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Existence and Asymptotic Behavior for Strong Solutions of the Navier–Stokes Equations in the Whole Space

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We shall consider the initial value problem for the nonstationary Navier–Stokes equations in the whole space, namely,

$$(0.1)\qquad \begin{cases} v' - \mu\Delta v + (v\cdot\nabla)v = f - \nabla p, & \text{in }]0,T[\times \mathbf{R}^n, \\ \nabla\cdot v = 0, & \text{in }]0,T[\times \mathbf{R}^n, \\ v = a(x), & \text{in } \mathbf{R}^n, \\ \lim_{|x|\to+\infty} v(t,x) = 0, & \text{for } t\in]0,T[, \end{cases}$$

where $T\in]0,+\infty]$, μ is a positive constant, $v' = \partial v/\partial t$, and

$$((v\cdot\nabla)v)_j = \sum_{i=1}^{n} v_i \frac{\partial v_j}{\partial x_i}, \qquad j = 1,\ldots,n.$$

The vector field $v(t,x)$ and the scalar field $p(t,x)$ are unknowns. The initial velocity $a(x)$ and the external forces $f(t,x)$ are given. The pressure is determined by the condition $\lim p(t,x) = 0$, as $|x|\to+\infty$. Moreover,

$$(0.2)\qquad \nabla\cdot f = 0 \ \text{a. e. in }]0,T[\qquad \text{and } \nabla\cdot a = 0.$$

The first condition (0.2) is not strictly necessary.

Our main concern will be the asymptotic behaviour of the solutions, and the core of the paper are the a priori estimates in §§1 and 3. Appendices and proofs concerning the existence of the solutions (estimates of §1 apart) are presented mainly for the sake of completeness. The reader acquainted with Navier–Stokes equations should skip §2 and appendices, or do them by different methods. It is worth noting that some technical difficulties can be avoided by assuming more regularity on f (as for instance, by assuming that $f\in L^2(0,T;L^\alpha)$, or that $f\in L^\infty(0,T;L^\alpha)$, instead of (1.1)).

By a solution of problem (0.1) we mean a divergence free vector field $v(t,x)\in L^1(0,T;L^2_{\text{loc}})$ such that

$$(0.1')\qquad \int_0^T\int [v\cdot\varphi' + \mu v\cdot\Delta\varphi + (v\cdot\nabla)\varphi\cdot v + f\cdot\varphi]\,dx\,dt = -\int a\varphi|_{t=0}\,dx,$$

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for every regular divergence free vector field $\varphi(t,x)$, with compact support respect to the space variables, and such that $\varphi(T,x) \equiv 0$.

In the sequel, $c, c_1, c_2, \ldots$, denote positive constants depending at most on α and n. The symbol c may be utilized (even in the same equation) to indicate distinct constants.

In §1 we establish some basic a priori estimates for the norm $|v(t)|_\alpha$ in $L^\alpha(\mathbf{R}^n)$, and we determine explicitly a lower bound T_α for the time of existence of the solution of (0.1) in the class above. The main *a priori estimate* in §1 is the following:

Theorem 0.1. *Let $\alpha > n$, and let a and f verify the assumptions* (0.2) *and* (1.1). *Let v be a solution of the Navier–Stokes equation* $(0.1)_{1,2,3}$, *belonging to the class* (1.2). *Then there exists a positive constant c such that*

$$\frac{d}{dt}|v(t)|_\alpha \leq c\mu^{-(n+\alpha)/(\alpha-n)}|v(t)|_\alpha^{1+2\alpha/(\alpha-n)} + |f(t)|_\alpha . \tag{0.3}$$

In particular, if $v \in L^q(0,T;L^\alpha)$, where

$$\frac{2}{q} + \frac{n}{\alpha} = 1, \tag{0.4}$$

one has

$$|v(t)|_\alpha \leq \exp\left(c\mu^{-(n+\alpha)/(\alpha-n)}\|v\|^q_{L^q(0,t;L^\alpha)}\right)\left(|v(0)|_\alpha + \|f\|_{L^1(0,t;L^\alpha)}\right), \tag{0.5}$$

for every $t \in [0,T]$. In particular, $v \in L^\infty(0,T;L^\alpha)$.

The second part of Theorem 0.1 follows by noting that $q = \frac{2\alpha}{\alpha-n}$, and by applying the estimate (0.3). We are grateful to P. Secchi for calling our attention to this fact.

The a priori estimate (0.5) *can be utilized to show that if a solution v of* (0.1) *belongs to the class $L^q(0,T;L^\alpha)$, then $v \in L^\infty(0,T;L^\alpha)$, and* (0.5) *holds.*

We leave the technical details to the interested reader. Note that the existence of a solution in the class $L^q(0,T;L^\alpha)$ is an open problem.

In §2 we assume that $a \in L^\alpha$ and $f \in L^1(0,T;L^\alpha)$, and we state two existence theorems. In Theorem 2.1 we prove that there exists a (unique) solution $v \in C_*([0,T_\alpha[;L^\alpha)$ of (0.1), such that $|v(t)|_\alpha \leq y(t)$, $\forall t \in [0,T_\alpha[$, where T_α is defined as the time of existence of the maximal solution $y(t)$ of the o. d. e. $y' = c\mu^{-(n+\alpha)/(\alpha-n)}y^{1+q} + |f(t)|_\alpha$, with initial data $y(0) = |a|_\alpha$. We denote by $C_*([0,T_\alpha[;L^\alpha)$ the space of the weakly continuous functions on $[0,T_\alpha[$ with values in L^α.

In Theorem 2.2 we assume that $a \in L^\alpha \cap L^2$ and $f \in L^1(0,T;L^\alpha \cap L^2)$, and we prove the existence of a (unique) solution $v \in C([0,T_\alpha[;L^2 \cap L^\alpha)$, such that $|v(t)|_\alpha \leq y(t)$.

Since we are mainly interested on finite energy solutions (in view of the results of §3), we prove the strong continuity only in Theorem 2.2. However, strong continuity could be proved also in Theorem 2.1.

