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On the singular limit for slightly compressible fluids

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Abstract. In this paper we study the motion of slightly compressible inviscid fluids. We prove that the solution of the corresponding system of nonlinear partial differential equations converges (uniformly) in the strong norm (that of the data space) to the solution of the incompressible equations, as the Mach number goes to zero (see Theorem 1.2). Actually, our proof applies to a large class of singular limit problems as shown in the Theorem 2.2.

Mathematics Subject Classification: 35B30, 35L45, 35L60, 76N10

1. Introduction

This paper is strongly related to that of Klainerman and Majda [KMa2]. For a discussion on the physical motivations and on the mathematical setting up the reader is referred to [Eb1,2; KMa1,2; Ma], and references there in.

Let us shortly introduce the problem studied below. We start by giving some notation, borrowed from [KMa2].

We denote by $\|u\|_l$, $l = 0, 1, 2, \ldots$, the norm it the Hilbert space $H^l = H^l(\Omega)$, by $\|u\|_{l,T}$ the norm in $L^\infty_T(H^l) \equiv L^\infty(0,T;H^l)$, and by $[u]_{l,T}$ the norm in $L^2_T(H^l) = L^2(0,T;H^l)$, $T > 0$. For convenience, we will study our equations in the space-periodic case. Hence Ω is the n-dimensional torus. We set $Q_T = \Omega \times [0,T]$. In the sequel k denotes a fixed integer such that $k \geq k_0 + 1$ where $k_0 = [n/2] + 1$. Moreover, u is the r-vector $(u_1, \ldots, u_r)$ and $\lambda > \lambda_0 \geq 0$ is a parameter. If $u = u(t,x)$, we denote by $u(0)$ the function $u(0,\cdot)$.

Let $B^i(u,\lambda)$, $i = 1, \ldots, n$ be $r \times r$ matrices, of class C^{k+1}, defined for each $\lambda > \lambda_0$ and each $u \in \mathscr{O}$. $\mathscr{O}$ is an open, regular, connected subset of $\mathbb{R}^r$. As in [KMa2], we assume that there are $n+1$ symmetric matrices $A^0(u,\lambda)$ and $A^i(u,\lambda)$, $i = 1, \ldots, n$, such that $A^0 B^i = A^i$ and that

$$(1.1) \qquad (A^0(u,\lambda)\xi, \xi) \geq m|\xi|^2\,, \qquad m > 0\,,$$

★ *Present address:* Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, I-56127 Pisa, Italy for all $u \in \mathcal{O}$ (shrink $\mathcal{O}$, if necessary) and all $\xi \in \mathbb{R}^r$. Moreover,

$$|B(u,\lambda)| \le c\lambda, \qquad |A^0(u,\lambda)| \le c, \tag{1.2}$$

$$\lambda |D_u A^0(u,\lambda)| \le c, \tag{1.3}$$

$$\sum_{j=1}^{k+1} |D_u^j B(u,\lambda)| \le c, \tag{1.4}$$

for all $u \in \mathcal{O}$, $\lambda > \lambda_0$. Here $B = (B^1, \ldots, B^n)$ and $Bu_x \equiv \sum_{i=1}^{n} B^i u_{x_i}$. Next, we consider the system of equations

$$\begin{cases} u_t^\lambda + B(u^\lambda, \lambda) u_x^\lambda = 0 & \text{in } Q_T \\ u^\lambda(0) = u_0^\lambda, \end{cases} \tag{1.5}$$

where $u_0^\lambda \in H^k$ and $\{u_0^\lambda(x) : x \in \Omega\} \subset \mathcal{O}_0$ for each $\lambda > \lambda_0$. $\mathcal{O}_0$ is a compact subset of $\mathcal{O}$.

We point out that we can assume, without loss of generality, that all the initial data that will be used in this paper (namely, u_0, u_0^λ, $u_0^{\lambda,\delta}$, $u_0^{\lambda,\varepsilon}$) take values in a compact subset $\mathcal{O}_1$ of $\mathcal{O}$ such that $\mathcal{O}_0 \Subset \mathcal{O}_1$. It readily follows that the solutions of the corresponding evolution equations take values in a compact subset $\mathcal{O}_2$ satisfying $\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \mathcal{O}$, at least until a finite time $T > 0$, which is independent of the parameters $\lambda, \delta, \varepsilon$. All that follows from the a priori estimates in [KMa2] and from the constructions done in the following Sects. 2 and 3.

In the sequel the symbols $C, C_0 C_1, \ldots,$ denote positive constants that are independent of λ. The same symbol may be used to denote distinct constants, even in the same formula.

The following result is due to Klainerman and Majda [KMa2]:

Theorem 1.1. *Assume that $\|u_0^\lambda\|_k \le C_0$, for all $\lambda > \lambda_0$. Then, there is a positive real T, independent of λ, and a unique solution $u^\lambda \in C_T(H^k) \cap C_T^1(H^{k-1})$ of (1.5). Moreover,*

$$\|u^\lambda\|_{k,T} + \lambda^{-1} \|\partial_t u^\lambda\|_{k-1,T} \le C. \tag{1.6}$$

Furthermore, if

$$\|B(u_0^\lambda, \lambda) u_{0,x}^\lambda\|_{k-1} \le C_0 \tag{1.7}$$

then

$$\|\partial_t u^\lambda\|_{k-1,T} \le C. \tag{1.8}$$

Note that $B(u_0^\lambda, \lambda) u_{0,x}^\lambda = -\partial_t u^\lambda(0)$. Hence (1.7) is also a necessary condition for having (1.8).

Next we describe the application of the above result to the Euler compressible and incompressible equations.

Consider a fluid filling Ω and obeying a law of state $p = p(\varrho)$. Denote by $\bar\varrho > 0$ the mean density of this fluid. By replacing $p(\varrho)$ by $p(\varrho) - p(\bar\varrho)$ one has $p(\varrho) = 0$ if and only if $\varrho = \bar\varrho$. Here and in the sequel we assume that $p \in C^{k+2}(\mathbb{R}^+; \mathbb{R})$ and that $p'(\varrho) > 0$ for each $\varrho > 0$. Let $\varrho(p)$ denote the inverse function of $p(\varrho)$, defined on the open interval $I \equiv p(\mathbb{R}^+)$. Set

$$g(p) \equiv \varrho'(p)/\varrho(p).$$

Clearly, $g(p) > 0$ for each $p \in I$. The equations of motion are

$$(1.9)\qquad \begin{cases} g(p)\,(\partial_t p + v\cdot\nabla p) + \nabla\cdot v = 0\,,\\ \varrho(p)\,(\partial_t v + (v\cdot\nabla)v) + \nabla p = 0\,,\\ v(0) = v_0(x)\,, \qquad p(0) = p_0(x)\,. \end{cases}$$

We are interested in considering a family of laws of state $p^\lambda(\varrho) = \lambda^2 p(\varrho)$ and in studying the behaviour of the solutions as the parameter λ goes to ∞. The parameter λ plays here the part of inverse of the Mach number; see [Ma].

