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Long Time Behavior for One-Dimensional Motion of a General Barotropic Viscous Fluid

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1. Introduction

In this paper we study one dimensional motion of a barotropic compressible viscous fluid in a bounded region with impermeable boundary, for a general pressure term. This and related problems have been studied by a number of authors, including KANEL [5], KAZHIKHOV [6], [7] ITAYA [4], KAZHIKHOV & SHELUKHIN [8]; the paper of SOLONNIKOV & KAZHIKHOV [12], in particular, can be consulted for more complete references. I wish to mention also the papers of SHELUKHIN [10], [11], which came to my attention only when this manuscript was finished. SHELUKHIN states quite complete results on bounded solutions for the case $p(v) \simeq k \log v$, and on periodic solutions for $p(v) \simeq k \log v$ and $p(v) = kv^{-1}$, $k > 0$. It should be noted that for pressures of the form $p(v) \simeq kv^{-\gamma}$, $\gamma > 1$, one does not expect to get the uniform estimate (1.6) since vacuum may occur as $t \to \infty$; see [3], section 5.

Without loss of generality we assume that the above bounded region is the interval (0, 1). In material Lagrangian coordinates, and after a normalization (see [8]), the equations of motion become

$$\begin{gathered} v_t - u_q = 0, \\ u_t = \mu(v^{-1}u_q)_q - p(v)_q + f\left(t, \int_0^q v(t, \xi)\, d\xi\right), \end{gathered} \tag{1.1}$$

where $q \in (0, 1)$, and $t \geqq 0$. Here the material Lagrangian coordinate q of a particle of fluid is the mass of the portion of fluid that occupies the region between the origin and the given particle. After the normalization, the total mass of the fluid is equal to 1. Hence the coordinate q runs over the interval (0, 1). The external force $f(t, x)$ is given as a known function of the Eulerian coordinates (t, x), $x \in (0, 1)$, where $x = \int_0^q v(t, \xi)\, d\xi$.

In the above problem, the boundary conditions are

$$u(t, 0) = u(t, 1) = 0 \quad \forall\, t \geqq 0, \tag{1.2}$$

and the initial conditions are

$$u(0, q) = u_0(q), \quad v(0, q) = v_0(q), \quad q \in (0, 1). \tag{1.3}$$

Without loss of generality, we suppose that

$$\int_0^1 v_0(q)\, dq = 1, \quad m^{-1} \leqq v_0(q) \leqq m \quad \forall\, q \in (0, 1). \tag{1.4}$$

We assume that a real function $p \in C^1(0, \infty)$ is given, such that $p(1) = 1$ and that

$$p'(s) < 0 \quad \forall\, s \in (0, +\infty). \tag{1.5}$$

Among other results, Kazhikhov shows in [7] (see also [6]) that for $f \equiv 0$ and for arbitrarily large initial data (u_0, v_0) there is a (unique) global bounded solution (u, v) to the above problem. Moreover, v satisfies the estimate

$$N_0^{-1} \leqq v(t, q) \leqq N_0 \quad \forall\, (t, q) \in Q_\infty, \tag{1.6}$$

where $N_0 \geqq 1$ is a constant which depends only on $p(\cdot)$, μ, m, $\|u_0\|$, and $\|v_0\|_1$. The main point of his proof is the global estimate (1.6). Following his argument, it is not difficult to extend the above result to the case in which there is an external force $f \in L^1(0, \infty; L^\infty(0, 1))$. However, this assumption is still too restrictive since it does not cover many significant cases, including time independent and time periodic external forces.

In the sequel we investigate, in particular, the existence of global solutions satisfying (1.6) in the presence of external forces which do not become small for large values of t. In view of the results proved in [3] § 5, however, we do not expect to get global solutions satisfying (1.6) in the presence of *arbitrary* (even constant) external forces f. In fact, in [3] I gave (for general pressure functions $p(\cdot)$) a complete characterization of the *time independent* external forces f for which a *stationary* solution of problem (1.1)–(1.3) (necessarily unique) exists. For brevity, assume that $p(s) = ks^{-\gamma}$, where k and γ positive constants. Then I showed in particular that a stationary solution exists for all time independent forces $f = \nabla F$ if $F \in L^\infty$ and $\gamma = 1$; and that, under the effect of suitable (even *constant*) external forces f, a vacuum may occur if $\gamma > 1$ though infinite density cannot appear, even for arbitrary $F \in L^\infty$. On other hand, if $\gamma < 1$, then vacuum cannot appear, and infinite density may occur only in the presence of *unbounded* forces f. This last result is proved in section 1 of reference [1].

In view of the above results, it is not surprising that evolutionary solutions of the problem (1.1)–(1.3) in many cases never develop infinite density (see Theorem 7.5, below). In particular, since vacuum and infinite density cannot occur if $0 < \gamma \leqq 1$ (for more general functions $p(\cdot)$, see [3]), I believe that in this case the solution of (1.1)–(1.3) satisfies (1.6) for arbitrarily large initial data and external (bounded) forces. However, the most significant cases correspond to values $\gamma > 1$. In general I expect in this case that there is a positive thereshold r_0 such that if

$$|f(t, x)| < r_0 \quad \forall\, (t, x) \in Q_\infty, \tag{1.7}$$

then solutions (for arbitrarily large initial data) are bounded and satisfy (1.6). In this paper I proved that there is a positive threshold r_0, and real positive decreasing functions $\varrho(r)$, $R(r)$, defined on $(0, r_0)$, satisfying $\lim_{r\to 0}\varrho(r) = \lim_{r\to 0} R(r) = \infty$, and such that the following condition holds: If $|f(t, x)| \leqq r \leqq r_0$, and if $\|u_0\| < \varrho(r)$, $\|(\log v_0)_q\| < \varrho(r)$, then there is a global solution $(u(t), v(t))$ of the problem (1.1)–(1.3) which satisfies (1.6). Moreover $\|u(t)\|$ and $\|(\log v(t))_q\|$ are uniformly bounded by $R(r)$, see Theorem 7.2. Furthermore, there is a positive function $R_1(r)$ $\left(\lim_{r\to 0} R_1(r) = 0\right)$ such that after a finite time $T^* = T^*(\|u_0\|, \|(\log v_0)_q\|, r)$ the solution $(u(t), \log v(t)_q)$ lies in the sphere of radius $R_1(r)$ in the space $L^2 \times L^2$(*). Hence, the "small" sphere of radius $R_1(r)$ is an absorbing set with respect to solutions which initial data that lie inside the "large" sphere of radius $\varrho(r)$, see Theorem 7.3. Specifically, stationary and periodic orbits that intersect the large sphere must lie entirely in the small one. It is worth noting that sets consisting of positive functions subjected to the condition $\int_0^1 v(q)\, dq = 1$ are bounded with respect to the norm $\|(\log v)_q\|$ if and only if they are bounded with respect to the norms $\|v\|_1$, $|v|_\infty$, and $|v^{-1}|_\infty$. In the following, we will use the quantity $\psi^2[u, v] \equiv (4/\mu)\,\|u\|^2 - 2(u, (\log v)_q) + \mu\,\|(\log v)_q\|^2$ instead of $\|u\|^2 + \|(\log v)_q\|^2$. By (2.9), these two quantities are equivalent.

We emphasize that the conclusions above are not results for small data. In fact, to any *arbitrarily large* real number ϱ corresponds a positive number $r = r(\varrho)$ such that any solution (u, v) of problem (1.1)–(1.3) is uniformly bounded and satisfies (1.6) provided its initial data $(u_0, (\log v_0)_q)$ belongs to the sphere of radius ϱ (in $L^2 \times L^2$) and f belongs to the sphere of radius $r = r(\varrho)$ (in $L^\infty(Q_\infty)$). In particular, if $r = 0$, then $\varrho = \infty$ (a result of KAZHIKHOV).

