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On the Motion of Non-homogeneous Fluids in the Presence of Diffusion

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1. INTRODUCTION

In this paper we study the motion of a non-viscous fluid consisting of two components, both incompressible, say, saturated salt water and water, taking into consideration the diffusion among these components. The equations of the model are obtained, for example, in Frank and Kamenetskii [6], and the viscous case is studied in Ignat'ev and Kuznetsov [8] and Kazhikhov and Smagulov [9, 10]. For a presentation of this kind of problems see also Kuznetsov [11]. Observe that in [8–10] the problem is solved only for fluids whose viscosity μ is sufficiently large; i.e.,

$$\mu > \frac{\lambda}{2} \operatorname*{osc}_{\Omega} \rho_0. \tag{*}$$

Since we get the existence and uniqueness of the solution for non-viscous fluid ($\mu = 0$), condition (*) can probably be suppressed.

Let $\rho_i = \text{const.} > 0$, $i = 1, 2$, be the characteristic densities of the components of the mixture, $v^{(1)} = v^{(1)}(t, x)$, $v^{(2)} = v^{(2)}(t, x)$ their velocities and $c = c(t, x)$, $d = d(t, x)$ the mass and volume concentration of the salt water.

The mean density of the mixture is

$$\rho(t, x) \equiv d(t, x)\rho_1 + [1 - d(t, x)]\rho_2$$

and we introduce the mean-volume and mean-mass velocities as follows

$$v(t, x) \equiv d(t, x)\, v^{(1)}(t, x) + [1 - d(t, x)]\, v^{(2)}(t, x),$$
$$w(t, x) \equiv c(t, x)\, v^{(1)}(t, x) + [1 - c(t, x)]\, v^{(2)}(t, x).$$

Then the equations of the motion are ($\dot{w}$ denotes the time derivative)

$$\rho[\dot{w} + (w \cdot \nabla)w - b] = -\nabla\pi \qquad \text{in} \quad Q_T \equiv]0, T[\times \Omega, \tag{1.1}$$

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$$\dot{\rho} + \operatorname{div}(\rho w) = 0 \qquad \text{in} \quad Q_T, \tag{1.2}$$

$$\operatorname{div} v = 0 \quad \text{in} \quad Q_T; \tag{1.3}$$

here Ω is a bounded connected open subset of $\mathbb{R}^3$, $\pi = \pi(t, x)$ is the (unknown) pressure and $b = b(t, x)$ is the external force field. The experimental Fick diffusion law is (see Frank and Kamenetskii [6])

$$v^{(1)} = w - \frac{\lambda}{c} \nabla c, \tag{1.4}$$

where $\lambda > 0$ is the diffusion constant. Since

$$c = \frac{d\rho_1}{\rho},$$

(1.4) is equivalent to

$$v^{(1)} = v - \frac{\lambda}{d} \nabla d$$

and to

$$w = v - \frac{\lambda}{\rho} \nabla \rho. \tag{1.5}$$

Then from (1.1) and (1.2) we obtain the Euler system

$$\begin{aligned} &\rho[\dot{v} + (v \cdot \nabla)v - b] - \lambda(v \cdot \nabla)\nabla\rho - \lambda(\nabla\rho \cdot \nabla)v \\ &\qquad + \frac{\lambda^2}{\rho}(\nabla\rho \cdot \nabla)\nabla\rho - \frac{\lambda^2}{\rho^2}(\nabla\rho \cdot \nabla\rho)\nabla\rho + \frac{\lambda^2}{\rho}\Delta\rho\,\nabla\rho + \nabla P = 0 \text{ in } Q_T, \end{aligned} \tag{E$_1$}$$

$$\dot{\rho} + v \cdot \nabla\rho - \lambda\,\Delta\rho = 0 \qquad \text{in } Q_T, \tag{E$_2$}$$

$$\operatorname{div} v = 0 \qquad \text{in } Q_T, \tag{E$_3$}$$

where $P = P(t, x) \equiv \pi - \lambda^2 \Delta\rho + \lambda v \cdot \nabla\rho$. Finally we consider the following boundary and initial conditions

$$v \cdot n = 0 \qquad \text{on }]0, T[\times \Gamma, \tag{E$_4$}$$

$$\frac{\partial\rho}{\partial n} = 0 \qquad \text{on }]0, T[\times \Gamma, \tag{E$_5$}$$

$$v|_{t=0} = a(x) \qquad \text{in } \Omega, \tag{E$_6$}$$

$$\rho|_{t=0} = \rho_0(x) \qquad \text{in } \Omega; \tag{E$_7$}$$

where $\Gamma = \partial\Omega$, $n = n(x)$ is the unit outward normal to Γ, $a = a(x)$ and $\rho_0 = \rho_0(x)$ are the initial velocity and density. The significance of the boundary conditions is that the fluid is isolated, i.e. there is no flux through the boundary.

