer satisfying $k \geqq k_0 + 1$.

The parameters n, k, ρ_0, v_0, and μ_0 are fixed. Positive constants that depend at most on these quantities are denoted by c. Below we also introduce the positive constants c_i, that play here an important role (see (1.6)–(1.8)). Positive constants that depend at most on c_1 are denoted by C_1; positive constants that depend at most on c_1 and c_2 by C_2; and positive constants that depend at most on c_1, c_2, c_3 by C_3. Distinct constants C_j are denoted by the same symbol provided that they depend on the same basic constants c_i.

Introduction

We study the dependence of solutions to the equations of motion of compressible fluids on the Mach number λ^{-1} and on the viscosity coefficients ν and μ. We assume that $\lambda \geqq 1$ (we shall be interested in letting λ tend to ∞) and that $\nu \in [0, \nu_0]$, $\mu \in [0, \mu_0]$ for arbitrary, but fixed, constants ν_0 and μ_0. Viscous and inviscid fluids are studied together, since 0 is an admissible value for the viscosity coefficients μ and ν.

We denote by v the velocity field, by ρ the density of the fluid, and by $p(\lambda, \rho)$ the pressure p as a function of the density ρ and the Mach number. It is worth noting that our results and proofs apply if $p(\lambda, \rho)$ enjoys the properties assumed in the papers [BV1,2]. In order to avoid technicalities, we assume here that

$$p(\lambda, \rho) = \lambda^2 p(\rho) \tag{1.1}$$

for a fixed function $p(\cdot)$. The main point is to assume that $\lim_{\lambda\to\infty} p'(\lambda, \bar\rho_0) = \infty$, where $\bar\rho_0$ is the "mean density" of the fluid. We assume that there are no external forces since their introduction into the equations does not give rise to any additional difficulties.

We assume that $p \in C^{k+2}(\mathbb{R}^+; \mathbb{R})$ and that $p'(s) > 0$ for all $s \in \mathbb{R}^+$. The equations of motion under the equation of state (1.1) are

$$\begin{aligned} &\rho_t + v\cdot\nabla\rho + \rho\nabla\cdot v = 0,\\ &\rho[v_t + (v\cdot\nabla)v] + \lambda^2 p'(\rho)\nabla\rho = \nu\Delta v + \mu\nabla(\nabla\cdot v),\\ &\rho(0) = \bar\rho_0 + \rho_0(x), \quad v(0) = v_0(x), \end{aligned} \tag{1.2}$$

where $\bar\rho_0$ is a fixed positive constant and $\bar\rho_0 + \rho_0(x) \geqq c_0 > 0$. We remark that if $\int_\Omega \rho_0(x)\,dx = 0$, then $\int_\Omega \rho(t,x)\,dx = \bar\rho_0$ for all $t \geqq 0$. This follows from the equation $(1.2)_1$. However, this hypothesis is not necessary here. Without loss of generality we assume that the "mean density" $\bar\rho_0$ is 1.

We find it convenient to make the change of variables

$$g = \log(\rho/\bar\rho_0). \tag{1.3}$$

Equations (1.2) are then equivalent to

$$\begin{aligned} &g_t + v\cdot\nabla g + \nabla\cdot v = 0,\\ &v_t + \lambda^2\phi'(g)\nabla g + (v\cdot\nabla)v = e^{-g}[\nu\Delta v + \mu\nabla(\nabla\cdot v)],\\ &v(0) = v_0(x), \quad g(0) = g_0(x), \end{aligned} \tag{1.4}$$

where, by definition, $\phi'(s) \equiv p'(e^s)$ for all $s \in \mathbb{R}$. Hence $\phi' \in C^{k+1}(\mathbb{R}; \mathbb{R}^+)$. Our proofs and results will be given in terms of the unknown g. If the reader wants to get the results in terms of ρ, the only rules to keep in mind are that $|g|_\infty$ is bounded if and only if $|\rho|_\infty$ and $|1/\rho|_\infty$ are bounded, and that $g \to 0$ in H^m is equivalent to $\rho \to \bar\rho_0$ in H^m. These facts follow easily from a lemma of MOSER (see Appendix, Lemma 4.5). More precisely, the results below hold by replacing $g_0(x)$ by $\rho_0(x)/\bar\rho_0$ and g by $(\rho - \bar\rho_0)/\bar\rho_0$ (hence, g_t by $\rho_t/\bar\rho_0$). The factor $1/\bar\rho_0$ can be dropped.

We are interested in studying the behaviour of (ρ, v) as (simultaneously) the Mach number λ^{-1} goes to zero, the viscosity ν converges to a value $\bar\nu \geqq 0$, $\bar\mu$ stays bounded, and (v_0, ρ_0) converges to $(w_0, \bar\rho_0)$. The limit equations are the equations of motion of an incompressible fluid with density $\bar\rho_0 = 1$ and viscosity $\bar\nu \geqq 0$, i.e.,

$$\begin{aligned} &\nabla\cdot w = 0,\\ &w_t + (w\cdot\nabla)w + \nabla\pi = \bar\nu\,\Delta w,\\ &w(0) = w_0(x), \end{aligned} \tag{1.5}$$

where $\nabla\cdot w_0 \equiv 0$. In reference [BV7] I studied the convergence of (v, ρ) to $(w, 1)$, i.e., the convergence of (v, g) to $(w, 0)$ as $(\lambda, \nu) \to (\infty, \bar\nu)$ and μ remains bounded. The results in [BV7] extend Theorem 2 in [KM1] by following similar ideas. However, these results are not quite satisfactory, in particular, from the mathematical point of view. In fact, the dynamical systems (1.4), (1.5) have solutions $(v(t), g(t))$ and $(w(t), 1)$

which describe continuous trajectories in the Hilbert space H^k, the data space. Hence, the natural and optimal result is to prove that trajectories converge to the limit trajectory in the H^k-norm, uniformly with respect to time. This is a significant accomplishment in the theory. We remark that convergence in $C(0, T; H^{k-\varepsilon})$ and in $L^\infty(0, T; H^k)$ weak-* are immediate consequences of the uniform a priori estimates leading to the existence theorem. Convergence in $C(0, T; H^k)$ requires deeper arguments.

In the next lemma we condense some of the results proved in [BV7]; see also [KM1].