An existence result, related to Theorem 2.1, was proved by Fabes, Jones and Riviere [2], by assuming that $a \in L^\alpha$ and $f \in L^q(0,T;L^\alpha)$, $q > 1$. Under these conditions, they show that there exists a (unique) solution in $L^p(0,T^*;L^\alpha)$, for some $T^* > 0$; however, the value $p = +\infty$ is not attained. Other interesting (related) existence results in the $\mathbf{R}^n$ case are proved by Kato [6] and, in the bounded domain case, by Giga and Miakawa [4]; see also Giga [5].

The uniqueness of the solution in the class $L^p(0,T;L^\alpha)$, with $n < \alpha < +\infty$ and $\frac{2}{p} + \frac{n}{\alpha} \le 1$, was proved by Fabes, Jones and Riviere [2].

In §3 we obtain some sharp estimates for the solution of (0.1) by assuming a smallness condition on the data. More precisely, we will prove the following results:

Theorem 0.2. *Given $\alpha > n$, there exist two positive constants c_1 and c_2, depending only on α and n, such that the following statement holds:*

> *Let $T \in]0,+\infty]$, and let $a \in L^\alpha \cap L^2$ and $f \in L^\infty(0,T;L^\alpha) \cap L^1(0,T;L^2)$ verify (0.2). Moreover, assume that the data a and f verify*

$$\left[|a|_2 + \|f\|_{L^1(0,T;L^2)}\right]^{2(\alpha-n)/\alpha(n-2)} |a|_\alpha \le c_1 \mu^{n(\alpha-2)/\alpha(n-2)}, \tag{0.6}$$

and that

$$\left[|a|_2 + \|f\|_{L^1(0,T;L^2)}\right]^{(6\alpha-2n)/\alpha(n-2)} \|f\|_{L^\infty(0,T;L^\alpha)} \le c_2 \mu^{2(\alpha n+\alpha-n)/\alpha(n-2)}. \tag{0.7}$$

Then, there exists a (unique) solution $v \in L^2(0,T;H^1) \cap C([0,T];L^\alpha \cap L^2)$ of the Navier–Stokes equation (0.1). *Moreover,*

$$\|v\|_{C([0,T[;L^\alpha)} \le c_1 \mu^{n(\alpha-2)/\alpha(n-2)} \left[|a|_2 + \|f\|_{L^1(0,T;L^2)}\right]^{-2(\alpha-n)/\alpha(n-2)}. \tag{0.8}$$

In the absence of external forces, we will prove the following decay estimate:

Theorem 0.3. *Given $\alpha > n$, there exist positive constants c_3, c_4 and c_5, depending only on α and n, such that if $f \equiv 0$, $a \in L^\alpha \cap L^2$, $\nabla \cdot a = 0$ and*

$$|a|_2^{2(\alpha-n)/\alpha(n-2)} |a|_\alpha \le c_3 \mu^{n(\alpha-2)/\alpha(n-2)}, \tag{0.9}$$

then there exists a (unique) solution $v \in L^2(0,+\infty;H^1) \cap C([0,+\infty[;L^\alpha \cap L^2)$ of problem (0.1). *Moreover,*

$$|v(t)|_\alpha \le |a|_\alpha \left[1 + c_4 \beta \mu |a|_2^{-\beta} |a|_\alpha^\beta t\right]^{-1/\beta}, \tag{0.10}$$

for every $t \in [0, +\infty[$, *where* $\beta = \frac{4\alpha}{(\alpha-2)n}$. *In particular,*

$$|v(t)|_\alpha \le c_5 |a|_2 \left(\frac{1}{\mu t}\right)^{(\alpha-2)n/4\alpha}, \qquad \forall t > 0. \tag{0.11}$$

Remarks

(i) Actually, the solution v in Theorem 0.3 belongs to $C^\infty(]0, +\infty[\times \mathbf{R}^n)$, since it is bounded in $L^\alpha(\mathbf{R}^n)$, for $\alpha > n$. By regularization, one can obtain estimates for stronger norms than $| \;|_\alpha$.

(ii) The uniqueness of the Leray–Hopf solution, in Theorems 0.2 and 0.3, follows from the uniqueness theorems of Prodi [14], Lions and Prodi [10], Foias [3], and Serrin [16]. See [9], Chapter 1, Theorem 6.9.

(iii) Conditions (0.6), (0.7), and (0.9) are invariant under scale change in space–time.

(iv) In view of results proved in [2], [6] it looks possible to replace in Theorems 0.2 and 0.3 the L^2–norm by an L^{α_0}–norm for $\alpha_0 < n$. However, we did not investigate in this direction.

At the end of §3 we prove that the statements in Theorems 0.2 and 0.3 hold again, by setting $\alpha = n$. In this particular case the formulas simplify considerably; see Theorem 3.3.

Some results, related to those presented in this paper, can be found in Fabes, Jones and Riviere [2], and in Kato [6]. In the latter paper some asymptotic estimates are given, especially in the case $a \in L^n$ and $f \equiv 0$. It is interesting to note that, by setting $p = 2$ and $q = n$ in estimate (1.5) of reference [6], one has $|v(t)|_n = 0(1/t^{(n-2)/4})$, as $t \to +\infty$, which is just the asymptotic behavior implied by our estimate (3.17). However, in [6] the result is proved under the assumption that the exponent $\frac{n-2}{4}$ is less than 1.

For other results, more or less related to ours, see, e.g., Giga and Miyakawa [4], Giga [5], Masuda [12], and Weissler [20]. See also [21].

The results proved in our paper were obtained independently of those of the papers above. The method utilized is quite different, too.

§1. In the sequel with the symbol L^α, $1 \le \alpha \le +\infty$, we will denote either $L^\alpha(\mathbf{R}^n)$ or $[L^\alpha(\mathbf{R}^n)]^n$. Both norms will be denoted $| \;|_\alpha$. Similarly, $W^{s,p}$, $s \in R$, $p \in [1, +\infty[$ will denote the Sobolev spaces $W^{s,p}(\mathbf{R}^n)$ and $[W^{s,p}(\mathbf{R}^n)]^n$, and $\| \;\|_{s,p}$ will denote the respective norms. For convenience, we set $W^s \equiv W^{s,2}$, $\| \;\|_s \equiv \| \;\|_{s,2}$. For definitions and properties see [7], [8], [11], [19]. We also define $H \equiv \{u \in L^2 \colon \nabla \cdot u = 0\}$ and $V \equiv \{u \in H^1 \colon \nabla \cdot u = 0\}$. In §2 we will utilize the Bessel potential spaces $H^{s,p}(\mathbf{R}^n)$ (see [7], [11], [19]). Recall that $H^{s,p} = W^{s,p}$ for any integer s. For a vector field v, we define

$$|\nabla v|^2 = \sum_{i,j=1}^n \left(\frac{\partial v_j}{\partial x_i}\right)^2.$$

Sometimes we will utilize abbreviated notations, such as $|\nabla v|_\alpha$ instead of $\big|\,|\nabla v|\,\big|_\alpha$, $L^p(X)$ instead of $L^p(0,T;X)$, and so on. Standard notation will be used without an explicit definition. Moreover, unless otherwise specified, the domain of integration with respect to the space variables is $\mathbf{R}^n$.