2. Main Results

We assume that Γ is a compact manifold of dimension 2, without boundary, and that Ω is locally situated on one side of Γ. Γ has a finite number of connected components $\Gamma_0, \Gamma_1, \dots, \Gamma_m$, such that $\Gamma_j (j = 1, \dots, m)$ are inside of Γ_0 and outside of one another. We prove the following results

THEOREM A. *Let* $\Gamma \in C^5$, $a \in H^3(\Omega)$ *with* $\operatorname{div} a = 0$ *in* Ω *and* $a \cdot n = 0$ *on* Γ, $\rho_0 \in H^4(\Omega)$, $\partial\rho_0/\partial n = 0$ *on* Γ, $(\partial/\partial n)(\lambda\Delta\rho_0 - a \cdot \nabla\rho_0) = 0$ *on* Γ, $\rho_0(x) > 0$ *for each* $x \in \bar{\Omega}$, $b \in L^1(0, T_0; H^2(\Omega)) \cap L^2(0, T_0; H^1(\Omega))$ *with* $\operatorname{rot} b \in L^1(0, T_0; H^2(\Omega))$. *Then there exist* $T_1 \in \,]0, T_0]$, $v \in L^\infty(0, T_1; H^3(\Omega))$ *with* $\dot{v} \in L^1(0, T_1; H^2(\Omega)) \cap L^2(0, T_1; H^1(\Omega))$, $P \in L^1(0, T_1; H^3(\Omega)) \cap L^2(0, T_1; H^2(\Omega))$ $\rho \in L^2(0, T_1; H^5(\Omega)) \cap C^0([0, T_1]; H^4(\Omega))$ *with* $\dot{\rho} \in L^2(0, T_1; H^3(\Omega))$ *such that* (v, P, ρ) *is a solution of* (E) *in* Q_{T_1}.

THEOREM B. *Suppose that* $\min_{\bar{\Omega}} \rho_0 > 0$ *and* $b \in L^1(0, T; L^\infty(\Omega))$. *Then the solution of* (E) *is unique in the class of functions* $v \in L^\infty(Q_T)$ *with* $Dv \in L^2(0, T; L^\infty(\Omega))$, $\dot{v} \in L^1(0, T; L^\infty(\Omega))$, $\rho \in L^\infty(Q_T)$ *with* $\nabla\rho \in L^4(0, T; L^\infty(\Omega))$, $D^2\rho \in L^2(0, T; L^\infty(\Omega))$. *The function P is unique up to an arbitrary function of t which may be added to it.*

For the sake of simplicity we assume in the proof of Theorem A that Ω is simply connected. Otherwise we argue as in [1, 2, 4].

3. Proof of Theorem A

Let $T \in \,]0, T_0]$ and $u \in L^\infty(0, T; H^3(\Omega))$ with $\dot{u} \in L^2(0, T; H^1(\Omega))$, such that $u(0, x) = a(x)$ in Ω, $\operatorname{div} u = 0$ in Q_T, $u \cdot n = 0$ on $\Sigma_T \equiv \,]0, T[\, \times \Gamma$, and

$$\|u\|_{3,T} \leqslant A, \qquad [\dot{u}]_{1,T} \leqslant A, \tag{3.1}$$

where A is a positive constant, which will be specified in (3.19). We denote by $\|\cdot\|_{k,T}$ and $[\cdot]_{k,T}$ the norm in $L^\infty(0, T; H^k(\Omega))$ and $L^2(0, T; H^k(\Omega))$, respectively.

We want to find a solution of

$$\begin{aligned} &\dot{\rho} + u \cdot \nabla\rho - \lambda \Delta \rho = 0 && \text{in } Q_T, \\ &\left.\frac{\partial \rho}{\partial n}\right|_{\Gamma} = 0 && \text{on } \Sigma_T, \\ &\rho|_{t=0} = \rho_0(x) && \text{in } \Omega. \end{aligned} \tag{3.2}$$

LEMMA 3.1. *There exists a unique solution of* (3.2), *such that* $\rho \in L^2(0, T, H^5(\Omega)) \cap C^0([0, T]; H^4(\Omega))$ *with* $\dot{\rho} \in L^2(0, T; H^3(\Omega))$, *and*

$$[\rho]_{5,T} + \|\rho\|_{4,T} + [\dot{\rho}]_{3,T} \leqslant \bar{c}(A). \tag{3.3}$$

Moreover, $0 < \min_{\bar{\Omega}} \rho_0 \leqslant \rho(t, x) \leqslant \max \rho_0$ *for each* $(t, x) \in \bar{Q}_T$.

Here and in the sequel, $\bar{c}(A)$ is a non-decreasing function of its argument, depending also at most on Ω, λ, T_0, and suitable norms of a, b, and ρ_0.