Denoting by ϱ^λ the inverse of the function p^λ, one has $\varrho^\lambda(p^\lambda) = \varrho(p^\lambda/\lambda^2)$, hence $g^\lambda(p^\lambda) \equiv \varrho'_\lambda(p_\lambda)/\varrho_\lambda(p_\lambda) = \lambda^{-2} g(p^\lambda/\lambda^2)$. Consequently, the equations of motion under the above λ-law of state and for initial data $v_0^\lambda(x)$, $p_0^\lambda(x)$ are

$$(1.10)\qquad \begin{cases} \lambda^{-1} g(\bar p^\lambda/\lambda)\,(\partial_t \bar p^\lambda + v^\lambda\cdot\nabla\bar p^\lambda) + \nabla\cdot v^\lambda = 0\,,\\ \varrho(\bar p^\lambda/\lambda)\,(\partial_t v^\lambda + (v^\lambda\cdot\nabla)v^\lambda) + \lambda\nabla\bar p^\lambda = 0\,,\\ v^\lambda(0) = v_0^\lambda(x)\,, \qquad \bar p^\lambda(0) = \bar p_0^\lambda(x) \equiv \lambda^{-1} p_0^\lambda(x)\,, \end{cases}$$

where the "true" pressure p^λ is replaced by $\bar p^\lambda \equiv \lambda^{-1}p^\lambda$. Note that $\bar p^\lambda(x) = 0$ if and only if $\varrho^\lambda(x) = \bar\varrho$.

The system (1.10) can be written in the above form (1.5), as follows. Denote by b^i_{kj} the "row k column j" element of the matrix $B^i \equiv E^i + \mathrm{diag}\{v_i/g, v_i/\varrho, v_i/\varrho, v_i/\varrho\}$, $i = 1,2,3$. The matrix E^i is defined by setting $e^i_{1,i+1} = \lambda/g$; $e^i_{i+1,1} = \lambda/\varrho$; and $e^i_{kj} = 0$ otherwise. Moreover, if $A^0 \equiv \mathrm{diag}\{g, \varrho, \varrho, \varrho\}$ one has $A^i = A^0 B^i$ where $A^i = F^i + \mathrm{diag}\{v_i, v_i, v_i, v_i\}$, $i = 1,2,3$. Here $f^i_{i+1,1} = f^i_{1,i+1} = \lambda$; and $f^i_{kj} = 0$ otherwise. Above, $g = g(\bar p^\lambda/\lambda)$ and $\varrho = \varrho(\bar p^\lambda/\lambda)$.

By using the above set up and by defining $u^\lambda \equiv (\bar p^\lambda, v^\lambda)$, $u_0^\lambda \equiv (\bar p_0^\lambda, v_0^\lambda)$ it readily follows that the system (1.10) has the form (1.5).

Now we assume (see [KMa2], Eq. (1.7)) that [1]

$$(1.11)\qquad \begin{cases} v_0^\lambda(x) = v_0(x) + \lambda^{-1} w_0^\lambda(x)\,,\\ \bar p_0^\lambda(x) = \lambda^{-1} p_0^\lambda(x)\,, \end{cases}$$

where

$$(1.12)\qquad \nabla\cdot v_0 \equiv 0 \quad \text{and} \quad \|w_0^\lambda\|_k + \|p_0^\lambda\|_k \le C\,.$$

Obviously, $\lim \|u_0^\lambda - u_0\|_k = 0$ as $\lambda \to \infty$, where $u_0 = (0, v_0)$. Moreover (1.7) holds since

$$(1.13)\qquad B(u_0^\lambda, \lambda)u^\lambda_{0,x} = (v_0^\lambda\cdot\nabla\bar p_0^\lambda + (\lambda/g_0^\lambda)\nabla\cdot v_0^\lambda, (v_0^\lambda\cdot\nabla)v_0^\lambda + (\lambda/\varrho_0^\lambda)\nabla\bar p_0^\lambda)\,,$$

where $\varrho_0^\lambda = \varrho(\bar p_0^\lambda/\lambda)$, $g_0^\lambda \equiv g(\bar p_0^\lambda/\lambda)$. Hence, by Theorem 1.1, one has the fundamental estimate

$$\|\bar p^\lambda\|_{k,T} + \|v^\lambda\|_{k,T} + \|\bar p_t^\lambda\|_{k-1,T} + \|v_t^\lambda\|_{k-1,T} \le C\,.$$

In particular, subsequences converge [2] in $L^\infty_T(H^k)$ or in $L^\infty_T(H^{k-1})$, with respect to the weak-* topologies, as $\lambda \to \infty$. It is not difficult to verify that the limit functions

[1] In fact, $\lim \lambda^{-1}\|w_0^\lambda\|_k = 0$ as $\lambda \to \infty$ plus $\|\nabla\cdot w_0^\lambda\|_{k-1} \le C$ would be sufficient here

[2] Here, and in the sequel, we use this short saying; the meaning is clear. Moreover we improperly will call $\{v^\lambda\}$, as $\lambda \to \infty$, a "sequence"

satisfy the incompressible Euler equations

$$\text{(1.14)} \qquad \begin{cases} \nabla \cdot v = 0\,, \\ \bar{\varrho}(\partial_t v + (v \cdot \nabla) v) + \nabla \pi = 0\,, \\ v(0) = v_0(x)\,, \end{cases}$$

for some $\pi(t, x)$. By the uniqueness of the regular solution of (1.14) it follows that the whole sequence $\{v^\lambda\}$, $\lambda \to \infty$, converges to v in the $L^\infty_T(H^k)$ weak-* topology. Similarly, v^λ_t converges to v_t and $\nabla p^\lambda = \nabla(\lambda \bar{p}_\lambda)$ converges to $\nabla \pi$, both in $L^\infty_T(H^{k-1})$ weak-*. However, $\partial_t p^\lambda(0)$ may blow up in H^{k-1} since (as $\lambda \to \infty$) it behaves like λw^λ_0. Clearly $\varrho^\lambda \equiv \varrho(\bar{p}^\lambda/\lambda) \to \bar{\varrho}$. Note that $\varrho^\lambda_0 \partial_t v^\lambda(0) + \nabla p^\lambda_0$ is convergent in H^{k-1}, but the same does not (necessarily) hold to $\partial_t v^\lambda(0)$ and ∇p^λ_0, separately. Hence, we can not expect that v^λ_t converges to v_t and that ∇p^λ converges to $\nabla \pi$ in $C_T(H^{k-1})$ (under the sole assumptions (1.11), (1.12)). However, we will prove that the trajectories $(\varrho^\lambda, v^\lambda)$ converge to that of the Euler incompressible equation, i.e. to $(\bar{\varrho}, v)$, in the strong norm H^k, uniformly in time. More precisely, we prove here the following result:

Theorem 1.2. *Let* $(\bar{p}^\lambda_0, v^\lambda_0)$, $\lambda > \lambda_0$, *be a family of initial data satisfying the assumptions* (1.11), (1.12), *and let* $(\bar{p}^\lambda, v^\lambda)$ *be the corresponding solution to the compressible Euler equations* (1.10). *Let* $\varrho^\lambda = \varrho(\bar{p}^\lambda/\lambda)$ *denote the density of the fluid. Then*

$$\text{(1.15)} \qquad \lim_{\lambda \to \infty} (\|v^\lambda - v\|_{k,T} + \|\varrho^\lambda - \bar{\varrho}\|_{k,T}) = 0\,,$$

where v *and* $\bar{\varrho}$ *are those appearing in the Euler incompressible equations* (1.14).

