The initial-boundary value problem for the non-barotropic compressible Euler equations: structural-stability and data dependence

by

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ABSTRACT. — We study the initial-boundary value problem for general compressible inviscid fluids. Let U_0, $U_0' \in H^k$ and U, $U' \in C(0, T; H^k)$ denote initial data and corresponding solutions, respectively. From the point of view of dynamical systems, a very basic problem is to prove that U' converges to U in $C(0, T; H^k)$ if U_0' converges to U_0 in H^k; this is proved in theorem 1.2 below. It must be pointed out that convergence in $C(0, T; H^{k-\varepsilon})$ and in $L^\infty(0, T; H^k)$ weak-* (easy consequences of the *a priori* estimates used to prove the existence theorem) have minor significance as part of the mathematical theory. We also show (theorem 1.3) that if $\rho'(p, S)$ approaches $\rho(p, S)$ in C^k then U' approaches U in the norm $C(0, T; H^k)$. In particular, small perturbations in the law of state generate small perturbations in the trajectory of the solution, with respect to the right metric.

Key words : Compressible Euler equations, sharp data dependence.

RÉSUMÉ. — Nous étudions le problème mixte pour des fluides non visqueux, dans le cas général. Soient U_0, $U_0' \in H^k$ et U, $U' \in C(0, T; H^k)$ respectivement les données initiales et les solutions correspondantes. Du

point de vue des systèmes dynamiques, un problème essentiel consiste à démontrer que U′ converge vers U dans C(0, T; H^k) lorsque U_0' converge vers U_0 dans H^k; on démontre ce résultat dans le théorème 1.2. En plus, on démontre (théorème 1.3) que si ρ′(p, S) converge vers ρ(p, S) dans C^k alors U′ converge vers U dans C(0, T; H^k).

1. INTRODUCTION

This paper follows previous work on the existence of regular local solutions to the equations of compressible inviscid fluids and on the *well-posedness*, in Hadamard's classical sense (continuous data dependence in the strong norm) of these equations. Here we show that our proof of the well-posedness theorem [BV4, 5] for barotropic fluids [*i.e.*, $p = p(\rho)$] can be extended to cover the non-barotropic case [*i.e.*, $p = p(\rho, S)$]. See theorem 1.2. Moreover, a structural-stability result holds. See theorem 1.3.

The existence of the solution to the mixed problem (for the Cauchy problem see [KMa2]) for the barotropic case was first proved by Ebin [E1] under the assumption that the initial velocity is subsonic and the initial density is nearly constant. The existence theorem without these assumptions was proved by us [BV1], [BV2] and (in an independent paper) by Agemi [A]. The existence of the solution [in spaces $\mathscr{L}^\infty(0, T; H^3)$] for the non-barotropic case was proved by Schochet [Sc1] by using a different approach which has, however, some ideas in common with the method followed in [BV2]; see also [Sc2]. It is worth nothing that Schochet's approach can be easily adapted to cover the case $\mathscr{L}^\infty(0, T; H^k)$, $k \geq 3$. Below, we prove the existence of the solution to this last problem in spaces $\mathscr{C}(0, T; H^k)$, by following our approach. See theorem 1.1.

Well-posedness for the mixed problem (barotropic case) was proved in reference [BV4] if $\Omega = \mathbb{R}_+^3$ and $k = 3$; and in reference [BV5] for arbitrarily large $k \geq 3$ and bounded regular Ω. The method followed in these references (introduced in [BV3] for first order hyperbolic systems) applies to a large class of problems; see [BV3, 4, 5, 6]. The lack of these basic results in the general theory of hyperbolic equations was certainly a main gap. The method relies on proving the strong continuous dependence of the solutions of hyperbolic *linear* systems on the coefficients of the differential operators, an interesting result by itself.

Below, we consider the system of equations that describes the motion of a compressible, inviscid fluid, namely

$$\left.\begin{aligned}
&\rho\, D\,(v)\, v + \nabla P = f,\\
&D\,(v)\, \rho + \rho\, \nabla \cdot v = 0,\\
&D\,(v)\, S = 0 \quad \text{in } Q_T;\\
&v \cdot \nu = 0 \quad \text{on } \Sigma_T; \qquad (v,\, P,\, S)\,(0) = (a,\, \phi,\, S_0).
\end{aligned}\right\} \tag{1.1}$$

where, for convenience, we set

$$D\,(v) \equiv \partial_t + v \cdot \nabla.$$

Moreover, $v \cdot \nabla = \sum_{i=1}^{3} v_i\, \partial_i$ and $\partial_i = \partial_{x_i}$. Here, v, P, S, ρ denote respectivelly velocity, pressure, entropy, and density of the fluid. Clearly, $v = (v_1,\, v_2,\, v_3)$ is a vector field and P, S, ρ are scalar fields. In equations (1.1), $\rho\,(t,\, x) = \rho\,(P\,(t,\, x),\, S\,(t,\, x))$ where $\rho = \rho\,(P,\, S)$ is a real, positive function, defined and of class C^{k+1}, $k \geq 3$, on a domain $\Lambda \subset \mathbb{R}^2$. By assumption, $\partial \rho / \partial P$ is positive over Λ. We assume that the initial data $(\phi\,(x),\, S_0\,(x))$ takes values on a compact subset $\Lambda_0 \subset \subset \Lambda$. Since solutions are continuous on Q_T and results are local in time, there is no loss in generality in assuming that $\Lambda = \mathbb{R}^2$. Hence, in the sequel, $\Lambda = \mathbb{R}^2$. The reader should note that we use the same symbol ρ to denote the function $\rho\,(P,\, S)$ of two real variables and the function $\rho\,(t,\, x) = \rho\,(P\,(t,\, x),\, S\,(t,\, x))$, defined on Q_T. This simplified notation will be used in other similar situations.