For the sake of convenience we define the quantities

$$N_\alpha(v) \equiv \int |\nabla v|^2\, |v|^{\alpha-2}\, dx,$$

$$M_\alpha(x) \equiv \int \left|\nabla\, |v|^{\alpha/2}\right|^2 dx.$$

These quantities will play a leading role in the sequel.

In this section we assume $\alpha > n$ (except that in Theorems 1.4 and 1.5, $\alpha > 2$ would suffice) and

$$a \in L^\alpha, \quad f \in L^1(0,T;L^\alpha). \tag{1.1}$$

Here we will establish some a priori estimates for solutions of $(0.1)_{1,2,3}$. In order to justify the calculations that follow, we assume in this section that

$$v \in L^1\big(0,T;W^{2,\alpha}\big), \quad v' \in L^1\big(0,T;L^\alpha\big). \tag{1.2}$$

Obviously, these assumptions are not strictly necessary. Assumption (1.2) implies further regularities for v and p. Specifically, since

$$\|\;\|_{1,\alpha} \le c|\;|_\alpha^{1/2}\,\|\;\|_{2,\alpha}^{1/2},$$

assumption (1.2) implies $v \in C\big([0,T];L^\alpha\big) \cap L^2\big(0,T;W^{1,\alpha}\big)$. On the other hand, a well–known Sobolev embedding theorem [8] implies $\nabla v \in L^1\big(0,T;L^\infty\big)$, hence from equation $(0.1)_1$ it follows that $\nabla p \in L^1\big(0,t;L^\alpha\big)$.

Moreover, since $v \in L^\infty(L^\alpha) \cap L^2(L^\infty)$, one has $v^2 \in L^2(L^\alpha)$. Consequently, by using Calderón–Zygmund's inequality [18], equation (1.10) yields $p \in L^2\big(0,T;L^\alpha\big)$.

We start by proving the following result:

Lemma 1.1. *Let v be a solution of $(0.1)_{1,2,3}$ belonging to the class (1.2). Then v verifies the estimates (1.5), (1.8) and*

$$\frac{1}{\alpha}\frac{d}{dt}|v|_\alpha^\alpha + \frac{\mu}{2}N_\alpha(v) + 4\mu\frac{\alpha-2}{\alpha^2}M_\alpha(v) \le \frac{(\alpha-2)^2}{2\mu}\int |p|^2\,|v|^{\alpha-2}\,dx + |f|_\alpha\,|v|_\alpha^{\alpha-1}. \tag{1.3}$$

Proof. Note, first, that

$$\big|\nabla\,|v|^{\alpha/2}\big| \le \frac{\alpha}{2}\,|v|^{\alpha/2-1}|\nabla v| \qquad \text{a. e. in } \mathbf{R}^n . \tag{1.4}$$

In order to prove (1.3), we multiply both sides of equation (0.1) by $|v|^{\alpha-2}v$, and integrate over $\mathbf{R}^n$. After suitable integration by parts (recall that $\nabla\cdot v = 0$) we obtain the identity

$$\frac{1}{\alpha}\frac{d}{dt}\,|v|_\alpha^\alpha + \mu N_\alpha(v) + 4\mu\,\frac{\alpha-2}{\alpha^2}\,M_\alpha(v) = \tag{1.5}$$

$$-\int \nabla p\cdot v|v|^{\alpha-2}\,dx + \int f\cdot v|v|^{\alpha-2}\,dx .$$

On the other hand, one has

$$-\int \nabla p\cdot v|v|^{\alpha-2}\,dx \tag{1.6}$$

$$= (\alpha-2)\sum_{i,j=1}^{n}\int p\frac{\partial v_j}{\partial x_i}\,v_iv_j|v|^{\alpha-2}\,dx$$

$$= \frac{2(\alpha-2)}{\alpha}\int p|v|^{\alpha/2-2}\big(\sum_{i=1}^{n} v_i\big)\left[\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|v|^{\alpha/2}\right)\right]dx .$$

From (1.5) and $(1.6)_1$, since

$$\left|\sum_{i,j} v_iv_j\frac{\partial v_j}{\partial x_i}\right| \le |v|^2\,|\nabla v|, \tag{1.7}$$

one gets

$$\frac{1}{\alpha}\frac{d}{dt}\,|v|_\alpha^\alpha + \mu N_\alpha(v) + 4\mu\,\frac{\alpha-2}{\alpha^2}\,M_\alpha(v) \tag{1.8}$$

$$\le (\alpha-2)\int |p|\,|\nabla v|\,|v|^{\alpha-2}\,dx + |f|_\alpha\,|v|_\alpha^{\alpha-1} .$$

Since

$$(\alpha-2)\int |p|\,|\nabla v|\,|v|^{\alpha-2}\,dx \le \frac{(\alpha-2)^2}{2\mu}\int p^2|v|^{\alpha-2}\,dx + \frac{\mu}{2}\,N_\alpha(v),$$

(1.3) follows. □

Lemma 1.2. *Let v be a solution of $(0.1)_{1,2,3}$ in the class (0.2). Then*

$$\frac{1}{\alpha}\frac{d}{dt}|v|_\alpha^\alpha + \frac{\mu}{2}N_\alpha(v) + 4\mu\frac{\alpha-2}{\alpha^2}M_\alpha(v) \leq c\frac{(\alpha-2)^2}{\mu}|v|_{\alpha+2}^{\alpha+2} + |f|_\alpha |v|_\alpha^{\alpha-1}. \tag{1.9}$$

Proof. Hölder's inequality gives

$$\int |p|^2 |v|^{\alpha-2}\, dx \leq |p|^2_{(\alpha+2)/2} |v|^{\alpha-2}_{\alpha+2}.$$

