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Existence results in Sobolev spaces for a stationary transport equation

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1. In the sequel Ω is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and ν is the unit outward normal to the boundary Γ.

We denote by L^p the Banach space $L^p(\Omega)$, $1 \leq p \leq +\infty$, endowed with the usual norm $|\ |_p$, and by $W^{k,p}$ the Sobolev space $W^{k,p}(\Omega)$, endowed with the usual norm $\|\ \|_{k,p}$. Moreover, $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}$, $W^{-1,q}$ is the dual space of $W_0^{1,p}$, $p \in]1, +\infty[$, $q = p/(p-1)$, and $W_0^{k,p} = W^{k,p} \cap W_0^{1,p}$, $k \geq 1$. These notations are also used for functional spaces whose elements are vector fields or matrices defined in Ω. If $h(x) = (h_{rs}(x))$, $r = 1, \ldots, R$, $s = 1, \ldots, S$, where h_{rs} are real functions defined in Ω, we set

$$|D^k h(x)|^2 = \sum_{|\alpha|=k} \sum_{r=1}^{R} \sum_{s=1}^{S} |D^\alpha h_{rs}(x)|^2 ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$.

If $h_{rs} \in X$, $r = 1, \ldots, R$, $s = 1, \ldots, S$, where X is a functional space, we write $h \in X$. Moreover, we will use the abbreviate notations

$$D_i h = \frac{\partial h}{\partial x_i} , \qquad \int h = \int_\Omega h(x)\, dx .$$

If $h \in W^{k,p}$ we set

$$|D^k h|_p = \left(\int |D^k h|^p \right)^{1/p} , \qquad \|h\|_{k,p} = \sum_{l=0}^{k} |D^l h|_p \cdot$$

Let $u = (u_1, \ldots, u_N)$, $w = (w_1, \ldots, w_N)$, $v = (v_1, \ldots, v_n)$. We define

$$u \cdot w = \sum_{j=1}^{N} u_j w_j , \qquad |u|^2 = u \cdot u , \qquad (v \cdot \nabla) u = \sum_{i=1}^{n} v_i D_i u ,$$

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$$\nabla v : \nabla^2 u = \sum_{i,l=1}^{n} (D_l v_i)(D_i D_l u) , \qquad [\nabla v] = \max_{|\xi|=1} \left| \sum_{i,l=1}^{n} (D_l v_i) \xi_i \xi_l \right| .$$

Moreover, $(u, w) = \int u \cdot w$.

In general, if X and Y are Banach spaces, $\mathcal{L}(X, Y)$ denotes the Banach space of all bounded linear maps from X into Y. We set $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Let us consider in Ω the equation

$$\lambda u + (v \cdot \nabla) u + au = f , \tag{1.1}$$

where $\lambda \in \mathbb{R}^+$, $v(x) = (v_1(x), \ldots, v_n(x))$, $a(x) = (a_{jk}(x))$, $j, k = 1, \ldots, N$, $f(x) = (f_1(x), \ldots, f_N(x))$ are given, and $u(x) = (u_1(x), \ldots, u_N(x))$ is the unknown. Here $(au)_j = \sum_{k=1}^{N} a_{jk} u_k$. One has the following result, which will be proved in the following sections:

THEOREM 1.1. – *Let $k \geq -1$ be an integer, and let $p \in]n/(k+2), +\infty[$. Assume that $\Gamma \in C^{k+3}$, $v \in W^{k+3,p}$, $a \in W^{k+2,p}$, and*

$$\nu \cdot v = 0 \qquad \text{on } \Gamma . \tag{1.2}$$

If

$$\lambda > \lambda_k \equiv c(\|v\|_{k+3,p} + \|a\|_{k+2,p}) , \tag{1.3}$$

then there exists a bounded linear map $B \in \mathcal{L}(W^{k,p})$ such that $u = Bf$ is a solution of (1.1), *for every $f \in W^{k,p}$. Moreover,*

$$(\lambda - \lambda_k) \|u\|_{k,p} \leq \hat{c} \|f\|_{k,p} . \tag{1.4}$$

If $k \geq 1$, then $u \in W_0^{k,p}$ if and only if $f \in W_0^{k,p}$. For $k \geq 0$, one has $\hat{c} = 1$ (c and $\hat{c}$ are suitable positive constants, depending only on Ω, n, N, p, k).

REMARKS 1.2. – (i) By definition, if $k = 0$ [resp. $k = -1$], u is said to be a solution of (1.1) if the weak formulation (2.18) [resp. (2.19)] holds.

If $k \geq 1$, the solution u is unique; see theorem 2.1.

