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Periodic Solutions for a Class of Autonomous Hamiltonian Systems.

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1. – Introduction.

In this paper we shall be concerned with the existence of T-periodic solutions of Hamiltonian systems $\dot{p} = -H'_q(p, q)$, $\dot{q} = H'_p(p, q)$ when H is of the form

$$H(p, q) = U(p) + V(q) \tag{1}$$

so that the above equations of motion became

$$\dot{p} = -V'(q), \quad \dot{q} = U'(p). \tag{2}$$

Hamiltonians of the form (1) occupy a central position in the general theory of Hamiltonian systems. Moreover, in applications to concrete problems, p and q play substantially distinct roles. In fact, in many classical problems, the term $U(p)$ has the form $(\frac{1}{2})|p|^2$ or, more in general, is a positive definite quadratic form. Hence $U(p)$ is strictly convex. On the contrary, a wide freedom in the choice of the potential $V(q)$ is required. For Hamiltonians of the special form $|p|^2/2 + V(q)$, Hamilton's equation reduces to Newton's equation $\ddot{q} + V'(q) = 0$. Here, the higher order term is a linear operator. The natural nonlinear generalization of the above class (which shall be our

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main model) consists in Hamiltonians of the form $(1/\alpha)|p|^\alpha + V(q)$.

Throughout this paper, c_i $(i \in \mathbb{N})$ denote positive constants. We shall prove the following result.

THEOREM A. *Let $U, V \in C^1(\mathbb{R}^n, \mathbb{R})$, U strictly convex, V everywhere nonnegative. Assume that there are positive constants $\alpha \in]1, +\infty[$, $\mu > \alpha/(\alpha - 1)$, and r such that the following conditions hold.*

$$(\mathrm{H}_1) \qquad c_1|p|^\alpha \leqq U(p) \leqq c_2|p|^\alpha , \qquad \text{for all } p \in \mathbb{R}^n ,$$

$$(\mathrm{H}_2) \qquad \alpha U(p) \leqq U'(p)\cdot p , \qquad \text{for all } p \in \mathbb{R}^n ,$$

$$(\mathrm{H}_3) \qquad 0 < \mu V(q) \leqq V'(q)\cdot q - c_3 , \qquad \text{for all } |q| \geqq r .$$

Then, for each $T > 0$, the problem (2) *has infinitely many T-periodic non trivial solutions.*

By setting $\alpha = 2$ and by considering the particular case $U(p) = (\frac{1}{2})|p|^2$, we reobtain a result of Benci (theorem 3.7 [B]), which in turn generalizes a result of Rabinowitz (theorem 2.61 [R1]). For $\alpha \neq 2$ theorem A is substantially different from all the results available to us. Note that (in theorem A): (i) the potential V is superquadratic at infinity when $1 < \alpha < 2$; (ii) the potential V could be subquadratic, quadratic or superquadratic at infinity, when $\alpha > 2$; (iii) no growth assumptions are made for small $|q|$; (iv) V is not necessarily convex. Remarks (i), (ii) and (iii) show that our assumptions are quite different from those made by Rabinowitz in his well known theorems on Hamiltonian systems (see [R3], [R4] for references).

Each one of the remarks (i)-(iv) show also that our assumptions are entirely different from those of Clarke's theorems 1.1 and 1.2 in reference [C2]. Note, in particular, that Clarke requires that $\mu < \alpha/(\alpha - 1)$, instead of $\mu > \alpha/(\hbar - 1)$. Our assumptions are also entirely different from those of Brezis and Coron theorem 2 [BC]. Hamiltonians of the particular form (1) satisfy the condition (6) of reference [BC] if $\alpha > 2$ and $\mu > 2$ (note that in theorem A, if $\alpha > 2$, μ can be smaller then 2); and under these assumptions theorem A gives T-periodic solutions for small T and theorem 2 in [BC] gives T-periodic solutions for large T. Note finally that, in references [BC] and [C2], the Hamiltonians are assumed to be convex but minimality of the period is proved.