On the other hand, by applying the divergence operator to both sides of equation (0.1) one gets

$$-\Delta p = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(v_i v_j). \tag{1.10}$$

By using the Calderón–Zygmund inequality [18], one obtains

$$|p|_{(\alpha+2)/2} \leq c|v|^2_{\alpha+2}. \tag{1.11}$$

Consequently,

$$\int |p|^2 |v|^{\alpha-2}\, dx \leq c|v|^{\alpha+2}_{\alpha+2}. \tag{1.12}$$

Equation (1.9) follows from (1.3) and (1.12). □

Lemma 1.3. *Let $w \in W^{1,\alpha}$. Then*

$$|v|^{\alpha+2}_{\alpha+2} \leq c|v|_\alpha^{\alpha-n+2}[M_\alpha(v)]^{n/\alpha}. \tag{1.13}$$

In particular,

$$|v|^{\alpha+2}_{\alpha+2} \leq c|v|_\alpha^{\alpha-n+2}[N_\alpha(v)]^{n/\alpha}. \tag{1.14}$$

Proof. Define $2^* = \frac{2n}{n-2}$. Since

$$\frac{\alpha}{2(\alpha+2)} = \frac{1-\vartheta}{2} + \frac{\vartheta}{2^*}, \qquad \text{for } \vartheta = \frac{n}{\alpha+2},$$

one gets

$$|g|_{2(\alpha+2)/\alpha} \leq |g|_2^{1-n/(\alpha+2)} |g|_{2^*}^{n/(\alpha+2)}. \tag{1.15}$$

On the other hand, by a well-known Sobolev's embedding theorem [8] one has $|g|_{2^*} \leq c|\nabla g|$. By applying this estimate, together with (1.15), to the function $g = |v|^{\alpha/2}$, one gets (1.13). Moreover, (1.13) and (1.4) yield (1.14). □

Theorem 1.4. *Let $\alpha > n$, and let v be a solution of $(0.1)_{1,2,3}$ in the class (1.2). Then*

$$\frac{1}{\alpha}\frac{d}{dt}|v|_\alpha^\alpha + \frac{\mu}{4}N_\alpha(v) \leq c\mu^{(-n+\alpha)/(\alpha-n)}|v|_\alpha^{\alpha(\alpha-n+2)/(\alpha-n)} + |f|_\alpha |v|_\alpha^{\alpha-1}. \tag{1.16}$$

In particular, (0.3) *holds.*

Proof. From (1.9) and (1.14) one obtains,

$$\frac{1}{\alpha}\frac{d}{dt}|v|_\alpha^\alpha + \frac{\mu}{2}N_\alpha(v) + 4\mu\frac{\alpha-2}{\alpha^2}M_\alpha(v) \leq \frac{c}{\mu}[N_\alpha(v)]^{n/\alpha}|v|_\alpha^{\alpha-n+2} + |f|_\alpha |v|_\alpha^{\alpha-1}.$$

By applying Young's inequality, with exponents α/n and $\alpha/(\alpha-n)$, to the first term on the right–hand side of the inequality above, one gets (1.16). The estimate (1.16) yields (0.3). □

Now we state an immediate consequence of (1.16). For convenience, define

$$q = \frac{2\alpha}{\alpha-n}, \qquad k = c_8\mu^{-(\alpha+n)/(\alpha-n)}.$$

Consider the following Cauchy problem for o. d. e.,

$$\begin{cases} y' = ky^{1+q} + |f(t)|_\alpha, & t > 0, \\ y(0) = |a|_\alpha. \end{cases} \tag{1.17}$$

Let T_α be the time existence of the maximal solution $y(t)$ of (1.17). One then has the following result.

Theorem 1.5. *Let $\alpha > n$, and assume that a and f verify (0.2) and (1.1). Let v be a solution of $(0.1)_{1,2,3}$ in the class (1.2), and let $y(t)$ and T_α be defined as above. Then*

$$|v(t)|_\alpha \leq y(t), \qquad \forall t \in [0, T_\alpha[. \tag{1.18}$$

Proof. Note that inequality (1.18) has the following meaning. Given $\tau \in]0, T_\alpha[$, if v is a solution of $(0.1)_{1,2,3}$ in $]0,\tau[$, which belongs to the class (1.2) in $]0,\tau[$, then (1.18) holds in $[0,\tau]$.

By defining $z(t) \equiv |v(t)|_\alpha$, from (1.16) one has $z' \leq kz^q + |f(t)|_\alpha$, $z(0) = |a|_\alpha$. The result follows by comparison theorems for o. d. e. □

§**2.** In this section we prove the existence theorems 2.1 and 2.2. For the reader's convenience, some auxiliary results are proved in the appendix.

Theorem 2.1. *Let $\alpha > n$, and assume that a and f verify* (0.2) *and* (1.1). *Let T_α be defined as in Theorem* 1.5. *Then, there exists a (unique) solution $v \in C_*([0,T_\alpha[;L^\alpha)$ of the Navier–Stokes equation* (0.1). *This solution satisfies inequality* (1.18).

Proof. The uniqueness follows from [2]. In view of the uniqueness, it is sufficient to argue on an arbitrary interval $[0,\tau]$, for $\tau \in [0,T_\alpha[$. Let a_n and f_n be regular functions (say, C^∞ functions, with compact support with respect to the space variables) verifying (0.2), and such that $a_n \to a$ in L^α, $f_n \to f$ in $L^1(0,T;L^\alpha)$. Denote by $T_{\alpha,n}$ the time existence (in Theorem 1.5) corresponding to the data a_n and f_n. Since $T_{\alpha,n} \to T_\alpha$ as $n \to +\infty$ we may assume that $T_{\alpha,n} \geq \tau$. Due to the regularity of the data a_n and f_n, it is well known that there exists a (unique) local *regular* solution v_n. In particular, $v_n \in L^\infty(H) \cap L^2(V)$. From the a priori estimate of Theorem 1.5, it follows that if v_n is regular in $[0,s[$, $0 \leq s \leq \tau$, then $v_n \in L^\infty(0,s;L^\alpha)$. On the other hand, if $v_n \in L^\infty(0,s;L^\alpha)$, then v_n is regular and in $[0,s]$. This is a well-known result (in line with Serrin's paper [15]), which can be proved by using a boot-strap method, together with (1.10) and with regularity results for the solutions of the heat equation.