The main point in the proofs of Theorems 1.2 and 2.2 is the general Theorem 2.1, which guarantees an uniform approximation result (in the strong norm) for the solutions u^λ of (1.5) by regular solutions $u^{\lambda,\delta}$.

The Theorem 2.2 applies to a large class of problems. In fact it shows that each problem that satisfies the hypothesis of Theorem 1.1 (for a couple of values k and $k+1$) enjoys the following property. If (as $\lambda \to \infty$) the solutions u^λ converge in the $C(0,T;H^0)$ norm to some limit then necessarily the convergence holds in the strong norm $C(0,T;H^k)$. This result holds under more general hypothesis. In fact, in the assumption (1.7) we can replace the H^{k-1} norm by the H^0 norm. This can be proved by arguing as in references [BV8]. Moreover, a combination of the proofs given in reference [BV8] with the proof of the Theorem 2.2 shows that convergence with respect to a variable viscosity can be introduced in this last theorem.

Remarks. (i) As remarked in reference [KL] "the continuous dependence in 'strong' topology of the solution on the data is the most difficult part in a theory dealing with nonlinear equations of evolution". It is also well known that the main difficulties arise in dealing with "mixed problems" in the hyperbolic case, as pointed out in the introduction of [K] where (with reference to mixed initial-boundary value problems) the author remarks that "the existence of solutions of (0.1) is not difficult to prove, but the continuity in the initial datum requires considerable efforts to prove". These opinions are not in contradiction with the fact that experts were aware that the answer to this basic problem should be positive. The point was just the lack of sufficiently general proofs. An approach, that roughly speaking requires some ellipticity, is developed just in reference [K]. In reference [BV3] we present a very general and completely distinct approach, little technical and easily adaptable to a large class of problems. The very simple basic idea is illustrated in reference [BV7], part I.

Nevertheless, in our opinion, there are not sufficiently strong heuristic reasons to lead us to believe that convergence in the strong topology for singular limit problems occurs. The main tool in the proof is an idea introduced in reference [BV3]. However, a valuable hint was given to us by the referee of [BV5], to whom we are grateful.

Finally, we remark that a simplified proof, which in general is not applicable in the presence of boundary conditions, is given in reference [BV8].

(ii) It seems advisable to recall that continuous dependence results in terms of Eulerian variables are not at all a consequence of (seemingly) similar results in terms of Lagrangian coordinates.

(iii) The above results of Klainerman and Majda on the incompressible limit have been extended and developed by Schochet [Sc1] for non barotropic fluids in bounded domains. It is worth noting that the presence of the boundary gives rise to serious obstacles (see also [Sc2,3]). It would be interesting to extend the method developed below to Schochet's approach. Or, alternatively, to get the same extension by using our approach to the compressible equations in bounded domains ([BV4, BV5, BV6], and references).

Other interesting results on the incompressible limit were obtained by Agemi [Ag], Asano [As], Ebin [Eb1,2], and Ukai [U].

(iv) For the *viscous*, time dependent, problem the reader is referred to [KMa1], [Ma], and references there in.

(v) Convergence of compressible *viscous* solutions to the incompressible one, for the *steady* equations, was studied by us in references [BV1,2].

The remaining of this paper is as follows. In Sect. 2 (after the necessary preparation) we state the Theorem 2.1, without proof. Then, by using this theorem, we give the complete proof of Theorem 1.2. Finally, in Sect. 3, we prove the Theorem 2.1.

2. The general theorem

Statement of Theorem 2.1. Here Ω can be the n-dimensional torus or the whole space $\mathbb{R}^n$. In the sequel we consider systems (1.5) enjoying the hypothesis (1.1) to (1.4). Moreover, we assume that

$$\lim_{\lambda\to\infty} \|u_0^\lambda - u_0\|_k = 0\,, \tag{2.1}$$

for some $u_o \in H^k$. In the sequel we will consider an auxiliary family $\{u_0^{\lambda,\delta} \in H^{k+1} : \lambda > \lambda_0, \delta \in]0,\delta_0]\}$, for some fixed $\delta_0 > 0$, such that, for each $\delta \in]0,\delta_0]$,

$$\begin{cases} \|u_0^{\lambda,\delta}\|_k \le C_0\,, & \forall\lambda > \lambda_0\,, \\ \|u_0^{\lambda,\delta}\|_{k+1} \le C_0(\delta)\,, & \forall\lambda > \lambda_0\,, \\ \|u_0^{\lambda,\delta} - u_0^\lambda\|_k \le \delta\,, & \forall\lambda > \lambda(\delta)\,, \\ \|u_0^{\lambda,\delta} - u_0^\lambda\|_{k-1} \le \delta\,, & \forall\lambda > \lambda_0\,. \end{cases} \tag{2.2}$$

Under the assumption (2.1) such a family $\{u_0^{\lambda,\delta}\}$ exists (proved below).

In the sequel we also consider the following two additional hypotheses on the family $\{u_0^{\lambda,\delta}\}$:

For each *fixed* $\delta \in]0,\delta_0]$ there is a function $u_0^\delta \in H^{k+1}$ such that

$$\lim_{\lambda\to\infty} \|u_0^{\lambda,\delta} - u_0^\delta\|_{k+1} = 0\,. \tag{2.3}$$

And

$$\|B(u_0^{\lambda,\delta},\lambda)u_{0,x}^{\lambda,\delta}\|_k \le C(\delta)\,, \qquad \forall \lambda > \lambda(\delta)\,. \tag{2.4}$$

We remark that in the fluidynamics case the assumptions (1.11), (1.12) are sufficient to guarantee the existence of a family $\{u_0^{\lambda,\delta}\}$ satisfying (besides (2.2)) (2.3) and (2.4).

The construction of the family $\{u_0^{\lambda,\delta}\}$, as well as proofs of related properties, will be done after the statements of Theorems 2.1 and 2.2. In order to state these results, we consider a family of initial data $u_0^{\lambda,\delta}$ satisfying (2.2) and the corresponding solutions $u^{\lambda,\delta}$ to the problems

$$\begin{cases} u_t^{\lambda,\delta} + B(u^{\lambda,\delta},\lambda)u_x^{\lambda,\delta} = 0\,, \\ u^{\lambda,\delta}(0) = u_0^{\lambda,\delta}\,. \end{cases} \tag{2.5}$$

These solutions satisfy the estimates

$$\begin{cases} \|u^{\lambda,\delta}\|_{k,T} + \lambda^{-1}\|u_t^{\lambda,\delta}\|_{k-1,T} \le C\,, \\ \|u^{\lambda,\delta}\|_{k+1,T} \le C(\delta)\,, \end{cases} \tag{2.6}$$

for each $\lambda > \lambda_0$ and each $\delta \in]0,\delta_0]$. This follows from Theorem 1.1. One has the following results.