In the sequel Ω denotes an open, bounded, connected subset of $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a differentiable manifold of class C^{k+2}. The integer $k \geq 3$ is fixed once for all. We denote by ν the unitary outward normal to the boundary Γ and by ∂_ν differentiation in the ν direction. Moreover, $Q_T = [0,\, T] \times \Omega$, $\Sigma_T = [0,\, T] \times \Gamma$. We set $g\,(P,\, S) \equiv \log \rho\,(P,\, S)$; $g_1\,(P,\, S) \equiv \partial g\,(P,\, S)/\partial P$; $g_2\,(P,\, S) \equiv \partial^2 g\,(P,\, S)/\partial P^2$; and $g_3\,(P,\, S) \equiv \partial g\,(P,\, S)/\partial S$. Note that $g_1 > 0$. The equations (1.1) take then the equivalent form

$$\left.\begin{aligned}
&D\,(v)\, v + e^{-g}\, \nabla P = e^{-g} f,\\
&D\,(v)\, P + g_1^{-1}\, \nabla \cdot v = 0,\\
&D\,(v)\, S = 0 \quad \text{in } Q_T;\\
&v \cdot \nu = 0 \quad \text{on } \Sigma_T; \qquad (v,\, P,\, S)\,(0) = (a,\, \phi,\, S_0).
\end{aligned}\right\} \tag{1.2}$$

Our results will be stated in terms of this last system.

Before stating the main results we introduce some notations. We denote by H^l, l nonnegative integer, the space $H^l\,(\Omega)$ endowed with the canonical norm $\|\,.\,\|_l$ defined by $\|u\|_l^2 = \sum \|\partial^\alpha u\|^2$, where the summation is extended over the multi-indices $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3)$ such that $0 \leq |\alpha| \leq l$ and $\|\,.\,\| = \|\,.\,\|_0$

denotes the L^2-norm in Ω. Moreover

$$|||u|||_l^2 = \sum_{j=0}^{l} \|\partial_t^j u\|_{l-j}^2, \qquad |||u|||_l'^2 = \sum_{j=0}^{l-1} \|\partial_t^j u\|_{l-j}^2.$$

We also use, on Γ, Sobolev fractionary spaces $H^{l-1/2}(\Gamma)$ denoted here by $\mathscr{H}^{l-1/2}$. The norm in this space is denoted by the symbol $\langle\langle\,\cdot\,\rangle\rangle_{l-1/2}$. Wet set

$$\langle\langle\langle u \rangle\rangle\rangle_{l-1/2}^2 = \sum_{j=0}^{l-1} \langle\langle \partial_t^j u \rangle\rangle_{l-j-1/2}^2.$$

In the sequel we use the notation $C_T^j(X) = C^j([0, T]; X)$, $L_T^p(X) = L^p(0, T; X)$, and so on. We define

$$\mathscr{C}_T(H^l) = \bigcap_{j=0}^{l} C_T^j(H^{l-j}), \qquad \mathscr{C}_T'(H^l) = \bigcap_{j=0}^{l-1} C_T^j(H^{l-j}),$$

$$\mathscr{L}_T^p(H^l) = \bigcap_{j=0}^{l} W_T^{j,p}(H^{l-j}), \qquad \mathscr{L}_T'^p(H^l) = \bigcap_{j=0}^{l-1} W_T^{j,p}(H^{l-j}),$$

$$\mathscr{C}_T(\mathscr{H}^{l-1/2}) = \bigcap_{j=0}^{l-1} C_T^j(\mathscr{H}^{l-j-1/2}), \qquad \mathscr{L}_T^p(\mathscr{H}^{l-1/2}) = \bigcap_{j=0}^{l-1} W_T^{j,p}(\mathscr{H}^{l-j-1/2}).$$

The norms in these functional spaces are the following:

$$|||u|||_{l,T}^2 = \sup_{0 \le t \le T} |||u(t)|||_l^2, \qquad |||u|||_{l,T}'^2 = \sup_{0 \le t \le T} |||u(t)|||_l'^2,$$

$$[u]_{l,T}^2 = \int_0^T |||u(t)|||_l^2 \, dt, \qquad [u]_{l,T}'^2 = \int_0^T |||u(t)|||_l'^2 \, dt,$$

$$\langle\langle\langle u \rangle\rangle\rangle_{l-1/2,T}^2 = \sup_{0 \le t \le T} \langle\langle\langle u(t) \rangle\rangle\rangle_{l-1/2}^2,$$

$$\langle u \rangle_{l-1/2,T}^2 = \int_0^T \langle\langle\langle u(t) \rangle\rangle\rangle_{l-1/2}^2 \, dt,$$

where "sup" denotes the essential supremum.

The above notation will be used both for scalar and for vector fields. This convention applies to all notation used in the sequel. In particular, we use notations like $v, g \in X$, even if v is a vector and g a scalar, and also $(v, g) \in X$ instead of $(v, g) \in X \times X$.

Given an arbitrary function $f(t, x)$ we denote by $f(t)$, for each fixed t, the function $f(t, .)$.

Obvious notation will be used without an explicit definition.

In the following, we often deal with positive "constants" that, in fact, depend (increasingly) on the norms of the coefficients of the differential operators used in the sequel. For convenience, we denote by $\lambda = \lambda(\cdot, \ldots, \cdot)$ generic real, nonnegative functions which are increasing

functions of each single (real, nonegative) variable. They will be called λ-functions. Since we are not particularly interested in their explicit form we often denote distinct λ-functions by the same symbol λ.

Some classes of λ-functions, particularly important in the sequel, will be denoted by specific symbols such as the α's and the β's defined in equation (2.4).

Now, we state our existence theorem for problem (1.2). We give a quite complete proof since many details will be used in proving the theorems 1.2 and 1.3 below.

