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On an Euler Type Equation in Hydrodynamics (*).

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Summary. – *See Introduction.*

1. – Introduction and main results.

Let Ω be a bounded connected open subset of $\boldsymbol{R}^3$, locally situated on one side of its boundary Γ. We assume that Γ, a differentiable manifold of class C^{k+2}, has a finite number of connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ such that Γ_j, $j = 1, 2, \ldots, m$, are inside of Γ_0 and outside of one another. We denote by $n = n(x)$ the unit outward normal to the boundary Γ.

We denote by $H^k(\Omega)$, k non negative integer, the Sobolev space of order k with the usual norm $\| \; \|_k$ and by $(\, , \,)$ and $\| \; \|$ the scalar product and the norm in $H^0(\Omega) = L^2(\Omega)$. We denote also by $H^k(\Omega)$ the space $(H^k(\Omega))^3$ of the vector fields $v = (v_1, v_2, v_3)$ such that $v_i \in H^k(\Omega)$, $i = 1, 2, 3$, and by $\|v\|_k$ the norm of the vector v in $(H^k(\Omega))^3$. The same convention applies to the other functional spaces and norms used in this paper.

Let $T > 0$ be given. We denote by $L^\infty(0, T; H^k)$ the Banach space of the (measurable) essentially bounded functions defined on $(0, T)$ with values in $H^k(\Omega)$. The norm in this space is denoted by $\| \; \|_{k,T}$. The subspace of the continuous [resp. lipschitz continuous] functions on the closed interval $[0, T]$ is denoted by $C(0, T; H^k)$ [resp. $\mathrm{Lip}(0, T; H^k)$]. As remarked above we write $L^\infty(0, T; H^k)$ instead of $L^\infty(0, T; (H^k)^3)$, and so on. Finally $[\;]_{k,T}$ denotes the usual norm in the space $L^1(0, T; H^k)$.

In this paper we consider the following system of equations

$$(1.1)\qquad \begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = - \nabla \pi & \text{in } Q_T \equiv \Omega \times]0, T[, \\ \operatorname{div} v = \theta & \text{in } Q_T, \\ v \cdot n = 0 & \text{on } \Sigma_T \equiv \Gamma \times]0, T[, \\ v|_{t=0} = a(x) & \text{in } \Omega, \end{cases}$$

where the scalar field $\theta(t, x)$ (verifying the compatibility conditions (1.5), (1.6)) and the initial velocity field $a(x)$ are given. We prove in this paper the existence of a

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unique vector field $v(t, x)$ and a scalar field $\pi(t, x)$ solutions of (1.1) in Q_T where T, given by (1.7), is maximum when $\theta \equiv 0$. The uniqueness of π is up to an arbitrary function of t. Remark that if $\theta(t, x) = 0$ on Q_T the equations (1.1) are those of the motion of an incompressible ideal fluid.

We introduce in this paper a combination of the « curl method » with a fixed point in the context of $L^\infty(0, T; H^k)$ spaces, which avoid the use of lagrangian coordinates and Hölder continuous functions, inadequate for our purposes. Really the equations (1.1), or more precisely the equivalent system of equations (1.1′) + (1.1″), will be utilized in a forthcoming paper (see [1]) in giving an existence theorem for the general motion of a compressible ideal fluid (1). To do this we use the solution $v = v[\theta]$ of (1.1) in a suitable hyperbolic system of two equations of the first order, whose unknowns are $\delta = \operatorname{div} v$ and a function of the density mass $\varrho(t, x)$; this last system is equivalent to a single second order hyperbolic linear equation. We complete the proof by proving the existence of a fixed point $\delta = \theta$.

Recently we have seen an independent paper of Ebin (see [3]) where a theorem is stated for subsonic initial velocity $a(x)$ and initial density near constant. It is interesting to point out that our hyperbolic system is equivalent to the hyperbolic equation (2.10) of [3], with the same boundary condition (2.4).

The method introduced in this paper is useful also to study problems in which $\operatorname{div} v = 0$. See for instance the subsequent paper [2].

Let us return to problem (1.1). We assume in this paper that

$$a(x) \in H^{k+2}(\Omega)\,, \qquad a\cdot n = 0 \quad \text{on } \Gamma, \tag{1.2}$$

where $k \geqslant 1$ is an integer. We define

$$\alpha(x) \equiv \operatorname{rot} a(x)\,, \qquad \gamma(x) \equiv \operatorname{div} a(x)\,. \tag{1.3}$$

Furthermore we assume that $\theta(t, x)$ verifies the following regularity conditions

$$\begin{cases} \theta \in L^\infty(0, T_0; H^{k+1})\,, \\ \dfrac{d\theta}{dt} \in L^\infty(0, T_0; H^k)\,, \end{cases} \tag{1.4}$$

and the following compatibility conditions

$$\int_\Omega \theta(t, x)\, dx = 0 \quad \text{in } [0, T_0]\,, \tag{1.5}$$

$$\theta|_{t=0} = \gamma(x) \quad \text{in } \Omega; \tag{1.6}$$

Recall that $\theta \in \operatorname{Lip}(0, T_0; H^k)$ as follows from (1.4).