We limit ourselves to give only the strictly necessary references.

For a complete bibliography and usefull comments we refer the reader to [R3].

2. – Proofs.

Without loss of generality we will assume that $V(0) = 0$. Let T be a fixed positive number and denote by $\| \ \|$ and $\| \ \|'$ the norms in $L^\beta(0, T; \mathbb{R}^n)$ and in $L^\alpha(0, T; \mathbb{R}^n)$, respectively. We set $\beta = \alpha/(\alpha - 1)$. Moreover,

$$E = \left\{u \in L^\beta(0, T; \mathbb{R}^n) : \int u = 0\right\},$$

where $\int u$ stands for $\int_0^T u(t)\,dt$. This abbreviated notation will be systematically used in the sequel. We set

$$B_\varrho = \{u \in E : \|u\| \leqq \varrho\}, \qquad \partial B_\varrho = \{u \in E : \|u\| = \varrho\}.$$

Define

$$Pu(t) = \int_0^t u(\tau)\,d\tau, \qquad \forall t \in [0, T]. \tag{3}$$

Clearly, $Pu(0) = Pu(T) = 0$, for every $u \in E$. The map P defines an isomorphism between E and the Sobolev space $W_0^{1,\beta}(0, T; \mathbb{R}^n)$.

The Legendre transform in $\mathbb{R}^n$ of $U(p)$ is defined by

$$G(u) = \operatorname{Sup}\{u \cdot p - U(p) : p \in \mathbb{R}^n\}.$$

We recall that $G'(u) = p$ if and only if $U'(p) = u$, and that

$$\begin{cases} c_4|u|^\beta \leqq G(u) \leqq c_5|u|^\beta, \\ G'(u)\cdot u \leqq \beta G(u), \\ |G'(u)| \leqq c_6|u|^{\beta-1}, \end{cases} \tag{4}$$

for all $u \in \mathbb{R}^n$. On the other hand, it readily follows, from (H^3), that

$$\begin{cases} V'(q)\cdot q \geqq \mu V(q) - c_7, \\ V(q) \geqq c_8|q|^\mu - c_9, \end{cases} \tag{5}$$

for all $q \in \mathbb{R}^n$.

One has the following result.

THEOREM 1. *Let (u, y) be a critical point of the functional*

$$f(u, y) = \int [G(u) - V(Pu + y)] , \tag{6}$$

which is defined on the Banach space $E \oplus \mathbb{R}^n$. Then, the pair $(p, q) = (G'(u), Pu + y)$ is a T-periodic solution of problem (2).

This result is proved by applying the « dual action principle » (see Clarke [C1] and Clarke and Ekeland [CE]) only just to those variables with respect to which the hamiltonian is convex. Before proving the lemma, let us introduce nome notations. The symbol $\langle\, ,\rangle$ denotes the duality pairing between the dual of a Banach space and the Banach space itself. The scalar product in $\mathbb{R}^n$ is denoted either by $x \cdot y$ or by $\langle x, y\rangle$. Furthermore, f' denotes the (Fréchet) derivative of f, and f'_u, f'_y denote the partial derivatives with respect to u and y, respectively.

PROOF OF THEOREM 1. By taking into account that Pv is a periodic function, one easily proves that

$$\langle f'_u(u, y), v\rangle = \int G'(u)\cdot v - V'(Pu + y)\cdot Pv = \int [G'(u) + PV'(Pu + y)]\cdot v \tag{7}$$

for every $u, v \in E$, $y \in \mathbb{R}^n$. Moreover,

$$\langle f'_y(u, y), x\rangle = -\left(\int V'(Pu + y)\right)\cdot x , \qquad \forall x \in \mathbb{R}^n . \tag{8}$$

In particular,

$$f'(u, y) = \left(G'(u) + PV'(Pu + y), -\int V'(Pu + y)\right) \in L^\alpha \oplus \mathbb{R}^n,$$

and

$$\langle f'(u, y), (v, x)\rangle = \int G'(u)\cdot v - \int V'(Pu + y)\cdot (Pv + x) . \tag{9}$$

Note that $f \in C^1(E \oplus \mathbb{R}^n, \mathbb{R}^n)$.