Proof. From well-known results on parabolic equations one gets a unique solution $\rho \in L^2(0, T; H^2(\Omega))$, with

$$[\rho]_{2,T} \leqslant \bar{c}(A).$$

To obtain further estimates, we look at the term $u \cdot \nabla\rho$ as a datum in Eq. (3.2) and we regularize the solution ρ in three steps. Since $u \cdot \nabla\rho \in L^2(0, T; H^1(\Omega))$, by using Theorem 3.2, Chap. 4 in Lions and Magenes [12] one gets that $\rho \in L^2(0, T; H^3(\Omega))$ with $\dot{\rho} \in L^2(0, T; H^1(\Omega))$, and, consequently, by interpolation $\rho \in C^0([0, T]; H^2(\Omega))$ (see Theorem 3.1, Chap. 1 in [12]). Moreover one has

$$[\rho]_{3,T} + \|\rho\|_{2,T} + [\dot{\rho}]_{1,T} \leqslant \bar{c}(A). \tag{3.4}$$

Hence $u \cdot \nabla\rho \in L^2(0, T; H^2(\Omega))$ with $(\partial/\partial t)(u \cdot \nabla\rho) \in L^2(0, T; L^2(\Omega))$. We use now Theorem 5.2, Chap. 4 in [12]. First we choose $H = H^2(\Omega)$, $\mathscr{H} = L^2(\Omega)$, $\beta = 1$ and we obtain $\rho \in L^2(0, T; H^4(\Omega))$ with $\dot{\rho} \in L^2(0, T; H^2(\Omega))$ and $\ddot{\rho} \in L^2(0, T; L^2(\Omega))$, where the corresponding norms are estimated by $\bar{c}(A)$. In particular $u \cdot \nabla\rho \in L^2(0, T; H^3(\Omega))$ with $(\partial/\partial t)(u \cdot \nabla\rho) \in L^2(0, T; H^1(\Omega))$ and from the theorem above (now with $H = H^3(\Omega)$, $\mathscr{H} = H^1(\Omega)$ and $\beta = 1$) we obtain the thesis.

For some remarks about these results see the Appendix. Finally, the maximum principle gives

$$\min_{\bar{\Omega}} \rho_0 \leqslant \rho(t, x) \leqslant \max_{\bar{\Omega}} \rho_0 \text{ for each } (t, x) \in \bar{Q}_T. \quad \blacksquare$$

Now let us consider the Neumann problem (3.5), formally obtained by taking the divergence of Eq. $(E)_1$ in Ω, and by taking the scalar product of $(E)_1$ with the outward normal n on Γ. If we write $(E)_1$ as

$$\frac{\nabla P}{\rho} + [\dot{u} + (u \cdot \nabla)u - b] + \frac{1}{\rho} \Lambda = 0,$$

where

$$\Lambda \equiv -\lambda(u \cdot \nabla)\nabla\rho - \lambda(\nabla\rho \cdot \nabla)u + \frac{\lambda^2}{\rho}(\nabla\rho \cdot \nabla)\nabla\rho$$
$$-\frac{\lambda^2}{\rho^2}(\nabla\rho \cdot \nabla\rho)\nabla\rho + \frac{\lambda^2}{\rho}\Delta\rho\,\nabla\rho,$$

we obtain

$$\Delta P - \frac{\nabla\rho}{\rho} \cdot \nabla P = -\rho\left[\sum_{i,j}(D_i u_j)(D_j u_i) - \operatorname{div} b\right]$$
$$-\operatorname{div}\Lambda + \frac{\nabla\rho}{\rho}\cdot\Lambda \equiv F \qquad \text{in } \Omega, \tag{3.5}$$
$$\frac{\partial P}{\partial n} = \rho\sum_{i,j}(D_i n_j)\, u_i u_j + \rho b \cdot n - \Lambda \cdot n \equiv G \qquad \text{on } \Gamma.$$

We remark that $F = -\rho \operatorname{div}[(u \cdot \nabla)u - b + \rho^{-1}\Lambda]$ and $G = -\rho[(u \cdot \nabla)u - b + \rho^{-1}\Lambda] \cdot n$.

LEMMA 3.2. *There exists a solution (unique up to an additive constant) of problem* (3.5) *such that* $\nabla P \in L^1(0, T; H^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ *and*

$$\int_0^T \|\nabla P\|_2\, dt \leqslant \bar{c}(A)\left(T + \int_0^T \|b\|_2\, dt\right),$$
$$[\nabla P]_{1,T} \leqslant \bar{c}(A)(T^{1/2} + [b]_{1,T}). \tag{3.6}$$

Proof. The compatibility condition for (3.5) is obviously verified as follows from the remark above. See also [3]. Moreover, one easily obtains from the previous estimates that

$$\|F(t)\|_0 + \|G(t)\|_1 \leqslant \bar{c}(A)[1 + \|b(t)\|_1].$$

Taking into account that

$$\left\|\frac{\nabla\rho}{\rho}\right\|_{2,T} \leqslant \bar{c}(A),$$

estimate $(3.6)_2$ follows by well-known regularization results for elliptic boundary value problems.