(1) The results on the continuity of the map $\theta(t, x) \to v(t, x)$, established in Section 4, will be useful in this context.

Let now the vector fields $u^{(l)}(x)$, $l = 1, 2, \ldots, N$ be those defined in section 2, and consider the following systems ($T \leqslant T_0$)

$$(1.1') \qquad \begin{cases} \operatorname{div} v = \theta & \text{in } Q_T, \\ \operatorname{rot} v = \zeta & \text{in } Q_T, \\ v \cdot n = 0 & \text{in } \Sigma_T, \\ (v|_{t=0} - a, u^{(l)}) = 0, & l = 1, 2, \ldots, N \\ \left(\dfrac{\partial v}{\partial t} + (v \cdot \nabla) v, u^{(l)}\right) = 0\,, & l = 1, 2, \ldots, N \end{cases}$$

and

$$(1.1'') \qquad \begin{cases} \dfrac{\partial \zeta}{\partial t} + (v \cdot \nabla)\zeta - (\zeta \cdot \nabla) v = -\theta\zeta & \text{in } Q_T, \\ \zeta|_{t=0} = \alpha(x) & \text{in } \Omega\,. \end{cases}$$

If v is a solution of these systems there exists a π such that $(1.1)_1$ holds since a vector field in Ω is a gradient if and only if it is irrotational and orthogonal in L^2 to the functions $u^{(l)}$. From this last property and from known results for the Neumann boundary problem, $(1.1)_4$ follows easily.

Reciprocally if (1.1) holds we apply the property just referred to get $(1.1'')_1$ and $(1.1')_5$. *Hence* (1.1) *is equivalent to* (1.1') *plus* (1.1'').

In this paper we prove the following results:

THEOREM 1.1. – Let $k \geqslant 1$ be an integer and let $a(x)$ and $\theta(t, x)$ verify the assumptions (1.2), (1.4), (1.5) and (1.6). Let $\alpha(x)$ and $\gamma(x)$ be defined by (1.3). Then there exists a constant $c_0 = c_0(k, \Omega)$ such that there exists a (unique) solution v of (1.1'), (1.1'') (or equivalently of (1.1)) in a time interval $[0, T]$ where T is given by

$$(1.7) \qquad T = \frac{c_0}{\|a\|_{k+2} + \|\theta\|_{k+1,T_0}}\,.$$

Moreover

$$(1.8) \qquad \begin{cases} v \in L^\infty(0, T; H^{k+2})\,, \\ \dfrac{dv}{dt} \in L^\infty(0, T; H^{k+1})\,, \end{cases}$$

and the following estimates hold:

$$(1.9) \qquad \|v\|_{k+2,T} \leqslant c_1(\|a\|_{k+2} + \|\theta\|_{k+1,T})\,,$$

$$(1.10) \qquad \left\|\frac{dv}{dt}\right\|_{k+1,T} \leqslant c_2 \left\|\frac{d\theta}{dt}\right\|_{k,T} + c_3\|v\|^2_{k+2,T}\,,$$

where c_1, c_2 and c_3 depends only on k and Ω.

For the applications we have in mind the following regularity result will be useful:

Theorem 1.2. – Under the assumptions of theorem 1.1, and if

$$\frac{d^2\theta}{dt^2} \in L^\infty(0, T; H^{k-1})$$

then

$$\frac{d^2 v}{dt^2} \in L^\infty(0, T; H^k) \tag{1.11}$$

and

$$\left\|\frac{d^2 v}{dt^2}\right\|_{k,T} \leqslant c_4 \left\|\frac{d^2\theta}{dt^2}\right\|_{k-1,T} + c_5 \|v\|_{k+2,T} \left\|\frac{dv}{dt}\right\|_{k+1,T} \tag{1.12}$$

where c_4, c_5 and c_6 depends only on k and Ω.

A uniqueness theorem and results on the continuity of the map $\theta \to v$ are stated and proved in section 4.

Finally we remark that we can add an external force field $-f(t, x)$ to the first side of equation (1.1). This is equivalent to adding the force $-f$ to the term $\partial v/\partial t +$ $+ (v \cdot \nabla) v$ in $(1.1')_5$ and $\operatorname{rot} f$ to the second side of $(1.1'')_1$. In this case results and proofs remain essentially the same. On following our proofs the reader easily verifies that the theorem 1.3 below holds. For the reader's convenience we state in section 5 the main equations, established for the case $f \equiv 0$ in the other sections; we denote corresponding equations by the same number, with an asterick in the case $f \neq 0$ (as for example (1.12) and (1.12)*).

Theorem 1.3. – Assume that in our equations we add an external force field $f(t, x)$ as described above. Let

$$f(t, x) \in L^1(0, T_0; H^{k+2}) \cap L^\infty(0, T_0; H^{k+1}) . \tag{1.13}$$

Then theorem 1.1 holds again with (1.7), (1.9) and (1.10) replaced by (1.7)*, (1.9)* and (1.10)* respectively (see section 5).