Lemma 1.1. *Assume that*

$$\|v_0\|_{k_0+1} \leqq c_1, \quad \lambda \|g_0\|_{k_0+1} \leqq c_1, \tag{1.6}$$

$$\|v_0\|_k \leqq c_2, \quad \lambda \|g_0\|_k \leqq c_2, \tag{1.7}$$

$$\lambda \|\nabla \cdot v_0\| \leqq c_3, \quad \lambda^2 \|\nabla g_0\|_0 \leqq c_3. \tag{1.8}$$

Then there is a positive constant T, depending only on c_1 (decreasingly), such that problem (1.4) *has a unique solution in* $[0, T]$. *Moreover,*

$$\begin{aligned}\lambda^2 \|g\|_{k,T}^2 + \|v\|_{k,T}^2 + \nu[\nabla v]_{k,T}^2 + \mu[\nabla \cdot v]_{k,T}^2 \\ \leqq C_1(\lambda^2 \|g_0\|_k^2 + \|v_0\|_k^2) \leqq C_2,\end{aligned} \tag{1.9}$$

$$\begin{aligned}\lambda^2 \|g_t\|_{0,T}^2 + \|v_t\|_{0,T}^2 + \lambda^{2k/(k-1)} \|\nabla g\|_{k-2,T}^2 \\ + \nu[\nabla v]_{0,T}^2 + \mu[\nabla \cdot v]_{0,T}^2 \leqq C_3.\end{aligned} \tag{1.10}$$

The above constants T, C_1, C_2, C_3 also depend on k, n, ν_0, μ_0 and on the particular function $\phi'(\cdot)$. However, we assume that these data are fixed once and for all. We note that related (but weaker) results have been proved in the literature by assuming that $k \geqq k_0 + 2$ and that $\|v_0 - w_0\|_k \leqq c_3/\lambda$, $\|g_0\|_k \leqq c_3/\lambda^2$, $\nu \|v_0 - w_0\|_{k+1} \leqq c$, and $\|\nabla \cdot v_0\|_k \leqq c_3/\lambda$ (compare with $(1.8)_1$). Note that these stronger hypotheses imply, in particular, that $(v_0, \lambda g_0) \to (w_0, 0)$ in H^k, an assumption made in theorem A below.

In order to state our main result in a clear form we introduce the following notation. We fix a set of constants $\{k, \nu_0, \mu_0, c_1, c_2, c_3\}$, where $k \geqq k_0 + 1$, $\nu_0, \mu_0 \in \mathbb{R}_0^+$, $c_1 \in \mathbb{R}^+$ and we define the corresponding set of admissible data (initial data and parameters)

$$\mathscr{X} \equiv \{X = (v_0, g_0, \lambda, \nu, \mu) \in H^k \times H^k \times [1, \infty[\times [0, \nu_0] \times [0, \mu_0]: \\ (1.6)\text{–}(1.8) \text{ hold}\}$$

endowed with the canonical product norm. Finally, we define a map S on $\mathscr{X}$ by setting $S(X) = (v, g)$, where (v, g) is the solution of the problem (1.4) corresponding to the particular data $X \in \mathscr{X}$.

Theorem A. *Let $\mathscr{X}$ and $S:\mathscr{X} \to C(0,T;H^k)$ be defined as above. Then*

$$\lim_{\substack{X\in\mathscr{X} \\ (v_0,\lambda g_0,\lambda,\nu)\to(w_0,0,\infty,\bar{\nu})}} \|(v,g)-(w,0)\|_{k,T}^2 + \lambda^2\|\nabla g\|_{k-1,T}^2 + \|g_t\|_{k-1,T}^2 + \bar{\nu}[v-w]_{k+1,T}^2 = 0, \tag{1.11}$$

where w is the solution of the problem (1.4). *If, moreover, $\mathscr{X}$ is replaced by $\{X\in\mathscr{X}:\bar{\mu}\leq\mu\leq\mu_0\}$ where $\bar{\mu}\in\mathbb{R}^+$, then* $\lim\ [\nabla\cdot v]_{k,T}^2 = 0$. *(Here, and in the sequel, the convergence of $(v_0,\lambda g_0)$ to $(w_0,0)$ is understood in the H^k norm.)*

Remarks. (i) The map $S:\mathscr{X}\to\mathscr{Y}\equiv C(0,T;H^k)$ is continuous on $\mathscr{X}$, i.e., (1.11) holds if λ converges to a finite limit $\bar{\lambda}$ (instead of to ∞). This result can also be proved for boundary-value problems. See references [BV3, Theorem 2.5], [BV4], [BV5]. However, we are interested here in the behaviour of the solution $S(X)\equiv(v,g)$ as $\lambda\to\infty$.

(ii) Let l, $0\leq l\leq k-1$, be fixed. If we replace assumption (1.8) by

$$\lambda\|\nabla\cdot v_0\|_l\leq c_3,\quad \lambda^2\|\nabla g_0\|_l\leq c_3, \tag{1.12}$$

then estimate (1.10) holds if we replace the norms $\|\cdot\|_{0,T}$ and $[\cdot]_{0,T}$ by $\|\cdot\|_{l,T}$ and $[\cdot]_{l,T}$, respectively. In this case $v_t, w_t\in C(0,T;H^l)$. However, it is (in general) false that $\lim\|v_t-w_t\|_{l,T}=0$. Similarly, if $\lim\nu\equiv\bar{\nu}>0$, then $v_t, w_t\in L^2(0,T;H^{k-1})$. However, it is false (in general) that $\lim[v_t-w_t]_{k-1,T}=0$. Convergence of v_t in the strong norm can be proved if one introduces additional conditions on the initial data. However, these conditions look quite artificial.

(iii) In the particular case in which $\nu_0,\mu_0=0$ (hence $\nu\equiv\mu\equiv0$ everywhere) Theorem A was proved in [BV8]. This result implies Theorem 1.2 of [BV6]. It is interesting to note that in [BV6] we used approximations of the solutions of (1.4) by two distinct systems (the ε- and the δ-approximation). Here, we show that the ε-approximation is superfluous. However, if we want to treat boundary-value problems (see, for instance, [BV3]), the ε-approximation is a very useful tool.

(iv) The main points in the proof of Theorem A can be easily extended in order to cover the general class of problems considered in reference [KM1] (in this direction, see Theorem 2.2 in [BV6]). An interesting application of the method followed here to the equations of magneto-fluid dynamics will be given in the forthcoming paper [R].

