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Structural Stability and Data Dependence for Fully Nonlinear Hyperbolic Mixed Problems

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Main Notation

Ω is an open, bounded, connected subset of $\mathbb{R}^n$, $n \geqq 2$, locally situated on one side of its boundary Γ, which is a differentiable manifold of class C^∞. In the sequel k denotes a fixed integer such that $k > \frac{n}{2} + 1$. We denote by $\nu = (\nu_1, \ldots, \nu_n)$ the unit outward normal to the boundary Γ, and by ∂_ν differentiation in the ν direction. We set $Q_T = [0, T] \times \Omega$, $\Sigma_T = [0, T] \times \Gamma$.

We denote by H^l, l a nonnegative integer, the Hilbert space $H^l(\Omega)$ endowed with the canonical norm $\|\cdot\|_l$ defined by $\|u\|_l^2 = \Sigma \|\partial^\alpha u\|^2$, where the summation is extended over the multi-indices α such that $0 \leqq |\alpha| \leqq l$, and $\|\cdot\| = \|\cdot\|_0$ denotes the L^2-norm in Ω. Moreover,

$$|||u|||_l^2 = \sum_{j=0}^{l} \|\partial_t^j u\|_{l-j}^2 .$$

On Γ we also use fractional Sobolev spaces $H^{l-1/2}(\Gamma)$. A half-integer index always denotes a Sobolev space over the boundary Γ. The norm in this space is denoted by the symbol $\langle\langle \cdot \rangle\rangle_{l-1/2}$. We set

$$\langle\langle\langle u \rangle\rangle\rangle_{l-1/2}^2 = \sum_{j=0}^{l-1} \langle\langle \partial_t^j u \rangle\rangle_{l-j-1/2}^2 .$$

In the sequel we use the notation $C_T^j(X) = C^j([0, T]; X)$, $L_T^p(X) = L^p(0, T; X)$, and so on. We define

$$\mathscr{C}_T(H^l) = \bigcap_{j=0}^{l} C_T^j(H^{l-j}),$$

$$\mathscr{L}_T^p(H^l) = \bigcap_{j=0}^{l} W_T^{j,p}(H^{l-j}), \quad \mathscr{L}_T^p(H^{l-1/2}) = \bigcap_{j=0}^{l-1} W_T^{j,p}(H^{l-j-1/2}).$$

The norms for these functional spaces are

$$|||u|||^2_{l,T} = \sup_{0 \leq t \leq T} |||u(t)|||^2_l,$$

$$[u]^2_{l,T} = \int_0^T |||u(t)|||^2_l \, dt, \quad \langle u \rangle^2_{l-1/2,T} = \int_0^T \langle\langle\langle u(t) \rangle\rangle\rangle^2_{l-1/2} \, dt,$$

where "sup" denotes the essential supremum and $p = 2$. The above notation will be used both for scalar and vector fields. This convention applies to all notations used in the sequel. In particular, we write $v, g \in X$ even if v is a vector, and g, a scalar.

Given an arbitrary function $f(t, x)$, we denote by $f(t)$, for each fixed t, the function $f(t, \cdot)$.

Introduction

In reference [BV1] I applied, to a simple but significant example, a new method for proving strong continuous dependence of solutions of differential equations on the data (see also [BV2]). In the present paper I apply this method to a class of nonlinear wave equations with a fully nonlinear boundary condition of Neumann-type. Very general existence results for these problems were proved by Shibata and coworkers [ShN, ShK], by using, in particular, an idea of Shibata (see [Sh]), namely, the reduction of the original problem to a suitable elliptic-hyperbolic system on the variables u and $\partial_t u$. By using Shibata's device and his own abstract theory Kato [K] proved similar existence results. In reference [K] the assumptions are less general than those in reference [ShK]; however, Kato proves the strong continuous dependence of the solutions on the initial data. One of the aims of this paper is to show that my method is general enough for proving the strong continuous dependence of the solution on the data (in a broad sense) under fully nonlinear (and nonhomogeneous) boundary conditions. See Theorem 1.1.

It is worth noting that in proving the continuous dependence theorem, by my method, there is no substantial distinction between the case in which the coefficients of the nonlinear equations depend only on ∇u (as below) and the case in which they depend on $(t, x, u, \nabla u)$ and possibly on $\partial_t u$. (This remark does not apply to the existence theorem.) Avoiding formal generality, we assume here the "simplified" situation. The case in which u is a vector instead of a scalar can also be treated by my method without significant alterations in the proofs. A more relevant generalization, from the mathematical and the physical point of view, is to allow here structural changes in the nonlinear differential operators and boundary conditions. See also [BV1, BV2].