If moreover

$$\frac{df}{dt} \in L^\infty(0, T; H^k) \tag{1.14}$$

then the theorem 1.2 holds with (1.12) replaced by (1.12)*.

In this paper we denote by c, c_0, c_1, ... positive contants depending at most on k and Ω.

Remark 1.4. – Assume that the conditions of theorem 1.3 are fulfilled with (1.4) replaced by the stronger condition

$$\theta \in L^\infty(0, T_0; H^{k+2}) , \qquad \frac{d\theta}{dt} \in L^\infty(0, T_0; H^{k+1}) .$$

Denote by $T^*(k)$, $k\geqslant 1$, the least upper bound of the $t_0\in\,]0, T_0]$ for which a solution, verifying (1.8) in $[0, t_0]$, exists in $[0, t_0]$. One can prove as in [5] that $T^*(k)$ is independent of k. Moreover, as in [5], a $C^\infty(\bar{\Omega}\times[0, T^*[)$ regularity result holds.

2. – Auxiliary results.

In the following $T\leqslant T_0$ is a positive real number, whose value will be fixed later by eq. (1.7).

By the hypothesis on the domain Ω it follows that if Ω is not simply- connected one can make it so by means of a finite number of cuts. If N is this number it is known that there exists N vector fields $u^{(l)}(x)$, $l=1, 2, \ldots, N$, defined on Ω and such that $\operatorname{div} u^{(l)}=0$, $\operatorname{rot} u^{(l)}=0$ in Ω, $u^{(l)}\cdot n=0$ on Γ, $(u^{(l)}, u^{(j)})=\delta_{lj}$ (see for instance [4]).

For convenience we put

$$\bar{B}\equiv\|\theta\|_{k+1,T_0}\,. \tag{2.1}$$

The system (1.1′), (1.1″) will be solved by using a fixed point theorem on $\operatorname{rot} v$. Thus we begin by studying the system

$$\begin{cases} \operatorname{div} v=\theta & \text{in } Q_T,\\ \operatorname{rot} v=\varphi & \text{in } Q_T,\\ v\cdot n=0 & \text{on } \Sigma_T,\\ \left(v|_{t=0}-a,\, u^{(l)}\right)=0\,, & l=1, 2, \ldots, N,\\ \dfrac{d}{dt}(v, u^{(l)})+\left((v\cdot\nabla)v, u^{(l)}\right)=0 \text{ classically in } [0, T], & l=1, 2, \ldots, N. \end{cases} \tag{2.2}$$

Let the vector field $\varphi(t, x)$ verify the hypothesis

$$\varphi\in L^\infty(0, T; H^{k+1})\cap C(0, T; H^k)\,, \tag{2.3}$$

$$\varphi|_{t=0}=\alpha(x) \quad \text{in } \Omega, \tag{2.4}$$

and

$$\operatorname{div}\varphi=0 \quad \text{in } \Omega, \qquad \int_{\Gamma_i}\varphi\cdot n\, d\Gamma=0 \quad i=1, 2, \ldots, m, \tag{2.5}$$

for every $t\in[0, T]$. Condition (2.5) is equivalent to the existence of a vector function w such that $\varphi=\operatorname{rot} w$; see [4], Prop. 1.3. We assume that φ belongs to the bounded set

$$\|\varphi\|_{k+1,T}\leqslant D\,, \tag{2.6}$$

where D is given by (3.1).

PROPOSITION 2.1. – Let θ verify (1.4), (1.5), (1.6) and let φ verify (2.3), (2.4), (2.5). Then (2.2) has a unique solution $v(t, x)$. Moreover $v(t, x)$ verifies $(1.1)_4$ and

$$v \in L^\infty(0, T; H^{k+2}) \cap C(0, T; H^{k+1}) \tag{2.7}$$

with (²)

$$\|v\|_{k+2,T} \leqslant c(\bar{B} + D + \|a\|) \exp [c(\bar{B} + D) T] . \tag{2.8}$$

PROOF. – The solutions of the elliptic system $(2.2)_{1,2,3}$ under the hypothesis (1.5), (2.5) are given by

$$v(t, x) = v^0(t, x) + \sum_{l=1}^{N} \theta_l(t) u^{(l)}(x) , \tag{2.9}$$

where v^0 is the particular solution of $(2.2)_{1,2,3}$ such that

$$(v^0, u^{(l)}) = 0 , \quad l = 1, 2, \dots, N. \tag{2.10}$$

Here t is a parameter. For the particular solution v^0 one has $\|v^0\|_{k+2} \leqslant c\ (\|\text{rot}\ v^0\|_{k+1} + \|\text{div}\ v^0\|_{k+1})$, hence

$$\|v^0\|_{k+2,T} \leqslant c(\bar{B} + D) . \tag{2.11}$$

Moreover

$$v^0 \in C(0, T; H^{k+1}) \tag{2.12}$$

since $\theta, \varphi \in C(0, T; H^k)$ (³).