Finally, we give some references to previous papers treating the incompressible limit for compressible fluids. In [Ag], [As], [BV6], [Eb 1, 2], [KM 1, 2], [M], [Sc 1, 2, 3], [U] the authors consider inviscid fluids. Viscous stationary fluids were studied in [BV1, 2]. Viscous nonstationary fluids were studied in [KM1] and, for the boundary-value problem, in a very recent work [Be]. There are also other directions of research in this same field. For instance, see [KLN] and [Sc4].

Preliminaries

The following result, a corollary of Theorem 1.5 and of the equation (1.18) in [BV7], is stated here just in the form needed in the sequel.

Proposition 2.1. *Under the assumptions of Theorem A,*

$$\lim_{\substack{X \in \mathscr{X} \\ (v_0, \lambda g_0, \lambda, \nu) \to (w_0, 0, \infty, \bar{\nu})}} \|v - w\|_{k-1,T}^2 + \|g\|_{k-1,T}^2 + \lambda^2 \|\nabla g\|_{k-2,T}^2 + \bar{\nu}[v - w]_{k,T}^2 = 0. \tag{2.1}$$

If, moreover, $\mu \in [\bar{\mu}, \mu_0]$ *for some* $\bar{\mu} > 0$, *then* $\lim [\nabla \cdot (v - w)]_{k-1,T}^2 = 0$.

The following system in which $\delta \in]0, 1]$ is a parameter plays a very important role in the sequel:

$$\begin{aligned} &g_t^\delta + (v^\delta \cdot \nabla) g^\delta + \nabla \cdot v^\delta = 0, \\ &v_t^\delta + \lambda^2 \phi'(g^\delta) \nabla g^\delta + (v^\delta \cdot \nabla) v^\delta = \tilde{e}(-g^\delta)[\nu \Delta v^\delta + \mu \nabla(\nabla \cdot v^\delta)], \\ &v^\delta(0) = v_0^\delta(x), \quad g^\delta(0) = g_0^\delta(x). \end{aligned} \tag{2.2}$$

For convenience, we use the notation $\tilde{e}(y) = e^y$. The parameters k, ν_0, μ_0, c_1, c_2, c_3 are fixed once and for all. The element $X = (v_0, g_0, \lambda, \nu, \mu)$ ranges over the set $\mathscr{X}$, which corresponds to the above values of parameters. We consider the Fourier series

$$u_0(x) = \sum_\xi \hat{u}_0(\xi) e^{2\pi i \xi \cdot x}$$

where the Fourier coefficients are given by

$$\hat{u}_0(\xi) = \int_\Omega e^{-2\pi i \xi \cdot x} u_0(x)\, dx$$

and where $\xi = (\xi_1, \ldots, \xi_n)$. The ξ_i's are nonnegative integers. We denote by $|\xi|$ the Euclidean norm of ξ. For each $s \in \mathbb{R}_0^+$, we have

$$\|u_0\|_s^2 = \sum_\xi (1 + |\xi|^2)^s \|\hat{u}_0(\xi)\|^2.$$

Next, given $\delta \in]0, 1]$ we define the operator

$$(T^\delta u_0)(x) = \sum_{|\xi| \leq 1/\delta} \hat{u}_0(\xi) e^{2\pi i \xi \cdot x} \tag{2.3}$$

and we set

$$v_0^\delta \equiv T^\delta v_0, \quad g_0^\delta \equiv T^\delta g_0. \tag{2.4}$$

The operator T^δ is a linear operator on H^s for each $s \in \mathbb{R}_0^+$. Moreover, $|||T^\delta|||_{s,s} \leqq 1$, where $|||\cdot|||_{s,m}$ denotes the canonical norm on bounded operators from H^s to H^m. On the other hand, T^δ commutes with the divergence operator. It readily follows that (v_0^δ, g_0^δ) satisfies conditions (1.6), (1.7), (1.8) for the same constants. In other words, if $(v_0, g_0, \lambda, \nu, \mu) \in \mathscr{X}$, then $(v_0^\delta, g_0^\delta, \lambda, \nu, \mu) \in \mathscr{X}$. Hence, by Lemma 1.1, it follows that

$$\begin{aligned}\lambda^2 \|g^\delta\|_{k,T}^2 + \|v^\delta\|_{k,T}^2 + \nu[\nabla v]_{k,T}^2 + \mu[\nabla\cdot v]_{k,T}^2 &\leqq C_2,\\ \lambda^2 \|g_t^\delta\|_{0,T}^2 + \|v_t^\delta\|_{0,T}^2 + \nu[\nabla v_t]_{0,T}^2 + \mu[\nabla\cdot v]_{0,T}^2 &\leqq C_3.\end{aligned} \tag{2.5}$$

Also note that

$$|||T^\delta|||_{s,m} \leqq (2/\delta)^{m-s}, \quad |||T^\delta - I|||_{m,s} \leqq \delta^{m-s} \tag{2.6}$$

if $0 \leqq s \leqq m$, where $s, m \in \mathbb{R}_0^+$. In particular,

$$\begin{aligned} &\lambda \|g_0^\delta\|_{k_0+1} \leqq c_1, &&\|v_0^\delta\|_{k_0+1} \leqq c_1,\\ &\lambda \|g_0^\delta\|_{k+1} \leqq 2c_2/\delta, &&\|v_0^\delta\|_{k+1} \leqq 2c_2/\delta,\\ &\lambda \|\nabla\cdot v_0^\delta\| \leqq c_3, &&\lambda^2 \|\nabla g_0^\delta\| \leqq c_3,\end{aligned} \tag{2.7}$$

$$\lambda \|g_0^\delta\|_{k+1} \leqq \frac{2}{\delta} \lambda \|g_0\|_k. \tag{2.8}$$

We also define $w_0^\delta = T^\delta w_0$. Note that

$$\|v_0^\delta - w_0^\delta\|_{k+1} \leqq \frac{2}{\delta} \|v_0 - w_0\|_k. \tag{2.9}$$

In particular, since $(v_0, \lambda g_0) \to (w_0, 0)$ in H^k, it follows that $(v_0^\delta, \lambda g_0^\delta) \to (w_0^\delta, 0)$ in H^{k+1} for each fixed δ.