THEOREM 1.1. — *Let* $k \geq 3$ *be a fixed integer,* Ω *be as above, and* $g \in C^{k+1}(\mathbb{R}^2; \mathbb{R})$ *satisfy* $g_1 \equiv \partial g / \partial P > 0$. *Assume that* $U_0 \equiv (a, \phi, S_0) \in H^k$, *that* $f \in \mathscr{L}^2(H^k)$ $(^1)$, *and that these data satisfy the compatibility conditions up to order* $k-1$, *for the system* (1.2). *Then, there is a positive* T *such that a (unique) solution* $U \equiv (v, P, S) \in \mathscr{C}_T(H^k)$ *of problem* (1.2) *exists in* Q_T. *Moreover,*

$$|||U|||'_{k,T} \leq \lambda_1, \qquad |||U|||_{k,T} \leq \lambda_4. \qquad (1.3)$$

The result is valid for any T *satisfying*

$$\lambda_2 T \leq 1, \qquad \lambda_3 [f]'_{k,T} \leq 1. \qquad (1.4)$$

Here, λ_1, λ_2, λ_3 *are suitable* λ-functions that depend only on $|||U_0|||_k$ and on $|||f(0)|||'_{k-1}$. The function λ_4 depends on these norms and on $[f]_{k,T}$.

Let us describe the main problems studied in the sequel:

(i) Assume that a sequence of data (U_0^n, f^n) is given, each pair satisfying the properties required in theorem 1.1. Assume that $U_0^n \to U_0$ in H^k as $n \to \infty$. Are the solutions U^n convergent to U, in the strong norm $\mathscr{C}_T(H^k)$?

(ii) Assume that a family of laws of state $\rho_n(\cdot, \cdot)$ is given, and that $\rho_n \to \rho$ as $n \to \infty$, in a suitable norm. Are the solutions U^n convergent to U in the strong norm?

In the sequel we prove that the answer to the above questions (put together) is affirmative. See theorem 1.2 and 1.3.

For convenience, in the following we replace the above parameter n by a "prime". We also remark that in theorems 1.2 and 1.3 below T_0 may be larger than the T guaranteed by theorem 1.1.

THEOREM 1.2. — *Let the data* $U_0 = (a, \phi, S_0)$ *and* f *be as in theorem* 1.1, *and assume that there is, for some* $T_0 > 0$, *a solution* $U \in \mathscr{C}_{T_0}(H^k)$ *of problem*

$(^1)$ Assume, without loss of generality, that f is defined for $t \in [0, \infty[$.

(1.2). *Consider the system*

$$\left.\begin{array}{c} D(v')\,v' + e^{-g'}\,\nabla P' = e^{-g'}f', \\ D(v')\,P' + g_1'^{-1}\,\nabla\cdot v' = 0, \\ D(v')\,S' = 0 \quad \text{in } Q_T; \\ v'\cdot\nu = 0 \quad \text{on } \Sigma_T; \quad (v',\,P',\,S')(0) = (a',\,\phi',\,S_0') \equiv U_0'. \end{array}\right\} \quad (1.2')$$

where, for brevety, we set $g' \equiv g(P', S')$ *and* $g_1' \equiv g_1(P', S')$. *There is a neighbourhood of* (U_0, f) *in* $H^k \times \mathscr{L}_{T_0}^2(H^k)$ *such that to each pair* (U_0', f') *in this neighbourhood that satisfies the compatibility conditions up to order* $k-1$ *for* $(1.2')$ *it corresponds a solution* $U' = (v', P', S') \in \mathscr{C}_{T_0}(H^k)$ *of the system* $(1.2')$ *in* Q_{T_0}. *Moreover, if one considers a sequence of problems* $(1.2')$ *and if*

$$\lim (U_0', f') = (U_0, f) \quad \text{in } H^k \times \mathscr{L}_{T_0}^2(H^k) \tag{1.5}$$

then

$$\lim U' = U \quad \text{in } \mathscr{C}_{T_0}(H^k). \tag{1.6}$$

In particular, if $[0, \tau'[$ *is the maximal interval of existence of the solution* U', *and if* $[0, \tau[$ *is that of* U, *one has* $\liminf \tau' \geqq \tau$.

In this statement the equation of state $g(\cdot, \cdot) \equiv \log \rho(\cdot, \cdot)$ is invariant. In fact, $g' = g(P', S')$ is distinct from $g = g(P, S)$ merely because $(P', S') \neq (P, S)$. However, the following sharp structural-stability theorem holds.

THEOREM 1.3. — *Let* $\rho'(P, S)$ *be real positive functions defined and of class* C^{k+1} *on* Λ, *and such that* $\partial\rho'/\partial P > 0$ *on* Λ. *Assume that*

$$\lim \rho'(P, S) = \rho(P, S) \tag{1.7}$$

in $C^k(\Lambda_1)$, *for each compact subset* $\Lambda_1 \subset\subset \Lambda$. *Then, the theorem* 1.2 *still holds if, in equation* $(1.2')$, *g' denotes* $\log\rho'(P', S')$ *instead of denoting* $\log\rho(P', S')$.

The proof of theorem 1.3 is a straightforward extension of that of theorem 1.2; the details are left to the reader (*see* [BV3] and especially [BV6] for similar details). Finally, an application of our method to the incompressible limit problem for the compressible Euler equations ([E2, 3], [KMa1, 2], [Ma], [Sc1, 2]) is given in [BV7].

In the sequel, in order to avoid unnecessary repetitions, we will partially apply to results proved in reference [BV5]. Hence, the reader is assumed to be well acquainted with that paper.

We denote by $\Gamma_0, \ldots, \Gamma_m$ the connected components of Γ. The Γ_j's, for $j \neq 0$, are inside Γ_0 and outside of one another. Just for convenience, we will assume that Ω is simply-connected. If not, we argue as done in reference [BV2]. For a brief discussion on this point see the remark 1 in the introduction of reference [BV5].