I prove the above results under weak regularity assumptions on the initial data, namely $(u_0, v_0) \in L^2 \times H^1$. In this case, in fact, the solutions describe continuous trajectories in the phase space $L^2 \times H^1$. I further prove *strong* continuous dependence of solutions on the initial data; see Theorem 5.3 and Corollary 5.4. Note that the results proved in this paper can be extended to stronger norms.

Some further results on the above problem are proved in [1], *e.g.* a stationary solution is necessarily stable. In the forthcoming paper [2] attracting properties of flows and periodic solutions are investigated.

I note that barotropic motions for viscous fluids are not totally realistic. Nevertheless, in any mathematical study of the equations describing compressible fluids, the first serious obstacle is the dependence of $p(v, T)$ on the specific volume v. The study of the simplified model (1.1) however makes it easy to understand the difficulties and the main points which arise. In this sense, the present work may be considered a contribution also to the mathematical study of the general thermally dependent case.

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* Because of (2.2), $v(t, q)$ is uniquely determined by $(\log v(t, q))_q$.

Notation. We set $I_T = [0, T]$, $I_\infty = [0, \infty)$, $Q_T = I_T \times (0, 1)$. The terms "local" and "global" are used here with respect to the time variable t. We denote integrals over $(0, 1)$ with respect to the q variable, *i.e.* integrals of the form $\int_0^1 g(q)\, dq$, by $\int g$. For $k \geqq 1$, we set

$$H^k(0, 1) = \{u \in L^2 : D_q u, \ldots, D_q^k u \in L^2\}, \quad \text{where } L^2 = L^2(0, 1).$$

We denote by (,) the scalar product in L^2. For $k \geq 1$, H_0^k is the closure of $\mathcal{D}(0, 1)$ in $H^k(0, 1)$ and $H^{-1}(0, 1)$ is the dual space of $H_0^1(0, 1)$. For convenience, we set $H^k = H^k(0, 1)$, $\mathcal{D} = \mathcal{D}(0, 1)$, and so on. Finally $W^{s,p}$, $s \in \mathbb{R}^+$, $p \in (1, \infty)$ is the usual Sobolev space, while $C^{0,\theta}$, $\theta \in (0, 1]$, denotes the space of Hölder (or Lipschitz)-continuous functions.

We use standard notations and conventions for function spaces consisting of functions defined on I with values in a Banach space X. In particular, we denote by $L^p_{\mathrm{loc}}(I_\infty; X)$ the space of functions that belong to $L^p(I_T; X)$ for every finite $T > 0$. For $g(t, q)$ defined on $I \times (0, 1)$, we denote by $g(t)$ the function $g(t, \cdot)$ of the variable q. Finally $L^1 \cap L^2(I; X) = L^1(I; X) \cap L^2(I; X)$.

The norms in the main functional spaces used here will be written as follows:

$$\begin{aligned} &|\ |_p && \text{norm in } L^p, \quad 1 \leq p \leq \infty, \\ &\|\ \| && \text{norm in } L^2, \\ &\|\ \|_k && \text{norm in } H^k, \quad k \geq 1, \\ &<\ > && \text{norm in } C^{0,1}. \end{aligned}$$

Moreover, for functions depending on both space and time,

$$\begin{aligned} &|\ |_{p;s,I} && \text{norm in } L^s(I; L^p), \\ &\|\ \|_{s,I} && \text{norm in } L^s(I; L^2), \\ &\|\ \|_{k;s,I} && \text{norm in } L^s(I; H^k), \\ &<\ >_{s,I} && \text{norm in } L^s(I; C^{0,1}), \quad 1 \leqq s \leqq \infty. \end{aligned}$$

We drop the symbol s if $s = \infty$. Finally, for convenience we set

$$|||\ |||_\infty = (10/\mu)^{\frac{1}{2}} \|\ \|_{L^\infty(Q_\infty)}.$$

If $f(t, x)$ is the external force field, we define

$$f[v]\,(t, q) = f\left(t, \int_0^q v(t, \xi)\, d\xi\right). \tag{1.8}$$

Hence, f is a function of the Euler coordinates (t, x) and $f[v]$ is a function of the Lagrange coordinates (t, q).

Positive constants that depend at most on the particular function $p(\cdot)$ and on μ are denoted by $c, c_0, c_1, \ldots$. The symbol c may denote different positive constants.

2. Some estimates

Here we establish some estimates for the solution of the problem (*cf.* (1.1))

$$v_t - u_q = 0,$$
$$u_t = \mu(v^{-1}u_q)_q - p(v)_q + g(t, q), \tag{2.1}$$

with boundary conditions and initial conditions given by (1.2), (1.3). We assume that (1.4) holds, and that $p \in C^1(0, \infty)$ satisfies (1.5). For convenience $p(1) = 1$. Here the forcing term g is given function of (t, q).

Some main tools used in this section can be found in papers by KANEL [5] and KAZHIKHOV [7]. We start by noting that $(1.1)_1$, (1.2), (1.4) yield

$$\int_0^1 v(t, q)\, dq = 1, \quad \forall t \geqq 0. \tag{2.2}$$

Define a real function $\pi(\cdot)$ on $(0, \infty)$ by $\pi'(s) = p(s)$, $\pi(1) = 1$, and assume that

$$(u_0, v_0) \in L^2 \times H^1, \quad g \in L^1(I_\infty; L^2). \tag{2.3}$$

Multiplying both sides of $(2.1)_2$ by u, integrating on $(0, 1)$, and taking into account the fact that $p(v)\, u_q = \pi(v)_t$ and $D_t \int v = 0$, one shows that

$$\tfrac{1}{2} \frac{d}{dt} \|u\|^2 + \frac{d}{dt} \int (v - \pi(v)) + \mu \int v^{-1}u_q^2 = \int gu. \tag{2.4}$$

On the other hand, since $v_t = u_q$ one has $v^{-1}u_q = (\log v)_t$. Hence $(v^{-1}u_q)_q = (\log v)_{qt}$. Therefore, multiplying both sides of equation $(2.1)_2$ by $(\log v)_q$ and integrating on (0, 1) easily yields

$$\frac{\mu}{2} \frac{d}{dt} \int (\log v)_q^2 + \int - vp'(v) (\log v)_q^2 = \frac{d}{dt} \int u(\log v)_q + \int v^{-1}u_q^2 - \int g(\log v)_q. \tag{2.5}$$

Finally, multiplying both sides of equation (2.4) by $4/\mu$ and adding the outcome to equation (2.5), one gets

$$\tfrac{1}{2} \frac{d}{dt} [\psi^2(t) + \phi^2(t)] + 3 \int v^{-1}u_q^2 + \int - vp'(v) (\log v)_q^2 = \frac{4}{\mu} \int gu - \int g(\log v)_q, \tag{2.6}$$

where we set $\psi^2(t) = \psi^2[u(t), v(t)]$ and $\phi^2(t) = \phi^2[v(t)]$. By definition

$$\psi^2[u, v] = \frac{4}{\mu} \|u\|^2 - 2(u, (\log v)_q) + \mu \|(\log v)_q\|^2, \tag{2.7}$$

and

$$\phi^2[v] = \frac{8}{\mu} \int [v - \pi(v)]. \tag{2.8}$$

Note that $s - \pi(s) > 0$ if $s \neq 1$. Since

$$\tfrac{1}{2}\left[\frac{4}{\mu}\|u\|^2 + \mu\,\|(\log v)_q\|^2\right] \leqq \psi^2[u, v] \leqq \tfrac{3}{2}\left[\frac{4}{\mu}\,\|u\|^2 + \mu\,\|(\log v)_q\|^2\right], \tag{2.9}$$

it follows that

$$\left|\frac{4}{\mu}(g, u) - (g, (\log v)_q)\right| \leqq \sqrt{10/\mu}\,\|g\|\,\psi(t). \tag{2.10}$$

Hence

$$\tfrac{1}{2}\frac{d}{dt}\,[\psi^2(t) + \phi^2(t)] + 3\int v^{-1}u_q^2 + \int - vp'(v)\,(\log v)_q^2 \leqq \sqrt{10/\mu}\,\|g\|\,\psi(t). \tag{2.11}$$