Analogously, we have

$$\|F(t)\|_1 + \|G(t)\|_2 \leqslant \bar{c}(A)[1 + \|b(t)\|_2]$$

and consequently $(3.6)_1$. ∎

Now consider the following equation (formally obtained by taking the curl of $(\mathrm{E})_1$ and writing $\operatorname{rot} u = \xi$):

$$\begin{aligned}
&\dot{\xi} + (u \cdot \nabla)\xi - \lambda \left(\frac{\nabla\rho}{\rho} \cdot \nabla\right)\xi \\
&\quad = (\xi \cdot \nabla)u - \lambda(\xi \cdot \nabla)\frac{\nabla\rho}{\rho} + \lambda\xi \operatorname{div}\left(\frac{\nabla\rho}{\rho}\right) \\
&\qquad + \beta + \frac{\nabla\rho}{\rho^2} \times \nabla P - \lambda \frac{\nabla\rho}{\rho^2} \\
&\qquad \times [\nabla\rho \times \operatorname{rot} u + (u \cdot \nabla)\nabla\rho + (\nabla\rho \cdot \nabla)u - \lambda\nabla\Delta\rho] \qquad \text{in } Q_T, \quad (3.7)\\
&\xi|_{t=0} = \alpha, \qquad \text{in } \Omega,
\end{aligned}$$

where by definition $\alpha \equiv \operatorname{rot} a$ and $\beta \equiv \operatorname{rot} b$.

LEMMA 3.3. *There exists a unique solution of* (3.7) *such that* $\xi \in L^\infty(0, T; H^2(\Omega))$ *with* $\dot{\xi} \in L^2(0, T; L^2(\Omega)) \cap L^1(0, T; H^1(\Omega))$. *Moreover*

$$\|\xi\|_{2,T} \leqslant \left[\|\alpha\|_2 + \int_0^T \|\beta\|_2\, dt + \bar{c}(A) \int_0^T \|b\|_2\, dt + \bar{c}(A)T\right] e^{\bar{c}(A)T}, \tag{3.8}$$

$$[\dot{\xi}]_{0,T} \leqslant \bar{c}(A)T^{1/2} + [b]_{1,T}. \tag{3.9}$$

Proof. Set

$$w = u - \lambda \frac{\nabla\rho}{\rho}.$$

Equation (3.7) can be written

$$\begin{aligned}
\dot{\xi} + (w \cdot \nabla)\xi &= M \cdot \xi + \gamma, \\
\xi|_{t=0} &= \alpha;
\end{aligned} \tag{3.10}$$

here $M(t, x)$ and $\gamma(t, x)$ are a matrix-valued and a vector-valued function, respectively.

It is easy to get from previous estimates that

$$\begin{aligned}&\|w\|_{3,T} \leqslant \bar{c}(A),\\ &\|M\|_{2,T} \leqslant \bar{c}(A), \qquad\qquad (3.11)\\ \int_0^T \|\gamma\|_2\,dt \leqslant \int_0^T &\|\beta\|_2\,dt + \bar{c}(A)\int_0^T \|b\|_2\,dt + \bar{c}(A)T.\end{aligned}$$

The solution of (3.10) can be obtained by the method of characteristics, solving the ordinary differential system

$$\begin{aligned}&\frac{dU(s;t,x)}{ds} = w(s, U(s;t,x)) \qquad \text{in }]0,T[\times Q_T,\\ &U(t;t,x) = x,\end{aligned} \qquad (3.12)$$

(which has a global solution since $w \cdot n|_\Gamma = 0$), and by using then the change of variable $(t,x) \to (t, U(0,t,x))$.

Now the proof follows as in [5], observing that

$$\int_\Omega [(w\cdot\nabla)\,D_iD_j\xi]\cdot D_iD_j\xi\,dx \leqslant \frac{\lambda}{2}\|\log\rho\|_4\,\|D_iD_j\xi\|_0^2.$$

One obtains

$$\begin{aligned}&\frac{d}{dt}\|\xi(t)\|_2 \leqslant c(\|w(t)\|_3 + \|\rho(t)\|_4 + \|M(t)\|_2)\,\|\xi(t)\|_2 + \|\gamma(t)\|_2,\\ &\|\xi(0)\|_2 = \|\alpha\|_2,\end{aligned}$$

and by comparison theorems and (3.11),

$$\|\xi\|_{2,T} \leqslant \left[\|\alpha\|_2 + \int_0^T \|\beta\|_2\,dt + \bar{c}(A)\int_0^T \|b\|_2\,dt + \bar{c}(A)T\right] e^{\bar{c}(A)T}.$$

Estimate (3.9) and the fact that $\dot{\xi} \in L^1(0,T;H^1(\Omega))$ follow directly from Eq. (3.7). ∎

Moreover if ξ is the solution of (3.7) we obtain

LEMMA 3.4. *For each* $t \in [0,T]$

$$\operatorname{div}\xi = 0 \qquad \text{a.e. } in\ \Omega,\ \int_{\Gamma_i} \xi\cdot n\,d\Gamma = 0\ \forall i = 1,\dots,m. \qquad (3.13)$$