Theorem 2.1. *Let Ω be the n-dimensional torus or the whole $\mathbb{R}^n$. Assume that the hypothesis* (1.1)–(1.4) *hold and let u_0, u_0^λ satisfy* (2.1). *Let $u_0^{\lambda,\delta}$ be a family of functions such that* (2.2) *holds (such functions exist, as proved below). Denote by u^λ and by $u^{\lambda,\delta}$ the solutions of problems* (1.5) *and* (2.5) *respectively. Let $\varepsilon \in]0,1]$ be given. Then, there are positive reals $C(\varepsilon)$, $\lambda(\varepsilon)$, and $\lambda_1(\delta)$ such that for each $\delta > 0$ one has*

$$\|u^{\lambda,\delta} - u^\lambda\|_{k,T} \le C_0(\varepsilon + C(\varepsilon)\delta)\,, \qquad \forall \lambda > \lambda(\varepsilon,\delta) \equiv \max\{\lambda(\varepsilon), \lambda_1(\delta)\}\,. \tag{2.7}$$

In Theorem 2.1 the assumptions on the data are much weaker then (1.11), (1.12). In fact, in the particular case of the system (1.10), the assumptions in Theorem 2.1 hold provided that v_0^λ and $\bar{p}_0^\lambda$ converge in H^k, as $\lambda \to \infty$. As explained below, this sole assumption guarantees the existence of a family $u_0^{\lambda,\delta} \equiv (\bar{p}_0^{\lambda,\delta}, v_0^{\lambda,\delta})$ satisfying (2.2).

The proof of Theorem 2.1 is postponed to Sect. 3.

Theorem 2.2. *Assume that the hypotheses of Theorem* 2.1 *are satisfied and that, for each fixed δ, the limit: $\lim_{\lambda\to\infty} u^{\lambda,\delta}$ exists in $C_T(H^0)$. Then, the sequence $\{u^\lambda\}$ must converge in $C_T(H^k)$, as $\lambda \to \infty$.*

Proof. Let $\sigma > 0$ be given. Fix $\varepsilon = \sigma C_0^{-1}$ and $\delta = \sigma C(\varepsilon)^{-1}$. Since

$$\|u^\lambda - u^\mu\|_{k,T} \le \|u^\lambda - u^{\lambda,\delta}\|_{k,T} + \|u^{\lambda,\delta} - u^{\mu,\delta}\|_{k,T} + \|u^{\mu,\delta} - u^\mu\|_{k,T}\,, \tag{2.8}$$

it follows from (2.7) that

$$\|u^\lambda - u^\mu\|_{k,T} \le 4\sigma + \|u^{\lambda,\delta} - u^{\mu,\delta}\|_{k,T}\,,$$

for each pair $\lambda, \mu > \lambda(\sigma) \equiv \lambda(\varepsilon,\delta)$. The thesis follows by using the $C_T(H^0)$ convergence assumption, $(2.6)_2$ and interpolation $\|\ \|_k^{k+1} \le c\|\ \|_0\,\|\ \|_{k+1}^k$. Note the following corollary to Klainerman and Majda's Theorem 1.1.

Lemma 2.3. *Assume* (2.1), (1.7), (2.2) *and* (2.4). *Then, to each fixed* $\delta \in]0, \delta_0]$ *their correspond* $C(\delta)$ *and* $\lambda(\delta)$ *such that*

$$\|u^{\lambda,\delta}\|_{k+1,T} + \|u_t^{\lambda,\delta}\|_{k,T} \le C(\delta) \tag{2.9}$$

if $\lambda > \lambda(\delta)$.

Construction of the family $u_0^{\lambda,\delta}$

Assume that $\Omega = [0, a]^n$ is the n-dimensional torus, and set $a = 1$ just for convenience. Note that the construction done below applies as well to the case in which $\Omega = \mathbb{R}^n$, by replacing the Fourier series by Fourier transforms.

In the following we assume that (2.1) holds. Consider Fourier series

$$u_0(x) = \sum_\xi \hat{u}_0(\xi)\, e^{2\pi i \xi \cdot x}$$

where

$$\hat{u}_0(\xi) = \int_\Omega e^{-2\pi i \xi \cdot x} u_0(x)\, dx\,.$$

Then

$$\|u_0\|_k^2 = \sum_\xi (1 + |\xi|^2)^k\, |\hat{u}_0(\xi)|^2$$

where $\xi = (\xi_1, \ldots, \xi_n)$, the ξ_i's are nonnegative integers and $|\xi|$ is the Euclidean norm. Analogous formulae hold for each $u_0^\lambda, \lambda > \lambda_0$. In this last case the Fourier coefficients of u_0^λ are denoted by $\hat{u}_0^\lambda$. Given $\delta > 0$ let $R(\delta)$ be such that

$$\sum_{|\xi| > R(\delta)} (1 + |\xi|^2)^k\, |\hat{u}_0(\xi)|^2 < \delta^2/4$$

and that $1 + R(\delta)^2 \ge c_0 \delta^{-1}$, where c_0 satisfies $\|u_0^\lambda\|_k^2 \le c_0$ for each $\lambda > \lambda_0$.

Next, for each $f(x) = \sum_\xi \hat{f}(\xi) \exp(2\pi i x \cdot \xi)$ we set

$$(T_\delta f)(x) \equiv \sum_{|\xi| \le R(\delta)} \hat{f}(\xi)\, e^{2\pi i \xi \cdot x}\,.$$

Define

$$u_0^{\lambda,\delta}(x) \equiv (T_\delta u_0^\lambda)(x)\,. \tag{2.10}$$

Note that $\|u_0^{\lambda,\delta}\|_k \le \|u_0^\lambda\|_k$. Hence $(2.2)_1$ holds. Clearly, $\|u_0^\lambda - u_0^{\lambda,\delta}\|_k^2 \le (\delta^2/2) + 2\|u_0 - u_0^\lambda\|_k^2$. In particular $(2.2)_3$ holds. On the other hand

$$\|u_0^{\lambda,\delta} - u_0^\lambda\|_{k-1}^2 \le c_0/R(\delta)^2\,.$$

Hence $(2.2)_4$ holds. Finally,

$$\|u_0^{\lambda,\delta}\|_{k+1}^2 \le (1 + R(\delta)^2)\, \|u_0^\lambda\|_k^2 \le C(\delta)$$

since the u_0^λ are uniformly bounded in H^k. Hence, $(2.2)_2$ holds. We have shown that a family $u_0^{\lambda,\delta}$ satisfying (2.2) exists if (2.1) holds. Moreover, this family satisfies (2.3).