In the sequel we replace the system (1.2) by the equivalent system

$$\left.\begin{array}{c} D(v)\zeta-(\zeta\cdot\nabla)v+(\nabla\cdot v)\zeta=H(v,P,S,f), \\ D(v)S=0, \\ D(v)^2P-h(P,S)\Delta P=F(v,P,S,f), \\ \nabla\cdot v=-g_1 D(v)P, \\ \zeta=\nabla\times v \quad \text{in } Q_T; \\ v\cdot\nu=0, \qquad \partial_\nu P=G \quad \text{on } \Sigma_T; \\ P(0)=\phi, \qquad \partial_t P(0)=-g_1^{-1}(\phi,S_0)\nabla\cdot a+a\cdot\nabla\phi; \\ v(0)=a; \qquad S(0)=S_0, \end{array}\right\} \quad (1.8)$$

where h, H, F, G are defined by the equations

$$\left.\begin{array}{c} h(P,S)=(g_1 e^g)^{-1} \\ H(v,P,S,f)=\nabla\times(e^{-g}f)+e^{-g}g_3\nabla S\times\nabla P, \\ F(v,P,S,f)=g_1^{-1}\{-e^{-g}\nabla g\cdot\nabla P-\nabla\cdot(e^{-g}f) \\ +\Sigma(\partial_i v_j)(\partial_j v_i)-g_1^{-2}g_2(\nabla\cdot v)^2\}. \\ G(v,P,S,f)=f\cdot\nu+e^g\Sigma(\partial_i v_j)v_i v_j, \end{array}\right\} \quad (1.9)$$

where, the indices i, j run from 1 to 3. Note that h is a positive function.

The system (1.8) is more adequate to our purposes than (1.2) by reasons similar to that described in the introduction of [BV5]. Let us prove the equivalence between these systems. The equivalence between the initial conditions follows trivially by using the equations. Next, note that the assumptions $(1.2)_2$, $(1.2)_3$, and $v\cdot\nu_{|\Sigma_T}=0$ are common to both systems. Hence, it is sufficient to show that under these assumptions the equation $(1.2)_1$ is equivalent to the equations $(1.8)_1$, $(1.8)_2$, and $\partial_\nu P_{|\Sigma_T}=G$. The proof is based on the fact that a vector field V vanishes in Ω if and only if it satisfies the linear system [2] $\nabla\times V=0$ and $\nabla\cdot V=0$ in Ω, $V\cdot\nu=0$ on Γ. Set $V=D(v)v+e^{-g}\nabla P-e^{-g}f$. By well known formulae in vector calculus one gets

$$\nabla\times V=D(v)\zeta-(\zeta\cdot\nabla)v+(\nabla\cdot v)\zeta-H \qquad (1.10)$$

where $\zeta=\nabla\times v$, and also

$$\nabla\cdot V=D(v)(\nabla\cdot v)+\Sigma(\partial_i v_j)(\partial_j v_i)+e^{-g}\nabla P-e^{-g}\nabla g\cdot\nabla P-\nabla\cdot(e^{-g}f).$$

By using $(1.2)_2$ and $(1.2)_3$ on gets

$$D(v)(\nabla\cdot v)=-g_1 D(v)^2 P-g_1^{-2}g_2(\nabla\cdot v)^2.$$

Hence

$$\nabla\cdot V=-g_1(D(v)^2 P-h\Delta P-F). \qquad (1.11)$$

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[2] If Ω is not simply-connected one has to take into account that this linear system admits a finite number of linearly independent solutions.

Finally, one has

$$V \cdot v = e^{-g} (\partial_v P - G) \qquad (1.12)$$

on Σ_T, since $[(v \cdot \nabla) v] \cdot v = -\Sigma (\partial_i v_j) v_i v_j$; see, for instance, [BV2] foot-note (4). We assume the normal v extended to a neighbourhood of Γ, as a C^{k+1} vector field.

The desired equivalence follows now from (1.10), (1.11), (1.12), since $V = 0$ if and only if the right hand sides of these equations vanish.

We remark that the regularity needed to justify the above manipulations is largely exceeded by the solution of the system (1.8) constructed in the sequel.

Finally we recall the following basic inequality (used here in the particular cases $q = l - 1$ or l, $m = 1$):

$$||| g \prod_{i=1}^{m} f_i |||_l^{[q]} \leq c ||| g |||_\beta^{[q]} \prod_{i=1}^{m} ||| f_i |||_{\alpha_i}^{[q]}, \qquad (1.13)$$

where $0 \leq q \leq l \leq r$; $r > n/2$; $\alpha_1, \ldots, \alpha_m$, $\beta \in [l, r]$; and $\beta + \sum_{i=1}^{m} \alpha_i = mr + l$. By definition

$$||| f |||_l^{[q]} \equiv \sum_{p=0}^{q} || \partial_t^p f ||_{l-p}.$$

This inequality is useful in order to estimate norms of products of functions. Since this technique is standard, we leave all that kind of manipulations to the reader.

2. PROOF OF THEOREM 1.1

We start by recalling a result concerning the linear equation

$$\left. \begin{aligned} D(v)^2 P - h \Delta P = F \quad &\text{in } Q_T; \\ \partial_v P = G, \quad \text{on } \Sigma_T; \qquad (P, \partial_t P)(0) = (\phi, \psi), \end{aligned} \right\} \qquad (2.1)$$

where, v, h, F, G, ϕ, ψ are given functions of (t, x) and, by assumption,

$$v, h \in \mathscr{L}_T'^{\infty}(H^k), \qquad (2.2)$$

$$v \cdot v = 0 \quad \text{on } \Sigma_T; \qquad h \geq m > 0 \quad \text{on } Q_T. \qquad (2.3)$$

In the sequel, the symbols α and β denote λ functions of type

$$\left. \begin{aligned} \alpha &= \alpha(m^{-1}, ||| v |||_{k-1, T}', ||| h |||_{k-1, T}'), \\ \beta &= \beta(m^{-1}, ||| v |||_{k, T}', ||| h |||_{k, T}'), \end{aligned} \right\} \qquad (2.4)$$

respectivelly.