Uniqueness can be proved also for the case $k = 0$. For the case $k = -1$, the weak solution exists and is unique if it is defined by using equation (2.33) instead of (2.19).

(ii) one can improve theorem 1.1, either by weakening the assumptions on the coefficients or by extending the statement to all values $p \in]1, +\infty[$. However, theorem 1.1 is proved here in the above version, since in our previous paper [2] we used it just in that form.

(iii) In theorem 2.1 and 2.3 we assume that $f=0$ on Γ and we look for solutions u verifying the boundary condition $u=0$ on Γ. It is interesting to note that problem (1.1) is well posed in the space L^p if u is assigned only on the set $\Gamma_1=\{x\in\Gamma: v\cdot\nu<0\}$ (this was shown, in the case $N=1$, by Fichera [5], [6]). At the light of this result, when assumption (1.2) holds one should not impose the boundary condition $u=0$. However, it is not difficult to show that (1.2) is a necessary and sufficient condition to get the solution u in $W_0^{k,p}$, for every $f\in W_0^{k,p}(k\geq 1)$.

Existence theorems in Sobolev spaces where previously established in the hilbertian case $p=2$ (K. O. Friedrichs [7], P. D. Lax and R. S. Phillips [9], J. J. Kohn and L. Nirenberg [8]) and in L^p (G. Fichera [5], [6]; see also O. A. Oleĭnik and E. V. Radekevič [12], and references contained therein).

The study done here (see also [3]) was motivated by our previous paper [2], on the stationary Navier-Stokes equations for compressible fluids. In reference [4], the reader can find a short revue on some of the ideas developed both here and in [2], and on the relationship between the two subjects. For subsequent developments and applications see [13], [14], [15].

2. For convenience, we assume in this section that $\Gamma\in C^4$. Let $p\in]1,+\infty[$ be fixed. We denote by $r=r(p)$ and $s=s(p)$ two reals such that: $r=p$ if $p>n$; $r>n$ if $p=n$; $r=n$ if $p<n$; and $s=p$ if $p>n/2$; $s>n/2$ if $p=n/2$; $s=n/2$ if $p<n/2$. By using Hölder's inequality and well known Sobolev's inequalities, one easily verifies that there exist positive constants $c=c(\Omega, n, N, p, r, s)$ such that

$$\text{(2.0)}\qquad \big||F|\,|w|\big|_p\leq c|F|_s\|w\|_{2,p}\,,\qquad \big||G|\,|Dw|\big|_p\leq c|G|_r\|w\|_{2,p}\,,$$

for every $w\in W_0^{2,p}$, $F\in L^s$, $G\in L^r$.

From (2.0) one easily deduces that there exist constants $c_i=$ $=c_i(\Omega, n, N, p, r, s)$ such that

$$\text{(2.1)}\qquad \begin{cases} |\nabla v:\nabla^2 w|_p\leq c_1|Dv|_\infty|\Delta w|_p\,,\\ |(\Delta v\cdot\nabla)\,w|_p\leq c_2|\Delta v|_r|\Delta w|_p\,,\\ |\Delta(aw)|_p\leq c_3(\|a\|_{2,s}+\|a\|_{1,r}+|a|_\infty)\,|\Delta w|_p\,, \end{cases}$$

for every $w\in W_0^{2,p}$. Note that $|\Delta w|_p$ and $\|w\|_{2,p}$ are equivalent norms in $W_0^{2,p}$.

Define

$$\text{(2.2)}\qquad \theta_2=\frac{1}{p}|div\, v|_\infty+2c_1|Dv|_\infty+c_2|\Delta v|_r+c_3(\|a\|_{2,s}+\|a\|_{1,r}+|a|_\infty)\,.$$

The following result is the core of our paper:

THEOREM 2.1. – *Let* $p \in]1, +\infty[$. *Assume that*

$$v \in W^{2,r} \cap W^{1,\infty}, \quad a \in W^{2,s} \cap W^{1,r} \cap L^{\infty}, \tag{2.3}$$

and that (1.2) *holds. Then, if* $\lambda > \theta_2$ *and* $f \in W_0^{2,p}$ *there exists a unique solution* $u \in W_0^{2,p}$ *of problem* (1.1).