In fact, define u_0^δ by dropping the λ's in Eq. (2.12). Since $\|u_0^{\lambda,\delta}-u_0^\delta\|_k^2 \le \|u_0^\lambda - u_0\|_k^2$, it follows that $u_0^{\lambda,\delta} \to u_0^\delta$ in H^k as $\lambda \to \infty$, uniformly with respect to δ. Moreover,

$$\|u_0^{\lambda,\delta} - u_0^\delta\|_{k+1}^2 \le (1 + |R(\delta)|^2)\, \|u_0^\lambda - u_0\|_k^2 \,.$$

Hence (2.3) holds, for each fixed $\delta \in]0, \delta_0]$.

Finally we consider the fluidynamics case. We assume that (1.11), (1.12) hold and we prove (2.2)–(2.4).

Set $u_0 \equiv (0, v_0)$, $u_0^\lambda \equiv (\bar{p}_0^\lambda, v_0^\lambda)$. Clearly (2.1) holds. We left to the reader the proof of (1.7) that can be easily done by taking into account (1.13). Next, define

$$\begin{cases} v_0^{\lambda,\delta} = v_0^\delta + \lambda^{-1} w_0^{\lambda,\delta}\,, \\ \bar{p}_0^{\lambda,\delta} = \lambda^{-1} p_0^{\lambda,\delta}\,, \end{cases} \tag{2.11}$$

and set $u_0^{\lambda,\delta} \equiv (\bar{p}_0^{\lambda,\delta}, v_0^{\lambda,\delta})$. Here,

$$v_0^\delta \equiv T_\delta v_0\,, \qquad w_0^{\lambda,\delta} \equiv T_\delta w_0^\lambda\,, \qquad p_0^{\lambda,\delta} \equiv T_\delta p_0^\lambda\,.$$

Moreover, $R(\delta)$ is defined in such a way that

$$\sum_{|\xi|>R(\delta)} (1 + |\xi|^2)^k\, |\hat{v}_0(\xi)|^2 < \delta^2/4$$

and that $\lim_{\delta\to 0} R(\delta) = \infty$. Note that $u_0^{\lambda,\delta}$ satisfy (2.4), for each fixed δ. For, the data in Eq. (2.11) satisfy (1.11), (1.12) when k is replaced by $k+1$. Moreover, $\nabla \cdot v_0(x) \equiv 0$ yields $\xi \cdot \hat{v}_0(\xi) = 0$, for each ξ. Hence $\nabla \cdot v_0^\delta(x) \equiv 0$.

Proof of Theorem 1.2. By Theorem 1.1 the solutions $u^\lambda \equiv (\bar{p}^\lambda, v^\lambda)$ of problem (1.10) satisfy

$$\|(\bar{p}^\lambda, v^\lambda)\|_{k,T} + \|\partial_t(\bar{p}^\lambda, v^\lambda)\|_{k-1,T} \le C\,. \tag{2.12}$$

By $L^\infty(H^l)$ weak-* compactness results, for $l = k-1, k$, it follows that (for suitable subsequences)

$$\begin{cases} (\bar{p}^\lambda, v^\lambda) \to (\bar{p}, v) & \text{in } L_T^\infty(H^k) \text{ weak-*}\,, \\ (\bar{p}_t^\lambda, v_t^\lambda) \to (\bar{p}_t, v_t) & \text{in } L_T^\infty(H^{k-1}) \text{ weak-*}\,, \end{cases} \tag{2.13}$$

for some $(\bar{p}, v)$. It readily follows that (as $\lambda \to \infty$) $\bar{p}^\lambda/\lambda \to 0$, hence $\varrho(\bar{p}^\lambda/\lambda) \to \varrho(0) \equiv \bar{\varrho}$ in $C_T(H^k)$. Moreover, $\nabla \cdot v^\lambda \to 0$ in $C_T(H^{k-1})$, hence $\nabla \cdot v \equiv 0$. Finally, $\lambda \nabla \bar{p}^\lambda \to -\bar{\varrho}(\partial_t + v \cdot \nabla) v$ in $L_T^\infty(H^{k-1})$ weak-*. In particular, $\nabla \bar{p}^\lambda \to 0$ in $C_T(H^{k-1})$, hence $\bar{p} = \bar{p}(t)$ is independent of x. Furthermore, $v(0) = v_0$, $\bar{p}(0) = 0$. Clearly, there is a function $\pi \in L_T^\infty(H^k)$ such that $\nabla \pi = -\bar{\varrho}(\partial_t + v \cdot \nabla) v$. We may assume that $\int \pi(t,x)\,dx = 0$, for each t. The above properties allow us to pass to the limit in Eq. (1.10) and to show that $(v, \nabla\pi)$ is a solution of Eq. (1.14). By the uniqueness of this solution it follows that the whole "sequence" $(v^\lambda, \partial_t v^\lambda, \lambda\nabla\bar{p}^\lambda, \nabla\bar{p}^\lambda, \varrho(\bar{p}^\lambda/\lambda))$ converges to $(v, \partial_t v, \nabla\pi, 0, \bar{\varrho})$ in $L_T^\infty(H^k) \times L_T^\infty(H^{k-1}) \times L_T^\infty(H^{k-1}) \times C_T(H^{k-1}) \times C_T(H^k)$ as $\lambda \to \infty$, where the convergence in the first three functional spaces is in the weak-* norms. By the way, note that $\varrho_\lambda = \varrho(\bar{p}^\lambda/\lambda)$ and $p_\lambda = \lambda\bar{p}^\lambda$ are the density and the "true" pressure of the λ-fluid.

Next, note that the $u_0^{\lambda,\delta}$ satisfy (2.3) and the hypotheses of the Theorem 2.1 and of the Lemma 2.3. By Eq. (2.9) it follows that for each *fixed* δ the solutions $u^{\lambda,\delta} \equiv (\bar{p}^{\lambda,\delta}, v^{\lambda,\delta})$ of problems (2.5) satisfy the estimate

$$(2.14) \qquad \|v^{\lambda,\delta}\|_{k+1,T} + \|\bar{p}^{\lambda,\delta}\|_{k+1,T} + \|v_t^{\lambda,\delta}\|_{k-1,T} + \|\bar{p}_t^{\lambda,\delta}\|_{k-1,T} \le C(\delta),$$

for $\lambda > \lambda(\delta)$. Note that this is just (2.12) if in this last equation k is replaced by $k+1$. Hence, the above weak-* convergence results hold if k is replaced by $k+1$. Since Ω is a bounded set, it follows in particular that

$$(2.15) \qquad (v^{\lambda,\delta}, \nabla\bar{p}^{\lambda,\delta}) \text{ is a Cauchy sequence in } C_T(H^k) \times C_T(H^{k-1})$$

as $\lambda \to \infty$ (this in the sole point in which boundedness of Ω is used).