The results stated above imply that the regular solution v_n exists in all $[0,\tau]$.

Since the sequence v_n is uniformly bounded in $L^\infty(0,\tau;L^\alpha)$ (by Theorem 1.5), there exists a subsequence which is weak-$*$ convergent to a function $v \in L^\infty(0,\tau;L^\alpha)$.[1] Clearly, the regular solution v_n solves the following weak formulation of the Navier–Stokes equation (0.1),

$$\text{(2.1)} \qquad \int_0^\tau \int [v_n \cdot \varphi' + \mu v_n \cdot \Delta\varphi + [(v_n \cdot \nabla)\varphi] \cdot v_n + f_n \cdot \varphi] \, dx \, dt = -\int a_n \cdot \varphi(0) \, dx,$$

where $\varphi(t,x)$ is any divergence-free test function, with compact support with respect to the space variables, and such that $\varphi(\tau,x) = 0$, $\forall x \in \mathbf{R}^n$.

To prove that the limit function v is a solution of the Navier–Stokes equation (2.1), with data a and f, we adapt to our case ($\alpha \neq 2$ and $\Omega = \mathbf{R}^n$) the method of Lions, described in [8], Chapter I, §6. We will prove (in Appendix A) the main point, namely, that there exist subsequences v_ν such that

$$\text{(2.2)} \qquad \lim_{\nu \to +\infty} v_\nu = v \qquad \text{in } L^p(0,\tau;L^\alpha(B_{\mathbf{R}})), \quad \forall \mathbf{R} > 0.$$

Here, $B_R \equiv \{x \in \mathbf{R}^n\colon |x| < \mathbf{R}\}$, and $p \in [1,+\infty[$ is arbitrarily chosen. Since the convergence in $L^1(]0,\tau[\times B_{\mathbf{R}})$ implies pointwise convergence for a

[1] Actually, by the uniqueness of the solution v [2], it follows that the sequence itself converges to v.

subsequence, we can assume that $v_\nu(t,x) \to v(t,x)$ almost everywhere in $]0,\tau[\times \mathbf{R}^n$. This is the main tool used to pass to the limit in the nonlinear term of equation (2.1).

Since $v \in L^\infty(0,\tau;L^\alpha) \cap C([0,\tau];X)$, where X is the Banach space $X \equiv W^{-1,\alpha} + W^{s-2,\alpha/2}$, $s < 1$, the weak continuity of $v(t)$ follows easily. Note that, as a consequence of $(0.1)_1$, one has $v' \in L^1(0,\tau;X)$; see $(4.3)_2$ and $(4.4)_2$ in Appendix A. □

In the next section we will be particularly interested on finite energy solutions. Hence, we establish here the following result:

Theorem 2.2. *Let $a \in H \cap L^\alpha$, $f \in L^1(0,T;H\cap L^\alpha)$, $\alpha > n$, and let T_α and $y(t)$ be defined as above. Then there exists a (unique) solution v of the Navier–Stokes equation* (0.1) *in the class $C([0,T_\alpha[;H\cap L^\alpha) \cap L^2(0,T_\alpha;V)$. Moreover,* (1.18) *holds.*

This result can be regarded as a consequence of Theorem 2.1 and energy estimate (2.3). However, it seems more natural to pass to the limit in equation (2.1) by using the energy estimate

$$\|v_n\|_{L^\infty(0,\tau;H)} + \mu\|v_n\|_{L^2(0,\tau;V)} \le |a_n|_2 + \|f_n\|_{L^1(0,\tau;H)}, \tag{2.3}$$

which is now available. In this case, the regular approximating data a_n and f_n verify the assumptions $a_n \to a$ in $H\cap L^\alpha$, $f_n \to f$ in $L^1(0,T;H\cap L^\alpha)$. By Theorem 1.5, one again has

$$\|v_n\|_{L^\infty(0,\tau;L^\alpha)} \le \text{constant independent of } n. \tag{2.4}$$

The proof of Theorem 2.2 follows the same ideas as in Theorem 2.1, except that for the compactness argument, which is now similar to that utilized (see [8]) for the usual Faedo–Galerkin procedure.[2] In fact, integrating by parts and by Sobolev's embedding theorem, it follows that the map

$$\varphi \to \int_0^\tau \int [(v_n\cdot\nabla)v_n]\cdot\varphi\,dx\,dt, \qquad \forall\varphi \in L^2(V),$$

defines a uniformly bounded family in $L^2(V')$. Here, we utilize (2.4), and also (2.3) if $n = 3$. By using $(0.1)_1$, it follows in particular that v_n is uniformly bounded in $L^1(V')$. Hence, for every $R > 0$, one has

$$\begin{cases} \|v_n\|_{L^2(0,\tau;V(B_R))} \le \text{constant}, \\ \|v_n'\|_{L^1(0,\tau;V'(B_R))} \le \text{constant}, \end{cases} \tag{2.5}$$

[2] Here, however, by using (2.4), we get stronger a priori bounds, which are independent of the dimension n.

uniformly with respect to n. By using (2.5), it is easy to prove that there exists a subsequence v_ν, strongly convergent to v in $L^2(0,\tau;L^2(B_{\mathbf{R}}))$, $\forall \mathbf{R} > 0$, and pointwise convergent, almost everywhere in $]0,\tau[\times \mathbf{R}^n$ (see the end of Appendix A). The uniqueness of the solution follows as in Prodi [14], Foias [3], Serrin [16]. See also [8], Chapter I, §6. The strong continuity of v will be proved in Appendix B.

§3. In this section we prove global estimates and decay properties for the norm $|v(t)|_\alpha$, $t \in [0,+\infty[$, $\alpha \geq n$, of the solution $v \in C([0,+\infty[;L^\alpha \cap L^2)$ of the Navier–Stokes equations, constructed in §2, Theorem 2.2.

The global a priori estimates of this section, together with the Local Existence Theorem 2.2, yield the global existence of the solutions. Obviously, the global estimates of this section are proved first for solutions belonging to the class (1.2), hence for the approximating solutions v_n utilized in Theorem 2.2. By passing to the limit when $n \to +\infty$, one shows that the estimates hold for the limit function v (argue as done for the local estimate (1.18) in Theorem 2.2). For clearness, and in order to avoid tedious repetitions, we will argue directly on the solution v.