Therefore

$$[\psi^2(t) + \phi^2(t)]^{\frac{1}{2}} \leqq [\psi^2(0) + \phi^2(0)]^{\frac{1}{2}} + \sqrt{10/\mu}\int_0^t \|g(s)\|\,ds. \tag{2.12}$$

On the other hand, since v satisfies $(1.4)_1$ there is a $q_1 = q_1(t) \in \,]0, 1[$ such that $v(t, q_1(t)) = 1$. Hence,

$$|(\log v(t, q))| \leqq \|(\log v(t))_q\| \leqq (2/\mu)^{\frac{1}{2}}\,\psi(t).$$

Consequently

$$N_0^{-1} \leqq v(t, q) \leqq N_0 \qquad \forall\,(t, q) \in Q_\infty, \tag{2.13}$$

where

$$N_0 \equiv \exp\{(2/\mu)^{\frac{1}{2}}\,[(\psi^2(0) + \phi^2(0))^{\frac{1}{2}} + (10/\mu)^{\frac{1}{2}}\,\|g\|_{1,I_\infty}]\}. \tag{2.14}$$

We use the letter C to denote positive constants which depend, at most, on the given function $p(\cdot)$ and on μ, $m\,\|u_0\|$, $\|v_0\|_1$, $\|g\|_{1,I_\infty}$ (and which are increasing functions of each one of the last four quantities). Then

$$\|v\|_{L^\infty(Q_\infty)} \leqq C, \qquad \|v^{-1}\|_{L^\infty(Q_\infty)} \leqq C, \tag{2.15}$$

and also

$$\|u\|_{\infty,I_\infty} \leqq C, \qquad \|v\|_{1;\infty,I_\infty} \leqq C. \tag{2.16}$$

By returning to equation (2.11), we see readily that

$$\|u\|_{L^2(I_\infty;H_0^1)} \leqq C, \qquad \|v_q\|_{2,I_\infty} \leqq C. \tag{2.17}$$

Equation $(2.1)_1$ shows that

$$\|v_t\|_{2,I_\infty} \leqq C. \tag{2.18}$$

Finally, we show that*

$$\|v\|_{C^{0,\theta}(\overline{Q_\infty})} \leqq C \qquad \forall\,\theta \in (0, \tfrac{1}{6}). \tag{2.19}$$

* Actually, we will exploit only the fact that $\theta > 0$.

In fact, one has $\|v\|_{L^\infty(Q_\infty)} \leqq 1 + \|v_q\|_{\infty,I_\infty}$ since $|v(t,q)| \leqq 1 + \|v_q(t)\|$. Hence it is sufficient to prove that a Hölder condition holds on the sets $Q_1 = [\tau, \tau+1] \times [0,1]$, uniformly with respect to τ, $\tau \geqq 0$. Let $p \in (2,\infty)$, and set $I = [\tau, \tau+1]$. Since $v \in H^1(I; L^2) \hookrightarrow W^{(1/2)+(1/p),p}(I; L^2)$ and since $v \in L^p(I; H^1)$, it readily follows (by interpolation) that

$$v \in W^{\left(\frac{1}{2}+\frac{1}{p}\right)\alpha,p}(I; H^{1-\alpha}) \qquad \forall\, p \in (2,\infty),\ \forall\, \alpha \in [0,1].$$

In particular, $v \in C^{0,(\alpha/2)-(1-\alpha)/p}(I; C^{0,(1/2)-\alpha})$. For $\alpha = \frac{1}{3}$, one gets $v \in C^{0,(1/6)-(2/3p)}(I; C^{0,1/6})$.

We now assume additional regularity of the data. Let

$$(u_0, v_0) \in H_0^1 \times H^1, \quad g \in L^1(I_\infty; L^2) \cap L^2(I_\infty; L^2). \tag{2.20}$$

Multiplying $(2.1)_2$ by u_{qq} and integrating on $(0,1)$, one gets

$$\frac{1}{2}\frac{d}{dt}\|u_q\|^2 + \mu \int v^{-1}u_{qq}^2 = \mu \int v^{-2}v_q u_q u_{qq} + \int p'(v)\, v_q u_{qq} - \int_0^1 g u_{qq}. \tag{2.21}$$

By use of the estimate $|u_q|_\infty \leqq \sqrt{2}\,\|u_q\|^{\frac{1}{2}}\|u_{qq}\|^{\frac{1}{2}}$* it readily follows that the first integral on the right-hand side of (2.21) is bounded by

$$c\,\|v_q\|\,\|u_q\|^{\frac{1}{2}}\,\|u_{qq}\|^{\frac{3}{2}} \leqq C\varepsilon^{-1}\|u_q\|^2 + \varepsilon^{\frac{1}{3}}\|u_{qq}\|^2.$$

Straightforward calculations (see [7]) now yield

$$\|u\|_{L^\infty(I_\infty; H_0^1)} \leqq C_1, \quad \|u\|_{2;2,I_\infty} \leqq C_1, \tag{2.22}$$

and also

$$\begin{aligned} \|v_t\|_{\infty,I_\infty} &\leqq C_1, \quad \|v_t\|_{1;2,I_\infty} \leqq C_1. \\ \|u_t\|_{2,I_\infty} &\leqq C_1, \end{aligned} \tag{2.23}$$

where the constants C_1 depend, in addition, on $\|u_0\|_1$ and on $\|g\|_{2,I_\infty}$.

The above estimates are sufficient to prove that under the hypotheses (1.4), (2.20) there is a unique solution (u,v) of the problem (2.1), (1.2), (1.3) on I_∞; see Theorem 1 in Kazhikhov's paper [7]. Solutions that belong to the above functional spaces will be called *strong solutions*.

3. The strong solution

Here we consider the case in which the external force f is given as a function of the Eulerian coordinates (t,x), *i.e.*, we study the system (1.1). Particular care is necessary in the presence of a discontinuous force. In fact, the usual proofs of existence of a solution are based on fixed point theorems which require continuity (with respect to a suitable topology) of the map $v \to f[v]$. I will use

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* Note that there is a $q_1 \in [0,1]$ such that $u_q(q_1) = 0$.

as point of departure the estimates and the existence result of section 2. We start with the following auxiliary result.

Lemma 3.1. *Let* $T \in (0, \infty)$ *be fixed and let* $\{v_n\}$, $n \in N$, *be a sequence of real continuous functions defined on* $\overline{Q_T}$, *which satisfy* (for some $N > 0$) *the estimate* $N^{-1} \leqq v_n(t, q) \leqq N$ *and the constraint* (2.2). *Assume that* $v_n \to v$ *uniformly on* $\overline{Q_T}$. *Let* $f \in L^1(I_T; L^1)$. *Then* $f[v_n]$ *converges to* $f[v]$ *weakly in measure and hence in* $\mathscr{D}'(Q_T)$.

Proof. For convenience, put $I = I_T$. Also for each fixed $t \in I$, set $x(q) = \int_0^q v(t, \xi)\, d\xi$. One has $N^{-1} \leqq dx/dq = v(t, q) \leqq N$. For each fixed t the map $q \to x(q)$ is a diffeomorphism of [0, 1] onto itself. The inverse map is given by $q(x) = \int_0^x [1/v(t, q(t, \eta))]\, d\eta$, and of course $dq/dx = 1/v(t, q(t, x))$. Similarly, for each fixed $t \in I$, we define diffeomorphisms $x_n \to q_n$ by setting

$$x_n(q_n) = \int_0^{q_n} v_n(t, \xi)\, d\xi, \qquad q_n(x_n) = \int_0^{x_n} [1/v_n(q_n(t, \eta))]\, d\eta .$$

Let us prove that

$$\lim_{n \to +\infty} q_n(t, y) = q(t, y), \qquad \text{uniformly on } \overline{Q_T}. \tag{3.1}$$

For convenience, we drop t from the notation. For each $y \in [0, 1]$ one has

$$y = \int_0^{q_n(y)} v_n(\xi)\, d\xi = \int_0^{q(y)} v(\xi)\, d\xi = \int_0^{q_n(y)} v(\xi)\, d\xi + \int_{q_n(y)}^{q(y)} v(\xi)\, d\xi .$$