If (u, y) is a critical point, it follows from (8) that

$$\int V'(Pu + y) = 0 . \tag{10}$$

Moreover, (7) shows that $\int [G'(u) + PV'(Pu + y)]\cdot v = 0$, $\forall v \in E$, or equivalently that there exists $z \in \mathbb{R}^n$ such that

$$G'(u) + PV'(Pu + y) = z , \qquad \forall t \in [0, T] . \tag{11}$$

Define

$$\begin{cases} p = G'(u) = z - PV'(Pu + y) , \\ q = Pu + y . \end{cases} \tag{12}$$

Due to (10), p and q are T-periodic.

Moreover, $\dot{p} = -V'(Pu + y) = -V'(q)$, and $\dot{q} = u = U'(p)$. //

Now, with the aid of Theorem 1, we will prove that the functional f has non trivial critical points. Hence Theorem A holds. Before proving Theorem A, let us make the following remarks:

REMARK 1. The above results also apply if

$$H(p, q) = U(p_1, \ldots, p_k, q_{k+1}, \ldots, q_n) + V(q_1, \ldots, q_k, p_{k+1}, \ldots, p_n) ,$$

where U and V are as in theorem 2, and $0 \leq k \leq n$. This is easily shown by doing the change of variables $q_j \to -p_j$, $p_j \to q_j$, $j = k + 1, \ldots, n$.

REMARK 2. It is worth noting that the functional $f(u, y)$ is invariant under the S^1-action of $\mathcal{A} = \{A_s : s \in \mathbb{R}\}$ which is defined on $E \oplus \mathbb{R}^n$ by

$$A_s(u, y) = \left(u(t + s),\ y + \int_0^s u(\tau)\, d\tau\right) . \tag{13}$$

One easily verifies that $A_{s+T}(u, y) = A_s(u, y)$ and that $A_r A_s(u, y) = A_{r+s}(u, y)$ (we assume that the elements $u \in E$ are extended as T-periodic functions over the entire real line). Moreover, straightforward calculations show that

$$(14)\qquad f(A_s(u,y)) = f(u,y)\,,\quad \forall (u,y)\in E\oplus \mathbb{R}^n\,,\quad \forall s\in\mathbb{R}\,.$$

The fixed points under the action of A are precisely the elements $(0, y)$, for $y\in\mathbb{R}^n$.

Due to the above S^1-invariance, it seems possible to apply Fadell, Husseini, Rabinowitz Theorem 3.14 [FHR] to show that f has an unbounded sequence of critical values. However the corresponding sequence of T-periodic solutions could coincide with some in the (T/m)-periodic solutions furnished by theorem $A (m\in\mathbb{N})$.

In the sequel we will prove theorem A by applying Rabinowitz's Theorem 5.3 [R4] to the functional f. Alternately, we could apply the theorem 1.1 in reference [R2]. In order to apply Rabinowitz's theorem it is sufficient to prove that f satisfies the following hypothesis.

(15) $f|_{\mathbb{R}^n}\leqq 0$,

(16) There are positive constants ϱ, θ such that $f(u, 0)\geqq\theta$ if $\|u\| = \varrho$.

(17) For each finite dimensional subspace $\tilde{E}$ of $E\oplus\mathbb{R}^n$ there exists a constant $R = R(\tilde{E})$ such that $f(u, y)\leqq 0$ wherever $\|u\| + |y|\geqq R$, $(u, y)\in\tilde{E}$ [1].

(18) The functional f verifies the Palais-Smale condition.

Condition (15) is trivially verified. Conditions (16), (17), and (18) will be proved in the sequel.