Proof. In fact ξ is a solution of (see also [5])

$$\dot{\xi} + w \operatorname{div} \xi = \beta - \operatorname{rot}(\xi \times w) - \operatorname{rot}\left(\frac{\nabla P}{\rho}\right)$$

$$+ \lambda \operatorname{rot}\left[\frac{1}{\rho}\nabla(u \cdot \nabla\rho)\right] - \lambda^2 \operatorname{rot}\left(\frac{1}{\rho}\nabla\Delta\rho\right) \qquad \text{in } Q_T, \tag{3.14}$$

$$\xi|_{t=0} = \alpha \qquad \text{in } \Omega,$$

and taking the divergence of $(3.14)_1$ we obtain

$$\frac{\partial}{\partial t}(\operatorname{div}\xi) + w \cdot \nabla(\operatorname{div}\xi) + (\operatorname{div} w)(\operatorname{div}\xi) = 0 \qquad \text{in } Q_T,$$

$$(\operatorname{div}\xi)|_{t=0} = \operatorname{div}\alpha = 0 \qquad \text{in } \Omega,$$

hence $\operatorname{div}\xi = 0$ a.e. in Ω for each $t \in [0, T]$. Moreover

$$\frac{d}{dt}\int_{\Gamma_i} \xi \cdot n \, d\Gamma = \int_{\Gamma_i} \dot{\xi} \cdot n \, d\Gamma = 0 \qquad \forall i = 1, ..., m,$$

since $\dot{\xi}$ is a curl (see (3.14)) and

$$\int_{\Gamma_i} \operatorname{rot} g \cdot n \, d\Gamma = 0$$

for every vector field g. Hence

$$\int_{\Gamma_i} \xi \cdot n \, d\Gamma = \int_{\Gamma_i} \alpha \cdot n \, d\Gamma = 0 \qquad \forall i = 1, ..., m. \quad \blacksquare$$

We can solve now the elliptic boundary value problem

$$\begin{aligned} \operatorname{rot} v &= \xi && \text{in } Q_T, \\ \operatorname{div} v &= 0 && \text{in } Q_T, \\ v \cdot n &= 0 && \text{on } \Sigma_T, \end{aligned} \tag{3.15}$$

and obtain that $v \in L^\infty(0, T; H^3(\Omega))$ with $\dot{v} \in L^2(0, T; H^1(\Omega)) \cap L^1(0, T; H^2(\Omega))$, and

$$\|v\|_{3,T} \leqslant c_* \left[\|\alpha\|_2 + \int_0^T \|\beta\|_2 \, dt + \bar{c}(A)\int_0^T \|b\|_2 \, dt + \bar{c}(A)T\right] e^{\bar{c}(A)T}, \tag{3.16}$$

$$[\dot{v}]_{1,T} \leqslant \bar{c}(A)T^{1/2} + c_*[b]_{1,T}, \tag{3.17}$$

where $c_* = c_*(\Omega)$. Moreover

$$\begin{aligned} &\operatorname{rot}(v|_{t=0} - a) = \xi|_{t=0} - \alpha = 0 && \text{in } \Omega, \\ &\operatorname{div}(v|_{t=0} - a) = 0 && \text{in } \Omega, \\ &(v|_{t=0} - a) \cdot n = 0 && \text{on } \Gamma, \end{aligned} \tag{3.18}$$

hence $v|_{t=0} = a$ in Ω.

We proceed to find a fixed point of the map $\Phi: u \to v$. Choose in fact

$$A > c_* \|\alpha\|_2. \tag{3.19}$$

From (3.16) and (3.17) it follows that there exists $T_1 \in]0, T_0]$ such that the set

$$S \equiv \{u \in L^\infty(0, T; H^3(\Omega)) \mid \dot{u} \in L^2(0, T; H^1(\Omega)), u|_{t=0} = a, \\ \operatorname{div} u = 0 \text{ in } Q_T, u \cdot n|_\Gamma = 0 \text{ on } \Sigma_T, u \text{ satisfies } (3.1)\}$$

satisfies $\Phi[S] \subset S$.

Moreover S is convex, and we want to prove that S is compact in $X \equiv C^0([0, T_1]; H^1(\Omega))$. From (3.1) S is bounded in $C^{1/2}([0, T_1]; H^1(\Omega)) \cap L^\infty(0, T_1; H^3(\Omega))$, which is compact in X by the Ascoli–Arzelà theorem.

So S is relatively compact in X, and one easily verifies that it is also closed in X; hence S is compact in X.

LEMMA 3.5. *$\Phi: S \to S$ is continuous, in the topology of X.*

Proof. Let $u, u^n \in S$, $u^n \to u$ in X. Then from (3.2) one obtains easily $\rho_n \to \rho$ in $C^0([0, T_1]; L^2(\Omega))$. By a compactness argument $\rho_n \to \rho$ in $C^0([0, T_1]]; H^3(\Omega))$.