Moreover

$$(\lambda - \theta)_2 |\Delta u|_p \leq |\Delta f|_p, \tag{2.4}$$

and estimates (2.12) *and* (2.15) *hold. In particular*

$$(\lambda - \theta_2)(|\Delta u|_p + |Du|_p + |u|_p) \leq |\Delta f|_p + |Df|_p + |f|_p .$$

PROOF – Let $\varepsilon > 0$, and consider the problem

$$\begin{cases} -\varepsilon \Delta u_\varepsilon + \lambda u_\varepsilon + (v \cdot \nabla) u_\varepsilon + a u_\varepsilon = f, & \text{in } \Omega, \\ (u_\varepsilon)|_\Gamma = 0 . \end{cases} \tag{2.5}$$

For λ sufficiently large, this problem has a unique solution $u_\varepsilon \in W_0^{4,p}$. Since $u_\varepsilon = 0$ on Γ, it follows from (1.2) that $(v \cdot \nabla) u_\varepsilon = 0$ on Γ. This relation, together with (2.5), yields

$$\Delta u_\varepsilon = 0 \qquad \text{on } \Gamma. \tag{2.6}$$

Let now δ be another positive parameter, and set $\Lambda = (\delta + |\Delta u_\varepsilon|^2)^{1/2}$.

For convenience, in the calculations which follow we will denote the solution u_ε of (2.5) by u.

By doing an integration by parts, and by taking in account equation (2.6), one gets

$$-\varepsilon \int \Delta(\Delta u) \cdot \Lambda^{p-2} \Delta u = \varepsilon \int \sum_{i=1}^{n} D_i(\Delta u) \cdot D_i(\Lambda^{p-2} \Delta u) .$$

On the other hand, the following identity

$$\sum_{i=1}^{n} D_i(\Delta u) \cdot D_i(\Lambda^{p-2} \Delta u) = \Lambda^{p-2} |D\Delta u|^2 + \frac{p-2}{4} \Lambda^{p-4} |\nabla(|\Delta u|^2)|^2 ,$$

holds. Hence,

$$-\varepsilon \int \Delta(\Delta u) \cdot \Lambda^{p-2} \Delta u = \varepsilon \int \Lambda^{p-2} |D\Delta u|^2 + \varepsilon \frac{p-2}{4} \int \Lambda^{p-4} |\nabla(|\Delta u|^2)|^2 . \tag{2.7}$$

This proves that the left hand side of (2.7) is non-negative, for $p \geq 2$. We get again this result if $p \in]1, 2]$, since

$$\sum_{i=1}^{n} D_i(\Delta u) \cdot D_i(\Lambda^{p-2}\Delta u) \geq [(p-1)|\Delta u|^2 + \delta]\Lambda^{p-4}\sum_{i=1}^{n}|D_i(\Delta u)|^2 .$$

Hence, the left hand side of (2.7) is non-negative for every $p \in]1, +\infty[$.
On the other hand, one has the identity

$$\Lambda^{p-2} D_i(\Delta u) \cdot \Delta u = \frac{1}{2}\Lambda^{p-2} D_i(|\Delta u|^2) = \frac{1}{2}\Lambda^{p-2} D_i(\delta + |\Delta u|^2) = \frac{1}{p} D_i \Lambda^p .$$

Hence, an integration by parts shows that

$$\int (v \cdot \nabla)\Delta u \cdot \Lambda^{p-2}\Delta u = -\frac{1}{p}\int (div\, v)\Lambda^p . \tag{2.8}$$

Equation (2.8) and the identity $\Delta[(v \cdot \nabla)u] = (v \cdot \nabla)\Delta u + 2\nabla v : \nabla^2 u + (\Delta v \cdot \nabla)u$, yield

$$\begin{aligned} \int \Delta[(v \cdot \nabla)u] \cdot \Lambda^{p-2}\Delta u = & -\frac{1}{p}\int (div\, v)\Lambda^p + \\ & + 2\int (\nabla v : \nabla^2 u)\Lambda^{p-2}\Delta u + \int [(\Delta v \cdot \nabla)u] \cdot \Lambda^{p-2}\Delta u . \end{aligned} \tag{2.9}$$

Let us now return to equation $(2.5)_1$. By applying the Δ operator to both sides of this equation, by taking the scalar product on $\mathbb{R}^N$ with $\Lambda^{p-2}\Delta u_\varepsilon$, by integrating in Ω, and by taking in account (2.9), it follows that

$$\begin{aligned} & -\varepsilon\int \Delta(\Delta u) \cdot \Lambda^{p-2}\Delta u + \lambda\int |\Delta u_\varepsilon|^2 \Lambda^{p-2} - \frac{1}{p}\int (div\, v)\Lambda^p \leq \\ & \leq 2\int |\nabla v : \nabla^2 u_\varepsilon|\,|\Delta u_\varepsilon|\Lambda^{p-2} + \int |(\Delta v \cdot \nabla)u_\varepsilon|\,|\Delta u_\varepsilon|\Lambda^{p-2} + \\ & \qquad + \int |\Delta(a u_\varepsilon)|\,|\Delta u_\varepsilon|\Lambda^{p-2} + \int |\Delta f|\,|\Delta u_\varepsilon|\Lambda^{p-2} . \end{aligned} \tag{2.10}$$