On the other hand, Theorem 2.1 shows that

$$(2.16) \qquad \|v^{\lambda,\delta} - v^{\lambda}\|_{k,T} + \|\nabla\bar{p}^{\lambda,\delta} - \nabla\bar{p}^{\lambda}\|_{k-1,T} \le C_0(\varepsilon + C(\varepsilon)\delta),$$

for each $\lambda > \lambda(\varepsilon,\delta)$. Now, given $\sigma > 0$, we fix $\varepsilon = \varepsilon(\sigma) \equiv \sigma/C_0$ and then fix $\delta = \delta(\sigma) \equiv \sigma C(\varepsilon)^{-1}$. It readily follows from (2.16) that

$$(2.17) \qquad \begin{aligned} &\|v^{\lambda} - v^{\mu}\|_{k,T} + \|\nabla\bar{p}^{\lambda} - \nabla\bar{p}^{\mu}\|_{k-1,T} \\ &\le 4\sigma + \|v^{\lambda,\delta} - v^{\mu,\delta}\|_{k,T} + \|\nabla\bar{p}^{\lambda,\delta} - \nabla\bar{p}^{\mu,\delta}\|_{k-1,T}, \end{aligned}$$

if $\lambda, \mu > \lambda(\sigma) \equiv \lambda(\varepsilon(\sigma), \delta(\sigma))$. Since $\delta = \delta(\sigma)$ is already fixed, it follows by (2.15) that there is a $\lambda_2(\sigma) \ge \lambda(\sigma)$ such that the left-hand side of (2.17) is bounded by 5σ if $\lambda, \mu > \lambda_2(\sigma)$. This ends the proof of Theorem 1.2.

3. Proof of Theorem 2.1

In the sequel, together to the parameter λ and the auxiliary parameter δ, we will use a second auxiliary parameter ε. A similar device was used in our previous papers [BV3–6]. Constants denoted by capital C (or $C_0, C_1, \ldots$) never depend on λ. Moreover, whenever such a constant is not uniform with respect to δ or ε, this fact will be displayed by writing $C(\delta), C(\varepsilon)$, or $C(\varepsilon,\delta)$.

For convenience we set $\partial_i = \partial_{x_i}$. Denote by $v^{\lambda} \equiv (\partial_1 u^{\lambda}, \ldots, \partial_n u^{\lambda})$ the rn vector whose components are the first order derivatives of the components of the solution u^{λ} of Eq. (1.5). By differentiation of this last equation with respect to each single x_i, we get

$$(3.1) \qquad \begin{cases} v_t^{\lambda} + \tilde{B}(u^{\lambda},\lambda)v_x^{\lambda} + \bar{B}_u(u^{\lambda},\lambda)(v^{\lambda},v^{\lambda}) = 0, \\ v^{\lambda}(0) = v_0^{\lambda} \equiv (\partial_1 u_0^{\lambda}, \ldots, \partial_n u_0^{\lambda}). \end{cases}$$

Here, $\tilde{B}v_x^{\lambda} = \sum_{i=1}^{n} \tilde{B}^i v_{x_i}^{\lambda}$, where each $\tilde{B}^i$, $i = 1, \ldots, n$, is the block matrix $\operatorname{diag}\{B^i, \ldots, B^i\}$, r times. We point out that by setting $\tilde{A}^0 = \operatorname{diag}\{A^0, \ldots, A^0\}$, where $A^0 = A^0(u^{\lambda},\lambda)$, one has $\tilde{A}^0\tilde{B}^i = \tilde{A}^i$, $i = 1, \ldots, n$, where $\tilde{A}^0, \tilde{B}^i, \tilde{A}^i$ satisfy the properties (1.1) to (1.4). In Eq. (3.1) the symbol $\bar{B}_u$ is formal, and has the following meaning. In each of the single nr scalar equations that make up the system $(3.1)_1$ there is a bilinear form over the vector v^{λ}. The coefficients of these bilinear forms are linear combinations with constant coefficients of the first order derivatives (with respect to the variables $u_j, j = 1, \ldots, r$) of the coefficients of the B^i's. This fact follows immediately from the construction of system (3.1). The symbol $\bar{B}_u(u^{\lambda},\lambda)(v^{\lambda},v^{\lambda})$

denotes the above bilinear forms (the index u stands for D_u). The particular form of each single coefficient is not important in the sequel. The point is that these coefficients satisfy (1.4).

Next, note that $v_0^\lambda \to v_0 \equiv (\partial_1 u_0, \dots, \partial_n u_0)$ in H^{k-1}. In particular, $\|v_0^\lambda\|_{k-1} \le C_0$. Clearly,

$$\|v^\lambda\|_{k-1,T} \le C\,. \tag{3.2}$$

Let now $\varepsilon \in]0,1]$ and fix $v_0^{\lambda,\varepsilon} \in H^{k+1}$ in such a way that

$$\begin{cases} \|v_0^{\lambda,\varepsilon}\|_k \le C_0(\varepsilon)\,, & \forall \lambda > \lambda_0\,, \\ \|v_0^{\lambda,\varepsilon} - v_0^\lambda\|_{k-1} \le \varepsilon\,, & \forall \lambda > \lambda(\varepsilon)\,, \end{cases} \tag{3.3}$$

for a suitable $\lambda(\varepsilon)$. The existence of the family $v_0^{\lambda,\varepsilon}$ is proved just as that of the family $u_0^{\lambda,\delta}$, by replacing δ by ε and k by $k-1$.

Next, we consider the system

$$\begin{cases} v_t^{\lambda,\varepsilon} + \tilde{B}(u^\lambda, \lambda) v_x^{\lambda,\varepsilon} + \bar{B}_u(u^\lambda,\lambda)(v^{\lambda,\varepsilon}, v^{\lambda,\varepsilon}) = 0\,, \\ v^{\lambda,\varepsilon}(0) = v_0^{\lambda,\varepsilon}\,. \end{cases} \tag{3.1$_\varepsilon$}$$

Note that this system is linear in the higher order derivatives since u^λ is fixed. The existence of the solution $v^{\lambda,\varepsilon}$ in the space $C_T(H^k) \cap C_T^1(H^{k-1})$ is easily shown by proving (by standard methods) the existence of a fixed point for the map $w \to v$, where v is the solution of the problem $v_t + \tilde{B}(u^\lambda,\lambda)v_x + \bar{B}_u(u^\lambda,\lambda)(w,w) = 0$, $v(0) = v_0^{\lambda,\varepsilon}$. The main point is to prove an a priori estimate for the norm $\|v^{\lambda,\varepsilon}\|_{k,T}$ of the solution of $(3.1)_\varepsilon$. The independence of this estimate with respect to λ is crucial in the sequel. In fact, one has

$$\|v^{\lambda,\varepsilon}\|_{k,T} \le C(\varepsilon)\,, \qquad \forall \lambda > \lambda_0\,. \tag{3.4}$$

Proof of (3.4). By Theorem 1.1,

$$\|u^\lambda\|_{k,T} + \lambda^{-1}\|u_t^\lambda\|_{k-1,T} \le C\,. \tag{3.5}$$

In order to prove (3.4) we argue as done by Klainerman and Majda in reference [KMa2] in order to prove their equation (2.11). Now, the rôle of the function g in [KMa2]'s Eq. (2.11) is played by the quadratic term in Eq. $(3.1)_\varepsilon$. In this way we prove that (see below, for notation)

$$\frac{d}{dt}\|v^{\lambda,\varepsilon}(t)\|_{E,t}^2 \le C\|v^{\lambda,\varepsilon}(t)\|_{E,k}^2 \tag{3.6}$$

provided that

$$\|\bar{B}_u(u^\lambda,\lambda)(v^{\lambda,\varepsilon}, v^{\lambda,\varepsilon})(t)\|_k \le C\|v^{\lambda,\varepsilon}(t)\|_k\,, \tag{3.7}$$

for each $\lambda > \lambda_0$ and each $\varepsilon \in]0,1]$. Then, from $(3.3)_1$ and (3.6), the estimate (3.4) follows.