Equations (2.7) show that (for each fixed δ) the solution (v^δ, g^δ) of problem (2.2) satisfies all the results proved in [BV7] if one replaces k by $k+1$ and c_2 by $2c_2/\delta$. Note that T depends only on c_1. Hence T is independent of the particular initial data (v_0, g_0) or (v_0^δ, g_0^δ) and of the particular values taken by the parameters k, λ, ν, μ, and δ. In particular, applying Proposition 2.1 (with k replaced by $k+1$) to the solutions (v^δ, g^δ) of the system (2.2) we get

Proposition 2.2. *Let $X \in \mathscr{X}$ and in* (2.2) *let the initial data be given by* (2.4). *Then, for each fixed δ,*

$$\lim_{(v_0, \lambda g_0, \lambda, \nu) \to (w_0, 0, \infty, \bar\nu)} \|v^\delta - w^\delta\|_{k,T}^2 + \|g^\delta\|_{k,T}^2 + \lambda^2 \|\nabla g^\delta\|_{k-1,T}^2 + \bar\nu[\nabla(v^\delta - w^\delta)]_{k,T}^2 = 0. \tag{2.10}$$

If $\mu \in [\bar\mu, \mu_0]$, $\bar\mu > 0$, then $\lim [\nabla\cdot v^\delta]_{k,T}^2 = 0$.

We denote by w^δ the solution of the problem

$$\begin{aligned} &\nabla \cdot w^\delta = 0, \\ &\partial_t w^\delta + (w^\delta \cdot \nabla) w^\delta + \nabla \pi^\delta = \bar{\nu} \Delta w^\delta, \\ &w^\delta(0) = w_0^\delta. \end{aligned} \tag{2.11}$$

The following estimates will be useful:

$$\begin{aligned} &\| v_0^\delta - v_0 \|_k^2 \leqq 2 \| w_0 - v_0 \|_k^2 + 2 \sum_{|\xi| > 1/\delta} (1 + |\xi|^2)^k |\hat{w}_0(\xi)|^2, \\ &\| g_0^\delta - g_0 \|_k^2 \leqq \| g_0 \|_k^2. \end{aligned} \tag{2.12}$$

Their proof is left to the reader.

Proof of Theorem A

Our next step is to prove

Theorem 3.1. *Let $0 \leqq m \leqq k$. For each $X \in \mathscr{X}$ and each $\delta > 0$,*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\lambda^2 \int_\Omega \phi'(g) \sum_{|\alpha| \leqq m} |D^\alpha \bar{g}|^2 \, dx + \| \bar{v} \|_m^2 \right) + \frac{1}{C_1} (\nu \| \nabla \bar{v} \|_m^2 + \mu \| \nabla \cdot \bar{v} \|_m^2) \\ &\leqq C_2 (\lambda^2 \| \bar{g} \|_m^2 + \| \bar{v} \|_m^2) + \delta_k^m C_2 [\lambda^2 \| \nabla g^\delta \|_k (|\bar{v}|_\infty \| \bar{g} \|_k + |\bar{g}|_\infty \| \bar{v} \|_k) \\ &\quad + \| v^\delta \|_{k+1} |\bar{v}|_\infty \| \bar{v} \|_k] + C_2 (\nu \| \nabla v^\delta \|_k + \mu \| \nabla \cdot v^\delta \|_k) \| \bar{v} \|_m \| \bar{g} \|_m \\ &\quad + \delta_k^m C_2 (\nu \| \nabla v^\delta \|_k^2 + \mu \| \nabla \cdot v^\delta \|_k^2) \| \bar{g} \|_m^2, \end{aligned} \tag{3.1}$$

where, by definition, δ_k^m is the Kronecker symbol and

$$\bar{g} = g^\delta - g, \quad \bar{v} = v^\delta - v,$$

$$\bar{\phi} = \phi'(g^\delta) - \phi'(g), \quad \bar{e} = \tilde{e}(-g^\delta) - \tilde{e}(-g).$$

Proof. Taking the termwise difference of the equations (2.2) and (1.4) we find that

$$\begin{aligned} &\bar{g}_t + v \cdot \nabla \bar{g} + \nabla \cdot \bar{v} = -\bar{v} \cdot \nabla g^\delta, \\ &\bar{v}_t + \lambda^2 \phi'(g) \nabla \bar{g} + (v \cdot \nabla) \bar{v} = -\lambda^2 \bar{\phi} \nabla g^\delta - (\bar{v} \cdot \nabla) v^\delta + \nu \tilde{e}(-g) \Delta \bar{v} + \nu \bar{e} \Delta v^\delta \\ &\qquad + \mu \tilde{e}(-g) \nabla (\nabla \cdot \bar{v}) + \mu \bar{e} \nabla (\nabla \cdot v^\delta). \end{aligned} \tag{3.2}$$

In the calculations that follow it is worth noting that the estimates $(1.9)_1$ and $(2.5)_1$ show that the quantities $\| v \|_{k,T}$, $\lambda \| g \|_{k,T}$, $\| \phi'(g) \|_{k,T}$, $\lambda \| \nabla \phi'(g) \|_{k-1,T}$, $\| v^\delta \|_{k,T}$, $\lambda \| g^\delta \|_{k,T}$, $\nu [v^\delta]_{k+1,T}$, and $\mu [\nabla \cdot v^\delta]_{k,T}$ are bounded by a constant C_2. The norms of $\phi'(g)$ and $\nabla \phi'(g)$ are estimated here by using well-known inequalities of Moser (see the Appendix). MOSER's lemma also shows that

$$
(3.3)\qquad \begin{aligned} &|\bar{\phi}|_\infty \leqq C_1 |g|_\infty, \quad \|\bar{\phi}\|_m \leqq C_1 \|g\|_m, \\ &|\bar{e}|_\infty \leqq C_1 |g|_\infty, \quad \|\bar{e}\|_m \leqq C_1 \|g\|_m \end{aligned}
$$

for suitable constants C_1. As in [BV7] we use the notation

$$
\tilde{D}^\alpha\{f\, g\} = D^\alpha(f\, g) - f(D^\alpha g).
$$

In order to carry out the calculations which follow, we need some useful inequalities. For the reader's convenience these inequalities are presented in the Appendix. We note that, for $0 \leqq |\alpha| \leqq m$,

$$
(3.4)\qquad \|D^\alpha(f\, g)\| \leqq c\|f\|_m \|g\|_{k-1} + c\delta_k^m |f|_\infty \|g\|_k.
$$

In fact, if $m \leqq k-1$, this inequality is obvious. If $m = k$, we have $D^\alpha(f\, g) = \tilde{D}^\alpha\{f\, g\} + fD^\alpha g$, and the result follows from (4.3).