Since $0 \leq |\Delta u_\varepsilon|\Lambda^{p-2} \leq \Lambda^{p-1}$, the Lebesgue's dominated convergence theorem applies as $\delta \to 0^+$. By taking in account that the first term on the left hand side of (2.10) is non-negative, and by passing to the limit as $\delta \to 0^+$ one gets

$$\begin{aligned} & \lambda\int |\Delta u_\varepsilon|^p - \frac{1}{p}\int (div\, v)|\Delta u_\varepsilon|^p \leq \\ & \leq \int (2|\nabla v : \nabla^2 u_\varepsilon| + |(\Delta v \cdot \nabla)u_\varepsilon| + |\Delta(a u_\varepsilon)| + |\Delta f|)|\Delta u_\varepsilon|^{p-1} . \end{aligned} \tag{2.11}$$

This shows, in particular, that $(\lambda-\theta_2)|\Delta u_\varepsilon|_p \leq |\Delta f|_p$. Since the u_ε are uniformly bounded in $W_0^{2,p}$, there exists a subsequence u_ε, weakly convergent to a limit u as $\varepsilon \to 0$ (actually, the all sequence u_ε converges to u). Clearly, u verifies (2.4). Moreover, by passing to the limit in $(2.5)_1$ as $\varepsilon \to 0$, it follows that u is a solution of (1.1). Note that $\varepsilon\Delta u_\varepsilon \to 0$ strongly, as $\varepsilon \to 0$. Uniqueness, and the estimates (2.12), (2.15) will be proved in the sequel. □

Let m_a and M_a be constants such that, for every $\xi \in \mathbb{R}^N$, $|\xi|=1$ the estimates

$$m_a \leq \sum_{j,k=1}^{N} a_{jk}(x)\,\xi_j\xi_k\,, \qquad \left|\sum_{j,k=1}^{N} a_{jk}(x)\,\xi_j\xi_k\right| \leq M_a\,,$$

hold a.e. in Ω.

Define $\theta_0 = (1/p)\,|div\, v|_\infty - m_a$. Clearly,

$$\theta_0 \leq (1/p)\,|div\, v|_\infty + M_a \leq c(\|v\|_{1,\infty} + |a|_\infty).$$

THEOREM 2.2. – *Let $v \in W^{1,\infty}$, $a \in L^\infty$, $f \in L^p$, let $u \in W_0^{1,p}$ be a solution of* (1.1) *[resp. $u \in W^{1,p}$ be a solution of* (1.1), *under the assumption* (1.2) *for the coefficient v], and let $\lambda > \theta_0$.*

Then

$$(\lambda-\theta_0)\,|u|_p \leq |f|_p\,. \tag{2.12}$$

In particular, the solution, if it exists, is unique.

PROOF. – By multiplying both sides of (1.1) by $(\delta+|u|^2)^{(p-2)/2}u$, by integrating in Ω, and by passing to the limit as $\delta \to 0^+$ one gets (2.12). □

Let $a \in W^{1,r} \cap L^\infty$. Since $\||Da|\,|w|\|_p \leq c\|a\|_{1,r}\|w\|_{1,p}$, and $\|w\|_{1,p} \leq c\,|Dw|_p$, one easily verifies that

$$|D(aw)|_p \leq c_4(\|a\|_{1,r} + |a|_\infty)\,|Dw|_p\,, \qquad \forall w \in W_0^{1,p}, \tag{2.13}$$

where $c_4 = c_4(\Omega, n, N, p, r)$. We set

$$\theta_1 = \frac{1}{p}|div\, v|_\infty + |[\nabla v]|_\infty + c_4(\|a\|_{1,r} + |a|_\infty)\,. \tag{2.14}$$

THEOREM 2.3. – *Let p, Γ and v be as in theorem* 2.1 *(actually, the assumption $v \in W^{2,r}$ can be dropped) and let $a \in W^{1,r} \cap L^\infty$. If $f \in W_0^{1,p}$ and $\lambda > \theta_1$ the problem* (1.1) *has a unique solution $u \in W_0^{1,p}$. Moreover*

$$(\lambda-\theta_1)|Du|_p \leq |Df|_p\,, \tag{2.15}$$

and (2.12) *holds.*

In particular, $(\lambda-\theta_1)\|u\|_{1,p} \leq \|f\|_{1,p}$.