A central point in the theory developed here is to provide a quite general method for proving the sharp dependence of the solutions to *linear* equations on the coefficients. In the specific case considered here the linear equation is (1.7) (which is a linearization of (1.2)), and the sharp dependence result is Theorem 1.2 below.

Results

Let $a = (a_1, \dots, a_n) \in C^{k+1}(\mathbb{R}^n; \mathbb{R}^n)$ and $b \in C^k(\mathbb{R}; \mathbb{R})$, where $k > \frac{n}{2} + 1$ is a fixed integer. We define $a_{ij} = \partial a_j/\partial p_j$, and we assume that $a_{ij}(p) = a_{ji}(p)$, $i, j = 1, \dots, n$, and that

$$a_{ij}(p)\,\xi_i\xi_j \geqq m(p)\,|\xi|^2 \quad \forall p, \xi \in \mathbb{R}^n. \tag{1.1}$$

Consider the following nonlinear hyperbolic mixed problem with a fully nonlinear boundary condition of Neumann type:

$$\begin{aligned} \partial_t^2 u - \partial_i(a_i(\nabla u)) &= f \quad \text{in } Q_T, \\ \nu_i a_i(\nabla u) + b(u) &= g \quad \text{on } \Sigma_T, \\ u(0) = u_0, \quad \partial_t u(0) &= u_1. \end{aligned} \tag{1.2}$$

Here and in the sequel the usual summation convention is employed. Assume that

$$(u_0, u_1) \in H^{k+1} \times H^k, \quad (f, g) \in \mathscr{L}^2(H^k) \times \mathscr{L}^2(H^{k+1/2}) \tag{1.3}$$

(where, in general, $\mathscr{L}^2(X)$ denotes $\mathscr{L}^2_{\text{loc}}(0, +\infty; X)$) and that the data satisfy the compatibility conditions up to order $k-1$ (see, for instance, [BV2] or [ShK]). It is well known that problem (1.2) has a unique local solution $u \in \mathscr{C}_T(H^{k+1})$; see [Sh, ShN, ShK, K]; (in [K] $g \equiv 0$). Above all, see [ShK] since the proof of Theorem 1.1 below applies to the general case treated in this reference. Actually (with suitable refinements; see [BV2]) the proof can be extended to the more general case in which the operator $\partial_t^2 u$ is replaced by $(\partial_t + v \cdot \nabla)^2 u$, and (in equation $(1.2)_\alpha$ below) the operator $\partial_t^2 u_\alpha$ is replaced by $(\partial_t + v_\alpha \cdot \nabla)^2 u_\alpha$. Here $v(t, x, z, p)$ is a vector field tangential to the boundary for $x \in \Gamma$ (and similarly for v_α).

Next, consider the problems

$$\begin{aligned} \partial_t^2 u_\alpha - \partial_i(a_i^\alpha(\nabla u_\alpha)) &= f_\alpha \quad \text{in } Q_T, \\ \nu_i a_i^\alpha(\nabla u) + b_\alpha(u) &= g_\alpha \quad \text{on } \Sigma_T, \\ u_\alpha(0) = u_0^\alpha, \quad \partial_t u_\alpha(0) &= u_1^\alpha, \end{aligned} \tag*{$(1.2)_\alpha$}$$

where $\alpha \in \mathbb{N}$, and a_α and b_α are as a and b above. In particular,

$$a_{ij}^\alpha(p)\,\xi_i\xi_j \geqq m_\alpha(p)\,|\xi|^2 \quad \forall p, \xi \in \mathbb{R}^n; \tag*{$(1.1)_\alpha$}$$

moreover, the data

$$(u_0^\alpha, u_1^\alpha) \in H^{k+1} \times H^k, \quad (f_\alpha, g_\alpha) \in \mathscr{L}^2(H^k) \times \mathscr{L}^2(H^{k+1/2}) \tag*{$(1.3)_\alpha$}$$

satisfy the compatibility conditions up to order $k-1$.

The main result of this paper is the following. Here $T_0 > 0$ is arbitrarily large, provided that the solution u of problem (1.2) exists on $[0, T_0]$.