Let l, $1 \leq l \leq k$, be a fixed integer. The compatibility conditions up to order $l-2$ for the system (2.1) can be written in the form

$$\partial_v \{ \partial_t^j P(0) \} = \partial_t^j G(0) \quad \text{on } \Gamma, \tag{2.5}$$

for $j = 0, 1, \ldots, l-2$, where $\{ \partial_t^j P(0) \}$ denotes the expression (in terms of ϕ, ψ, and F) formally obtained by solving the equations (2.1) for $\partial_t^j P(0)$. These expressions involve data and coefficients, but not eventual solutions. One has the following result ([BV5], theorem 1.1).

THEOREM 2.1. — *Assume that*

$$(\phi, \psi) \in H^l \times H^{l-1}, \quad F \in \mathscr{L}_T^2(H^{l-1}), \quad G \in \mathscr{L}_T^2(\mathscr{H}^{l-1/2}), \tag{2.6}$$

and that the hypotheses (2.2), (2.3), (2.5) *are satisfied. Let* $1 \leq l \leq k-1$. *Then, there is a solution* $P \in \mathscr{C}_T(H^l)$ *of problem* (2.1). *Moreover, for suitable* α *and* β *having the form* (2.4), *one has*

$$|||P(t)|||_l^2 + [P]_{l,T}^2 \leq \alpha e^{\beta t} (\|\phi\|_l^2 + \|\psi\|_{l-1}^2 + |||F(0)|||_{l-2}^2)$$
$$+ \beta e^{\beta t} ([F]_{l-1,t}^2 + \langle G \rangle_{l-1/2,t}^2), \tag{2.7}$$

for each $t \in [0, T]$. *If* $l=k$, *the solution* P *exists, belongs to* $C_T'(H^k)$ *and satisfies* (2.7) *provided that we replace the left hand side of this equation by* $|||P(t)|||_k'^2 + [P]_{k,t}'^2$; *alternativelly,* P *belongs to* $\mathscr{C}_T(H^k)$ *and* (2.7) *holds without modification if* $v \in \mathscr{C}_T(H^{k-1})$ *and if* α *can depend on the full norm* $|||v|||_{k-1,T}$.

For the proof of the above result see [BV5], § 2. In this last reference the term $\nabla \cdot (h \Delta P)$ replaces the term $h \Delta P$, in equation (2.1). However the proof applies as well to the case under consideration here. Alternatively, by setting $\tilde{L}P \equiv D(v)^2 P - h \Delta P$, it follows that $LP = \tilde{L}P - \nabla h \cdot \nabla P$, if L is the operator considered in reference [BV5]. The equation (2.3) in this last reference shows that

$$|||P(t)|||_{l,\gamma}^2 + \gamma [P]_{l,\gamma,t}^2 \leq \alpha (\|P(0)\|_l^2 + \|\partial_t P(0)\|_{l-1}^2$$
$$+ |||\tilde{L}P(0)|||_{l-2}^2 + |||(\nabla h \cdot \nabla P)(0)|||_{l-2}^2)$$
$$+ \beta \gamma^{-1} ([\tilde{L}P]_{l-1,\gamma,t}^2 + [\nabla h \cdot \nabla P]_{l-1,\gamma,t}^2 + [P]_{l,\gamma,t}'^2 + \langle \partial_v P \rangle_{l-1/2,\gamma,t}^2) \tag{2.8}$$

if $\gamma \geq \bar{\beta}$, for suitables α, β, and $\bar{\beta}$. By using inequalities (1.13) one shows that $[\nabla h \cdot \nabla P]_{l-1,\gamma,t} \leq c |||h|||_{k,t} [P]_{l,\gamma,t}$ and that

$$|||(\nabla h \cdot \nabla P)(0)|||_{l-2} \leq c |||\nabla h(0)|||_{k-2} |||\nabla P(0)|||_{l-1}.$$

Moreover $(2.1)_t$ shows that

$$|||P(0)|||_l' \leq \alpha (\|P(0)\|_l + \|\partial_t P(0)\|_{l-1} + |||\tilde{L}P(0)|||_{l-2}').$$

Hence, the terms containing $\nabla h \cdot \nabla P$ can be eliminated from the right hand side of (2.8), by eventually increasing α and β. Now, we prove (2.7) by arguing as in [BV5] in order to get (1.7) from (2.3). $\square$

Proof of theorem 1.1. — The proof follows that of the theorem 1.3 in [BV5] section 5, to which the reader is refered. However, to the reader's

convenience, we give below an overview of the main points. Details will be given only in connection to the proof of equations (2.18) below, since this is the sole point that requires some additional manipulation.

Concerning notations, we remark that the roles played in [BV5] by g and q are played here by P and Q, and the roles played in [BV5] by the λ-functions P and Q are now played by α and β, respectively.