PROOF. – (i) Assume that $a \equiv 0$, and denote by $\bar{\theta}_2$ the right hand side of (2.2) (for $a \equiv 0$). Assume that $\lambda > \bar{\theta}_2$. Let $f_m \in W_0^{2,p}$ be a sequence, such that $f_m \to f$ in $W_0^{1,p}$ as $m \to +\infty$, and denote by u_m the solution of $\lambda u_m + (v \cdot \nabla) u_m = f_m$. Set, for convenience, $\Lambda = (\delta + |Du_m|^2)^{1/2}$, where $\delta > 0$. One has

$$\int \sum_{l=1}^{n} [(v \cdot \nabla) D_l u_m] \cdot \Lambda^{p-2} D_l u_m = \frac{1}{p} \int (v \cdot \nabla) \Lambda^p . \tag{2.16}$$

By taking the scalar product in $\mathbb{R}^n$ *of both sides of the equation* $\lambda D_l u_m + (v \cdot \nabla) D_l u_m + [(D_l v) \cdot \nabla] u_m = D_l f_m$ *with* $\Lambda^{p-2} D_l u_m$, *by adding for* $l = 1, \ldots, n$, by integrating in Ω, and by taking in account equation (2.16), one gets

$$\lambda \int |Du_m|^2 \Lambda^{p-2} - \frac{1}{p} |div\, v|_\infty \int \Lambda^p \leqslant \leqslant \int [\nabla v] \, |Du_m|^2 \Lambda^{p-2} + \int |Df| \, |Du_m| \, \Lambda^{p-2} . \tag{2.17}$$

By passing to the limit as $\delta \to 0^+$, we show that $(\lambda - \bar{\theta}_1) |Du_m|_p \leqslant |Df_m|_p$, where θ_1 denotes the right hand side of (2.14) (for $a \equiv 0$). It easily follows the existence of $u \in W_0^{1,p}$, solution of $\lambda u + (v \cdot \nabla) u = f$, such that $(\lambda - \bar{\theta}_1)|Du|_p \leqslant Df|_p$.

(ii) Here, we extend the result proved in part (i) to the case in which $\lambda > \bar{\theta}_1$. Fix $\bar{\lambda} > \bar{\theta}_2$, and denote by $u = Tw$ the solution of problem $\bar{\lambda} u + (v \cdot \nabla) u = f + (\bar{\lambda} - \lambda) w$, for an arbitrary $w \in W_0^{1,p}$. If $\bar{u} = T\bar{w}$, one has $(\bar{\lambda} - \bar{\theta}_1) |D(u - \bar{u})|_p \leqslant (\bar{\lambda} - \lambda) |D(w - \bar{w})|_p$. Hence, T is a contraction in $W_0^{1,p}$. The fixed point u is a solution of $\lambda u + (v \cdot \nabla) u = f$. Moreover $(\bar{\lambda} - \bar{\theta}_1) |Du|_p \leqslant |D(f + (\bar{\lambda} - \lambda) u)|_p$. Hence $(\lambda - \bar{\theta}_1) |Du|_p \leqslant |Df|_p$.

(iii) Finally, we assume that $a \not\equiv 0$, and that $\lambda > \theta_1$ (hence, $\lambda > \bar{\theta}_1$). Let $w \in W_0^{1,p}$, and denote by $u = Tw$ the solution of $\lambda u + (v \cdot \nabla) u = f - aw$.

Let $\bar{u} = T\bar{w}$. One has $(\lambda - \bar{\theta}_1) |D(u - \bar{u})|_p \leqslant |D[a(\bar{w} - w)]|_p \leqslant \leqslant c_4(\|a\|_{1,r} + |a|_\infty) |D(w - \bar{w})|_p$. This shows that T is a contraction in $W_0^{1,p}$ since $\lambda > \theta_1$. The fixed point $u = Tu$ is a solution of (1.1). Moreover, $(\lambda - \bar{\theta}_1) |Du|_p \leqslant |D(f - au)|_p \leqslant |Df|_p + c_4(\|a\|_{1,r} + |a|_\infty) |Du|_p$. Hence, (2.15) holds. □

Assume that $v \in W^{1,\infty}$, $a \in L^\infty$, $u, f \in L^p$.

We say that u is a weak solution of (1.1) if

$$\lambda(u, \varphi) - (\sum_{i=1}^{n} D_i(v_i \varphi), u) + (au, \varphi) = (f, \varphi) \qquad \forall \varphi \in W_0^{1,q} , \tag{2.18}$$

where $q = p/(p-1)$, $\varphi = (\varphi_1, \ldots, \varphi_N)$.