We will now determine continuously differentiable real functions $\theta_l(t)$ in such a way that the vector field $v(t, x)$ given by (2.9) also solves equations $(2.2)_{4,5}$. It follows easily that

$$((v \cdot \nabla) v, u^{(l)}) = \alpha^{(l)}(t) + \sum_{j=1}^{N} a_j^{(l)}(t) \theta_j(t) + \sum_{i,j=1}^{N} \alpha_{ij}^{(l)}(t) \theta_i(t) \theta_j(t) \tag{2.13}$$

where

$$\alpha^{(l)}(t) = -\int_\Omega \theta v^0 \cdot u^{(l)} dx - \int_\Omega [(v^0 \cdot \nabla) u^{(l)}] \cdot v^0 dx , \tag{2.14}$$

(²) If Ω is simply connected, i.e. $N = 0$, one has obviously $\|v\|_{k+2,T} \leqslant c(\bar{B} + D)$ and the proof is finished.

(³) Recall that θ is a scalar and φ a vector.

$$a_j^{(l)}(t) = -\int_\Omega \theta u^{(j)}\cdot u^{(l)}\,dx - \int_\Omega [(v^0\cdot\nabla)\,u^{(l)}]\cdot u^{(j)}\,dx - \int_\Omega [u^{(j)}\cdot\nabla)\,u^{(l)}]\cdot v^0\,dx\,, \tag{2.15}$$

$$\alpha_{ij}^{(l)} = -\int_\Omega [(u^{(i)}\cdot\nabla)\,u^{(l)}]\cdot u^{(j)}\,dx\,. \tag{2.16}$$

Hence

$$\alpha^{(l)}(t),\ \alpha_j^{(l)}(t)\in C(0,T;\boldsymbol{R})\qquad i,j,l=1,2,\ldots,N. \tag{2.17}$$

On the other hand using the identity

$$(v,u^{(l)}) = \theta_l(t) \tag{2.18}$$

one gets from $(2.2)_{4,5}$

$$\frac{d\theta_l}{dt} = \alpha^{(l)} + \sum_{j=1}^N \alpha_j^{(l)}\theta_j + \sum_{i,j=1}^N \alpha_{ij}^{(l)}\theta_i\theta_j\,,\qquad \theta_l(0) = (a,u^{(l)})\,, \tag{2.19}$$

$l=1,2,\ldots,N$. This system of ordinary differential equations have a unique local solution. We will see now that the solution is global. Put

$$v^1 = \sum_{l=1}^N \theta_l(t)\,u^{(l)}(x)\,. \tag{2.20}$$

Multiplying $(2.2)_5$ by $\theta_l(t)$ and adding in l one gets

$$\frac{1}{2}\frac{d}{dt}\|v^1(t)\|^2 + \big((v\cdot\nabla)\,v,\,v^1\big) = 0\,. \tag{2.21}$$

On the other hand, with integration by parts, one see easily that

$$\big((v\cdot\nabla)\,v,\,v^1\big) = -\tfrac{1}{2}\big(\theta,|v^1|^2\big) - (\theta v^0,v^1) - \big([(v^0+v^1)\cdot\nabla]\,v^1,v^0\big)\,.$$

Hence if we define

$$y(t) = \sum_{l=1}^N (\theta_l(t_l))^2 \tag{2.22}$$

it follows in particular that

$$\frac{dy}{dt}\leqslant c(\bar B+D)\,y(t) + c(\bar B+D)^2\sqrt{y(t)}\,,\qquad y(0) = \sum_{k=1}^N (a,u^{(k)})^2\,.$$

Thus

$$y(t)\leqslant\Big[c(\bar B+D)^2 + \sum_{l=1}^N (a,u^{(l)})^2\Big]\exp\,[c(\bar B+D)\,t]\,, \tag{2.23}$$

and the solution of (2.19) is global. Moreover

$$\|v^1(t)\|^2_{k+2} \leqslant c\, y(t)\,, \tag{2.24}$$

where c depends only on the quantities $\|u^{(l)}\|_{k+2}$, hence on Ω and k. From (2.24), (2.23) and (2.11) one gets (2.8). □

We consider now the equation (1.1″). One has the following result:

Proposition 2.2. – Let θ verify $(1.4)_1$ and v verify $(2.2)_3$ and (2.7). Then (1.1″) has a unique solution $\zeta(t, x)$. Moreover

$$\begin{cases} \zeta \in L^\infty(0, T; H^{k+1})\,, \\ \dfrac{d\zeta}{dt} \in L^\infty(0, T; H^k)\,, \end{cases} \tag{2.25}$$

and

$$\begin{cases} \|\zeta\|_{k+1,T} \leqslant \|\alpha\|_{k+1} \exp\left[cT(\|v\|_{k+2,T} + \|\theta\|_{k+1,T})\right], \\ \left\|\dfrac{d\zeta}{dt}\right\|_{k,T} \leqslant c(\|v\|_{k+1,T} + \|\theta\|_{k,T})\,\|\zeta\|_{k+1,T}\,. \end{cases} \tag{2.26}$$

By using proposition 2.1 and 2.2 one gets the following result:

Theorem 2.3. – Let θ and φ verify the hypothesis of proposition 2.1 and let v be the corresponding solution of (2.2). Then (1.1″) has a unique solution ζ and moreover

$$\begin{cases} \|\zeta\|_{k+1,T} \leqslant \|\alpha\|_{k+1} \exp\left[cT(\bar{B} + D + \|a\|)\, e^{c(\bar{B}+D)T}\right], \\ \left\|\dfrac{d\zeta}{dt}\right\|_{k,T} \leqslant c(\bar{B} + D + \|a\|) \exp\left[c(\bar{B} + D)\,T\right] \|\zeta\|_{k+1,T}\,. \end{cases} \tag{2.27}$$

Proof of proposition 2.2. – The construction of a solution of the linear system (1.1″) with the aid of the method of characteristics is classical and we leave it to the reader. This method gives also the uniqueness. We prove now the estimates (2.26) (4). Let D^β, β a multindex, be a derivative of order $|\beta| \leqslant k+1$. Apply D^β to both sides of $(1.1'')_1$, multiply it by $D^\beta\zeta$ and integrate over Ω. Recalling that

$$\big((v\cdot\nabla)\, D^\beta\zeta,\, D^\beta\zeta\big) = -\tfrac{1}{2}\big(\operatorname{div} v,\, |D^\beta\zeta|^2\big) \tag{2.28}$$

one gets easily

$$\frac{1}{2}\frac{d}{dt}\|D^\beta\zeta\|^2 \leqslant c\|D^\beta\zeta\| \sum_{|\alpha|+|\gamma|=|\beta|} \big\||D^\alpha(Dv)|\,|D^\gamma\zeta|\big\| + c\|D^\beta\zeta\| \sum_{|\alpha|+|\gamma|=|\beta|} \big\||D^\alpha\theta|\,|D^\gamma\zeta|\big\|\,.$$

(4) We can use (2.26) as an a priori estimate, and afterwards prove the existence of a solution by an approximating method. Alternatively one can prove (2.26) by estimating the solution constructed by the method of characteristics.

By adding in β, for $|\beta|\leq k+1$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\zeta(t)\|^2_{k+1}\leq c(\|v(t)\|_{k+2}+\|\theta(t)\|_{k+1})\|\zeta(t)\|^2_{k+1}$$

since H^{k+1} is an algebra. Recalling that $\zeta(0)=\alpha$, we apply the Gronwall's lemma to get $(2.26)_1$.

On the other hand from $(1.1'')_1$ it follows that for almost all t one has

$$\left\|\frac{\partial\zeta}{\partial t}\right\|_k\leq c(\|v\|_{k+1}+\|\theta\|_k)\|\zeta\|_{k+1}\,,$$

and this gives $(2.26)_2$. □

LEMMA 2.4. – Under the assumptions of theorem 2.3 the solution ζ of $(1.1'')$ verifies

$$\operatorname{div}\zeta=0\quad\text{ in }\Omega,\qquad \int_{\Gamma_i}\zeta\cdot n\,d\Gamma=0\quad i=1,2,\dots,m,$$

for every $t\in[0,T]$.

PROOF. – From $(1.1'')_1$, $(2.2)_1$ and from the general formulae

$$(v\cdot\nabla)\zeta-(\zeta\cdot\nabla)v=v\operatorname{div}\zeta-\zeta\operatorname{div}v-\operatorname{rot}(v\wedge\zeta)\tag{2.30}$$

it follows that

$$\frac{\partial\zeta}{\partial t}+v\operatorname{div}\zeta=\operatorname{rot}(v\wedge\zeta)\,.\tag{2.31}$$

Applying the divergence to both sides of (2.31), using the general formulae

$$\operatorname{div}(v\operatorname{div}\zeta)=(v\cdot\nabla)\operatorname{div}\zeta+(\operatorname{div}\zeta)(\operatorname{div}v)\tag{2.32}$$

and using also $(1.1'')_2$ one gets

$$\begin{cases}\dfrac{\partial(\operatorname{div}\zeta)}{\partial t}+(v\cdot\nabla)\operatorname{div}\zeta=-\theta\operatorname{div}\zeta\,,\\ (\operatorname{div}\zeta)|_{t=0}=0\,;\end{cases}$$

this trasport equation has a unique solution $\operatorname{div}\zeta=0$. Finally utilizing (2.31) one has

$$\frac{d}{dt}\int_{\Gamma_i}\zeta\cdot n\,d\Gamma=-\int_{\Gamma_i}(\operatorname{div}\zeta)v\cdot n\,d\Gamma+\int_{\Gamma_i}\operatorname{rot}(v\wedge\zeta)\cdot n\,d\Gamma=0\,,$$

hence

$$\int_{\Gamma_i}\zeta\cdot n\,d\Gamma=\int_{\Gamma_i}\alpha\cdot n\,d\Gamma=0\,.\quad\square$$

3. – Construction of the fixed point.