Theorem 1.1. *Let a, b, u_0, u_1, f, g, and $a^\alpha, b_\alpha, u_0^\alpha, u_1^\alpha, f_\alpha, g_\alpha$ $(\alpha \in \mathbb{N})$ satisfy the above assumptions and let $u \in \mathscr{C}_{T_0}(H^{k+1})$ be a solution of (1.2) in Q_{T_0}.*

Assume that

$$
(1.4)\qquad
\begin{aligned}
&\lim_{\alpha\to\infty} a^{\alpha}(p) = a(p) \quad \text{in } C^{k+1}, \text{ uniformly on compact subsets of } \mathbb{R}^n,\\
&\lim_{\alpha\to\infty} b_{\alpha}(z) = b(z) \quad \text{in } C^{k}, \text{ uniformly on compact subsets of } \mathbb{R},
\end{aligned}
$$

that

$$
(1.5)\qquad
\begin{aligned}
&\lim_{\alpha\to\infty} (u_0^{\alpha}, u_1^{\alpha}) = (u_0, u_1) \quad \text{in } H^{k+1}\times H^{k},\\
&\lim_{\alpha\to\infty} (f_{\alpha}, g_{\alpha}) = (f, g) \quad \text{in } \mathscr{L}^2_{T_0}(H^k)\times \mathscr{L}^2_{T_0}(H^{k+1/2}),
\end{aligned}
$$

and that the assumptions (1.9), (1.10) *hold* (*see Remark 1.3*). *Then for sufficiently large values of* α *the problem* $(1.2)_{\alpha}$ *admits a solution* $u_{\alpha} \in \mathscr{C}_{T_0}(H^{k+1})$ *on the whole interval* $[0, T_0]$. *Moreover,*

$$
(1.6)\qquad \lim_{\alpha\to\infty} u_{\alpha} = u \quad \text{in } \mathscr{C}_{T_0}(H^{k+1}).
$$

In particular, if $[0, \tau_{\alpha}[$ *is the maximal interval of existence of the solution* u_{α} *and* $[0, \tau[$ *that of* u, *then* $\liminf_{\alpha\to\infty} \tau_{\alpha} \geqq \tau$.

One of the main tools in the proof of Theorem 1.1 is the following perturbation result for linear equations. Let

$$
h_{ij}, h'_{ij} \in \mathscr{L}^{\infty}_T(H^k), \qquad h_{ij} = h_{ji}, \qquad h'_{ij} = h'_{ji},
$$

for $i, j = 1, \dots, n$. Moreover, assume that

$$
h_{ij}(t, x)\,\xi_i\xi_j \geqq m|\xi|^2, \qquad h'_{ij}(t, x)\,\xi_i\xi_j \geqq m|\xi|^2,
$$

for each $\xi \in \mathbb{R}^n$ and each $(t, x) \in Q_T$, where m is a positive constant.

For convenience, set $h = \{h_{ij} : i, j = 1, \dots, n\}$, $\|h\|^2 = \Sigma_{i,j}\|h_{ij}\|^2$, and so on. Let (w_0, w_1), $(w_0', w_1') \in H^k\times H^{k+1}$ and (F, G), $(F', G') \in \mathscr{L}^2_T(H^{k-1})\times \mathscr{L}^2_T(H^{k-1/2})$. Finally consider the linear problems

$$
(1.7)\qquad
\begin{aligned}
\partial_t^2 w - \partial_i(h_{ij}\partial_j w) &= F \quad \text{in } Q_T,\\
\nu_i h_{ij}\partial_j w &= G \quad \text{on } Q_T,\\
w(0) = w_0, \quad \partial_t w(0) &= w_1,
\end{aligned}
$$

$$
(1.7)'\qquad
\begin{aligned}
\partial_t^2 w' - \partial_i(h'_{ij}\partial_j w') &= F' \quad \text{in } Q_T,\\
\nu_i h'_{ij}\partial_j w' &= G' \quad \text{on } Q_T,\\
w_0'(0) = w_0', \quad \partial_t w'(0) &= w_1',
\end{aligned}
$$

and assume that the data satisfy the compatibility conditions up to order $k-2$ for these systems. In the sequel R denotes generic real, nonnegative functions, which depend increasingly on each of the single arguments. These arguments are, at most, m^{-1}, $|||h|||_{k,T}$, $|||h'|||_{k,T}$, $\|w_0\|_k$, $\|w_1\|_{k-1}$, $\|w_0'\|_k$, $\|w_1'\|_{k-1}$, $|||F(0)|||_{k-2}$, $|||F'(0)|||_{k-2}$, $[F]_{k-1,T}$, and $[F']_{k-1,T}$. The same symbol R may denote distinct "constants" of the above type, even in the same equation. We have the following result.