The proof of theorem 1.1 consists on showing the existence of a solution of problem (1.8). This is done as follows, by a fixed point argument. Consider the following set up [recall definitions (1.9)]:

$$\nabla \cdot v = \vartheta \quad \text{and} \quad \nabla \times v = \xi \quad \text{in } Q_T; \qquad v \cdot \nu = 0 \quad \text{on } \Sigma_T. \quad (2.9)$$

$$D(v) S = 0 \quad \text{in } Q_T; \qquad S(0) = S_0. \quad (2.10)$$

$$g = g(Q, S), \quad g_1 = g_1(Q, S), \quad g_2 = g_2(Q, S), \quad g_3 = g_3(Q, S). \quad (2.11)$$

$$\left. \begin{array}{c} D(v)\zeta - (\zeta \cdot \nabla) v + (\nabla \cdot v) \zeta = H(v, Q, S, f) \quad \text{in } Q_T, \\ \zeta(0) = \nabla \times a, \end{array} \right\} \quad (2.12)$$

$$\left. \begin{array}{c} D(v)^2 P - h(Q, S) \Delta P = F(v, Q, S, f) \quad \text{in } Q_T, \\ \partial_\nu P = G(v, Q, S, f) \quad \text{on } \Sigma_T; \\ P(0) = \phi; \qquad \partial_t P(0) = g_1^{-1}(\phi, S_0)(\nabla \cdot a + a \cdot \nabla \phi). \end{array} \right\} \quad (2.13)$$

$$\delta = -g_1 D(v) P. \quad (2.14)$$

Let now ϑ, ζ, Q be given functions defined on Q_T (ζ is a vector field, ϑ and Q are scalars) that satisfy suitable conditions, specified later on. By solving the elliptic system (2.9) we get v. Then, the transport equation (2.10) gives S. At this stage, we can define g, g_1, g_2, and g_3, by using (2.11). Next, we solve the hyperbolic mixed problem (2.13), which gives P. Finally (2.14) gives δ. The above procedure defines a map $\mathscr{S}$, by setting $\mathscr{S}(\vartheta, \xi, Q) = (\delta, \zeta, P)$. Solving the system (1.8) is equivalent to proving the existence of a fixed point for $\mathscr{S}$ in a suitable set $\mathbb{K}$. The set $\mathbb{K}$ is defined by

$$\mathbb{K}(A, T) = \left\{ (\vartheta, \xi, Q) \in \mathscr{L}_T^\infty(H^{k-1}) \times \mathscr{L}_T^\infty(H^{k-1}) \right.$$
$$\left. \times \mathscr{L}_T'^\infty(H^k) : (2.16), (2.17), (2.18) \text{ holds} \right\} \quad (2.15)$$

where

$$|||\vartheta|||_{k-1, T} \le A, \qquad |||\xi|||_{k-1, T} \le A, \qquad |||Q|||'_{k, T} \le A, \quad (2.16)$$

$$\left. \begin{array}{c} \partial_t^j \vartheta(0) = \nabla \cdot \{ \partial_t^j v(0) \}, \qquad \partial_t^j \xi(0) = \nabla \times \{ \partial_t^j v(0) \}, \\ \partial_t^j Q(0) = \{ \partial_t^j P(0) \} \quad \text{for } j = 0, \ldots, k-2, \end{array} \right\} \quad (2.17)$$

and

$$\left. \begin{array}{c} \displaystyle\int_\Omega \vartheta(t, x) \, dx = 0; \\[3mm] \nabla \cdot \xi(t, x) = 0; \qquad \displaystyle\int_{\Gamma_i} \xi(t, x) \, d\Gamma = 0, \qquad i = 1, \ldots, m, \end{array} \right\} \quad (2.18)$$

for each $t \in [0, T]$. The functions $\{\partial_t^j P(0)\}$ and $\{\partial_t^j v(0)\}$ are defined (in Ω) in terms of a, ϕ, S_0, f, by solving *formally* the equations (1.8) for $\partial_t^j P(0)$ and $\partial_t^j v(0)$ respectivelly. Note that from (2.17) it follows that the solution $v(t)$ of problem (2.9) satisfies the equation $\partial_t v(0) = \{\partial_t v(0)\}$, for $j = 0, \ldots, k-2$, independently of the particular element $(\vartheta, \xi, Q) \in \mathbb{K}(A, T)$. Similar relations hold for the solutions $S(t)$, $P(t)$ of problems (2.10), (2.13).

Suitable estimates for the solutions v, ζ, P, and δ of the above systems (2.10), (2.12), (2.13), (2.14) are obtained as in section 5 of reference [BV5]. In particular, the solution P of problem (2.13) is estimated by using equation (2.7). Here, the positive lower bound condition (2.3) for h becomes $h(Q, S) \equiv [g_1(Q, S) \exp g(Q, S)]^{-1} \geq m$ on Q_T. It readily follows that m^{-1} is a λ-function that depends only on $\|Q\|_{k-1, T}$ and $\|S\|_{k-1, T}$. Furthermore, if $l = k-1$ or if $l = k$, the $||| \cdot |||'_{l, T}$ norms of $h(Q, S)$ and $h^{-1}(Q, S)$ are bounded from above by λ-functions that depend only on the $||| \cdot |||'_{l, T}$ norms of Q and S. In particular, the α's and the β's [*see* (2.4)] are now λ-functions depending only on the $||| \cdot |||'_{l, T}$ norms of the functions v, Q, and S. For more details, see [BV5]. Estimates for S follow easily from equations (2.10) by arguing as in [BV5] section 4. One gets, for $l = k-1$ and for $l = k$,

$$||| S(t) |||_l'^2 \leq \lambda (||| v(0) |||'_{k-1}) e^{c \, ||| v' |||_{k, t} \, t} \| S_0 \|_l^2. \tag{2.19}$$