In this section v and ζ are the solutions of (2.2) and (1.1″) respectively. Furthermore

$$D = 2\|\alpha\|_{k+1}\,. \tag{3.1}$$

From (2.27), it follows that there exists a constant $c_0 = c_0(k, \Omega)$ such that if T is given by (1.7) then

$$\begin{cases} \|\zeta\|_{k+1,T} \leqslant D\,, \\ \left\|\dfrac{d\zeta}{dt}\right\|_{k,T} \leqslant D_1\,, \end{cases} \tag{3.2}$$

where

$$D_1 = c(\bar{B} + D + \|a\|)\,D\,. \tag{3.3}$$

In this section T is given by (1.7). We define now

$$S = \{\varphi \in L^\infty(0, T; H^{k+1}) \cap C(0, T; H^k)\colon \\ \|\varphi\|_{k+1,T} \leqslant D \text{ and } \varphi \text{ verifies (2.5) on } [0, T] \text{ and (2.4)}\}\,, \tag{3.4}$$

and we denote by F the operator

$$\zeta = F[\varphi]\,,$$

which is the product of the composition of the operator «$\varphi \to v$» defined by (2.2) with the operator «$v \to \zeta$» defined by (1.2″).

LEMMA 3.1. – The operator F has a fixed point in S.

PROOF. – We will utilize the Schauder's fixed point theorem in the space $X \equiv C(0, T; H^k)$. We easily see that S is closed and convex in X; furthermore from (3.2) and lemma 2.4 one has

$$F(S) \subset \left\{\zeta \in S\colon \left\|\frac{d\zeta}{dt}\right\|_{k,T} \leqslant D_1\right\}. \tag{3.5}$$

In particular $F(S)$ is bounded in Lip $(0, T; H^k)$, and from Ascoli-Arzela's theorem $F(S)$ is relatively compact in X.

Let now $\varphi, \varphi_n \in S$ $(n = 1, 2, \ldots)$, $\varphi_n \to \varphi$ in X, and denote by v_n^0 the particular solution of the elliptic system $(2.2)_{1,2,3}$, with data φ_n, for which (2.10) holds. Since $v_n^0 \to v^0$ in $C(0, T; H^{k+1})$ it follows that the coefficients of the system (2.19) constructed by using the v_n^0 converges uniformly in $[0, T]$ to the corresponding coefficients constructed by using v^0; hence the same holds for the corresponding solutions $\theta_l^{(n)}$, θ_l. It follows now from (2.9) that the solutions v_n of (2.2) converges in $C(0, T; H^{k+1})$ to the solution v of (2.2), where v_n corresponds to the data φ_n and v to the data φ.

Let now ζ_n be the solution of (1.1″) when v is replaced by v_n. By taking the difference between the two equation and multiplying it by $\zeta - \zeta_n$ we obtain after integrating over Ω that

$$\frac{1}{2}\frac{d}{dt}\|\zeta-\zeta_n\|^2 \leqslant c(\|\zeta_n\|_2\|v-v_n\|_1 + \|v\|_3 + \|\theta\|_2)\|\zeta-\zeta_n\|^2 + c\|\zeta_n\|_2\|v-v_n\|_1\|\zeta-\zeta_n\|$$

where $(2.2)_{1,3}$ and the Sobolev's embedding theorems are used. Denoting by C a constant depending at most on k, Ω, D, $\bar{B}$ and $\|a\|$ one has

$$\frac{d}{dt}\|\zeta-\zeta_n\|^2 \leqslant C(\|\zeta-\zeta_n\|^2 + \|v-v_n\|_1\|\zeta-\zeta_n\|)\,, \qquad \|(\zeta-\zeta_n)(0)\| = 0\,.$$

Since $\|v - v_n\|_1 \to 0$ uniformly on $[0, T]$ it follows that $\zeta_n \to \zeta$ in $L^\infty(0, T; L^2)$, hence in X. Thus F is continuous on S and Schauder's theorem applies. □

PROOF OF THEOREM 1.1. – Let $\zeta = \varphi$ be the fixed point of lemma 3.1. By proposition 2.1 and theorem 2.3 the corresponding v verifies (1.1′), (1.1″) on $[0, T]$ [5] and also verifies $(1.1)_4$. We obtain (1.9) from (2.8), (3.1), (1.7). We prove now (1.10) and $(1.1)_1$. Equations (2.9), (2.10) give $\operatorname{div} v^0 = \theta$, $\operatorname{rot} v^0 = \zeta$ in Q_T, $v^0 \cdot n = 0$ in Σ_T and $(v^0, u^{(l)}) = 0$ in $[0, T]$, $l = 1, \dots, N$. By differentiating with respect to t one obtains corresponding equations for $\partial v^0/\partial t$, with θ an ζ replaced by $\partial\theta/\partial t$ and $\partial\zeta/\partial t$ respectively. Hence

$$\left\|\frac{dv^0}{dt}\right\|_{k+1,T} \leqslant c\left(\left\|\frac{d\theta}{dt}\right\|_{k,T} + \left\|\frac{d\zeta}{dt}\right\|_{k,T}\right),$$

and using $(1.1'')_1$ one gets in particular

$$\left\|\frac{dv^0}{dt}\right\|_{k+1,T} \leqslant c\left\|\frac{d\theta}{dt}\right\|_{k,T} + c\|v\|_{k+1,T}\|v\|_{k+2,T}\,. \tag{3.6}$$

In particular it follows from (3.6) that $(1.8)_2$ holds, consequently $(2.2)_5$ yields $(1.1')_5$. Moreover $(2.2)_5$ gives

$$\frac{d\theta_l(t)}{dt} = -\left((v\cdot\nabla)v, u^{(l)}\right),$$

hence

$$\left\|\frac{d\theta_l}{dt}\right\|_{L^\infty(0,T)} \leqslant c\|v\|_{0,T}\|v\|_{1,T} \tag{3.7}$$

where c depends only on k and Ω (via the quantities $\|u^{(l)}\|_2$).