Hence

$$N^{-1}\, |q_n(y) - q(y)| \leqq \int_0^1 |v_n(\xi) - v(\xi)|\, d\xi$$

and in particular

$$\|q_n - q\|_{C(\overline{Q_T})} \leqq N\, \|v_n - v\|_{C(\overline{Q_T})} .$$

This proves (3.1). It readily follows that

$$\lim_{n \to +\infty} v_n(t, q_n(t, y)) = v(t, q(t, y)), \tag{3.2}$$

uniformly on Q_T. Let now $\phi \in C(\overline{Q_T})$. By using the change of variables $q = q_n(t, y)$, one proves that

$$\int_0^1 f[v_n]\,(t, q)\, \phi(t, q)\, dq = \int_0^1 f(t, x)\, \phi(t, q_n(t, y))\, [v_n(t, q_n(t, y))]^{-1}\, dy .$$

By integrating both sides of the above equation on I, passing to the limit as $n \to \infty$, and taking into account (3.1) and (3.2), it readily follows that

$$\lim_{n \to +\infty} \int_{Q_T} f[v_n]\,(t, q)\, \phi(t, q)\, dq\, dt = \int_{Q_T} f(t, y)\, \phi(t, q(t, y))\, [v(t, q(t, y))]^{-1}\, dy\, dt .$$

Finally, by using the change of coordinates $y = x(t, q)$ one proves that the last integral is equal to $\int_{Q_T} f[v](t, q)\,\phi(t, q)\,dq\,dt$. □

Theorem 3.2. *Let* $(u_0, v_0) \in H_0^1 \times H^1$, $f \in L^1 \cap L^2(I_\infty; L^\infty)$, *and assume that* (1.4) *holds. Then there is a strong solution* (u, v) *of problem* (1.1)–(1.3) *satisfying the estimates* (2.13)–(2.19), (2.22), (2.23)*.

Proof. Since $\|g(t)\| \leq |g(t)|_\infty$, the statements and estimates of section 2 are still satisfied if $L^1(I_\infty; L^2)$ and $L^2(I_\infty; L^2)$ are replaced by $L^1(I_\infty; L^\infty)$ and $L^2(I_\infty; L^\infty)$. Denote by N_0 the constant defined by the equation (2.14) with $\|g\|_{1,I_\infty}$ replaced by $|f|_{\infty;1,I_\infty}$. In order to apply compactness arguments, we start by considering solutions on I_T, where $T < +\infty$ is arbitrarily large. For convenience, we set $I_T = I$. Define

$$\mathbb{K} = \{w \in C(\overline{Q_T}) : w(0) = v_0,\ (2.2) \text{ holds on } I, \text{ and } N_0^{-1} \leq w(t, q) \leq N_0 \text{ on } Q_T\}. \tag{3.3}$$

$\mathbb{K}$ is a closed, convex, bounded subset of $C(\overline{Q_T})$. Since $|f(t)|_\infty = |f[w](t)|_\infty$ (the L^∞-norm on the left-hand side concerns the x variable, and the one on the right-hand side concerns the q variable) it follows that $|f[w]|_{\infty;1,I} = |f|_{\infty;1,I}$ and that $|f[w]|_{\infty;2,I} = |f|_{\infty;2,I}$ for each $w \in \mathbb{K}$. In particular, for each $w \in \mathbb{K}$ the problem

$$\begin{aligned} v_t - u_q &= 0, \\ u_t &= \mu(v^{-1}u_q)_q - p(v)_q + f[w](t, q), \end{aligned} \tag{3.4}$$

with conditions (1.2), (1.3), has a unique strong solution $(u, v) = Sw$. This solution satisfies the estimates of section 2 with $\|g\|_{1,I}$ and $\|g\|_{2,I}$ replaced by $|f|_{\infty;1,I}$ and $|f|_{\infty;2,I}$, respectively. In particular $v \in \mathbb{K}$. Therefore $S'(\mathbb{K}) \subset \mathbb{K}$, where the map S' is defined by setting $S'w = v$, $\forall\, w \in \mathbb{K}$. If $S'v = v$, then $(u, v) = Sv$ is a strong solution of the problem (1.1)–(1.3). Hence the proof of Theorem 3.2 will be complete if we show that S' has a fixed point in $\mathbb{K}$. Let us show that S' is a completely continuous map with respect to the $C(\overline{Q_T})$ topology. Assume that $w, w_n \in \mathbb{K}$ and that $w_n \to w$ in $C(\overline{Q_T})$ as $n \to +\infty$. Denote by $(3.4)_n$ the system obtained by replacing w by w_n in the equation (3.4) and let the solution of $(3.4)_n$ be written $(u_n, v_n) = Sw_n$. By Lemma 3.1 it follows that $f[w_n] \to f[w]$ in the distributional sense. A compactness argument shows that $f[w_n] \to f[w]$ weakly in $L^2(I; L^2)$. On the other hand, the solutions (u_n, v_n) of the system $(3.4)_n$ satisfy the estimates proved in Section 2, uniformly with respect to n. Compactness arguments show that we can pick subsequences that converge with respect to suitable topologies. These topologies are strong enough to prove, by passing to the limit on the equations $(3.4)_n$, that the limits of the

* In equation (2.14) and in the definition of the constants C and C_1 the norms $\|g\|_{1,I_\infty}$ and $\|g\|_{2,I_\infty}$ should be replaced by $|f|_{\infty;1,I_\infty}$ and $\|f\|_{\infty,I_\infty}$, respectively.

above subsequences are solutions of (3.4) and satisfy (1.2), (1.3). Since the solution (u, v) of this problem is unique, the entire sequence (u_n, v_n) converges to (u, v). In particular $v_n = S'w_n$ converges uniformly on Q_T to $v = S'w$, as follows from the uniform estimate (2.19). Finally, $S'(\mathbb{K})$ is relatively compact since it is bounded in $C^{0,\theta}(\overline{Q_T})$.

Now we extend the above result to I_∞. If $f(t, x)$ is regular (see Section 5) the solution of the problem (1.1)–(1.3) is unique. In particular, since the solution on Q_{T_2} overlaps that on Q_{T_1} $(T_1 < T_2)$, the result holds on I_∞. In the general case, we consider the sequence of solutions (u_n, v_n) which were constructed above for the problem (1.1)–(1.3) on $I_n = [0, n]$. Since $|f[v_n](t)|_\infty = |f(t)|_\infty$, the solutions (u_n, v_n) satisfy the estimates of Section 2 uniformly with respect to n. Hence for each fixed n_0 there are subsequences that converge on I_{n_0} to a solution of (1.1)–(1.3). A diagonalization procedure yields a solution on I_∞. □

Note that $u \in C(I_\infty; H_0^1)$, that $v \in C^{0,1/2}(I_\infty; H^1)$, and that the corresponding norms are bounded by constants of type C_1.

Remark 3.3. If $f \in L^2(I_\infty; L^2)$, an argument similar to the one used in the above proof shows that there is a local strong solution of the problem (1.1)–(1.3).

Sketch of the proof. Consider the set

$$\mathbb{K}_h = \{w \in C(\overline{Q_h}) : w(0) = w_0, \text{(2.2) holds on } I_h, \text{ and } (2m)^{-1} \leq w(t, q) \leq 2m \text{ on } Q_h\}.$$

One has

$$(2m)^{-1} \|f(t)\|^2 \leq \|f[w](t)\|^2 \leq 2m \|f(t)\|^2 .$$

Hence $f[w]$ is uniformly bounded in $L^2(I_n; L^2)$ as w varies over $\mathbb{K}$. As above, the solution $(u, v) = Sw$ of the problem (3.4), (1.2), (1.3) on I_h satisfies the estimates of Section 2, uniformly with respect to $w \in \mathbb{K}_h$. In particular, the $C^{0,\theta}(\overline{Q_T})$ norm of $v = S'w$ is uniformly bounded. Hence, if h is sufficiently small, one has $2m^{-1} \leq v(t, q) \leq 2m$ on Q_h. Therefore $S'(\mathbb{K}_h) \subset \mathbb{K}_h$. Arguing as in the proof of Theorem 3.2, we show the existence of a solution of (1.1)–(1.3) on I_h.