Theorem 1.2. *Assume that the above conditions on h, w_0, w_1, F, G and on h', w_0', w_1', F', G' hold. Let w and w' be $\mathscr{C}_T(H^k)$ solutions on $[0, T]$ of problems (1.7) and (1.7)′ respectively. Then, given $\varepsilon > 0$, there exists a real positive $\Lambda(\varepsilon)$ that depends only on ε, on T, and on the particular functions h, w_0, w_1, F, and G, such that, for each $t \in [0, T]$,*

$$
\begin{aligned}
(1.8)\quad & |||w(t) - w'(t)|||_k^2 \\
& \leqq R e^{Rt}\{\varepsilon + \|w_0 - w_0^\alpha\|_k^2 + \|w_1 - w_1^\alpha\|_{k-1}^2 + |||F(0) - F'(0)|||_{k-2}^2 \\
& \quad + \langle\langle G(0) - G'(0)\rangle\rangle_{k-3/2}^2 + |||h(0) - h'(0)|||_{k-1}^2 + [F - F']_{k-1,t}^2 \\
& \quad + \langle G - G'\rangle_{k-1/2,t}^2 + [h - h']_{k,t}^2 + [w - w']_{k,t}^2 + \Lambda(\varepsilon)\,[h - h']_{k-1,t}^2\}.
\end{aligned}
$$

This theorem was proved in reference [BV2, Theorem 1.2], where the situation is more general since ∂_t is replaced by $\partial_t + v \cdot \nabla$ or by $\partial_t + v' \cdot \nabla$. The fact that $v \equiv v' \equiv 0$ simplifies the proofs substantially. On the other hand, in [BV2] the coefficients h_{ij} have the particular form $h_{ij}(t, x) = \delta_{ij} h(t, x)$, where $h(t, x) \geqq m > 0$, and similarly for h_{ij}'. However, the new situation in the present paper does not require any important change in the proof of the above result.

Remark 1.3. The starting point of the proof of Theorem 1.1 is the existence theorem for the solution of problem (1.2). To this end we refer to Theorem 1.1 of [ShK]. In order to apply this theorem literally we make the following assumptions:

$$
(1.9)\qquad (f, g),\ (f_\alpha, g_\alpha) \in \mathscr{L}_{T_0}^\infty(H^k) \times \mathscr{L}_{T_0}^\infty(H^{k+1/2})
$$

and the corresponding norms are uniformly bounded as $\alpha \to \infty$;

$$
(1.10)\qquad a,\ a^\alpha \in C^{k_0+1}(\mathbb{R}^n; \mathbb{R}^n), \qquad b,\ b_\alpha \in C^{k_0}(\mathbb{R}; \mathbb{R})
$$

and the corresponding norms are uniformly bounded (on compact subsets) as $\alpha \to \infty$. Here, $k_0 \geqq k$ is an integer such that the Existence Theorem 1.1 of [ShK] applies to (1.2) under the assumption (1.10) for a and b. Note that in reference [ShK] the authors assume, for convenience, that $a, b \in C^\infty$. Obviously, there is a finite $k_0 = k_0(k)$ for which the theorem already holds. We presume that $k_0 = k$, i.e., that (1.10) is superfluous. Here (1.9) also seems superfluous, since $\mathscr{L}^2$ (instead of $\mathscr{L}^\infty$) should be sufficient in order to prove the existence theorem.

2. Proof of Theorem 1.1

Preliminaries. Note that the data and the solution $u \in \mathscr{C}_{T_0}(H^{k+1})$ of problem (1.2) are fixed once and for all. Let us, once and for all, fix a positive constant Λ_0 such that the norms $\|u\|_{C(\bar{Q}_{T_0})}$, $\|\nabla u\|_{C(\bar{Q}_{T_0})}$, and $|||u|||_{k+1,T_0}$ are bounded by Λ_0. Let us also fix $K > 0$ such that $[f]_{k,T_0} \leqq K$ and $\langle g\rangle_{k+1/2,T_0} \leqq K$. Note that $\|u_0\|_{k+1} \leqq \Lambda_0$, $\|u_1\|_k \leqq \Lambda_0$.