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[5] For the moment $(1.1')_5$ holds only in the weak sense $(2.2)_5$.

From (2.9), (3.6) and (3.7) one has the estimate (1.10).
Finally from $(1.1')_5$ and from

$$\operatorname{rot}\left(\frac{\partial v}{\partial t} + (v\cdot\nabla)\,v\right) = \frac{\partial\zeta}{\partial t} + (v\cdot\nabla)\zeta - (\zeta\cdot\nabla)v + \theta\zeta = 0$$

one obtains the existence of $\pi(t, x)$ such that $(1.1)_1$ holds. The uniqueness of the solution follows from lemma 4.1, proved in the next section. □

PROOF OF THEOREM 1.2. – By differentiating with respect to t the equation $(1.1'')_1$ one easily gets

$$\left\|\frac{d^2\zeta}{dt^2}\right\|_{k-1,T} \leqslant c\|v\|_{k+2,T}\left\|\frac{dv}{dt}\right\|_{k+1,T}. \tag{3.9}$$

On the other hand by differentiating twice the system of equations satisfied by v^0 (see the beginning of the proof of theorem 1.1) one gets

$$\left\|\frac{d^2 v^0}{dt^2}\right\|_{k,T} \leqslant c\left(\left\|\frac{d^2\zeta}{dt^2}\right\|_{k-1,T} + \left\|\frac{d^2\theta}{dt^2}\right\|_{k-1,T}\right). \tag{3.10}$$

Moreover, by differentiating with respect to t the equality stated after equation (3.6) one obtains $d^2\theta_l/dt^2 = -\,d((v\cdot\nabla)v, u^{(l)})/dt$; hence in particular

$$\left\|\frac{d^2\theta_l}{dt^2}\right\|_{L^\infty(0,T)} \leqslant c\|v\|_{k+2,T}\left\|\frac{dv}{dt}\right\|_{k+1,T}. \tag{3.11}$$

From (2.9), (1.9), (1.10) and from the estimates just obtained one gets (1.12). □

4. – Uniqueness of the solution and continuous dependence on θ.

LEMMA 4.1. – Let v_1 and v_2 be the solutions of (1.1) corresponding to θ_1 and θ_2 respectively. Assume that [6]

$$\theta_1,\ \theta_2 \in L^\infty(0, T; L^2), \tag{4.1}$$

and that $(i = 1, 2)$

$$\begin{cases} v_i \in L^\infty(0, T; L^\infty), \\ \nabla v_i \in L^1(0, T; L^\infty), \\ \dfrac{\partial v_i}{\partial t} \in L^1(0, T; L^2). \end{cases} \tag{4.2}$$

(6) These assumptions are not the best possible, but are sufficient for our purposes.

Then

$$\|v_1 - v_2\|^2_{L^\infty(0,T;L^2)} \leqslant C\|\theta_1 - \theta_2\|_{L^\infty(0,T;L^2)}, \tag{4.3}$$

where C depends only on Ω and on the norms corresponding to the assumptions (4.1), (4.2). In particular the solution is unique in the class (4.2) for data in the class (4.1).

PROOF. – One easily sees that

$$\left\| \frac{\partial v_i}{\partial t} + (v_i \cdot \nabla) v_i \right\|_{L^1(0,T;L^2)} \leqslant C .$$

Moreover from $(1.1)_1$ it follows that $-\nabla \pi_i \in L^2(\Omega)$ $(i = 1, 2)$ for almost all $t \in [0, T]$. By choosing π_i with mean value zero in Ω it follows that $\|\pi_i\| \leqslant c\|\nabla \pi_i\|$ for almost all $t \in [0, T]$, where c depends only on Ω. Hence

$$\|\pi_i\|_{L^1(0,T;L^2)} \leqslant C . \tag{4.4}$$

Let $w = v_2 - v_1$. From (1.1) we deduce that $(dw/dt) + (v_2 \cdot \nabla) w + (w \cdot \nabla) v_1 = -\nabla(\pi_2 - \pi_1)$ and $\operatorname{div} w = \theta_2 - \theta_1$ in Q_T, $w \cdot n = 0$ on Σ_T and $w|_{t=0} = 0$; with standard calculations we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 + \left(|\nabla v_1| - \frac{1}{2}\theta_2, |w|^2\right) = (\pi_2 - \pi_1, \theta_2 - \theta_1) .$$

Now from the assumptions on ∇v_1 and θ_2 and from (4.4) one obtains

$$y'(t) \leqslant a(t) y(t) + b(t)\mu(t) , \qquad y(0) = 0$$

where $y(t) = \|w(t)\|^2$, $\mu(t) = \|(\theta_2 - \theta_1)(t)\|$ and the coefficients $a(t)$ and $b(t)$ belong to $L^1(0, T)$.