4. Existence of weak solutions

We say that (u, v) is a *weak solution* on I_T of problem (1.1), (1.2) if the following three conditions hold:

$$u \in L^2(I_T; H_0^1), \quad v \in L^\infty(I_T; H^1); \tag{4.1}$$

there is a constant N such that $N^{-1} \leq v(t, q) \leq N$ on Q_T; and (u, v) is a solution of (1.1) in the sense of distributions. A weak solution of problem (1.1)–(1.3) is, by definition, a weak solution of (1.1), (1.2) such that $\lim_{t \to 0} u(t) = u_0$ and $\lim_{t \to 0} v(t) = v_0$ (in the sense of distributions).

Theorem 4.1. *Let* $(u_0, v_0) \in L^2 \times H^1$ *and* $f \in L^1 \cap L^2(I_\infty; L^\infty)$, *and assume that* (1.4) *holds. Then there is a weak solution* (u, v) *of problem* (1.1)–(1.3) *on* I_∞. *This solution satisfies the estimates* (2.13)–(2.19)*.

Proof. Let $\{u_0^{(n)}\}$, $n \in N$, be a sequence in H_0^1 that converges in L^2 to u_0. Denote by (u_n, v_n) the strong solution of the equations $(1.1)_n$–$(1.3)_n$, these equations being obtained by replacing u_0 by $u_0^{(n)}$ in the equations (1.1)–(1.3). By Theorem 3.2, the solutions (u_n, v_n) satisfy the estimates (2.13)–(2.19) uniformly with respect to n. Actually, the constants C depend on $\|u_0^{(n)}\|$, but $\|u_0^{(n)}\| \to \|u_0\|$ as $n \to \infty$ (obvious details are left to the reader). In particular, there is a subsequence u_n, v_n such that $u_n \to u$ weakly in $L^2(I_\infty; H_0^1)$ and *-weak in $L^\infty(I_\infty; L^2)$; $(u_n)_t \to u_t$ weakly in $L^2(I_\infty; H^{-1})$**; $(v_n)_t \to v_t$ weakly in $L^2(I_\infty; L^2)$; and $v_n \to v$ uniformly on Q_T for each finite T. The lower semicontinuity of the norms shows that (u, v) satisfies (2.13)–(2.19). Moreover, by passing to the limit in the equations $(1.1)_n$–$(1.3)_n$ it readly follows that u, v is a solution of (1.1)–(1.3). The convergence of $f[v_n]$ to $f[v]$ follows from Lemma 3.1.

Remark 4.2. If $f \in L^2(I_\infty; L^2)$ there is a *local* weak solution of the problem (1.1)–(1.3) in the interval I_h. This is proved as above, by using Remark 3.3 instead of Theorem 3.2. Note that the "smallness" of h is determined by m and by the $C^{0,\theta}(Q_T)$-norm of v. Since this last quantity depends on $\|u_0\|$ but not on $\|u_0\|_1$, it readily follows that h is independent of n.

5. Regularity and well-posedness for weak solutions

We first show that for all initial data $(u_0, v_0) \in L^2 \times H^1$ the solution $(u(t), v(t))$, $t \in I_\infty$, describes a continuous trajectory in the space $L^2 \times H^1$. We further prove that if the initial data (u_0', v_0') converge to (u_0, v_0) in $L^2 \times H^1$, then the solution $(u'(t), v'(t))$ converges tothe solution $(u(t), v(t))$ in $L^2 \times H^1$. On compact intervals I_T the convergence is uniform. There is no loss of generality in considering an arbitrary bounded interval I_T instead of I_∞.

Theorem 5.1. *Let* $f \in L^2(I; L^2)$ *and let* (u, v) *be a weak solution of the problem* (1.1), (1.2). *Then*

$$(u, v) \in C(I; L^2 \times H^1), \quad (u_t, v_t) \in L^2(I; H^{-1} \times L^2)$$
$$v \in C^{0,\theta}(Q_T), \quad \forall\, \theta \in (0, 1/6). \tag{5.1}$$

In particular, if (u, v) *is a weak solution of* (1.1)–(1.3), *then*

$$(u(t), v(t)) \to (u_0, v_0) \quad \text{in } L^2 \times H^1 \text{ as } t \to 0.$$

* In equation (2.14) and in the definition of the constants C, the norm $\|g\|_{1,I_\infty}$ should be replaced by $|f|_{\infty;1,I_\infty}$.

** Equation $(2.1)_2$ shows that $\|u_t\|_{L^2(I_\infty; H^{-1})} \leq C$.

Proof. The assertion for v_t follows from $(4.1)_1$, since $v_t = u_q$. On the other hand, for almost all $t \in I$ the maps $q \to x = \int_0^q v(t, \xi)\, d\xi$ are diffeomorphisms since $H^1 \hookrightarrow C^{0,\frac{1}{2}}$ and $N^{-1} \leqq dx/dq \leqq N$. Therefore $f[v] \in L^2(I; L^2)$. It readily follows from equation $(1.1)_2$ that $u_t \in L^2(I; H^{-1})$. This result, together with $(4.1)_1$, shows that $u \in C(I; L^2)$; [9], Chapter 1, Theorem 3.1. The Hölder continuity of v follows as in the proof of (2.19).

Setting*

$$w = \mu(\log v)_q - u, \tag{5.2}$$

one has $w \in L^\infty(I; L^2)$. Since $(v^{-1}u_q)_q = (\log v)_{qt}$, equation (1.1) yields $w_t = -p'(v)\, v_q + f[v]$, in the sense of distributions.

Hence $w_t \in L^2(I; L^2)$ and so $w \in C(I; L^2)$. Therefore $(\log v)_q \in C(I; L^2)$. It readily follows that $v_q = v\,(\log v)_q \in C(I; L^2)$. □

Lemma 5.2. *Let $f \in L^2(I; L^2)$ and let (u, v) and (u', v') be weak solutions of (1.1), (1.2) on I such that $N^{-1} \leqq v(t, q) \leqq N$ and $N^{-1} \leqq v'(t, q) \leqq N$. Then, for each $t \in I$, one has*

$$|v(t) - v'(t)|_\infty \leqq N \|(\log v)_q - (\log v')_q\| \leqq \mu^{-1} N(\|u - u'\| + \|w - w'\|), \tag{5.3}$$

where w is given by (5.2) and $w' = \mu(\log v')_q - u'$.

Proof. Let t be fixed and then (for convenience) dropped from the notation. Since $v(q)$ and $v'(q)$ satisfy (2 2) and are continuous on [0,1], there is a constant $q_1 \in (0, 1)$ such that $v(q_1) = v'(q_1)$. Hence

$$\log v(q) - \log v'(q) = \int_{q_1}^q (\log v - \log v')_q\, d\xi. \tag{5.4}$$

On the other hand,

$$|\log v(q) - \log v'(q)| \geqq N^{-1}\, |v(q) - v'(q)| \qquad \forall\, q \in [0, 1]. \tag{5.5}$$

Therefore,

$$|v - v'|_\infty \leqq N\, |\log v - \log v'|_\infty \qquad \forall\, t \in I.$$

By using (5.4), one then shows that (5.3) holds. □

Theorem 5.3. *Assume that $p'(\cdot)$ is locally Lipschitz-continuous on $(0, +\infty)$ and that*

$$f \in L^2(I; L^2) \cap L^1(I; C^{0,1}). \tag{5.6}$$

Let (u, v) and (u', v') be two weak solutions of (1.1), (1.2) on I. Then the function $\|u(t) - u'(t)\|^2 + \|w(t) - w'(t)\|^2$ is absolutely continuous on I. Moreover

$$\frac{d}{dt}(\|u - u'\|^2 + \|w - w'\|^2) + k_0\, \|u_q - u'_q\|^2 \leqq \lambda(t)\, (\|u - u'\|^2 + \|w - w'\|^2), \tag{5.7}$$