From assumptions (1.4) and (1.5) it follows that the norms $\|u_0^\alpha\|_{k+1}$, $\|u_1^\alpha\|_k$, $\|w_0^\alpha\|_{C(\bar\Omega)}$, and $\|\nabla w_0^\alpha\|_{C(\bar\Omega)}$ are bounded by $2\Lambda_0$, and the norms $[f_\alpha]_{k,T_0}$ and $\langle g_\alpha\rangle_{k+1/2,T_0}$ are bounded by $2K$ (for sufficiently large values of $\alpha\in\mathbb{N}$). Without loss of generality we assume that all the above properties hold for any $\alpha\in\mathbb{N}$.

Our first step is to show that it is sufficient to prove the conclusion of Theorem 1.1 for a suitable value $T\in]0, T_0]$ (defined in the sequel) which is independent of α. In problems $(1.2)_\alpha$ let us replace the functions $a^\alpha(p)$ by functions $\tilde a^\alpha(p)$ and the functions $b_\alpha(z)$ by functions $\tilde b_\alpha(z)$ such that $\tilde a^\alpha(p)=a^\alpha(p)$ and $\tilde b_\alpha(z)=b_\alpha(z)$ if $|p|\leqq 3\Lambda_0$, $\tilde a^\alpha(p)=a(p)$, and $\tilde b_\alpha(z)=b(z)$ if $|p|\geqq 4\Lambda_0$. Since $\tilde a_{ij}^\alpha\to a_{ij}^\alpha$ uniformly on $B(0;4\Lambda_0)\equiv\{p:|p|\leqq 4\Lambda_0\}$ as $\alpha\to\infty$, we assume, without loss of generality, that for each $\alpha\in\mathbb{N}$

$$\tilde a_{ij}^\alpha(p)\,\xi_i\xi_j\geqq \tfrac{1}{2}\,m(p)\,|\xi|^2\qquad \forall\xi, p\in\mathbb{R}^n, \tag{2.1}$$

where $m(p)$ is as in (1.1).

Let $\tilde u_\alpha$ denote the solution of problem $(\widetilde{1.2})_\alpha$ where $(\widetilde{1.2})_\alpha$ denotes the equation $(1.2)_\alpha$ with the coefficients a^α and b^α replaced by $\tilde a^\alpha$ and $\tilde b^\alpha$, respectively. The initial data and the "external forces" remain unchanged; note that these data satisfy the compatibility conditions for the system $(\widetilde{1.2})_\alpha$ since the value of the coefficients does not change for $|p|\leqq 3\Lambda_0$. By construction, the following quantities are uniformly bounded with respect to $\alpha\in\mathbb{N}$: The C^{k+1} norms of the functions $\tilde a^\alpha$ and the C^k norms of the functions $\tilde b_\alpha$, on compact subsets of $\mathbb{R}^n$; the ellipticity constant in equation (2.1); the $H^{k+1}\times H^k$ norms of (u_0^α, u_1^α); and the $\mathscr{L}^2_{T_0}(H^k)\times\mathscr{L}^2_{T_0}(H^{k+1/2})$ norms of the external forces (f^α, g^α). These quantities determine a positive lower bound for the time of existence of solutions and an upper bound for their $|||\cdot|||_{k+1,T}$ norms. Hence there are positive reals T and K_0 (independent of α) and there are solutions $\tilde u_\alpha\in\mathscr{C}_T(H^{k+1})$ of problems $(\widetilde{1.2})_\alpha$ in $[0, T]$ that satisfy $|||\tilde u^\alpha|||_{k+1,T}\leqq K_0$. Since $\|v\|_{C(\bar Q_T)}\leqq\|v(0)\|_{C(\bar\Omega)}+c_0T|||v|||_{k,T}$, it follows that

$$\|\nabla\tilde u^\alpha\|_{C(\bar Q_T)}\leqq 2\Lambda_0+c_0TK_0,$$

and similarly for $\|\tilde u^\alpha\|_{C(\bar Q_T)}$. By choosing T as the minimum of the previous T and $(c_0K_0)^{-1}\Lambda_0$, we have

$$\|\tilde u^\alpha\|_{C(\bar Q_T)}\leqq 3\Lambda_0,\qquad \|\nabla\tilde u^\alpha\|_{C(\bar Q_T)}\leqq 3\Lambda_0.$$

Since $\tilde a^\alpha(p)=a^\alpha(p)$ and $\tilde b_\alpha(z)=b_\alpha(z)$ for $|p|\leqq 3\Lambda_0$, it follows that $\tilde u^\alpha$ is, in fact, a solution of $(1.2)_\alpha$ in $[0, T]$. Hence from now on we denote $\tilde u^\alpha$ by the symbol u^α.