COROLLARY 2.4. – *Under the hypothesis of theorem 2.3 on v, a, Γ and λ, there exists a bounded linear map $G \in \mathcal{L}(L^p)$ such that $u = Gf$ is a weak solution of* (1.1), *for every $f \in L^p$. Moreover,* (2.12) *holds.*

PROOF. – Let $u_m \in W_0^{1,p}$ be the solution of $\lambda u_m + (v \cdot \nabla) u_m + a u_m = f_m$, where $f_m \in W_0^{1,p}$ is a sequence which converges to f in L^p. From (2.12) it follows that $(\lambda - \theta_0)|u_m - u_n|_p \leq |f_m - f_n|_p$. Hence, the sequence u_m is convergent to a function u in L^p, as $m \to +\infty$. One easily verifies that u is the desired solution. □

REMARKS. – The assumptions $v \in W^{2,r}$ and $a \in W^{1,r}$ are superfluous. Moreover, the map G exists for every $\lambda > \theta_0$. Finally, from the existence theorem for the adjoint problem $\lambda\varphi - (v \cdot \nabla)\varphi - (div\, v)\varphi + a^*\varphi = g$, it follows an uniqueness result for the above solution u, at least for sufficiently large values of λ.

Now, we turn out to the study of equation (1.1) in spaces $W^{-1,q}$, $q \in]1, +\infty[$. Set $p = q/(q-1)$, denote by $\langle\ ,\ \rangle$ the duality pairing between $W_0^{1,p}$ and the dual space $W^{-1,q}$.

DEFINITION 2.5. – *Let $f \in W^{-1,q}$. We say that $u \in W^{-1,q}$ is a weak solution of equation* (1.1) *if*

$$\left\langle \lambda\varphi - \sum_{i=1}^{n} D_i(v_i\varphi) + a^*\varphi, u \right\rangle = \langle \varphi, f \rangle, \qquad \forall \varphi \in \mathcal{D}(\Omega), \tag{2.19}$$

where $\varphi = (\varphi_1, \ldots, \varphi_N)$, and a^* is the transpose matrix of a.

Let v, a, Γ be as in theorem 2.3. From this theorem it follows that there exists a positive constant c_5 depending only on Ω, n, N, p, r such that if

$$\lambda > \theta_1^* \equiv c_5(\|v\|_{2,r} + \|v\|_{1,\infty} + \|a\|_{1,r} + |a|_\infty), \tag{2.20}$$

then there exists a bounded linear map $B \in \mathcal{L}(W_0^{1,p})$ such that $\varphi = Bg$ is the (unique) solution of the equation

$$\lambda\varphi - (v \cdot \nabla)\varphi - (div\, v)\varphi + a^*\varphi = g, \tag{2.21}$$

for every $g \in W_0^{1,p}$. Obviously, the operator B is invertible. Set $A = B^{-1}$, and denote by $D(A)$ the domain of A *i.e.* the range of B. The operator A is closed in $W_0^{1,p}$, moreover,

$$D(A) = \{\varphi \in W_0^{1,p} : (v \cdot \nabla)\varphi \in W_0^{1,p}\}. \tag{2.22}$$

In particular, one has $\mathcal{D}(\Omega) \subset D(A)$. Hence, $D(A)$ is dense in $W_0^{1,p}$.

Denote by A^* the adjoint of A. Since $A^{-1} = B \in \mathcal{L}(W_0^{1,p})$ a well known result on Functional Analysis guarantees that $(A^*)^{-1} = B^* \in \mathcal{L}(W^{-1,q})$ (moreover, $\|B^*\| = \|B\|$).

Consequently, the equation $A^*u = f$ has a unique solution $u = B^*f$, for each $f \in W^{-1,q}$. This equation is equivalent to $\langle A\varphi, u\rangle = \langle \varphi, f\rangle$, $\forall \varphi \in D(A)$; hence it is equivalent to

$$\left\langle \lambda\varphi - \sum_{i=1}^{n} D_i(\varphi_i v) + a^*\varphi,\, u\right\rangle = \langle \varphi, f\rangle \qquad \forall \varphi \in D(A)\,. \tag{2.23}$$

In conclusion, $u = B^*f$ is a weak solution of (1.1), $\|u\|_{-1,q} \leq \|B\| \|f\|_{-1,q}$, and (2.24) holds. We have then proved the following result:

THEOREM 2.6. – *Let $q \in]1, +\infty[$, $p = q/(q-1)$, $r = r(p)$. Let v, a, and Γ be as in theorem 2.3. Assume that λ verifies (2.20). Let $B^* \in \mathcal{L}(W^{-1,q})$ be defined as above. Then, $u = B^*f$ is a weak solution of equation (1.1) for every $f \in W^{-1,q}$ (actually, (2.23) holds). Moreover*

$$(\lambda - \theta_1^*)\, \|u\|_{-1,q} \leq \|f\|_{-1,q}\,. \tag{2.24}$$

3. In this section we prove theorem 1.1. We start from the main a priori bound:

PROPOSITION 3.1. – *Let p, $k \geq 0$, v, f, and a be as in theorem 1.1. There exists $c = c(\Omega, n, N, p, k)$ such that if λ verifies (1.3) and if $u \in W^{k+1,p}$ is a solution of (1.1), then*

$$(\lambda - \lambda_k)\, \|u\|_{k,p} \leq \|f\|_{k,p}. \tag{3.1}$$

PROOF. – Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index, $|\alpha| = k$. By using an abbreviate notation, the application of the operator D^α to both sides of equation (1.1) yields

$$\lambda D^\alpha u + (v \cdot \nabla)\, D^\alpha u + \sum [(D^k v)(Du) + \ldots + (Dv)(D^k u)] + \\ + \sum [(D^k a)\, u + \ldots + a(D^k u)] = D^\alpha f\,. \tag{3.2}$$

Set $\Lambda = (\delta + |D^k u|^2)^{1/2}$, where δ is a positive parameter, and $|D^k u|^2 = \sum |D^\alpha u_j|^2$; this summation is extended to all α such that $|\alpha| = k$, and to all j, $1 \leq j \leq N$. By multiplying both sides of equation (3.2) by $\Lambda^{p-2} D^\alpha u$, by adding side for all index α such that $|\alpha| = k$, by integrating in Ω, and by taking in account that the third term [resp. last term] on the left hand side of (3.2) is bounded by $c\|v\|_{k+3,p}\|u\|_{k,p}$ [resp. $c\|a\|_{k+2,p}\|u\|_{k,p}$], it follows that

$$\lambda\int \Lambda^{p-2}|D^k u|^2 \leqslant \frac{1}{p}|div\, v|_\infty\, |\Lambda|_p^p +$$
$$+\, c(\|v\|_{k+3,p} + \|a\|_{k+2,p})\,\|u\|_{k,p}\,|\Lambda|_p^{p-1} + |D^k f|_p\,|\Lambda|_p^{p-1}\,.$$

By passing to the limit as $\delta\to 0^+$ one gets

$$\lambda|D^k u|_p \leqslant c(\|v\|_{k+2,p} + \|a\|_{k+2,p})\,\|u\|_{k,p} + |D^k f|_p\,. \tag{3.3}$$

Clearly, (3.3) holds for every integer k_0 such that $0\leqslant k_0\leqslant k$. By adding side by side all that estimates, when $k_0 = 0, 1, \ldots, k$, one gets (3.1). □

PROOF OF THEOREM 1.1. – Theorem 2.3 together with the first statement of theorem 1.1 show that $u=0$ on Γ if and only if $f=0$ on Γ. The proof of the first statement of theorem 1.1 will be divided in three steps, as follows.

(i) Here, we prove the statement of theorem 1.1 for the values $k=-1, 0, 1, 2$.

Let I be a ball such that $\bar{\Omega}\subset I$ and fix linear maps $T_1\in \in \mathcal{L}(W^{k+3,p}, W_0^{k+3,p}(I))$, $T_2\in\mathcal{L}(W^{k+2,p}, W^{k+2,p}(I))$, and $T_3\in \in\mathcal{L}(W^{k,p}, W_0^{k,p}(I))$, such that $(T_1 v)_{|\Omega} = v$, $(T_2 a)_{|\Omega} = a$, $(T_3 f)_{|\Omega} = f$; Here, $W_0^{0,p} = L^p$, and $W_0^{-1,p} = W^{-1,p}$. In case that $k=0$ we define the map T_3 by setting $(T_3 f)(x) = f(x)$, if $x\in\Omega$, $(T_3 f)(x) = 0$ if $x\notin\Omega$. We put $\tilde{v} = T_1 v$, $\tilde{a} = T_2 a$, $\tilde{f} = T_3 f$. Note that if $k=0$ then $|\tilde{f}|_{p,I} = |f|_p$ (this allows us to choose $\hat{c}=1$, in this case). By using Sobolev's inequalities, one easily proves that the coefficients $\tilde{v}$ and $\tilde{a}$ verify, in the ball I, the assumptions of theorem 2.6 if $k=-1$ (*); of corollary 2.4 if $k=0$; of theorem 2.3 if $k=1$; of theorem 2.1 if $k=2$. Denoting by $\hat{u}$ the solution of the equation $\lambda\hat{u} + (\tilde{v}\cdot\nabla)\,\hat{u} + \tilde{a}\hat{u} = \tilde{f}$ in I (whose existence is guaranteed by one of the above theorems, depending on the value of k), one easily verifies that $u = \hat{u}|_\Omega$ is a solution of (1.1), and that all the desired properties hold. We leave the quite obvious details to the reader.