Let us now consider (3.2). Applying the operator D^α to $(3.2)_1$ we obtain

$$
D^\alpha \bar{g}_t + (v\cdot\nabla)D^\alpha \bar{g} + \tilde{D}^\alpha\{v\cdot\nabla\bar{g}\} + \nabla\cdot D^\alpha \bar{v} = -D^\alpha(\bar{v}\cdot\nabla g^\delta).
$$

Next, we multiply both sides of this equation by $\lambda^2\phi'(g)\, D^\alpha\bar{g}$ and integrate the result over Ω (this leads to the symmetrization of system (3.2)). Using, in particular, (4.2), (4.3), and (3.4), and doing standard manipulations (see [BV7]), we prove that

$$
(3.5)\qquad \begin{aligned} &\frac{\lambda^2}{2}\frac{d}{dt}\int_\Omega \phi'(g)(D^\alpha\bar{g})^2\,dx - \lambda^2\int_\Omega \phi'(g)(D^\alpha\bar{v})\cdot\nabla D^\alpha\bar{g}\,dx \\ &\quad \leqq C_2\lambda^2\|\bar{g}\|_m^2 + C_2\lambda\|\bar{v}\|_m\|\bar{g}\|_m + \delta_k^m C_2\lambda^2\|\nabla g^\delta\|_k|\bar{v}|_\infty\|\bar{g}\|_k. \end{aligned}
$$

Next, we apply D^α to $(3.2)_2$, multiply the result by $D^\alpha\bar{v}$ and integrate the product over Ω. Using inequalities in the Appendix and devices similar to those in [BV7] we show that

$$
(3.6)\qquad \begin{aligned} &\tfrac{1}{2}\|D^\alpha\bar{v}\|^2 + \lambda^2\int_\Omega \phi'(g)\nabla D^\alpha\bar{g}\cdot D^\alpha\bar{v}\,dx + \lambda^2\int_\Omega \tilde{D}^\alpha\{\phi'(g)\nabla\bar{g}\}\cdot D^\alpha\bar{v}\,dx \\ &\qquad - C_2\|D^\alpha\bar{v}\|^2 + \int_\Omega \tilde{D}^\alpha\{(v\cdot\nabla)\bar{v}\}\cdot D^\alpha\bar{v}\,dx \\ &\quad \leqq C_2\lambda\|\bar{\phi}\|_m\|\bar{v}\|_m + c\delta_k^m\lambda^2\|\nabla g^\delta\|_k|\bar{\phi}|_\infty\|\bar{v}\|_k + C_2\|\bar{v}\|_m^2 \\ &\qquad + c\delta_k^m\|v^\delta\|_{k+1}|\bar{v}|_\infty\|\bar{v}\|_k - \nu\int_\Omega \tilde{e}(-g)|\nabla D^\alpha\bar{v}|^2\,dx \\ &\qquad + C_2\nu\|\nabla D^\alpha\bar{v}\|\,\|D^\alpha\bar{v}\| \\ &\qquad + \nu\int_\Omega \tilde{D}^\alpha\{\tilde{e}(-g)\Delta\bar{v}\}\cdot D^\alpha\bar{v}\,dx + \nu\int_\Omega D^\alpha(\bar{e}\Delta v^\delta)\cdot D^\alpha\bar{v}\,dx \\ &\qquad - \mu\int_\Omega \tilde{e}(-g)\,|\nabla\cdot D^\alpha\bar{v}|^2\,dx + C_2\mu\|\nabla\cdot D^\alpha\bar{v}\|\,\|D^\alpha\bar{v}\| \\ &\qquad + \mu\int_\Omega \tilde{D}^\alpha\{\tilde{e}(-g)\nabla(\nabla\cdot\bar{v})\}\cdot D^\alpha\bar{v}\,dx + \mu\int_\Omega D^\alpha(\bar{e}\nabla(\nabla\cdot v^\delta))\cdot D^\alpha\bar{v}\,dx. \end{aligned}
$$

Next, we note that

$$(3.7)\qquad \nu\left|\int_\Omega D^\alpha(\bar e\Delta v^\delta)\cdot D^\alpha\bar v\,dx\right|$$
$$\leqq C_2\nu\|\nabla v^\delta\|_k\|\bar g\|_m\|\bar v\|_m+\delta_k^m C_2\nu\|\nabla v^\delta\|_k^2\|\bar g\|_k^2$$
$$+\frac{\nu}{4}\delta_k^m\int_\Omega \tilde e(-g)|\nabla D^\alpha\bar v|^2\,dx.$$

If $0\leqq m\leqq k-1$, this is obvious. If $l=k$, we split the integral on the left-hand side of (3.7) into

$$\int_\Omega \tilde D^\alpha\{\bar e\Delta v^\delta\}\cdot D^\alpha\bar v\,dx-\int_\Omega\nabla\bar e\cdot\nabla D^\alpha v^\delta\cdot D^\alpha\bar v\,dx-\int_\Omega\bar e\cdot(\nabla D^\alpha v^\delta)\cdot\nabla D^\alpha\bar v\,dx$$

and apply our standard devices. An inequality similar to (3.7) holds for the corresponding μ-term. Now, we estimate some terms in equation (3.6) by using, in particular, (3.4), (3.3) and (3.7). We obtain

$$(3.8)\qquad \frac{1}{2}\frac{d}{dt}\|D^\alpha\bar v\|^2+\lambda^2\int_\Omega\phi'(g)\nabla D^\alpha\bar g\cdot D^\alpha\bar v\,dx+\frac{1}{C_1}(\nu\|\nabla\bar v\|_m^2+\mu\|\nabla\cdot\bar v\|_m^2)$$
$$\leqq C_2\lambda\|\bar g\|_m\|\bar v\|_m+C_2\|\bar v\|_m^2+\delta_k^m C_2\lambda^2\|\nabla g^\delta\|_k|\bar g|_\infty\|\bar v\|_k$$
$$+c\,\delta_k^m\|v^\delta\|_{k+1}|\bar v|_\infty\|\bar v\|_k+C_2(\nu+\mu)\|\bar v\|_m^2$$
$$+C_2(\nu\|\nabla v^\delta\|_k+\mu\|\nabla\cdot v^\delta\|_k)\|\bar g\|_m\|\bar v\|_m$$
$$+\delta_k^m C_2(\nu\|\nabla v^\delta\|_k^2+\mu\|\nabla\cdot v^\delta\|_k^2)\|\bar g\|_k.$$