Let us show that it is sufficient to prove the conclusion of Theorem 1.1 for the above value of T (instead of for T_0). In fact, assume that the conclusion holds for $[0, T]$. Then, $u^\alpha\to u$ in $\mathscr{C}_T(H^{k+1})$. Hence, for sufficiently large α,

$$|||u^\alpha|||_{k+1,T}\leqq 2K_0,\qquad \|u^\alpha\|_{C(\bar Q_T)}\leqq 2\Lambda_0,\qquad \|\nabla u^\alpha\|_{C(\bar Q_T)}\leqq 2\Lambda_0;$$

moreover, $(u^\alpha(T), \partial_t u^\alpha(T))$ converges to $(u(T), \partial_t u(T))$ in $H^{k+1} \times H^k$, as $\alpha \to \infty$. Hence Theorem 1.1 in the restricted form (i.e., with T_0 replaced by T) can be applied to the interval $[T, 2T]$. It follows that the above results hold on this last interval (hence on $[0, 2T]$) for sufficiently large values of α. Next, we apply the restricted conclusion to $[2T, 3T]$, and so on.

Proof of the restricted conclusion. According to the above argument, we assume that $t \in [0, T]$. We have shown that u^α and ∇u^α can be assumed to be uniformly bounded in the $C(\bar{Q}_T)$ norm by the constant $3\Lambda_0$. Hence we assume, without loss of generality, that the convergence in (1.3) is uniform in $\mathbb{R}^n$ instead of on compact subsets only. Moreover (recall (2.1)),

$$a_{ij}^\alpha(p)\, \xi_i \xi_j \geqq \frac{m(p)}{2} |\xi|^2 \quad \forall\, \xi, p \in \mathbb{R}^n, \tag{2.2}$$

uniformly with respect to $\alpha \in \mathbb{N}$. Finally, as shown above,

$$|||u^\alpha|||_{k+1,T} \leqq K_0. \tag{2.3}$$

Set $w \equiv \partial_t u$ and differentiate the equations (1.2) with respect to t to get

$$\begin{aligned} &\partial_t^2 w - \partial_i(a_{ij}(\nabla u)\, \partial_j w) = F \quad \text{in } Q_T, \\ &\nu_i a_{ij}(\nabla u)\, \partial_j w = G \quad \text{on } \Sigma_T, \\ &w(0) = w_0, \quad \partial_t w(0) = w_1, \end{aligned} \tag{2.4}$$

where $F = \partial_t f$, $G = \partial_t g - b'(u)\, u_1$, $w_0 = u_1$, $w_1 = F(0) - \partial_i a_i(\nabla u)$, and the compatibility conditions are satisfied up to order $k-2$. On the other hand, when we set $w_\alpha = \partial_t u_\alpha$, a similar argument for the system $(1.2)_\alpha$ yields

$$\begin{aligned} &\partial_t^2 w_\alpha - \partial_i(a_{ij}(\nabla u_\alpha)\, \partial_j w_\alpha) = F_\alpha \quad \text{in } Q_T, \\ &\nu_i a_{ij}^\alpha(\nabla u_\alpha)\, \partial_j w_\alpha = G_\alpha \quad \text{on } \Sigma_T, \\ &w_\alpha(0) = w_0^\alpha, \quad \partial_t w_\alpha(0) = w_1^\alpha, \end{aligned} \tag{2.4$_\alpha$}$$

where $F_\alpha = \partial_t f_\alpha$, $G_\alpha = \partial_t g_\alpha - b_\alpha'(u_\alpha)\, u_1^\alpha$, $w_0^\alpha = u_1^\alpha$, $w_1^\alpha = f_\alpha(0) - \partial_i a_i^\alpha(\nabla u_\alpha)$, and the compatibility conditions are satisfied up to order $k-2$. The pair of systems (2.4) and $(2.4)_\alpha$ satisfies the hypothesis of Theorem 1.2. Note that functions of type R in equation (1.8) are uniformly bounded here. Note, in particular, that

$$a_{ij}^\alpha(\nabla u_\alpha(t, x))\, \xi_i \xi_j \geqq \tfrac{1}{2} m |\xi|^2,$$

where $m = \inf\{m(p) : |p| \leqq 3\Lambda_0\}$. Furthermore $\Lambda(\varepsilon)$ depends only on ε since in equation (2.4) the real T, the coefficients $a_{ij}(\nabla u(t, x))$ and $b(u(t, x))$, the data, and the solution $w = \partial_t u$ are fixed. From (1.8) we easily get