* The same device is used in Shelukhin's paper [11].

where

$$\lambda(t) = k_1(1 + \langle f(t)\rangle + \|u(t)\| + \|w(t)\| + \|u_q\|^2).$$

Here k_0 and k_1 are positive constants which depend only on μ, $p(\cdot)$ and N. In particular, if (u_0, v_0) and (u_0', v_0') belong to $L^2 \times H^1$ and satisfy the assumption (1.4), *and if (u, v) and (u', v') are weak solutions of problem* (1.1)–(1.3) *with initial data (u_0, v_0) and (u_0', v_0'), then*

$$\|u(t) - u'(t)\|^2 + \|w(t) - w'(t)\|^2 + k_0 \int_0^t \|u_q(t) - u_q'(t)\|^2 \, dt$$
$$\leqq (\|u_0 - u_0'\|^2 + \|w_0 - w_0'\|^2) \exp\left[\int_0^t \lambda(s)\, ds\right], \tag{5.8}$$

where $w_0 = \mu(\log v_0)_q - u_0$ *and* $w_0' = \mu(\log v_0')_q - u_0'$.

Proof. We denote by $\langle , \rangle$ the duality pairing between H^{-1} and H_0^1. Equation $(1.1)_2$ is an equation in $L^2(I; H^{-1})$. Forming the difference of this last equation and the similar equation satisfied by the couple u', v', and "multiplying" by an arbitrary element $\phi \in L^2(I; H_0^1)$, one gets

$$\langle (u' - u)_t, \phi\rangle + \mu \int v'^{-1}(u' - u)_q \phi_q$$
$$= \mu \int [(v' - v)/vv'] \, u_q \phi_q + \int [p(v') - p(v)] \, \phi_q + \int (f[v'] - f[v]) \, \phi. \tag{5.9}$$

By setting $\phi = u' - u$ in the equation (5.9), one gets

$$\tfrac{1}{2} \frac{d}{dt} \|u' - u\|^2 + \mu N^{-1} \|u_q' - u_q\|^2 \leqq \mu N^2 \, |v' - v|_\infty \, \|u\|_q \, \|u_q' - u_q\|$$
$$+ \left(\max_{N^{-1} \leqq s \leqq N} |p'(s)|\right) |v' - v|_\infty \|u_q' - u_q\| + \|f[v'] - f[v]\| \, \|u' - u\|.$$

Recall that $\|w(t)\|^2$ is absolutely continuous on I and $d\|w(t)\|^2/dt = \langle w'(t), w(t)\rangle$ a.e. on I, if $w \in L^2(I; H_0^1)$ and $w' \in L^2(I; H^{-1})$.

Also, using the inequality $|f[v'] - f[v]|_\infty \leqq \langle f(t)\rangle \, |v' - v|_\infty$, taking (5.3) into account, and using the Cauchy-Schwarz inequality, we obtain

$$\tfrac{1}{2} \frac{d}{dt} \|u' - u\|^2 + k \|u_q' - u_q\|^2 \leqq k'(1 + \langle f(t)\rangle + \|u_q\|^2)\,(\|u' - u\|^2 + \|w' - w\|^2) \tag{5.10}$$

a.e. on I_T, where k and k' are positive constants depending only on μ, $p(\cdot)$ and N.

On the other hand,

$$w_t = \mu^{-1} v p'(v)\,(w + u) - f[v] \quad \text{and} \quad w'(t) = \mu^{-1} v' p'(v')\,(w' + u') - f[v']. \tag{5.11}$$

It readily follows that, for every $\psi \in L^2(I; L^2)$,

$$\int (w' - w)_t \, \psi = \mu^{-1} \int v'p'(v') \, [(u' - u) + (w' - w)] \, \psi$$
$$+ \mu^{-1} \int [v'p'(v') - vp'(v)] \, (u + w) \, \psi + \int (f[v] - f[v']) \, \psi,$$

a.e. on I. By setting $\psi = w' - w$ and using (5.3) one proves that

$$\tfrac{1}{2}\frac{d}{dt}\|w' - w\|^2 \leq k(1 + \|u + w\| + \langle f(t)\rangle)(\|u' - u\|^2 + \|w' - w\|^2). \quad (5.12)$$

By adding the equations (5.10) and (5.11), one gets (5.7). Equation (5.8) then follows by using standard comparison theorems in ordinary differential equations. □

Corollary 5.4. *Under the assumptions of Theorem 5.3, a weak solution of problem* (1.1) (1.3) *is unique. Moreover, the solution depends continuously and strongly on the initial data; that is, if* (u_0', v_0') *converges to* (u_0, v_0) *in* $L^2 \times H^1$*, then* $(u'(t), v'(t))$ *converges to* $(u(t), v(t))$ *in* $C(I; L^2 \times H^1)$ *and* u' *converges to* u *in* $L^2(I; H_0^1)$.

The proof is a consequence of equation (5.8). The convergence of v' to v in $C(\overline{Q_T})$ follows from (5.3).

Theorems 4.1 and 5.1, together with Corollary 5.4, give the following result.

Theorem 5.5. *Let* $p'(\cdot)$ *be locally Lipschitz-continuous and assume that* $(u_0, v_0) \in L^2 \times H^1$, *that* v_0 *satisfies* (1.4), *and that* $f \in L^1 \cap L^2(I_\infty; L^\infty) \cap L^2_{\text{loc}}(I_\infty; C^{0,1})$. *Then there is a unique global weak solution of problem* (1.1)–(1.3). *This solution belongs* (*for* $I = I_\infty$) *to the function spaces indicated in* (4.1) *and* (5.1) *and satisfies the estimates* (2.13)–(2.19).* *If* $(u_0', v_0') \in L^2 \times H^1$, *if* v_0' *satisfies* (1.4), *and if* (u_0', v_0') *converges to* (u_0, v_0) *in* $L^2 \times H^1$, *then the global solution* (u', v') *of problem* (1.1)–(1.3) *with initial data* (u_0', v_0') *converges to* (u, v) *in* $C([0, T]; L^2 \times H^1)$ *for every* $T > 0$. *Moreover,* u' *converges to* u *in* $L^2(0, T; H_0^1)$ *for every* $T > 0$.

6. Preliminaries for Section 7

Here we introduce some real functions which will be used in the next section. The function $p(\cdot)$ is assumed to be of class C^1 and satisfy (1.5). Define real nonnegative functions $\alpha : [1, \infty) \to (0, \infty)$ and $\beta : [1, \infty) \to (0, \infty)$ by

$$\alpha(N) = \min_{N^{-1} \leq s \leq N} \{-sp'(s)\} \quad \forall\, N \in [1, \infty), \quad (6.1)$$

$$\beta(N) = c \min\{N^{-1}, \alpha(N)\} \quad \forall\, N \in [1, \infty). \quad (6.2)$$

The above functions are strictly positive, continuous, and decreasing.

Also define $F: [0, \infty) \to [0, \infty)$ by

$$F(y) = y\beta(\exp(c_0 y)) \quad \forall\, y \in [0, \infty). \quad (6.3)$$

Clearly F is a continuous, nonnegative function with $F(0) = 0$, $F(y) > 0$ if $y > 0$, and $F(\infty) = \lim_{y \to \infty} F(y) = 0$. We denote by F_0 the maximum of F on

* In equation (2.14) and in the definition of the constants C the norm $\|g\|_{1,I_\infty}$ should be replaced by $|f|_{\infty;1,I_\infty}$.

$[0, \infty]$; Clearly $F_0 > 0$. Set

$$y_0 = \min \{y \in (0, \infty) : F(y) = F_0\},$$

and define a function $R: (0, F_0] \to [y_0, \infty)$ by

$$R(r) = \min \{y \in [y_0, \infty) : F(y) = r\} \quad \forall\, r \in (0, F_0). \tag{6.4}$$

R is a lower semicontinuous, strictly decreasing function with $\lim_{r \to 0} R(r) = \infty$. Note that $F([y_0, R(r)) \subset (r, F_0]$.