From the Neumann problem (3.5) one gets $\nabla P_n \to \nabla P$ in $L^2(0, T_1; L^2(\Omega))$. To show this result we write equation (3.5) in the form

$$\begin{aligned} &\operatorname{div}\left(\frac{1}{\rho}\nabla P\right) = \operatorname{div} h && \text{in } \Omega, \\ &\frac{1}{\rho}\nabla P \cdot n = h \cdot n && \text{on } \Gamma, \end{aligned}$$

where $h \equiv -(u \cdot \nabla)u + b - (1/\rho)\Delta$, and we verify that $h^n \to h$ in $L^2(0, T_1; L^2(\Omega))$. Finally, by evaluating

$$\frac{d}{dt}\|\xi^n - \xi\|_0^2,$$

from Eq. $(3.7)_1$ one gets that $\xi^n \to \xi$ in $C^0([0, T_1]; L^2(\Omega))$, hence $v^n \to v$ in X. ∎

Then from Schauder's theorem, we get the existence of a fixed point $u = v = \Phi[u]$.

Consider now the fixed point $u = v$ and the corresponding functions ρ, P constructed in (3.2), (3.5); they are solutions of system $(E)_1 - (E)_7$. In fact from $\xi = \operatorname{rot} v$ and $(3.7)_1$, $\operatorname{div} v = 0$ and $(3.5)_1$, $v \cdot n|_\Gamma = 0$ and $(3.5)_2$ one easily verifies that

$$\begin{aligned} \operatorname{rot} V &= 0 \quad && \text{in } Q_{T_1}, \\ \operatorname{div} V &= 0 \quad && \text{in } Q_{T_1}, \\ V \cdot n &= 0 \quad && \text{on } \Sigma_{T_1}, \end{aligned}$$

where V is the left-hand side of $(E)_1$ divided by ρ; hence $V = 0$ in Q_{T_1}.

4. Proof of Theorem B

Let $\tilde{v}, \tilde{\rho}, \tilde{P}$ and v, ρ, P be two solutions of $(E)_1$–$(E)_7$. Set $u \equiv \tilde{v} - v$, $\eta \equiv \tilde{\rho} - \rho$, $\sigma \equiv \tilde{P} - P$: from equation $(E)_1$ and $(E)_2$ one obtains

$$\begin{aligned} &\tilde{\rho}[\dot{u} + (\tilde{v} \cdot \nabla)u + (u \cdot \nabla)v] + \nabla\sigma + \eta[\dot{v} + (v \cdot \nabla)v - b] \\ &\quad - \lambda(u \cdot \nabla)\nabla\tilde{\rho} - \lambda(v \cdot \nabla)\nabla\eta - \lambda(\nabla\eta \cdot \nabla)\tilde{v} - \lambda(\nabla\rho \cdot \nabla)u \\ &\quad + \frac{\lambda^2}{\rho}(\nabla\eta \cdot \nabla)\nabla\tilde{\rho} + \frac{\lambda^2}{\rho}(\nabla\rho \cdot \nabla)\nabla\eta - \frac{\lambda^2}{\rho\tilde{\rho}}\eta(\nabla\tilde{\rho} \cdot \nabla)\nabla\tilde{\rho} - \frac{\lambda^2}{\rho^2}(\nabla\eta \cdot \nabla\rho)\nabla\tilde{\rho} \\ &\quad - \frac{\lambda^2}{\rho^2}(\nabla\eta \cdot \nabla\tilde{\rho})\nabla\tilde{\rho} - \frac{\lambda^2}{\rho^2}(\nabla\rho \cdot \nabla\rho)\nabla\eta + \frac{\lambda^2}{\tilde{\rho}^2\rho}\eta(\nabla\tilde{\rho} \cdot \nabla\tilde{\rho})\nabla\tilde{\rho} \\ &\quad + \frac{\lambda^2}{\tilde{\rho}\rho^2}\eta(\nabla\tilde{\rho} \cdot \nabla\tilde{\rho})\nabla\tilde{\rho} + \frac{\lambda^2}{\rho}\Delta\tilde{\rho}\,\nabla\eta - \frac{\lambda^2}{\rho\tilde{\rho}}\Delta\tilde{\rho}\eta\,\nabla\tilde{\rho} + \frac{\lambda^2}{\rho}\Delta\eta\,\nabla\rho = 0, \end{aligned} \tag{4.1}$$

$$\dot{\eta} + v \cdot \nabla\eta - \lambda\,\Delta\eta = -u \cdot \nabla\tilde{\rho}. \tag{4.2}$$

Take the scalar product in $L^2(\Omega)$, denoted by (,), of Eq. (4.1) with u, and of Eq. (4.2) with η. Since