(2.5)

$$\begin{aligned}|||w(t) - w_\alpha(t)|||_k^2 \leqq C\{&\varepsilon + \|u_0 - u_0^\alpha\|_{k+1}^2 + \|u_1 - u_1^\alpha\|_k^2 + |||f(0) - f_\alpha(0)|||_{k-1}^2 \\ &+ \langle\langle g(0) - g_\alpha(0)\rangle\rangle_{k-1/2}^2 \\ &+ \langle\langle (b(u)\, w)\,(0) - (b_\alpha(u_\alpha) w_\alpha)\,(0)\rangle\rangle_{k-3/2}^2 \\ &+ |||a_{ij}(\nabla u(0)) - a_{ij}(\nabla u_\alpha(0))|||_{k-1}^2 + [f - f_\alpha]_{k,t}^2 \\ &+ \langle g - g_\alpha\rangle_{k+1/2,t}^2 + \langle b(u)\, w - b(u_\alpha)\, w_\alpha\rangle_{k-1/2,t}^2 \\ &+ [a_{ij}(\nabla u) - a_{ij}^\alpha(\nabla u_\alpha)]_{k,t}^2 + [w - w_\alpha]_{k,t}^2 \\ &+ \Lambda(\varepsilon)\, [a_{ij}(\nabla u) - a_{ij}^\alpha(\nabla u_\alpha)]_{k-1,t}^2\},\end{aligned}$$

for each $t \in [0, T]$. Here and in the sequel we use some well-known inequalities for H^l-norms of products of functions; see, for instance, Appendix A in [BV1]. We easily verify that, for each $t \in [0, T]$,

$$\text{(2.6)} \qquad |||a_{ij}(\nabla u) - a_{ij}^\alpha(\nabla u_\alpha)|||_l^2 \leqq C|||u - u_\alpha|||_{l+1}^2 + C\|a_{ij} - a_{ij}^\alpha\|_{C^l}^2,$$

where $l \leqq k$, $C^l = C^l(B_0; \mathbb{R})$, and $B_0 \equiv \{p \in \mathbb{R}^n : |p| \leqq 3\Lambda_0\}$. Below, on applying this estimate for $l = k - 1$ and $t = 0$ (in order to bound the seventh term on the right-hand side of (2.5)), we should take into account that the first term on the right-hand side of (2.6) is bounded by a constant C times $\|u_0 - u_0^\alpha\|_k^2 + \|u_1 - u_1^\alpha\|_{k-1}^2 + |||f(0) - f_\alpha(0)|||_{k-2}^2$.

On the other hand, since $\langle\cdot\rangle_{k-1/2,t} \leqq c[\cdot]_{k,t}$, easy manipulations show that the tenth term on the right-hand side of (2.5) is bounded by a constant C times $[u - u_\alpha]_{k,t}^2 + \|b - b_\alpha\|_{C^k}^2 + [w - w_\alpha]_{k,t}^2$ and, similarly, the sixth term is bounded by C times $\|u_0 - u_0^\alpha\|_{k-1}^2 + \|b - b_\alpha\|_{C^{k-1}}^2 + \|w_0 - w_o^\alpha\|_{k-1}^2$. By using the above bounds together with (2.5), we easily prove that

(2.7)

$$\begin{aligned}|||w(t) - w_\alpha(t)|||_k^2 \leqq C\{&\varepsilon + \|u_0 - u_0^\alpha\|_{k+1}^2 + \|u_1 - u_1^\alpha\|_k^2 + |||f(0) - f_\alpha(0)|||_{k-1}^2 \\ &+ \langle\langle g(0) - g_\alpha(0)\rangle\rangle_{k-1/2}^2 \\ &+ \|a - a^\alpha\|_{C^{k+1}}^2 + \|b - b_\alpha\|_{C^k}^2 + [f - f_\alpha]_{k,t}^2 \\ &+ \langle g - g_\alpha\rangle_{k+1/2,t}^2 + [u - u_\alpha]_{k+1,t}^2 \\ &+ \Lambda(\varepsilon)\,([u - u_\alpha]_{k,t}^2 + \|a - a^\alpha\|_{C^k}^2)\},\end{aligned}$$

for each $t \in [0, T]$. Here, the C^l norms always concern the sets where $|p| \leqq 3\Lambda_0$ and $|z| \leqq 3\Lambda_0$.