Define (see [5], [7]) a real function π on $(0, \infty)$ as $\pi(s) = 1 + \int_1^s p(t)\, dt$ and define also a nonnegative continuous function M by setting

$$M^2(N) = \max_{N^{-1} \leqq s \leqq N} (s - \pi(s)) \quad \forall\, N \in [1, \infty). \tag{6.5}$$

The properties of $\pi(s)$ show that $M(1) = 0$ and that M is strictly increasing. Also set

$$G^2(y) = y^2 + c_1 M^2(\exp(c_0 y)) \quad \forall\, y \in [0, \infty); \tag{6.6}$$

Clearly $G(0) = 0$, G is a strictly increasing, continuous function, and $G(\infty) = \infty$. Denote by G^{-1} the inverse function of G, and define

$$\varrho(r) = G^{-1}(R(r)) \quad \forall\, r \in (0, F_0]. \tag{6.7}$$

The function ϱ is strictly decreasing, and

$$\lim_{r \to 0} \varrho(r) = \infty.$$

Since $R^2(r) = G^2(\varrho(r))$ it follows that $\varrho(r) < R(r)$. Put

$$r_0 = \sup \{r \in (0, F_0] : R(r) \geqq G(y_0)\} \tag{6.8}$$

(the set used here is nonempty since $\lim_{r \to 0} R(r) = \infty$). If R is continuous, then r_0 is the unique solution of the equation $R(r_0) = G(y_0)$ since $R(F_0) = y_0 < G(y_0) < R(0) = \infty$.

In the following we will restrict the domain of the function $\varrho(r)$ to the interval $(0, r_0)$. Note that

$$\varrho(r) > r_0 \quad \forall\, r \in (0, r_0), \tag{6.9}$$

since $G(\varrho(r)) = R(r) > G(y_0)$ and G is strictly increasing.

The constants F_0, y_0, r_0 and the functions $\alpha, \beta, F, R, M, G, \varrho$, depend only on c_0, c_1, c_2 and on the particular function $p(\cdot)$. In turn, in the next section c_0, c_1, c_2 will depend only on μ and on $p(\cdot)$.

7. Global properties

In this section we prove some global estimates for the weak solution (u, v) of problem (1.1)–(1.3) which was constructed by a limit process in Section 4. Since the regularity of strong solutions is sufficient to justify the calculations made in this section, by passing to the limit one verifies that our estimates hold also for the weak solution (u, v). In order to simplify the exposition we show the calculations as if the weak solution (u, v) were itself a strong solution. In the following we assume that

$$(u_0, v_0) \in L^2 \times H^1, \quad f \in L_\infty(I; L^\infty), \tag{7.1}$$

and that (1.4) holds. Before going further, the reader should recall the definitions (2.7), (2.8) and the estimates (2.9), (2.11). The estimate (2.11) yields

$$\tfrac{1}{2} \frac{d}{dt} [\psi^2(t) + \phi^2(t)] + 3 \int v^{-1} u_q^2 + \int -v p'(v) (\log v)_q^2 \leq |||f|||_\infty \psi(t), \tag{7.2}$$

since $\|f[v]\| \leq |f[v]|_\infty = |f|_\infty$. Set

$$N(t) \equiv \exp \|(\log v(t))_q\|. \tag{7.3}$$

By arguing as in the proof of (2.13), it readily follows that

$$N^{-1}(t) \leq v(t, q) \leq N(t) \qquad \forall\, q \in [0, 1]. \tag{7.4}$$

Since $\|u\| \leq \|u_q\|$, one has (for each fixed t)

$$3 \int v^{-1} u_q^2 + \int -v p'(v) (\log v)_q^2 \geq \psi^2(t) \min \left\{ \frac{\mu}{2} N^{-1}, \frac{2}{3\mu} \alpha(N) \right\},$$

where α is given by the equation (6.1). Define β by setting $c = \min \{\mu/2, 2/(3\mu)\}$ in (6.2). It follows that

$$\tfrac{1}{2} \frac{d}{dt} [\psi^2(t) + \phi^2(t)] + \beta(N(t)) \psi^2(t) \leq |||f|||_\infty \psi(t).$$

Since β is a decreasing function and

$$N(t) \leq \exp [(2/\mu)^{\frac{1}{2}} \psi(t)], \tag{7.5}$$

we obtain

$$\tfrac{1}{2} \frac{d}{dt} [\psi^2(t) + \phi^2(t)] + F(\psi(t)) \psi(t) \leq |||f|||_\infty \psi(t), \tag{7.6}$$

where F is defined by setting $c_0 = (2/\mu)^{\frac{1}{2}}$ in equation (6.3).

Lemma 7.1. *Let M and G be defined by equations* (6.5) *and* (6.6), *respectively, with c_0 as above and $c_1 = 8/\mu$. Then*

$$\psi^2(t) + \phi^2(t) \leq G^2(\psi(t)). \tag{7.7}$$

Proof. It follows from (2.8) that

$$\phi^2(t) \leq (8/\mu) \max_{0 \leq q \leq 1} [v(t, q) - \pi(v(t, q))].$$

Since

$$\exp [-(2/\mu)^{\frac{1}{2}} \psi(t)] \leq v(t, q) \leq \exp [(2/\mu)^{\frac{1}{2}} \psi(t)],$$

it follows that

$$\phi^2(t) \leqq (8/\mu)\, M^2(\exp\,[(2/\mu)^{\frac{1}{2}}\,\psi(t)]), \tag{7.8}$$

which yields (7.7). □

Theorem 7.2. *Let* u_0, v_0, *and* f *be as in* (7.1), *and let* v_0 *satisfy* (1.4). *Assume that*

$$|||f|||_\infty < r, \tag{7.9}$$

and that

$$\psi^2(0) + \phi^2(0) < R^2(r) \tag{7.10}$$

for some $r \in (0, r_0)$. *Then*

$$\psi^2(t) + \phi^2(t) < R^2(r) \qquad \forall\, t \in [0, \infty). \tag{7.11}$$

Moreover, the condition (7.10) *is satisfied if*

$$\psi^2(0) < \varrho^2(r). \tag{7.12}$$

Proof. Since G is strictly increasing and $G(\varrho(r)) = R(r)$, Lemma 7.1 shows that if $\psi^2(t) < \varrho^2(r)$ then

$$\psi^2(t) + \phi^2(t) < R^2(r). \tag{7.13}$$

This proves the last assertion of the theorem. Let us prove the first assertion. Assume that (7.11) is false, and let t_0 be the smallest nonnegative real number for which $\psi^2(t_0) + \phi^2(t_0) = R^2(r)$. Since $y_0 < \varrho(r)$, one gets

$$y_0^2 < \psi^2(t_0) \leqq \psi^2(t_0) + \phi^2(t_0) = R^2(r).$$

By the definition of t_0, it follows that there is an $\varepsilon > 0$ such that

$$y_0^2 < \psi^2(t) \leqq \psi^2(t) + \phi^2(t) < R^2(r) \qquad \forall\, t \in (t_0 - \varepsilon, t_0).$$

In particular, $\psi(t) \in (y_0, R(r))$. Since $F([y_0, R(r)) \subset (r, F_0]$, one has

$$F(\psi(t)) > r \qquad \forall\, t \in (t_0 - \varepsilon, t_0). \tag{7.14}$$

On the other hand, (7.6) and (7.9) yield

$$\tfrac{1}{2}\frac{d}{dt}[\psi^2(t) + \phi^2(t)] \leqq -[F(\psi(t)) - r]\,\psi(t). \tag{7.15}$$

It readily follows that $\psi^2(t) + \phi^2(t) > R^2(r)$ $\forall\, t \in (t_0 - \varepsilon, t_0)$. Hence there is a $\tau_0 \in (0, t_0)$ for which $\psi^2(\tau_0) + \phi^2(\tau_0) = R^2(r)$. This contradicts the definition of t_0. □

For each $r \in (0, r_0)$ we put

$$\varrho_1(r) = \max\,\{y \in (0, y_0) : F(y) = r\} \tag{7.16}$$

and

$$R_1(r) = G(\varrho_1(r)). \tag{7.17}$$

The functions ϱ_1 and R_1 are strictly increasing. Moreover, $\varrho_1(r) < y_0$, $0 < \varrho_1(r) < R_1(r)$, and $\lim_{r\to 0} \varrho_1(r) = \lim_{r\to 0} R_1(r) = 0$.