Above, $\|\;\|_{E,k}$ is defined by (see [KMa2])

$$\|v\|_{E,k}^2 \equiv \sum_{|\alpha| \le k} \|D^\alpha v\|_E^2\,, \qquad \|v\|_E^2 \equiv \int (\tilde{A}^0 v, v)\,dx\,.$$

The norms $\| \ \|_{E,k}$ and $\| \ \|_k$ are uniformly equivalent since the functions $u^\lambda(t,x)$ that appear in the matrices $\tilde{A}^0(u^\lambda, \lambda)$ take values in the set $\mathscr{O}$, for all $(t,x) \in Q_T$ and all $\lambda > \lambda_0$ (recall (1.1)).

In order to prove (3.7) we start by remarking that (for each t)

$$\|\bar{B}_u(u^\lambda, \lambda)\,(v^{\lambda,\varepsilon}, v^{\lambda,\varepsilon})\|_k \le c\|\bar{B}_u(u^\lambda, \lambda)\|_k\,\|v^{\lambda,\varepsilon}\|_k^2\,.$$

Next, as in [KMa2] Lemma 3, we use (3.5) and (1.4) to show that $\|\bar{B}_u(u^\lambda, \lambda)\|_k \le C$. Finally, $\|v^{\lambda,\varepsilon}\|_k^2 \le C\|v^{\lambda,\varepsilon}\|_k$, since $\|v^{\lambda,\varepsilon}\|_{k-1} \le C$. This last estimate is proved by arguing as done below (in a more complicated context) in order to prove (3.9) □

Next, we consider the system

$$\text{(3.1')}\qquad \begin{cases} v_t^{\lambda,\delta} + \tilde{B}(u^{\lambda,\delta}, \lambda)v_x^{\lambda,\delta} + \bar{B}_u(u^{\lambda,\delta}, \lambda)\,(v^{\lambda,\delta}, v^{\lambda,\delta}) = 0\,, \\ v^{\lambda,\delta}(0) = v_0^{\lambda,\delta} \equiv (\partial_1 u_0^{\lambda,\delta}, \ldots, \partial_n u_0^{\lambda,\delta})\,, \end{cases}$$

obtained by differentiation, with respect to the x variables, of system (2.5). By definition $v^{\lambda,\delta} \equiv (\partial_1 u^{\lambda,\delta}, \ldots, \partial_n u^{\lambda,\delta})$. From (2.6) one shows (for each $\delta \in]0, \delta_0]$) that $\|v^{\lambda,\delta}\|_{k-1,T} \le C$ for each $\lambda > \lambda_0$ and that $\|v^{\lambda,\delta}\|_{k,T} \le C(\delta)$ for each $\lambda > \lambda(\delta)$.

Next, we take the difference between the Eqs. (3.1′) and $(3.1)_\varepsilon$. We get

$$\text{(3.8)}\qquad \begin{cases} w_t + \tilde{B}(u^{\lambda,\delta})w_x = g\,, \\ w(o) = v_0^{\lambda,\delta} - v_0^{\lambda,\varepsilon}\,, \end{cases}$$

where $w \equiv v^{\lambda,\delta} - v^{\lambda,\varepsilon}$ and, by definition,

$$\begin{aligned} g \equiv [\tilde{B}(u^\lambda) - \tilde{B}(u^{\lambda,\delta})]v_x^{\lambda,\varepsilon} - \bar{B}_u(u^{\lambda,\delta})\,(v^{\lambda,\delta}, w) - \bar{B}_u(u^{\lambda,\delta})\,(v^{\lambda,\varepsilon}, w) \\ + (\bar{B}_u(u^\lambda) - \bar{B}_u(u^{\lambda,\delta}))\,(v^{\lambda,\varepsilon}, v^{\lambda,\varepsilon})\,. \end{aligned}$$

Here, all the terms $B(\cdot, \lambda)$ are estimated at the same value λ, hence we drop this symbol. Next, by arguing as done in [KMa2] in order to prove the equation that appears in the fourth row p. 638 in that reference (for $s = k-1$), we show that

$$\text{(3.9)}\qquad \frac{d}{dt}\,\|w(t)\|_{E,k-1} \le C(\|w(t)\|_{E,k-1} + \|g(t)\|_{k-1})\,,$$

where C is independent of $(\varepsilon, \delta, \lambda)$. We point out that a modification in the above [KMa2]'s argument is needed here. In fact, by following exactly their proof one finds on the right-hand side of (3.9) the additional term $\|u^{\lambda,\delta}(t)\|_{k-1}\,\|w(t)\|_{E,k}$. However, Eq. (3.9) is valid without this term. In fact this undesired term comes from the second term in the summation in the right-hand side of [KMa2]'s Eq. (2.14); here $|\alpha| = k-1$. Instead of estimating this term by using the point (ii) in [KMa2]'s Lemma 2 we use the estimate

$$\text{(3.10)}\qquad \|\partial_x^\alpha(fg) - f\partial_x^\alpha g\|_0 \le c\|\partial_x f\|_{k-1}\,\|g\|_{k-2}\,;$$

see, for instance, [BV3], Appendix A, Corollary A.3. Hence

$$\|\partial_x^\alpha(\tilde{B}w_x) - \tilde{B}\partial_x^\alpha w_x\|_0 \le c\|\partial_x\tilde{B}\|_{k-1}\,\|w\|_{k-1}\,.$$

On the other hand, by $(2.6)_1$ and (1.4),

$$\|\partial_x\tilde{B}\|_{k-1} \le c\|(D_u\tilde{B})\,(u^{\lambda,\delta}, \lambda)\|_{k-1}\,\|\partial_x u^{\lambda,\delta}\|_{k-1} \le C\,.$$

Hence the H^{k-1} norm of the undesired term is bounded by $\|w(t)\|_{k-1}$. Hence it is bounded by the first term on the right-hand side of (3.9). □

The next step is to estimate conveniently the norm $\|g(t)\|_{k-1}$. We want to show that (for each t)

$$\|g\|_{k-1} \leq C(\|w\|_{k-1} + \|u^\lambda - u^{\lambda,\delta}\|_{k-1} + \|v^{\lambda,\varepsilon}\|_k \, \|u^\lambda - u^{\lambda,\delta}\|_{k-1}) . \tag{3.11}$$

From (1.4), (2.6), and $\|v^{\lambda,\delta}\|_{k-1,T} \leq C$, it follows that the second term in the right-hand side of the definition of g is bounded by $C\|w(t)\|_{k-1}$. Next, by arguing for the system $(3.1)_\varepsilon$ as done above for the system (3.8) we show that $y(t) \equiv \|v^{\lambda,\varepsilon}(t)\|_{k-1}$ satisfies the differential inequality $y'(t) \leq C(y(t) + y^2(t))$, moreover $y(0) \leq C$. Hence $\|v^{\lambda,\varepsilon}(t)\|_{k-1} \leq C$.[3] It follows that the third term in the right-hand side of the definition of g is bounded by $C\|w(t)\|_{k-1}$.