Lemma 3.1. *Let $\alpha > 2$. Then*

$$N_\alpha(v) \geq c|v|_2^{-(4\alpha)/(\alpha-2)n}|v|_\alpha^{\alpha+(4\alpha)/(\alpha-2)n}. \tag{3.1}$$

Proof. From (1.4) and from Sobolev's embedding theorem ($|g|_{2^*} \leq c|\nabla g|_2$, $2^* = \frac{2n}{n-2}$), one gets

$$N_\alpha(v) \geq c|v|^\alpha_{\alpha n/(n-2)}. \tag{3.2}$$

Furthermore, if $\vartheta = \frac{4}{[4+(\alpha-2)n]}$, one has

$$\frac{1}{\alpha} = \frac{\vartheta}{2} + \frac{1-\vartheta}{\dfrac{\alpha n}{n-2}}.$$

Consequently

$$|v|_\alpha \leq |v|_2^{4/(4+(\alpha-2)n)}\,|v|_{\alpha n/(n-2)}^{(\alpha-2)n/(4+(\alpha-2)n)}. \tag{3.3}$$

From (3.2) and (3.3), one gets (3.1). □

Let now v be as in Theorem 1.4. By using (1.16) and (3.1), a straightforward calculation gives

$$y' \leq c_8\left[c_9\mu|v|_2^{-\beta} - \mu^{-(\alpha+n)/(\alpha-n)}y^\gamma\right]y^{1+\beta} + |f|_\alpha, \tag{3.4}$$

where for convenience we define $y(t) \equiv |v(t)|_\alpha$, $\beta = \frac{4\alpha}{(\alpha-2)n}$, $\gamma = \frac{2\alpha^2(n-2)}{n(\alpha-2)(\alpha-n)}$. Let $T \in]0, +\infty]$. It is well known that for every $t \in [0,T]$, one has

$$|v(t)|_2 \le |a|_2 + \int_0^T |f(\tau)|\, d\tau \equiv K. \tag{3.5}$$

If $K = 0$, then $v(t) = 0$, $\forall t \ge 0$. Hence we assume that $K > 0$. From (3.4) one gets

$$y' \le -c_8\left[c_9 \mu K^{-\beta} - \mu^{-(n+\alpha)/(\alpha-n)} y^\gamma\right] y^{1+\beta} + |f|_\alpha. \tag{3.6}$$

Let us prove now the following result:

Lemma 3.2. *Assume that* (3.6) *holds. If*

$$y(0)^\gamma \le \frac{c_9}{2} K^{-\beta} \mu^{2\alpha/(\alpha-n)} \tag{3.7}$$

and

$$|f(t)|_\alpha \le c_8 \mu \frac{c_9}{4} K^{-\beta}\left[\mu^{2\alpha/(\alpha-n)} \frac{c_9}{2} K^{-\beta}\right]^{(1+\beta)/\gamma} \tag{3.8}$$

almost everywhere in $[0,T]$, *then*

$$y(t)^\gamma \le \frac{c_9}{2} \mu^{2\alpha/(\alpha-n)} K^{-\beta}, \qquad \forall t \in [0,T]. \tag{3.9}$$

Proof. For $t = 0$, (3.9) holds. Moreover, by using (3.6) and (3.8), one easily shows that whenever (3.9) holds with the equal sign, then $y'(t) < 0$. This proves the lemma. □

Theorem 0.2 follows from Lemma 3.2, by setting $c_1 = (c_9/2)^{1/\gamma}$, $c_2 = c_8(c_9/4)(c_9/2)^{(1+\beta)/\gamma}$.

Let us now consider the homogeneous case $f \equiv 0$. By setting $c_3 = (c_9/2)^{1/\gamma}$, assumption (0.9) is nothing but (3.7), since $K = |a|_2$. Hence, from (3.6), it follows that

$$y' \le -c_4 \mu |a|_2^{-\beta} y^{1+\beta},$$

for every $t \in [0,T]$, where for convenience we put $c_4 = c_8 c_9/2$. Consequently, by comparison theorems for o. d. e., one gets

$$y(t) \le y(0)\left[1 + c_4 \mu \beta |a|_2^{-\beta} y(0)^\beta t\right]^{-(1/\beta)}.$$

This yields (0.10) and (0.11). □

Remark 3.3. In a bounded domain Ω (with the boundary condition $v = 0$ on $\partial\Omega$), by using the following Poincaré inequality $|g|_2 \le c(\Omega,n)|\nabla g|_2$, $\forall g \in H_0^1(\Omega)$, one gets $N_\alpha(v) \ge c(\alpha,n,\Omega)|v|_\alpha^\alpha$ (compare with (3.2)). Hence, from (1.16), one would obtain

$$\frac{d}{dt}|v|_\alpha + c|v|_\alpha \le c_\mu |v|_\alpha^{(3\alpha-n)/(\alpha-n)} + |f|_\alpha, \tag{3.10}$$

which would immediately give quite a strong estimate for $|v(t)|_\alpha$; in particular, if $f \equiv 0$, one would have an exponential decay for $|v(t)|_\alpha$. However, some devices must be introduced in order to obtain estimates like (1.9) (not obtainable from (1.12) alone).

In the remainder of this section we present the asymptotic estimates for the limit case $\alpha = n$ (here, the positive constants c depend only on n). We wish to point out that these estimates will be proved only for sufficiently regular solutions (say, in the class (1.2)). However, one can apply these L^n estimates, together with the uniform estimate in $L^\infty(H) \cap L^2(V)$, to a sequence of regular approximate solution v_n, in order to get (by a compactness argument) a weak solution $v \in L^\infty(H) \cap L^2(V) \cap L^\infty(L^n)$, verifying the L^n estimate under consideration. Alternatively, one can utilize the methods introduced by Kato (see for instance [6]) to get the existence of the solution.[3]

By starting from (1.9) and (1.14), we obtain

$$\frac{1}{n}\frac{d}{dt}|v|_n^n + \frac{\mu}{2}N_n(v) \le \frac{c_{10}}{2\mu}N_n(v)\,|v|_n^2 + |f|_n\,|v|_n^{n-1},$$

where c_{10} is a suitable constant. Hence

$$\frac{1}{n}\frac{d}{dt}|v|_n^n \le -\frac{\mu}{2}N_n(v)\left[1 - \frac{c_{10}}{\mu^2}|v|_n^2\right] + |f|_n\,|v|_n^{n-1}. \tag{3.11}$$