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\tilde{\rho}u, u) &= \frac{1}{2}(\dot{\tilde{\rho}}u, u) + (\tilde{\rho}\dot{u}, u) \\ &= \frac{\lambda}{2}(\Delta\tilde{\rho}\,u, u) - \frac{1}{2}(\tilde{v} \cdot \nabla\tilde{\rho}\,u, u) + (\tilde{\rho}\dot{u}, u) \end{aligned}$$

and

$$\frac{1}{2}(\tilde{v} \cdot \nabla\tilde{\rho}\, u, u) + (\tilde{\rho}(\tilde{v} \cdot \nabla)u, u) = 0,$$

one gets

$$\frac{1}{2}\frac{d}{dt}[(\tilde{\rho}u, u) + (\eta, \eta)] + \lambda(\nabla\eta, \nabla\eta)$$
$$\leqslant c(t)[(\tilde{\rho}u, u) + (\eta, \eta)] + \lambda(\nabla\eta, \nabla\eta) + \frac{\lambda}{2}(\Delta\eta, \Delta\eta), \tag{4.3}$$

where $c(t) \in L^1(0, T)$, and depends also on λ.

Moreover, taking the gradient of Eq. (4.2) and taking the scalar product with $\nabla\eta$, we have

$$\frac{1}{2}\frac{d}{dt}(\nabla\eta, \nabla\eta) + \lambda(\Delta\eta, \Delta\eta)$$
$$\leqslant \|Dv\|_\infty(\nabla\eta, \nabla\eta) + \frac{1}{2\lambda \min_{\bar{\Omega}} \rho_0}\|D\tilde{\rho}\|_\infty^2(\tilde{\rho}u, u) + \frac{\lambda}{2}(\Delta\eta, \Delta\eta), \tag{4.4}$$

where $\|\cdot\|_\infty$ is the norm in $L^\infty(\Omega)$.

Hence from (4.3) and (4.4) one obtains

$$\frac{1}{2}\frac{d}{dt}[(\tilde{\rho}u, u) + (\eta, \eta) + (\nabla\eta, \nabla\eta)] + \lambda(\Delta\eta, \Delta\eta)$$
$$\leqslant c(t)[(\tilde{\rho}u, u) + (\eta, \eta) + (\nabla\eta, \nabla\eta)] + \lambda(\Delta\eta, \Delta\eta),$$

and by Gronwall's lemma one sees that $u = 0$ and $\eta = 0$ in Q_T. From (4.1) it is clear that $\nabla\sigma = 0$, i.e., $\tilde{P} = P$ in Q_T up to an arbitrary function of t. ∎

Remark. We can obtain analogous results for the simpler model (see for instance Graffi [7])

$$\begin{aligned} \rho[\dot{v} + (v \cdot \nabla)v - b] + \nabla\pi &= 0 && \text{in } Q_T, \\ \dot{\rho} + v \cdot \nabla\rho &= \lambda\Delta\rho && \text{in } Q_T, \\ \operatorname{div} v &= 0 && \text{in } Q_T, \end{aligned} \tag{E}$$

with the same initial and boundary conditions $(E)_4$–$(E)_7$. In this case in Theorem A it is sufficient to have $\Gamma \in C^4$, $\rho_0 \in H^3(\Omega)$, and the condition $(\partial/\partial n)(\lambda\, \Delta\rho_0 - a \cdot \nabla\rho_0) = 0$ on Γ can be dropped.

Another case which can be solved under these hypotheses is that obtained by neglecting the terms in λ^2 in $(E)_1$ (see [9, 10]).

5. Appendix

Consider the parabolic initial-boundary value problem

$$\begin{aligned} \dot{U} - \lambda \Delta U &= f \qquad \text{in } Q_T, \\ \frac{\partial U}{\partial n} &= 0 \qquad \text{on } \Sigma_T, \\ U|_{t=0} &= 0 \qquad \text{in } \Omega, \end{aligned} \tag{5.1}$$

where λ is a positive constant, and $f \in L^2(0, T, H^3(\Omega))$ with $\dot{f} \in L^2(0, T; H^1(\Omega))$ and $f(0, x) \equiv 0$ in Ω.

We want to apply Theorem 5.2, Chapter 4 in [12], with $H = H^3(\Omega)$, $\mathscr{H} = H^1(\Omega)$, $\beta = 1$, $A = -\lambda\Delta$ and $D(A) = H^5_N(\Omega) \equiv \{U \in H^5(\Omega) \mid \partial U/\partial n = 0$ on $\Gamma\}$. One obtains easily the following estimates

$$\|U\|_5 \leqslant c(\|(A+p)U\|_3 + |p|\,\|U\|_3), \qquad U \in D(A), \operatorname{Re} p > 1. \tag{5.2}$$

$$|p|\,\|U\|_1 \leqslant \|(A+p)U\|_1, \qquad U \in D(A), \operatorname{Re} p \geqslant 0, \tag{5.3}$$

$$\|U\|_3 \leqslant c\,\|(A+p)U\|_1 \qquad U \in D(A), \operatorname{Re} p > 1. \tag{5.4}$$

From (5.2) and (5.4) one gets

$$\|U\|_5 \leqslant c\,\|(A+p)U\|_3 + c\,|p|\,\|(A+p)U\|_1, \tag{5.5}$$

and from (5.3) one has

$$|p|^2\,\|U\|_1 \leqslant |p|\,\|(A+p)U\|_1. \tag{5.6}$$

Since

$$\|U\|_{D(A)} \equiv \|U\|_3 + \|AU\|_3 \simeq \|U\|_5,$$

from (5.5) and (5.6) one obtains at once estimate (5.11) in Theorem 5.2, Chapter 4 of [12].