Next, we make similar computations, replacing differentiation with respect to t by differentiation with respect to each single tangential direction. This is done locally, on "small" neighbourhoods of points of the boundary, by using a partition of unity and also a change of local coordinates to rectify the boundary. (These well-known devices are left to the reader.) In view of this, let us denote by y a generical tangential direction. By setting $w = \partial_y u$ and $w_\alpha = \partial_y u_\alpha$, and by differentiating (1.2) and $(1.2)_\alpha$ with respect to y, we prove (2.7) just as we did above for t. By addition of the $n-1$ estimates concerning the tangential variables y together with that concerning the variable t we get, with obvious notation,

(2.8) $\|\partial_t u(t) - \partial_t u_\alpha(t)\|_k^2 + \|\nabla_y u(t) - \nabla_y u_\alpha(t)\|_k^2 \leqq$ right-hand side of (2.7).

Finally, equation $(1.2)_1$ yields

$$\partial_n^2 u = a_{nn}^{-1}(\nabla u) \left\{ \partial_t^2 u - \sum_{i,j}^{*} a_{ij}(\nabla u)\, \partial_i \partial_j u - f \right\}$$

where, for convenience, we assume x_n to be the normal direction. Here, Σ^* means that the pair of indices (n, n) does not appear in the summation. An expression similar to the above one holds for $\partial_n^2 u_\alpha$. By taking the difference $\partial_n^2 u - \partial_n^2 u_\alpha$ and by doing straightforward manipulations, we get

$$\text{(2.9)} \qquad |||\partial_n^2 u(t) - \partial_n^2 u_\alpha(t)|||_{k-1}^2 \leqq C\{|||u(t) - u_\alpha(t)|||_{k+1}^{*2} + |||f(t) - f_\alpha(t)|||_{k-1}^2 + \|a - a^\alpha\|_{C^k}^2\}$$

for each t, where $(|||\cdot|||_{k+1}^*)^2 = |||\cdot|||_{k+1}^2 - \|\partial_n^{k+1}\cdot\|^2$. From (2.8) it follows that the left-hand side of (2.9) is bounded by the right-hand side of (2.7). It readily follows that

$$\begin{aligned} \text{(2.10)} \quad |||u - u_\alpha|||_{k+1,T}^2 \leqq C_0\{&\varepsilon + \|u_0 - u_0^\alpha\|_{k+1}^2 \\ &+ \|u_1 - u_1^\alpha\|_k^2 + |||f(0) - f_\alpha(0)|||_{k-1}^2 \\ &+ \langle\langle g(0) - g_\alpha(0)\rangle\rangle_{k-1/2}^2 + \|a - a^\alpha\|_{C^{k+1}}^2 \\ &+ \|b - b_\alpha\|_{C^k}^2 + [f - f_\alpha]_{k,T}^2 + \langle g - g_\alpha\rangle_{k+1/2,T}^2 \\ &+ \Lambda(\varepsilon)\,([u - u_\alpha]_{k,T}^2 + \|a - a^\alpha\|_{C^k}^2)\}, \end{aligned}$$

where Gronwall's lemma was previously used for dropping the term $[u - u_\alpha]_{k+1,t}^2$ from the right-hand side of the above equation.

Next, we show that

$$\text{(2.11)} \qquad \lim_{\alpha\to 0}\, [u - u_\alpha]_{k,T}^2 = 0.$$

Since $\mathscr{C}_T(H^{k+1}) \hookrightarrow C^{2,\beta}(\bar{Q}_T)$ for some $\beta = \beta(n, k) > 0$, it follows that there are subsequences u_α convergent in $C^2(\bar{Q}_T)$, in $\mathscr{L}_T^\infty(H^{k+1})$ weak-* and, in particular, in $\mathscr{L}_T^2(H^k)$. Since these limits are the solution of (1.2), equation (2.11) follows. Actually, $u_\alpha \to u$ in $\mathscr{C}_T(H^{k+1-\varepsilon})$ for arbitrarily small $\varepsilon > 0$, a trivial consequence of the $\mathscr{L}_T^\infty(H^{k+1})$ uniform estimate furnished by the existence theorem.

Finally, let $\delta > 0$ be given and fix $\varepsilon = \varepsilon(\delta) > 0$ such that $C_0\varepsilon < \delta/2$. Now $\Lambda(\varepsilon)$ is fixed. Then by (1.4), (1.5), and (2.11) there is an integer $M \in \mathbb{N}$ such that the right-hand side of (2.10) is less than δ if $\alpha > M$. $\square$

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