From (3.11) and (3.1) it follows that

$$\frac{d}{dt}|v|_n^n \le -\frac{\mu}{2}c_{11}|v|_2^{-(4/(n-2))}|v|_n^{(n+2)/(n-2)}\left[1 - \frac{c_{10}}{\mu^2}|v|_n^2\right] + |f|_n,$$

provided $c_{10}\mu^{-2}|v|_n^2 \le 1$. Recalling (3.5), one shows that if $|a|_n \le \mu(2c_{10})^{-1/2}$ and if

$$\|f\|_{L^\infty(0,T;L^n)} \le \frac{\mu}{8}c_{11}K^{-4/(n-2)}\left(\frac{\mu}{\sqrt{2c_{10}}}\right)^{(n+2)/(n-2)},$$

then $|v(t)|_n \le \mu(2c_{10})^{-1/2}$, $\forall t \in [0,T]$. In fact, $\frac{d}{dt}|v|_n < 0$, whenever $|v|_n = \mu(2c_{10})^{-1/2}$. This proves the first part of the following result.

[3] For uniqueness results in $L^\infty(L^n(\Omega))$, we refer the reader to [17].

Theorem 3.3. *Let $a \in L^n \cap L^2$ and $f \in L^1(0,T;L^2) \cap L^\infty(0,T;L^n)$ verify (0.2). Assume that v is a sufficiently regular (say, in class (1.2)) solution of (0.1). Then there exist positive constants c_6 and c_{12} such that if*

$$|a|_n \leq c_6 \mu \tag{3.13}$$

and

$$\Big[|a|_2 + \|f\|_{L^1(0,T;L^2)}\Big]^{4/(n-2)} \|f\|_{L^\infty(0,T;L^n)} \leq c_{12}\mu^{2n/(n-2)}, \tag{3.14}$$

one has

$$|v(t)|_n \leq c_6\mu, \qquad \forall t \in [0,T]. \tag{3.15}$$

Moreover, if $f = 0$, and if (3.13) holds, then

$$|v(t)|_n \leq |a|_n \Big[1 + c\mu |a|_2^{-4/(n-2)} |a|_n^{4/(n-2)} t\Big]^{-(n-2)/4}, \tag{3.16}$$

for every $t \in [0, +\infty[$. In particular,

$$|v(t)|_n \leq c|a|_2 \left(\frac{1}{\mu t}\right)^{(n-2)/4}, \qquad \forall t > 0. \tag{3.17}$$

In order to prove the statement concerning the case $f \equiv 0$, we remark that if a verifies $|a|_n \leq \mu(2c_{10})^{-1/2}$, then

$$\frac{d}{dt}|v|_n \leq -\frac{c_{11}\mu}{4}|a|_2^{-4/(n-2)}|v|_n^{1+4/(n-2)}, \qquad \forall t > 0.$$

Now (3.16) follows, by using comparison theorems for o. d. e. □

Remark. Note that the estimates proved in Theorem 3.3 are just those proved in Theorems 0.2 and 0.3, by setting there $a = n$.

§4. Appendix A. In this appendix we prove the statement (2.2). We start by establishing an auxiliary lemma, whose proof is given for the reader's convenience. For the sake of brevity we utilize here some results on parabolic semigroups. More direct computations could be done, by using the heat potentials in the whole space.

Lemma 4.1. *Let u be a solution of the heat equation $u' - \Delta u = f$ in $]0,T[\times \mathbf{R}^n$, with zero initial data. Assume that $1 \leq p \leq +\infty$ and $1 < q < +\infty$. If $f \in L^p(0,T;L^q)$, then $u \in L^p(0,T;W^{s,q})$, $\forall s \in [0,2[$. If $f \in L^p(0,t;W^{-1,q})$, then $u \in L^p(0,T;W^{s,q})$, $\forall s \in [0,1[$.*

Proof. By a well-known device, we can replace $-\Delta$ by $A \equiv -\Delta + 1$. Since $-A$ is the generator of a holomorphic semigroup in L^q, and $0 \in \rho(A)$, one has $\|A^\vartheta e^{tA}\| \leq ct^{-\vartheta}$, $0 < \vartheta < 1$. (see [13]). Hence,

$$|A^\vartheta u(t)|_q \leq \int_0^T \frac{c}{|t-s|^\vartheta} |f(s)|_q \, ds, \qquad \forall t \in [0,T].$$

By utilizing well-known results on the convolution of functions, one shows that $u \in L^p\big(D(A^\vartheta)\big)$. The first statement in the lemma follows, since $D(A^\vartheta) = H^{2\vartheta,q} \hookrightarrow W^{2\vartheta-\varepsilon,q}$, for $\varepsilon > 0$ (see [7], [11], [19]). The second statement follows from the first one, by using the isomorphism $A^{-1/2}$, from $W^{-1,q}$ onto L^q. □

Let now v_n be defined as in the proof of Theorem 2.1. We want to show that there exists a subsequence v_ν verifying (2.2). Let p_n be the pressure corresponding to the regular solution v_n, and consider the solutions u_n and w_n of the equations

$$\begin{cases} u_n' - \mu\Delta u_n = -\nabla p_n + (v_n \cdot \nabla)v_n & \text{in }]0,\tau[\times \mathbf{R}^n, \\ u_n = 0 & \text{for } t = 0, \end{cases} \tag{4.1}$$

and

$$\begin{cases} w_n' - \mu\Delta w_n = f_n & \text{in }]0,\tau[\times \mathbf{R}^n, \\ w_n = a_n & \text{for } t = 0, \end{cases} \tag{4.2}$$

respectively. Note that it is possible to consider each scalar equation separately. Clearly, $v_n = u_n + w_n$. Since the sequence v_n is uniformly bounded in $L^\infty(0,\tau;L^\alpha)$, the terms

$$(v_n \cdot \nabla)v_n = \sum_i \frac{\partial}{\partial x_i}(v_{n,i} v_n)$$

are uniformly bounded in $L^\infty(0,\tau;W^{-1,\alpha/2})$. The same holds for ∇p_n, as a consequence of (1.10) and of the Calderón–Zygmund inequality. By Lemma 4.1, one has