Hence there exists a unique solution of (5.1) and one has

$$[U]_{5,T} + [\dot{U}]_{3,T} + [\ddot{U}]_{1,T} \leqslant c([f]_{3,T} + [\dot{f}]_{1,T}), \tag{5.7}$$

where the constant c does not depend on T.

Now we consider problem (5.1) with an arbitrary initial condition $U|_{t=0} = U_0$ and without the assumption $f(0, x) \equiv 0$. By assuming that the (necessary) compatibility conditions

$$\left.\frac{\partial U_0}{\partial n}\right|_\Gamma = 0, \qquad \left.\frac{\partial}{\partial n}(\lambda \Delta U_0 + f(0))\right|_\Gamma = 0$$

are satisfied, one can find a function $W \in L^2(0, +\infty; H^5_N(\Omega))$ with $\dot{W} \in L^2(0, +\infty; H^3_N(\Omega))$, $\ddot{W} \in L^2(0, +\infty; H^1(\Omega))$ such that

$$W|_{t=0} = U_0, \qquad \dot{W}|_{t=0} = f(0) + \lambda \Delta U_0.$$

Moreover

$$[W]_{5,T} + [\dot{W}]_{3,T} + [\ddot{W}]_{1,T} \leqslant c(\|U_0\|_4 + \|f(0)\|_2),$$

where the constant c does not depend on T. By taking $\Psi \equiv U - W$ one reduces the problem under consideration to the first one. Hence in the general case the solution U verifies the estimate

$$\begin{aligned}[U]_{5,T} &+ [\dot{U}]_{3,T} + [\ddot{U}]_{1,T} \\ &\leqslant c\{\|U_0\|_4 + \|f(0)\|_2 + [f]_{3,T} + [\dot{f}]_{1,T}\},\end{aligned}$$

where the constant c does not depend on T.

References

1. H. Beirão da Veiga, On an Euler type equation in hydrodynamics, *Ann. Mat. Pura Appl.* **125** (1980), 279–295.
2. H. Beirão da Veiga and A. Valli, On the motion of a non-homogeneous ideal incompressible fluid in an external force field, *Rend. Sem. Mat. Univ. Padova* **59** (1978), 117–145.
3. H. Beirão da Veiga and A. Valli, On the Euler equations for non–homogeneous fluids, I. *Rend. Sem. Mat. Univ. Padova* **63** (1980), 151–168.
4. H. Beirão da Veiga and A. Valli, On the Euler equations for non-homogeneous fluids. I. Ω not simply-connected, preprint Università di Trento, 1980.
5. H. Beirão da Veiga and A. Valli, Existence of C^∞ solutions of the Euler equations for non-homogeneous fluids, *Comm. Partial Differential Equations* **5** (1980), 95–107.
6. D. A. Frank-Kamenetskii, "Diffusion and Heat Transfer in Chemical Kinetics," 2nd ed., Plenum, New York/London, 1969, [Translated from russian].
7. D. Graffi, Il teorema di unicità per i fluidi incompressibili, perfetti, eterogeneii, *Rev. Univ. Mat. Argentina* **17** (1955), 73–77.
8. V. N. Ignat'ev and B. G. Kuznetsov, A model for the diffusion of a turbolent boundary layer in a polymer, *Čisl. Metody Meh. Splošn. Sredy* **4** (1973), 78–87. [Russian]
9. A. V. Kazhikhov and Sh. Smagulov, The correctness of boundary-value problems in a diffusion model of an inhomogeneous liquid, *Sov. Phys. Dokl.* **22** (1977), 249–259 (previously in *Dokl. Akad. Nauk SSSR* **234** (1977), 330–332 [in Russian]).
10. A. V. Kazhikhov and Sh. Smagulov, The correctness of boundary-value problems in a certain diffusion model of an inhomogeneous fluid, *Čisl. Metody Meh. Splošn. Sredy* **7** (1976), 75–92. [Russian]
11. B. G. Kuznetsov, Quelques problèmes en dynamique des solutions, *in* "Étude numérique des grands systèmes" (J. L. Lions and G. I. Marchouk, Eds.), Dunod, Paris, 1978.
12. J. L. Lions and E. Magenes, "Problèmes aux limites non homogènes et applications," Vols. 1 and 2, Dunod, Paris, 1968.