Theorem 7.3. *Let u_0, v_0 and f be as in (7.1), and let v_0 satisfy $(1.4)_1$. Assume that*

$$|||f|||_\infty < r \tag{7.18}$$

and that

$$\psi^2(0) + \phi^2(0) < R_1^2(r), \tag{7.19}$$

for some $r \in (0, r_0)$. *Then*

$$\psi^2(t) + \phi^2(t) < R_1^2(r) \qquad \forall\, t \in [0, \infty). \tag{7.20}$$

Moreover, the assumption (7.19) is satisfied if

$$\psi^2(0) < \varrho_1^2(r). \tag{7.21}$$

Proof. As in the proof of Theorem 7.2,

$$\psi^2(t) < \varrho_1^2(r) \Rightarrow \psi^2(t) + \phi^2(t) < R_1^2(r). \tag{7.22}$$

This proves the last assertion of Theorem 7.3. Let us prove the first assertion. From (7.6) it follows that

$$\tfrac{1}{2}\frac{d}{dt}[\psi^2(t) + \phi^2(t)] \leqq -(F(\psi(t)) - |||f|||_\infty)\,\psi(t). \tag{7.23}$$

Assume that (7.20) is false, and denote by t_0 the smallest nonnegative real number for which $\psi^2(t_0) + \phi^2(t_0) = R_1^2(r)$. From (7.22) it follows that $\psi(t_0) \geqq \varrho_1(r)$. On the other hand (since $\varrho_1(r) < y_0 < \varrho(r)$ and since G is increasing) one has $R_1(r) < R(r)$. Therefore, $\psi(t_0) \in [\varrho_1(r), R(r)]$. The definitions of $\varrho_1(r)$ and $R(r)$ show that $F([\varrho_1(r), R(r)]) \subset [r, F_0]$. Consequently $F(\psi(t_0)) \geqq r$, and so $(F(\psi(t_0)) - |||f|||_\infty)\,\psi(t_0) > 0$. The left-hand side of the equation (7.23) is therefore negative in a neighborhood of t_0. This contradicts the definition of t_0. □

Note that

$$\tfrac{1}{2}\frac{d}{dt}[\psi^2(t) + \phi^2(t)] \leqq -(r - |||f|||_\infty)\,\varrho_1(r) \tag{7.24}$$

whenever $\psi(t) \in [\varrho_1(r), R(r)]$. Set

$$T^* = \frac{\psi^2(0) + \phi^2(0) - R_1^2(r)}{2(r - |||f|||_\infty)\,\varrho_1(r)} \tag{7.25}$$

Then one has the following result.

Theorem 7.4. *Assume that the hypotheses of Theorem 7.2 are satisfied. Then the inequalities (7.23) and (7.26) hold as long as $\psi^2(t) + \phi^2(t) \geqq R_1^2(r)$. In particular, $\psi^2(t) + \phi^2(t) < R_1^2(r)$ for all $t \in [T^*, \infty)$**.

* It follows that $\lim_{t\to\infty} [\psi^2(t) + \phi^2(t)] = 0$ for all initial data (u_0, v_0) if $f \equiv 0$ (see also [7]).

Proof. If (7.19) holds, then $T^* = 0$. If (7.19) does not hold, equation (7.22) shows that $\psi(t) \geqq \varrho_1(t)$ as long as $\psi^2(t) + \phi^2(t) \geqq R_1^2(r)$. On the other hand, $\psi(t) \leqq R(r)$, $\forall\, t \in [0, \infty]$. Hence equation (7.24) shows that

$$\psi^2(t) + \phi^2(t) \leqq \psi^2(0) + \phi^2(0) - 2(r - |||f|||_\infty)\, \varrho_1(r)\, t, \tag{7.26}$$

as long as $\psi^2(t) + \phi^2(t) \geqq R_1^2(r)$. In particular, if (7.20) is not satisfied by any $t \in [0, T^*]$, then (7.26) must be satisfied on $[0, T^*]$. This leads to a contradiction. Theorem 7.3 thus shows that when $\psi^2(t) + \phi^2(t)$ enters the "sphere" of radius $R_1^2(r)$, it remains there forever. □

In order to interpret the above results, the reader shoulds recall that $\lim \varrho(r) = \lim R(r) = \infty$, and that $\lim R_1(r) = 0$, as $r \to 0$.

On the other hand, (2.9) and the inequalities $\exp(-\|(\log v)_q\|) \leqq |v|_\infty$, $|v^{-1}|_\infty \leqq \exp \|(\log v)_q\|$ show that the quantity $\psi^2[u, v]^*$ is equivalent to $\|u\|^2 + \|v\|_1^2 + \|v^{-1}\|_1^2$ (and also to $\|u\|^2 + \|v\|_1^2 + |v|_\infty^2 + \|v^{-1}\|_1^2 + |v^{-1}|_\infty^2$). In fact, there are continuous strictly increasing functions g_1 and g_2 satisfying $g_1(0) = g_2(0) = 0$ and $g_1(\infty) = g_2(\infty) = \infty$, such that

$$g_1(\|u\|^2 + \|v\|_1^2 + \|v^{-1}\|_1^2) \leqq \psi[u, v] \leqq g_2(\|u\|^2 + \|v\|_1^2 + \|v^{-1}\|_1^2).$$

This allows us to use (in the theorems of this section) "balls" of suitable radius $\varrho'(r)$ and $R'(r)$ with respect to the quantity $\|u\|^2 + \|v\|_1^2 + \|v^{-1}\|_1^2$ instead of "balls" of radius $\varrho(r)$ and $R(r)$ with respect to the quantities $\psi^2[u, v]$ and $\psi^2[u, v] + \phi^2[v]$. A similar remark holds for $R_1(r)$. Clearly, $\lim \varrho'(r) = \lim R'(r) = \infty$ and $\lim R_1'(r) = 0$ as $r \to 0$.

Theorem 7.5. *Assume that $p(\cdot)$ satisfies the additional assumption***

$$\liminf_{s \to 0^+} - sp'(s) > 0. \tag{7.27}$$

Let (u, v) be a weak solution of problem (1.1)–(1.3), *and assume that there is a constant N_0 such that*

$$v(t, q) \leqq N_0 \quad \forall\, (t, q) \in Q_\infty. \tag{7.28}$$

Then there is a positive constant N^{-1} such that $N^{-1} \leqq v(t, q)$, $\forall\, (t, q) \in Q_\infty$.

Proof. From (7.2), (7.27) and (7.28) it follows that

$$\tfrac{1}{2}[\psi^2(t) + \phi^2(t)]_t + c\psi^2(t) \leqq |||f|||_\infty\, \psi(t).$$

Assume that $\psi^2(t) + \phi^2(t) > G^2(c^{-1}\, |||f|||_\infty)$ for some t. It follows from (7.7) that $\psi(t) > c^{-1}\, |||f|||_\infty$. Hence $[\psi^2(t) + \phi^2(t)]_t < 0$. This shows, in particular, that $\psi^2(t) + \phi^2(t)$ is uniformly bounded. By (7.5) the quantity $N(t)$ is also uniformly bounded. Hence $N^{-1}(t)$ is bounded from below by a positive constant N^{-1}. The thesis follows now from (7.4). □

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* As well as $\psi^2[u, v] + \phi^2[v]$; *cf.* (7.7).

** It is worth noting that $p(s) = ks^{-\gamma}$, where k and γ are positive constants, satisfies this assumption.

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