Finally, we add termwise (3.5) and (3.8) for all α such that $0\leqq|\alpha|\leqq m$. This yields, in particular, equation (3.1). □

Next, fix a real number β_0 satisfying $0<\beta_0<k_0-(n/2)$. Clearly, $0<\beta_0<1$. Since $k_0-\beta_0>n/2$, we have $|\cdot|_\infty\leqq c\|\cdot\|_{k_0-\beta_0}$. Well-known interpolation results for L^2-Sobolev spaces show that

$$(3.9)\qquad |\cdot|_\infty\leqq c\|\cdot\|_{k_0-1}^{\beta_0}\|\cdot\|_{k_0}^{1-\beta_0}.$$

Theorem 3.2. *For each $X\in\mathscr{X}$ and each $\delta>0$,*

$$(3.10)\qquad \lambda^2|\bar g|_{\infty,T}^2+|\bar v|_{\infty,T}^2\leqq C_2\delta^{2(k-k_0+\beta_0)}.$$

Proof. Set

$$(3.11)\qquad G_m^2(t)=\lambda^2\int_\Omega\phi'(g)\sum_{|\alpha|\leqq m}|D^\alpha\bar g|^2\,dx+\|\bar v\|_m^2,$$

and let m satisfy $0\leqq m\leqq k-1$. Clearly $1/C_1\leqq\phi'(g)\leqq C_1$ for a suitable constant of type C_1. Moreover, $\nu[\nabla v^\delta]_{k,T}^2+\mu[\nabla\cdot v^\delta]_{k,T}^2\leqq C_2$, by $(2.5)_1$. Hence, from (3.1) it follows, by straightforward calculations, that

$$G_m^2(t)\leqq C_2G_m^2(0)\quad\forall t\in[0,T].$$

Consequently,

$$\lambda^2 \|\bar g\|^2_{m,T} + \|\bar v\|^2_{m,T} \leqq C_2(\lambda^2 \|g_0^\delta - g_0\|^2_m + \|v_0^\delta - v_0\|^2_m). \tag{3.12}$$

Using this inequality for $m = k_0$ and $m = k_0 - 1$, and taking into account (3.9), we prove that

$$\begin{aligned} \lambda^2 |\bar g|^2_{\infty,T} + |\bar v|^2_{\infty,T} &\leqq C_2(\lambda^2 \|g_0^\delta - g_0\|^2_{k_0-1} + \|v_0^\delta - v_0\|^2_{k_0-1})^{\beta_0} \\ &\quad \times (\lambda^2 \|g_0^\delta - g_0\|^2_{k_0} + \|v_0^\delta - v_0\|^2_{k_0})^{1-\beta_0}. \end{aligned} \tag{3.13}$$

Next, by applying $(2.6)_2$ for $m = k$ and $s = k_0 - 1$, we get

$$\lambda^2 \|g_0^\delta - g_0\|^2_{k_0-1} + \|v_0^\delta - v_0\|^2_{k_0-1} \leqq \delta^{2(k-k_0+1)}(\lambda^2 \|g_0\|^2_k + \|v_0\|^2_k). \tag{3.14}$$

Again by $(2.6)_2$, we have

$$\lambda^2 \|g_0^\delta - g_0\|^2_{k_0} + \|v_0^\delta - v_0\|^2_{k_0} \leqq \delta^{2(k-k_0)}(\lambda^2 \|g_0\|^2_k + \|v_0\|^2_k). \tag{3.15}$$

The estimates (3.13), (3.14) and (3.15) show that (3.10) holds. □

Corollary 3.3. *For each $X \in \mathscr{X}$ and each $\delta > 0$,*

$$(\lambda |\bar g|_{\infty,T} + |\bar v|_{\infty,T})(\lambda \|g^\delta\|_{k+1,T} + \|v^\delta\|_{k+1,T}) \leqq C_2 \delta^{k-k_0-1+\beta_0}. \tag{3.16}$$

Proof. From $(2.6)_1$ for $m = k + 1$ and $s = k$ and from estimate (1.9) with k replaced by $k + 1$, it readily follows that

$$\lambda^2 \|g^\delta\|^2_{k+1,T} + \|v^\delta\|^2_{k+1,T} \leqq C_2/\delta^2. \tag{3.17}$$

This estimate together with (3.10) shows that (3.16) holds. □

Theorem 3.4. *For each $X \in \mathscr{X}$ and $\delta > 0$,*

$$\begin{aligned} \lambda^2 \|\bar g\|^2_{k,T} + \|\bar v\|^2_{k,T} + \int_0^T (\nu \|\nabla \bar v\|^2_k + \mu \|\nabla \cdot \bar v\|^2_k) \\ \leqq C_2(\lambda^2 \|g_0^\delta - g_0\|^2_k + \|v_0^\delta - v_0\|^2_k + \delta^{2\beta_0}). \end{aligned} \tag{3.18}$$

Proof. Define $G_k^2(t)$ by (3.11). Equation (3.1) and the estimates already proved show that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} G_k^2(t) + \frac{1}{C_1}(\nu \|\nabla \bar v\|^2_k + \mu \|\nabla \cdot \bar v\|^2_k \\ \leqq C_2 G_k^2(t) + C_2(\lambda \|\nabla g^\delta\|_k + \|v^\delta\|_{k+1}) \\ \times (\lambda |\bar g|_\infty + |\bar v|_\infty) G_k(t) + C_2 h(t) G_k^2(t), \end{aligned} \tag{3.19}$$

where $h(t) \equiv \nu(1 + \|\nabla v^\delta\|^2_k) + \mu(1 + \|\nabla \cdot v^\delta\|^2_k)$ satisfies $\int_0^T h(t)\,dt \leqq C_2$. Hence, by (3.16), it readily follows that

$$G_k(t) \leqq C_2(G_k(0) + \delta^{\beta_0}). \tag{3.20}$$

By taking into account the explicit expression of $G_k(0)$, by integrating (3.19) on $[0, T]$, and by using (3.20), we show (3.18). □