Lemma 1. *Under the hypothesis of theorem* A *the condition* (16) *is fullfilled.*

Proof. We shall denote by $|\;|_\infty$ the usual norm on the space $L^\infty(0, T; \mathbb{R}^n)$. To show that

$$\int[G(u) - V(Pu)]\geqq\theta\qquad\text{for all } u\in\partial B_\varrho$$

(1) In particular the assumption (I5) of Theorem 5.3 [R4] holds. See also Remark 5.5 (iii) there.

it is sufficient to prove that, for every $u \in \partial B_\varrho$, one has

$$c_4\int[|u|^\beta - V(Pu)] \geq \theta .$$

Let c_{10} be a positive constant such that $|Pv|_\infty \leq c_{10}\|v\|$ for all $v \in E$. By assuming that $\varrho \leq c_{10}^{-1}$ one gets, for every $t \in [0, T]$,

$$|V(Pu(t))| \leq |Pu(t)||\omega(P(u(t))| \leq |Pu(t)|^\beta |\omega(Pu(t))| ,$$

where $\lim\limits_{|q|\to 0} \omega(q) = \omega(0) = 0$. It readily follows that

$$\left|\int V(Pu)\right| \leq c_{11} \max_{0\leq t\leq T} |\omega(Pu(t))| \, \|u\|^\beta .$$

In particular,

$$c_4\int[|u|^\beta - V(Pu)] \geq \left(c_4 - c_{11} \max_{0\leq t\leq T} |\omega(Pu(t))| \, \varrho^\beta\right) .$$

Since $|Pu|_\infty \leq c_{10}\varrho$ we conclude that

$$c_4 - c_{11} \max_{0\leq t\leq T} |\omega(Pu(t))| > 0$$

if $\varrho = \|u\|$ is small enough. //

LEMMA 2. *Under the assumptions of theorem* A, *condition* (17) *is fulfilled.*

PROOF. One easily verifies that

$$[(u, y)] \equiv \|Pu + y\|_\mu \tag{19}$$

is a norm in $E \oplus \mathbb{R}^n$, where $\| \ \|_\mu$ stands for the usual norm in the space $L^\mu(0, T; \mathbb{R}^n)$. Let $u_1, \ldots, u_k$ be linearly independent vectors in E, and denote by E_k the subspace generated by these vectors. Set $\tilde{E} = E_k \oplus \mathbb{R}^n$. Since $\tilde{E}$ is finite dimensional, there exists a positive constant $K = K(\tilde{E})$ such that

$$K(\|u\| + |y|) \leq \|Pu + y\|_\mu , \quad \forall (u, y) \in \tilde{E} . \tag{20}$$

By using $(5)_2$, $(4)_1$ and (20) one proves that

$$f(u, c) \leqq c_5\|u\|^\beta - c_8\|Pu + y\|^\mu - c_9 T \leqq$$
$$\leqq c_5(\|u\| + |y|)^\beta - c_8 K^\mu(\|u\| + |y|)^\mu ,$$

for every $(u, y) \in \tilde{E}$. The thesis follows, since $\mu > \beta$. //

Finally we prove the Palais-Smale condition.

LEMMA 3. *Let $(u_m, y_m) \in E \oplus \mathbb{R}^n$ be a sequence such that*

$$f(u_m, y_m) \leqq M , \quad \forall m \in \mathbb{N} ,$$

and $f'(u_m, y_m) \to 0$ as $m \to +\infty$. Then (u_m, y_m) is a bounded sequence in $E \oplus \mathbb{R}^n$. Moreover, there exists a convergent subsequence in $E \oplus \mathbb{R}^n$.

PROOF. In the sequel we denote by $E' = \{w \in L^\alpha(0, T; \mathbb{R}^n) : \int w = 0\}$ the dual space of E, and by $\|P\|$ the norm of the linear operator $P: E \to L^\beta(0, T; \mathbb{R}^n)$. For convenience, we set $\varepsilon_m = f'_u(u_m, y_m)$, $\delta_m = f'_y(u_m, y_m)$. By assumption one has $\|\varepsilon_m\|_{E'} \to 0$, $|\delta_m| \to 0$, as $m \to +\infty$.