Next, we consider the first term on the right-hand side of the definition of g. In order to bound its H^{k-1} norm by the last term on the right-hand side of (3.11) it is sufficient to show that

$$\|\tilde{B}(u^\lambda) - \tilde{B}(u^{\lambda,\delta})\|_{k-1} \leq C\|u^\lambda - u^{\lambda,\delta}\|_{k-1} . \tag{3.12}$$

Setting for convenience $u = u^\lambda$ and $u' = u^{\lambda,\delta}$, and dropping the symbol λ, one has

$$\partial_x^{k-1}\tilde{B}(u) = (D_u^{k-1}\tilde{B})(u)(\partial_x u)^{k-1} + \ldots + (D_u\tilde{B})(u)\partial_x^{k-1}u .$$

Hence

$$\begin{aligned} \partial_x^{k-1}[\tilde{B}(u) - \tilde{B}(u')] = {} & [(D_u^{k-1}\tilde{B})(u) - (D_u^{k-1}\tilde{B})(u')](\partial_x u)^{k-1} \\ & + (D_u^{k-1}\tilde{B})(u')[(\partial_x u)^{k-1} - (\partial_x u')^{k-1}] + \ldots . \end{aligned} \tag{3.13}$$

By using the mean value theorem in the phase space $\mathscr{O} \subset \mathbb{R}^r$ together to (1.4) it readily follows that the H^0 norm of the first term on the right-hand side of (3.13) is bounded by $C\|u - u'\|_\infty \, \|\partial_x u\|_\infty^{k-2} \, \|\partial_x u\|_0$, hence by $C\|u - u'\|_{k-1}$. Here, $\| \, \|_\infty$ denotes the norm in $L^\infty(\Omega)$. The H^0 norm of the second term on the right-hand side of (3.13) is bounded by $C\|(\partial_x u - \partial_x u')((\partial_x u)^{k-2} + \ldots + (\partial_x u')^{k-2})\|_0$, hence by $C\|\partial_x u - \partial_x u'\|_{k-2} \, \|(\partial_x u)^{k-2} + \ldots + (\partial_x u')^{k-2}\|_1$ which, in turn, is bounded by $C\|u - u'\|_{k-1}$. In a similar way, each term that is part of the right-hand side of (3.13) is bounded by $\|u^\lambda - u^{\lambda,\delta}\|_{k-1}$. We left details to the reader. The estimates for H^l norms of products of functions proved in [KMa1] Lemma A.1 or in [BV3] Appendix A, are useful here. Similar manipulations show that the H^{k-1} norm of the last term in the right-hand side of the definition of g is bounded by $\|u^\lambda - u^{\lambda,\delta}\|_{k-1}$, since $\|v^{\lambda,\varepsilon}\|_{k-1,T} \leq C$. hence (3.11) holds. □

Next, by (3.9), (3.11), and Gronwall's lemma it follows that

$$\|v^{\lambda,\delta}(t) - v^{\lambda,\varepsilon}(t)\|_{k-1} \leq C\{\varepsilon + \|u_0^\lambda - u_0^{\lambda,\delta}\|_k + (1 + \|v^{\lambda,\varepsilon}\|_{k,T})\,[u^\lambda - u^{\lambda,\delta}]_{k-1,t}\} \tag{3.14}$$

for each $\lambda > \lambda(\varepsilon)$, where C is independent of $(\varepsilon, \delta, \lambda)$. Here, the assumption $(3.3)_2$ leads to the condition $\lambda > \lambda(\varepsilon)$.

On the other hand, by taking the difference between the Eqs. (3.1) and $(3.1)_\varepsilon$ (instead of (3.1′) and $(3.1)_\varepsilon$, as above) we get (3.8) and g both without the δ's. In

[3] We point out that there is a positive lower bound T for the times of existence of all the solutions of the equations considered in this paper; in fact the norms of all the initial data are uniformly bounded

particular, the first and the third terms in the right-hand side of the definition of g do not appear. A quite simplified version of the above manipulations shows that

$$\|v^\lambda(t) - v^{\lambda,\varepsilon}(t)\|_{k-1} \le C(\varepsilon + [u^\lambda - u^{\lambda,\delta}]_{k-1,t}), \tag{3.15}$$

for each $\lambda > \lambda(\varepsilon)$. From (3.14), (3.15), and the definitions of v^λ and $v^{\lambda,\delta}$, it follows that

$$\|u^\lambda(t) - u^{\lambda,\delta}(t)\|_k \le C\{\varepsilon + \|u_0^\lambda - u_0^{\lambda,\delta}\|_k + (1 + \|v^{\lambda,\varepsilon}\|_{k,T})\,[u^\lambda - u^{\lambda,\delta}]_{k-1,t}\}, \tag{3.16}$$

for $\lambda > \lambda(\varepsilon)$. Use also the estimate (3.17), proved below. □

Next we show that

$$\|u^\lambda - u^{\lambda,\delta}\|_{k-1,T} \le C\delta, \qquad \forall \lambda > \lambda_0. \tag{3.17}$$

By taking the difference between the Eqs. (2.5) and (1.5) and by setting $w = u^{\lambda,\delta} - u^\lambda$, we get

$$\begin{cases} w_t + B(u^{\lambda,\delta}, \lambda) w_x = [B(u^\lambda, \lambda) - B(u^{\lambda,\delta}, \lambda)] u_x^\lambda, \\ w(0) = u_0^{\lambda,\delta} - u_0^\lambda. \end{cases} \tag{3.18}$$

This system is similar to (3.8) if now, in the definition of g, we take into account only the first term. This leads to Eqs. (3.9) and (3.11) for $w(t)$, where now (3.11) becames $\|g(t)\|_{k-1} \le C\|w(t)\|_{k-1}$ (since the term $\|v^{\lambda,\varepsilon}(t)\|_k$ is replaced by $\|u^\lambda(t)\|_k$). By taking into account $(2.3)_4$ we prove (3.17). □

From (3.16), with the help of (3.17), we prove that

$$\|u^\lambda - u^{\lambda,\delta}\|_{k,T} \le C(\varepsilon + \|u_0^\lambda - u_0^{\lambda,\delta}\|_k + \delta + \|v^{\lambda,\varepsilon}\|_{k,T}\delta), \tag{3.19}$$

for each $\lambda > \lambda(\varepsilon)$, where C is independent of $(\varepsilon, \delta, \lambda)$. By taking into account (3.4) and $(2.2)_3$, we prove (2.7). □

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