Proof of Theorem A. From (3.18) and (2.12) it follows that

$$\lambda^2 \|\tilde{g}\|_{k,T}^2 + \|\tilde{v}\|_{k,T}^2 + \nu[\nabla \tilde{v}]_{k,T}^2 + \mu[\nabla \cdot \tilde{v}]_{k,T}^2$$
$$\leqq C_2\Big(\lambda^2 \|g_0\|_k^2 + \|w_0 - v_0\|_k^2 + \sum_{|\xi| > 1/\delta} (1 + |\xi|^2)^k |\hat{w}_0(\xi)|^2 + \delta^{2\beta_0}\Big).$$

In particular,

$$\lambda^2 \|\tilde{g}\|_{k,T}^2 + \|\tilde{v}\|_{k,T}^2 + \bar{\nu}[\nabla \tilde{v}]_{k,T}^2 + \bar{\mu}[\nabla \cdot \tilde{v}]_{k,T}^2 \tag{3.21}$$
$$\leqq C_2(\lambda^2 \|g_0\|_k^2 + \|w_0 - v_0\|_k^2 + \hat{h}(\delta) + |\nu - \bar{\nu}|),$$

where $\hat{h}(\delta)$ depends only on δ (w_0 and k are fixed) and satisfies $\lim_{\delta \to 0} \hat{h}(\delta) = 0$. For convenience, we set $\bar{\mu} = 0$ if $\mu \in [0, \mu_0]$. Given $\varepsilon > 0$, we fix $\delta = \delta(\varepsilon)$ such that $C_2 \hat{h}(\delta) \leqq \varepsilon/2$. Next, (3.21) shows that there is a neighbourhood $U = U(\varepsilon)$ of $(w_0, 0, \infty, \bar{\nu})$ such that if $X = (v_0, g_0, \lambda, \nu, \mu) \in \mathscr{X}$ and $(v_0, \lambda g_0, \lambda, \nu) \in U$, then

$$\lambda^2 \|g^\delta - g\|_{k,T}^2 + \|v^\delta - v\|_{k,T}^2 + \bar{\nu}[\nabla(v^\delta - v)]_{k,T}^2 + \bar{\mu}[\nabla \cdot (v^\delta - \bar{v})]_{k,T}^2 < \varepsilon. \tag{3.22}$$

On the other hand, Proposition 2.2 shows that (here $\delta = \delta(\varepsilon)$)

$$|||v^\delta - w^\delta|||^2 + \|g^\delta\|_{k,T}^2 + |||\lambda g^\delta|||_\nabla^2 < \varepsilon \tag{3.23}$$

if $X \in \mathscr{X}$ and if $(v_0, \lambda g_0, \lambda, \nu) \in V = V(\varepsilon)$, where $V(\varepsilon)$ is a suitable neighbourhood of $(w_0, 0, \infty, \bar{\nu})$. For convenience we set

$$|||v|||^2 \equiv \|v\|_{k,T}^2 + \bar{\nu}[v]_{k+1,T}^2 + \bar{\mu}[\nabla \cdot v]_{k,T}^2, \quad |||g|||_\nabla^2 \equiv |||\nabla g|||_{k-1,T}^2.$$

Let $X = (v_0, g_0, \lambda, \nu, \mu)$ and $\tilde{X} = (\tilde{v}_0, \tilde{g}_0, \tilde{\lambda}, \tilde{\nu}, \tilde{\mu})$ be arbitrary data in $\mathscr{X}$ such that $(v_0, \lambda g_0, \lambda, \nu)$ and $(\tilde{v}_0, \tilde{\lambda}\tilde{g}_0, \tilde{\lambda}, \tilde{\nu})$ belong to $U \cap V$. Moreover, set $(v, g = S(X)$, $(\tilde{v}, \tilde{g}) = S(\tilde{X})$. Let $X^\delta = (v_0^\delta, g_0^\delta, \lambda, \nu, \mu)$, $\tilde{X}^\delta = (\tilde{v}_0^\delta, \tilde{g}_0^\delta, \tilde{\lambda}, \tilde{\nu}, \tilde{\mu})$, and set $(v^\delta, g^\delta) = S(X^\delta)$, $(\tilde{v}^\delta, \tilde{g}^\delta) = S(\tilde{X}^\delta)$. Clearly

$$|||v - \tilde{v}|||^2 + |||\lambda g - \tilde{\lambda}\tilde{g}|||_\nabla^2 \tag{3.24}$$
$$\leqq c(|||v - v^\delta|||^2 + |||v^\delta - w^\delta|||^2 + |||w^\delta - \tilde{v}^\delta|||^2 + |||\tilde{v}^\delta - \tilde{v}|||^2$$
$$+ |||\lambda g - \lambda g^\delta|||_\nabla^2 + |||\lambda g^\delta|||_\nabla^2 + |||\tilde{\lambda}\tilde{g}^\delta|||_\nabla^2 + |||\tilde{\lambda}\tilde{g}^\delta - \tilde{\lambda}\tilde{g}|||_\nabla^2).$$

Hence, from (3.22) and (3.23) it follows that $|||v - \tilde{v}||| + |||\lambda g - \lambda \tilde{g}|||_\nabla < c\varepsilon$. In other words, given $\varepsilon > 0$, there is a neighbourhood $U \cap V$ of $(w_0, 0, \infty, \bar{\nu})$ such that this last estimate holds whenever the data $X, \tilde{X} \in \mathscr{X}$ satisfy the assumption that $(v_0, \lambda g_0, \lambda, \nu)$, $(\tilde{v}_0, \tilde{\lambda}\tilde{g}_0, \tilde{\lambda}, \tilde{\nu}) \in U \cap V$. Since the spaces $C(0, T; H^k)$ and $L^2(0, T; H^{k+1})$ are complete and since a basis of fundamental neighbourhoods of $(w_0, 0, \infty, \bar{\nu})$ is countable (not essential), the convergence of v in the norm $|||\cdot|||$ and the convergence of $\lambda \nabla g$ in the norm $\|\cdot\|_{k-1,T}$ follow. Strong convergence of g_t is now a consequence of $(1.4)_1$. Theorem A is proved. □

Remark. The last step of this proof can be carried out in a more direct and elegant way by using

$$\begin{aligned}&|||v - w|||^2 + |||\lambda g|||_{\nabla}^2\\ &\quad \leqq c(|||v - v^\delta|||^2 + |||v^\delta - w^\delta|||^2 + |||w^\delta - w|||^2 + |||\lambda g - \lambda g^\delta|||_{\nabla}^2 + |||\lambda g^\delta|||_{\nabla}^2)\end{aligned}$$

instead of (3.24). In this case, however, we must prove that

$$\lim_{\delta\to 0} \|w^\delta - w\|_{k,T}^2 + \bar{v}[w^\delta - w]_{k+1,T}^2 = 0. \tag{3.25}$$