By using formulae (9) with $(u, y) = (v, x) = (u_m, y_m)$, and by taking into account $(4)_2$ and $(5)_1$, it readily follows

$$\langle \varepsilon_m, u_m \rangle + \langle \delta_m, y_m \rangle \leqq \beta \int G(u_m) - \mu \int V(Pu_m + y_m) + c_7 T .$$

The above estimate, the assumption

$$\int G(u_m) - \int V(Pu_m + y_m) \leqq M ,$$

the boundedness of the sequences $\|\varepsilon_m\|_{E'}$ and $|\delta_m|$, and the condition $\mu > \beta$, imply that

$$(21) \qquad \begin{cases} \int V(Pu_m + y_m) \leqq c_{12} + c_{13}(\|u_m\| + |y|_m) , \\ \int G(u_m) \leqq M + c_{12} + c_{13}(\|u_m\| + |y_m|) . \end{cases}$$

From $(4)_1$ and $(21)_2$ it follows that

$$\|u_m\|^\beta \leq c_{14} + c_{15}\big(\|u_m\| + |y_m|\big) . \tag{22}$$

On the other hand,

$$\int |y_m|^\beta \leq 2^{\beta-1}\int (1 + |Pu_m + y_m|^\mu) + 2^{\beta-1}\|P\|^\beta \|u_m\|^\beta .$$

This inequality, together with $(5)_2$, $(21)_1$ and (22) yields

$$|y_m|^\beta \leq c_{16} + c_{17}\big(\|u_m\| + |y_m|\big) . \tag{23}$$

The estimates (22), (23) show that $\|u_m\|$ and $|y_m|$ are uniformly bounded.

Now we prove the second part of the lemma. From (7) one gets

$$\langle \varepsilon_m , v\rangle = \int [G'(u_m) + PV'(Pu_m + y_m)]\cdot v ,$$

for every $v \in E$. Hence

$$\left|\int [G'(u_m) + PV'(Pu_m + y_m)]\cdot v\right| \leq \|\varepsilon_m\|_{E'} \|v\| . \tag{24}$$

On the other hand, from (8) it follows that $|\int V'(Pu_m + y_m)| = |\delta_m|$, and from (4) it follows

$$|\int G'(u_m)| \leq c_9 T^{1/\beta} \|u_m\|^{\beta-1} .$$

Consequently, the mean value of $G'(u_m) + V'(Pu_m + y_m)$ is uniformly bounded with respect to m. Hence, along a suitable subsequence, one has

$$\lim_{m\to+\infty} \frac{1}{T}\int [G'(u_m) + V'(Pu_m + y_m)] = \xi_0 \in \mathbb{R}^n . \tag{25}$$

Equations (24) and (25) imply that

$$\lim_{m\to+\infty} \|G'(u_m) + PV'(Pu_m + y_m) - \xi_0\|' = 0 . \tag{26}$$

Therefore, by setting $z_m = G'(u_m)$, $\xi_0 - PV'(Pu_m + y_m) = z$, one has $z_m \to z$ in L^α. Moreover, $u_m = U'(z_m)$, a.e. in $]0, T[$. A well known

Krasnoselskii's theorem shows that U' is a continuous map from L^α into L^β (note that assumption (H1) implies that $|U'(p)| \leq c|p|^{\alpha-1}$, $\forall p \in \mathbb{R}^n$; argue as in [E], lemma 1). Hence, $u_m \to U'(z)$ in L^β. The convergence of y_m along some subsequence is obvious. //

The existence of infinitely many T-periodic solutions follows by a well known argument, since each (T/m)-periodic solution $(m \in \mathrm{N})$ is T-periodic. We don't know if our solution has T as the minimal period.

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