Denote by d generic λ-functions of the form

$$d = d(\| a \|_k, \| \phi \|_k, \| S_0 \|_k, ||| f(0) |||'_{k-1}).$$

Arguing as in [BV5] we show that if $A \geq d$, for a suitable d, the set $\mathbb{K}(A, T)$ is not empty. Moreover, for suitable values $A = A(d)$ and $T = T(d, f)$, the functions δ, ζ, and P satisfy the assumptions (2.16), (2.17) (in those ones ϑ, ξ, Q should be replaced by δ, ζ, P), moreover $\mathcal{S}$ is a contraction of $\mathbb{K}$ into $\mathbb{K}$ with respect to the $H^0 \times H^0 \times H^1$ norm. In order to end the proof of theorem 1.1, by proving the existence on $\mathbb{K}$ of a fixed point for $\mathcal{S}$, it remains to be shown that δ and ζ satisfy (2.18). However, ζ satisfies (2.18) but δ does not (in general); this fact will require an apropriate device. Let us show that ζ satisfies (2.18). The well known identity $\nabla \times (v \times \zeta) = (\zeta \cdot \nabla) v - (v \cdot \nabla) \zeta + (\nabla \cdot \zeta) v - (\nabla \cdot v) \zeta$ shows that $(2.12)_1$ can be written in the form

$$\partial_t \zeta + (\nabla \cdot \zeta) v - \nabla \times (v \times \zeta) = \nabla \times (e^{-g} f) + e^{-g} \nabla g \times \nabla Q. \tag{2.20}$$

Since $\nabla \cdot (a \times b) = b \cdot (\nabla \times a) - a \cdot (\nabla \times b)$ it follows, by applying the divergence operator to both sides of (2.20), that

$$D(v)(\nabla \cdot \zeta) + (\nabla \cdot v)(\nabla \cdot \zeta) = 0 \quad \text{in } Q_T.$$

Since $(\nabla \cdot \zeta)(0) = \nabla \cdot (\nabla \times a) = 0$, one proves that $\nabla \cdot \zeta = 0$ on Q_T. On the other hand, the equation $e^{-g} \nabla g \times \nabla Q = \nabla \times (e^{-g} \nabla Q)$ together with (2.20)

shows that $\partial_t \zeta$ is a curl in Ω, for each fixed t. Consequently, for each $i=1, \ldots, m$, the derivative with respect to t of the integral $\int_{\Gamma_i} \zeta \cdot v \, d\Gamma$ vanishes on $[0, T]$. Since these integrals vanishes for $t=0$, they vanishes on $[0, T]$.

Next, we introduce a suitable device in order to overcome the possibility that the mean value of $\delta(t)$ does not vanish. We start by proving the following auxiliar result.

LEMMA 2.2. — *Set* $P=Q$ *in equations* (2.9)-(2.14). *Then*

$$\frac{d}{dt} \int_\Omega \delta \, dx = \int_\Omega [\vartheta - g_1^{-2} g_2 (\vartheta + \delta)] (\delta - \vartheta) \, dx. \qquad (2.21)$$

Proof. — By using the divergence theorem, the equation $(2.13)_2$, the identity

$$\nabla \cdot [(v \cdot \nabla) v] = \Sigma (\partial_i v_j)(\partial_j v_i) + (v \cdot \nabla)(\nabla \cdot v), \qquad (2.22)$$

and the equation $[(v \cdot \nabla) v] \cdot v = -\Sigma (\partial_i v_j) v_i v_j$ on Σ_T, one shows that

$$\int_\Omega \nabla \cdot [(v \cdot \nabla) v + e^{-g} \nabla P - e^{-g} f] \, dx = 0. \qquad (2.23)$$

On the other hand, (2.14) and $(2.13)_1$ yield

$$\partial_t \delta + v \cdot \nabla \delta = e^{-g} \Delta P + e^{-g} \nabla g \cdot \nabla Q + \nabla \cdot (e^{-g} f)$$
$$- \Sigma (\partial_i v_j)(\partial_j v_i) + g_1^{-2} g_2 (\vartheta^2 - \delta^2).$$

By integrating both sides of this last equation in Ω, and by using (2.22) and (2.23), it readily follows (2.21). □

We end the proof of the theorem 1.1 as follows. Define the linear operator

$$\pi u = u - |\Omega|^{-1} \int_\Omega u(y) \, dy$$

and set $\bar{\mathscr{S}} = (\pi \times \mathrm{id} \times \mathrm{id}) \circ \mathscr{S}$, *i.e.* $\bar{\mathscr{S}}(\vartheta, \xi, Q) = (\pi\delta, \zeta, P)$, where $(\delta, \zeta, P) = \mathscr{S}(\vartheta, \xi, Q)$. One easily shows that $\bar{\mathscr{S}}(\mathbb{K}) \subset \mathbb{K}$. Hence, by the contraction map principle, $\bar{\mathscr{S}}$ has a fixed point in $\mathbb{K}$ (*see* [BV5], for details). Let be $(\vartheta, \xi, Q) = (\pi\delta, \zeta, P)$. Obviously, $(\xi, Q) = (\zeta, P)$. It remains to show that $\vartheta = \delta$. Since $\vartheta = \pi\delta$, one has $\delta - \vartheta = y(t)$, where $y(t) = |\Omega|^{-1} \int_\Omega (\delta - \vartheta) \, dx$. It follows from (2.21) that $y'(t) = g(t) y(t)$, for a suitable (regular) function $g(t)$. Since $y(0) = 0$, $y(t)$ must vanishes on $[0, T]$. Hence $\delta = \vartheta$. □

3. PROOF OF THEOREM 1.2.

Fix a constant c_0 such that the norms $|||U|||_{k, T_0}$, $|||f|||_{k-1, T_0}$, and $[f]_{k, T_0}$ are bounded by $c_0 - 1$ and let λ_2, λ_3, and λ_4 (*see* theorem 1.1) denote the values of these λ-functions for all the arguments equal to c_0. Fix $T > 0$ such that $\lambda_2 T \le 1$, and such that $\lambda_3 [f]_{k, [t_0, t_0 + T]} \le 1/2$ for every $t_0 \in [0, T[$ (if $t_0 + T > T_0$, replace $t_0 + T$ by T_0). Note that T and that the norms (of the data and of the solutions) used below depend only on c_0. These quantities, as well as other quantities that depend only on that ones, will be denoted by c or, if necessary, by $c_1, c_2, \ldots$ One easily shows that the thesis, stated in theorem 1.2 for the whole interval $[0, T_0]$, follows easily from the corresponding result in the interval $[0, T]$.