$$\begin{cases} \|u_n\|_{L^\infty(W^{s,\alpha/2})} \leq \text{constant}, \\ \|u_n'\|_{L^\infty(W^{s-2,\alpha/2})} \leq \text{constant}, \end{cases} \tag{4.3}$$

where $s < 1$ and the constants are independent of n. On the other hand, one has

$$\begin{cases} \|w_n\|_{L^p(W^{1,\alpha})} \leq \text{constant}, \\ \|w_n'\|_{L^1(W^{-1,\alpha})} \leq \text{constant} \end{cases} \tag{4.4}$$

for every $p \in [1,2[$. The estimate $(4.4)_1$ is proved by using an argument similar to that utilized in the proof of Lemma 4.1, and by recalling that $L^p(D(A^{1/2})) = L^p(W^{1,\alpha})$. The estimate $(4.4)_2$ follows from $(4.4)_1$ and $(4.2)_1$.

Define $B_{\mathbf{R}} = \{x \in \mathbf{R}^n \colon |x| \le \mathbf{R}\}$. Clearly, the estimates (4.3) and (4.4) hold with $\mathbf{R}^n$ replaced by $B_{\mathbf{R}}$. Moreover, the embeddings $W^{s,\alpha/2}(B_{\mathbf{R}}) \hookrightarrow L^\alpha(B_{\mathbf{R}})$, $s > n/\alpha$, and $W^{1,\alpha}(B_{\mathbf{R}}) \hookrightarrow L^\alpha(B_{\mathbf{R}})$, are compact. Consequently, well–known compactness theorems (see Lions [8], Chapter I, §5, and Aubin [1]) show that the sequence v_n is relatively compact in $L^p(0,\tau;L^\alpha)$, $1 \le p < 2$. Actually, this result holds for every $p \in [1,+\infty[$, since in addition the sequence v_n is bounded in $L^\infty(L^\alpha)$.

Finally, fix a sequence of radius $\mathbf{R}_m$ such that $\lim \mathbf{R}_m = +\infty$ as $m \to +\infty$, and select convergent subsequences (successively, with respect to m) in $L^p(0,\tau;L^\alpha(B_{\mathbf{R}_m}))$. The diagonal subsequence verifies the desired property (2.2). □

§**5. Appendix B.** Here we prove that the solution v in Theorem 2.2 belongs to $C([0,\tau];L^\alpha)$, for every $\tau \in [0,T_\alpha[$. We start by proving the following result:

Lemma 5.1. *Let a, f and v be defined as in Theorem 2.2, let $q \in [1,2[$, $\beta \in [2,\alpha[$, and assume that $\nabla v \in L^p(0,\tau;L^\beta)$. Define γ by the equation $1/\gamma = (1/\alpha) + (1/\beta)$, and let $s \in]n/\alpha,1[$. Moreover, if $\gamma > n$, assume that $s > n/\gamma$. Finally, define β_1 by the equation*

$$\frac{1}{\beta_1} = \frac{1}{\gamma} - \frac{s}{n} = \frac{1}{\beta} + \left(\frac{1}{\alpha} - \frac{s}{n}\right).$$

One then has

$$(5.1)\qquad \begin{cases} \nabla v \in L^p\left(0,\tau;L^{\beta_1}\right) & \textit{if } \dfrac{1}{\beta_1} \ge \dfrac{1}{\alpha}, \\ \nabla v \in L^p(0,\tau;L^\alpha) & \textit{if } \dfrac{1}{\beta_1} < \dfrac{1}{\alpha}. \end{cases}$$

Proof. Let $v = u + w$, where u and w are the solutions of the linear equations (4.1), (4.2) after dropping the indices n. Since $v \in L^\infty(L^\alpha)$, one has $(v\cdot\nabla)v \in L^p(L^\gamma)$. Moreover, $-\Delta p = \operatorname{div}(v\cdot\nabla)v$ implies $\nabla p \in L^p(L^\gamma)$. From Lemma 4.1, one deduces that $\nabla u \in L^p(W^{s,\gamma})$.

If $1/\beta_1 > 0$, then by Sobolev's embedding theorem one has $W^{s,\gamma} \hookrightarrow L^{\beta_1}$. Hence $(5.1)_1$ holds for ∇u. Similarly, if $1/\beta_1 = 0$, then $W^{s,\gamma} \hookrightarrow L^\infty$, hence $(5.1)_2$ holds for ∇u. Finally, if $1/\beta_1 < 0$, then $W^{s,\gamma} \hookrightarrow L^\infty$, and $(5.1)_2$ holds again for ∇u. Equation (5.1) holds also for ∇w, since $\nabla w \in L^p(L^\alpha \cap L^2)$ (argue as for the proof of $(4.4)_1$). □

We prove now that $v \in C\big([0,\tau[; L^\alpha \cap L^2\big)$. By starting from the value $\beta = 2$, and by applying successively Lemma 5.1, one shows that $\nabla v \in L^p(L^\alpha)$, $\forall p \in [1,2[$. Consequently, $(v \cdot \nabla)v$ and ∇p belong to $L^p(L^q)$, $\forall p \in [1,2[$, $\forall q \in]1,\alpha/2]$. By using Lemma 4.1 we show that ($v = u + w$, as in the proof of Lemma 5.1),

$$u \in L^p(W^{s,p}) \cap W^{1,p}(W^{s-2,q}), \qquad \forall 0 \le s < 2.$$

Hence,

$$u \in W^{1-\vartheta,p}(W^{s-2(1-\vartheta),q}), \qquad \forall 0 \le \vartheta \le 1.$$

By choosing $q = \frac{\alpha}{2}$, $\frac{n}{2\alpha} < \vartheta < \frac{1}{2}$, $s = 2(1-\vartheta) + \frac{n}{\alpha}$, $\frac{1}{1-\vartheta} < p < 2$, well-known embedding theorems yield $u \in C(L^\alpha)$. By choosing $q = \frac{2\alpha}{2+\alpha}$, one gets $u \in C(L^2)$. Hence $u \in C(L^\alpha \cap L^2)$. On the other hand, well-known results on the Cauchy problem for parabolic equations give $w \in C(L^\alpha \cap L^2)$. □

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