Since $\|w_0^\delta - w_0\|_k^2 \leqq \sum_{|\xi|>1/\delta}(1 + |\xi|^2)^k |\hat{w}_0(\xi)|^2$, we have

$$\lim_{\delta\to 0} \|w_0^\delta - w_0\|_k = 0. \tag{3.26}$$

The proof of (3.25) is as follows. Taking the termwise difference of equations (2.11) and (1.5) and by setting $z = w^\delta - w$ we obtain

$$\begin{aligned}&\nabla\cdot z = 0,\\ &\partial_t z + (w\cdot\nabla)z + \nabla\pi' = -(z\cdot\nabla)w^\delta + \bar{v}\Delta z,\\ &z(0) = w_0^\delta - w_0.\end{aligned} \tag{3.27}$$

Next, we apply the operator D^α to both sides of (3.27), multiply the result by $D^\alpha z$, integrate the product over Ω, and sum the result with respect to α for $|\alpha| \leqq k$. Calculations similar to some of the calculations above show that

$$\frac{1}{2}\frac{d}{dt}\|z\|_k^2 + \frac{\bar{v}}{2}\|\nabla z\|_k^2 \leqq c\,\|w\|_k\,\|z\|_k^2 + c\,\|w^\delta\|_{k+1}|z|_\infty\,\|z\|_k. \tag{3.28}$$

Note that this equation is a simplification of (3.8). Next (compare with (3.16)) we prove that

$$|z|_{\infty,T}\,\|w^\delta\|_{k+1,T} \leqq C\delta^{k-k_0-1+\beta_0}\,\|w_0^\delta - w_0\|_k. \tag{3.29}$$

From (3.28) and (3.29) it readily follows that

$$\|w^\delta - w\|_{k,T} \leqq C\,\|w_0^\delta - w_0\|_k. \tag{3.30}$$

We use (3.29) and (3.30) in order to estimate the right-hand side of (3.28). Next, we integrate the equation obtained over $[0, T]$. This shows that

$$\|w^\delta - w\|_{k,T}^2 + \bar{v}[\nabla(w^\delta - w)]_{k,T}^2 \leqq C\,\|w_0^\delta - w_0\|_k^2$$

which, together with (3.26), yields (3.25). □

We point out the following by-product of our arguments. Set $H_\sigma^k = \{w_0 \in H^k : \nabla\cdot w_0 \equiv 0\}$. We have

Theorem 3.5. *The map $(w_0, \bar{v}) \to w$, where w is the solution of problem (1.5), is norm-continuous on $H_\sigma^k \times [0, v_0]$ with values in $C(0, T; H_\sigma^k)$. Moreover, this map is norm-continuous on $H_\sigma^k \times [\bar{v}_1, v_0]$ with values in $L^2(0, T; H_\sigma^{k+1})$ if $\bar{v}_1 > 0$.*

We note that, if $\bar{v} > 0$ is *fixed*, then the continuity of the map $w_0 \to w$ from H^k_σ to $C(0, T; H^k_\sigma) \cap L^2(0, T; H^{k+1}_\sigma)$ is trivial.

We remark that the counterpart of Theorem 3.5 for the equations for *compressible* fluids is also a by-product of our proof. It is sufficient to argue as in the proof of Theorem 1.1, but with a fixed λ instead of $\lambda \to \infty$. We may also leave $p(\lambda, \rho) \to p(\lambda_0, \rho)$, i.e., $\lambda \to \lambda_0 < \infty$. This corresponds to structural stability in the strong norm.

Appendix

For the reader's convenience we state here some useful results. For references and proofs see [BV7, Appendix].

Here Ω is the n-dimensional torus, an open bounded regular subset of $\mathbb{R}^n$, or $\mathbb{R}^n$ itself, or $\mathbb{R}^n_+ \equiv \{x : x_n > 0\}$.

Lemma 4.1. *Let $r > n/2$. If $0 \leqq s \leqq r$, then*

$$\| f\, g \| \leqq c \| f \|_{r-s} \| g \|_s. \tag{4.1}$$

If $0 \leqq l$ and $0 \leqq s \leqq r - l$, then

$$\| f\, g \|_l \leqq c \| f \|_{r-s} \| g \|_{l+s}. \tag{4.2}$$

Lemma 4.2. *Let $r > n/2$, $0 \leqq l \leqq r$, $l \leqq l_1 \leqq r$ for $i = 1, \ldots, m$, and $l_1 + \ldots + l_m = l + (m-1)r$. Then*

$$\| f_1 \ldots f_m \|_l \leqq c \| f_1 \|_{l_1} \ldots \| f_m \|_{l_m}. \tag{4.3}$$

Lemma 4.3. *Let $k > 1 + n/2$ and $1 \leqq l \leqq k$. If $|\alpha| \leqq l$, then*

$$\| D^\alpha \{ f\, g \} \| \leqq c \| Df \|_{k-1} \| g \|_{l-1}. \tag{4.4}$$

Lemma 4.4. *Let $|\alpha| \leqq l$. Then*

$$\| \tilde{D}^\alpha \{ f\, g \} \| \leqq c (|Df|_\infty \| g \|_{l-1} + |g|_\infty \| Df \|_{l-1}). \tag{4.5}$$

Lemma 4.5. *Let $\psi \in C^r(\mathbb{R}; \mathbb{R})$, $r \geqq 1$. Then there are increasing functions $\beta_1 \in C^\infty(\mathbb{R}^+_0; \mathbb{R}^+)$ and $\beta_2 \in C^\infty(\mathbb{R}^+_0 \times \mathbb{R}^+_0; \mathbb{R}^+)$ such that*

$$\| D^\alpha \psi(g) \|^2 \leqq \beta_1(|g|_\infty) \| g \|^2_r, \tag{4.6}$$

$$\| D^\alpha \psi(g) - D^\alpha \psi(f) \|^2 \leqq \beta_2(|g|_\infty, |f|_\infty) \| g - f \|^2_r \tag{4.7}$$

for each α, $1 \leqq |\alpha| \leqq r$.

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Centro Linceo Interdisciplinare "B. Segre"
Accademia Nazionale dei Lincei
Rome

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