(ii) Here, we prove the first statement of theorem 1.1 for $k\geqslant 3$ and $p>n$. We assume that the thesis hold for a value $k\geqslant 2$, and we will prove it for the value $k+1$ (by step (i), the thesis holds for the values $k=1, 2$).

Let $v\in W^{k+4,p}$, $v\cdot\nu = 0$ on Γ, $a\in W^{k+3,p}$, $f\in W^{k+1,p}$, and assume that λ verifies (1.3). By the induction hypothesis, there exists a (unique) solution $u\in W^{k,p}$ of (1.1). Moreover (1.4) holds.

(*) Here we use the statement of theorem 2.6 with the roles of p and q exchanged.

From (1.1) one has, for each index l, $1 \leqslant l \leqslant n$

$$\lambda D_l u + (v \cdot \nabla) D_l u + a D_l u + [(D_l v) \cdot \nabla] u = D_l f - (D_l a) u .$$

This is again a system of type (1.1), on the nN variables $D_l u_j$. By the induction hypothesis, there exist $\bar{c} \equiv c(nN, k)$, and $\hat{c}(nN, k)$ such that for every $\lambda > \lambda_k \equiv \bar{c}(\|v\|_{k+3,p} + \|a\|_{k+2,p})$, one has $Du \in W^{k,p}$ and

$$(3.4) \qquad (\lambda - \bar{\lambda}_k) \|Du\|_{k,p} \leqslant \hat{c}(nN, k)(\|Df\|_{k,p} + \|a\|_{k+1,p} \|u\|_{k,p}) .$$

Hence $u \in W^{k+1,p}$. Moreover, the estimate (1.4) for the value $k+1$ follows from that estimate for the value k together with (3.4). Set, for instance, $c(N, k+1) = \max\{c(N, k), c(nN, k)\} + \hat{c}(nN, k)$, $\hat{c}(N, k+1) = \max\{\hat{c}(N, k), \hat{c}(nN, k)\}$.

(iii) Here, the main point is to prove the first statement of theorem 1.1 for $k \geqslant 3$, $p > n/(k+2)$. However, in order to show that $\hat{c} = 1$ if $k \geqslant 1$, we will assume that $k \geqslant 1$ and $p > n/(k+2)$. Fix a real p_0 such that $p_0 \geqslant p$, and $p_0 > n$, and let $v \in W^{k+4,p_0}$, $a \in W^{k+3,p_0}$ (this additional assumption will be droped later on).

Let $\{f_m\}$, $m \in N$, be a sequence of functions belonging to W^{k+1,p_0}, and such that $f_m \to f$ in $w^{k,p}$. Let λ verify the additional assumption

$$(3.5) \qquad \lambda > c(\Omega, n,. N, p_0, k+1)(\|v\|_{k+4,p_0} + \|a\|_{k+3,p_0}) \equiv \mu$$

and let $u_m \in W^{k+1,p}$ be the solution of $\lambda u_m + (v \cdot \nabla) u_m + a u_m = f_m$, whose existence was proved before. By proposition 3.1, the estimate (3.1) holds for the pair u_m, f_m. It follows that $\{u_m\}$ converges in $W^{k,p}$ to a function u. Clearly, u is a solution of (1.1), and u verifies (3.1).

Now we drop the additional condition (3.5). Assume that $\lambda \in]\lambda_k, \mu]$, fix a value $\bar{\lambda} > \mu$, and consider the problem $\bar{\lambda} \mu + (v \cdot \nabla) u + au = f + (\bar{\lambda} - \lambda) w$, $w \in W^{k,p}$. By arguing as in the part (ii) of the proof of theorem (2.3) one easily proves the thesis.

Finally, we drop the additional assumptions on v and a. Let v and a be as stated in theorem 1.1, and consider sequences $v_m \in W^{k+4,p_0}$, $a_m \in W^{k+3,p_0}$, such that $v_m \cdot \nu = 0$ on Γ, $v_m \to v$ in $W^{k+3,p}$, and $a_m \to a$ in $W^{k+2,p}$, as $m \to +\infty$. Let u_m be the solution of $\lambda u_m + (v_m \cdot \nabla) u_m + a_m u_m = f$, where λ verifies the assumption (1.3) (*).

By (3.1), one has $(\lambda - \lambda_k) \|u_m\|_{k,p} \leqslant \|f\|_{k,p}$. Let u be the weak limit in $W^{k,p}$ of a subsequence of the sequence $\{u_m\}$. Then, u is the desired solution of problem (1.1). □

(*) Hence, for sufficiently large values of m, (1.3) holds if v and a are replaced by v_m and a_m respectively.

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