Instead of studying directly the systems (1.2) and (1.2'), we consider the equivalent systems (1.8) and (1.8'). We use the following convention:

CONVENTION. − We denote by (1.8') the equation obtained by replacing everywhere in equation (1.8) the elements v, P, S, ζ, f, G, a, ϕ, S_0, by v', P', S', ζ', f', G', a', ϕ', S_0' respectively. This convention applies as well to any other equation.

The theorem 1.2 in reference [BV5] applies to the difference $P - P'$, where P and P' are the solutions of problems $(1.8)_3$, $(1.8)_6$, $(1.8)_7$ and $(1.8')_3$, $(1.8')_6$, $(1.8')_7$ respectively. Straightforward calculations [*see* [BV5], equation (6.1)] show that to each $\varepsilon > 0$ it corresponds a positive real number $\Lambda(\varepsilon)$, that depends only on ε and on the solution U, such that

$$|||P(t) - P'(t)|||_k^2 \le c \{ \varepsilon + ||U_0 - U_0'||_k^2 + |||f(0) - f'(0)|||_{k-1}^2$$
$$+ [f - f']_{k, t}^2 + [U - U']_{k, t}^2 + \Lambda(\varepsilon) [U - U']_{k-1, t}^2 \}. \quad (3.1)$$

On the other hand, by applying the theorem 4.1 in reference [BV5] to the difference between the solution ζ of $(1.8)_1$ and the solution ζ' of $(1.8')_1$, one easily shows that

$$|||\zeta(t) - \zeta'(t)|||_{k-1}^2 \le c \{ \varepsilon + ||U_0 - U_0'||_k^2 + [f - f']_{k, t}^2$$
$$+ [U - U']_{k, t}^2 + \Lambda(\varepsilon) (|||v(0) - v'(0)|||_{k-2}^2) + [v - v']_{k-1, t}^2 \}, \quad (3.2)$$

where $\Lambda(\varepsilon)$ is as above. In order to prove a suitable estimate for $S - S'$, we argue as follows: we differentiate both sides of equations $(1.8)_2$ and $(1.8')_2$ with respect to each variable x_i, $i = 1, 2, 3$, and we apply the theorem 4.1 in reference [BV5] (replace in this theorem ζ by $\partial_i S$, H by $-(\partial_i v) \cdot \nabla S$, α by S_0, and similarly for the variables with primes). This procedure yields the estimate for $|||(S - S')(t)|||_k^2$. Then, we use the equation $D(v)(S - S') = (v' - v) \cdot \nabla S'$ in order to obtain the estimate for the full norm. These calculations show that, given a positive ε, there is a $\Lambda(\varepsilon)$,

that depends only on ε, S, and v, such that

$$|||(S-S')(t)|||_k^2 \leq c \{ \varepsilon + ||(S-S')(0)||_k^2$$
$$+ |||(v, S)(0) - (v', S')(0)|||_{k-1}^2 + [(v, S) - (v', S')]_{k,t}^2$$
$$+ \Lambda(\varepsilon)(|||(v-v')(0)|||_{k-2}^{'2} + [v-v']_{k-1,t}^2) \}. \quad (3.3)$$

Elliptic regularisation shows that

$$|||(v-v')(t)|||_k^{'2} \leq c(|||(\zeta-\zeta')(t)|||_{k-1}^2 + ||\nabla \cdot (v-v')(t)||_{k-1}^2). \quad (3.4)$$

Moreover, arguing as for proving the estimate (6.4) in reference [BV5], one shows that

$$|||\nabla \cdot (v-v')(t)|||_{k-1}^2 \leq c(|||(P-P')(t)|||_k^2$$
$$+ |||(v, S)(0) - (v', S')(0)|||_{k-1}^2 + [(v, S) - (v', S')]_{k,t}^2). \quad (3.5)$$

These two last estimates together with (3.1) and (3.2) yield

$$|||(v-v')(t)|||_k^2 \leq c \{ \varepsilon + ||U_0 - U_0'||_k^2$$
$$+ |||(v, S)(0) - (v', S')(0)|||_{k-1}^2 + |||(f-f')(0)|||_{k-1}^2$$
$$+ [f-f']_{k,t}^2 + [U-U']_{k,t}^2 + \Lambda(\varepsilon)(|||(v-v')(0)|||_{k-2}^2 + [U-U']_{k-1,t}^2), \quad (3.6)$$

where $\Lambda(\varepsilon)$ depends only on ε and on the particular solution U (but not on U'); note that

$$|||v-v'|||_k \leq c(|||v-v'|||_k' + |||f-f'|||_{k-1}$$
$$+ |||S-S'|||_{k-1} + |||P-P'|||_k),$$

by equations $(1.2)_1$ and $(1.2')_1$.

Finally, from equations (3.1), (3.3), and (3.6), it follows that

$$|||(U-U')(t)|||_k^2 \leq c \{ \varepsilon + ||U_0 - U_0'||_k^2 + |||(f-f')(0)|||_{k-1}^2 + [f-f']_{k,t}^2$$
$$+ [U-U']_{k,t}^2 + \Lambda(\varepsilon)(||U_0 - U_0'||_{k-1}^2$$
$$+ |||(f-f')(0)|||_{k-2}^2 + [U-U']_{k-1,t}^2) \}, \quad (3.7)$$

where the equations (1.2) and (1.2') have been used to express the derivatives $\partial_t^j(v, S)(0)$ and $\partial_t^j(v', S')(0)$ in terms of U_0 and $\partial_t^j f(0)$, and of U_0' and $\partial_t^j f'(0)$, respectively.

Finally, we end the proof of the theorem 1.2 by arguing as in the proof of the theorem 1.4 in the section 6 of reference [BV5]. □

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