By comparison theorems for ordinary differential equations it follows that $y(t) \leqslant C\mu(t)$. □

Let now $B > 0$ and $B_1 > 0$ be arbitrarily gived and define

$$T = \frac{c_0}{B + \|a\|_{k+2}} \tag{4.5}$$

where c_0 is the constant that appears in (1.7). Define also

$$R = \left\{\theta : \theta \text{ verifies (1.4), (1.5), (1.6) in } [0, T] \text{ and } \|\theta\|_{k+1,T} \leqslant B, \quad \left\|\frac{d\theta}{dt}\right\|_{k,T} \leqslant B_1\right\}. \tag{4.6}$$

Let now v be the solution of (1.1′), (1.1″) (i.e. (1.1)), which exists by theorem 1.1, and define Φ_1 by

$$v = \Phi_1[\theta]\,. \tag{4.7}$$

One has the following result:

LEMMA 4.2. – The operator

$$\Phi_1\colon R \to C(0, T; H^{k+1})$$

is continuous, when R is provided with the X norm.

PROOF. – Let $\theta_n,\ \theta \in R$, $\theta_n \to \theta$ in $C(0, T; H^k)$, and put $v_n = \Phi_1[\theta_n]$, $v = \Phi_1[\theta]$. From (1.9), (1.10) one easily gets that

$$\|v_n\|_{\mathrm{Lip}(0,T;\,H^{k+1})} \leqslant C\,, \qquad \|v_n(t)\|_{k+2} \leqslant C \qquad \forall t \in [0, T], \tag{4.8}$$

where the constants are independent of n. Hence by Ascoli-Arzela's theorem the set $\{v_n\}$ is relatively compact in $C(0, T; H^{k+1})$. On the other hand $v_n \to v$ in $L^\infty(0, T; L^2)$, by lemma 4.1. These two statements imply the result. □

Define now

$$R_1 = \left\{\theta \in R\colon \left\|\frac{d^2\theta}{dt^2}\right\|_{k-1,T} \leqslant B_2\right\},$$

where B_2 is a positive constant. One has the following result:

LEMMA 4.3. – Let $\theta_n \in R_1$, $\theta_n \to \theta$ in $C(0, T; H^k)$. Then

$$\frac{dv_n}{dt} \to \frac{dv}{dt} \qquad \text{in } C(0, T; H^k)$$

where $v_n = \Phi_1[\theta_n]$, $v = \Phi_1[\theta]$.

PROOF. – The proof is analogue to that of the preceding lemma; instead of (4.8) we use now the estimates

$$\left\|\frac{dv_n}{dt}\right\|_{\mathrm{Lip}(0,T;\,H^k)} \leqslant C\,, \qquad \left\|\frac{dv_n}{dt}\right\|_{k+1} \leqslant C\,, \qquad \text{for each } t \in [0, T], \tag{4.9}$$

which follow from (1.10) and (1.12). □

In the case of an external force field $f(t, x)$ (see the end of section 1) the lemmas 4.2 and 4.3 hold again as one easily verifies. One has then:

LEMMA 4.4. – Assume that one adds an external force field $f(t, x)$ to the equation (1.1) and let f verify the assumption (1.13). Furthermore replace (4.5) by

$$T = \frac{\bar{c}_0}{B + \|a\|_{k+2} + [f]_{k+2,T_0}} \tag{4.10}$$

where $\bar{c}_0 = \bar{c}_0(k, \Omega)$ is now the constant appearing in (1.7)*. Then the statement of lemma 4.2 holds. Moreover, if f verifies the assumption (1.14) lemma 4.3 holds.

5. – Motion in an external force field.

We give here without any comments (see the end of section 1) the modified main equations corresponding to the case when an external force field $f(t, x)$ verifying (1.13) is added to the equation (1.1):

$$(1.1)_1^* \quad \frac{\partial v}{\partial t} + (v\cdot\nabla)v - f = -\nabla\pi\,,$$

$$(1.1')_5^* \quad \left(\frac{\partial v}{\partial t} + (v\cdot\nabla)v - f,\, u^{(l)}\right) = 0\,,$$

$$(1.1'')_1^* \quad \frac{\partial \zeta}{\partial t} + (v\cdot\nabla)\zeta + (\zeta\cdot\nabla)v = -\theta\zeta + \operatorname{rot} f\,,$$

$$(1.7)^* \quad T = \frac{\bar{c}_0}{\|a\|_{k+2} + \|\theta\|_{k+1,T_0} + [f]_{k+2,T_0}}\,,$$

$$(1.9)^* \quad \|v\|_{k+2,T} \leqslant c(\|a\|_{k+2} + \|\theta\|_{k+1,T} + [f]_{k+2,T})$$

$$(1.10)^* \quad \left\|\frac{dv}{dt}\right\|_{k+1,T} \leqslant c\left\|\frac{d\theta}{dt}\right\|_{k,T} + c\,\|v\|^2_{k+2,T} + c\|f\|_{k+1,T}\,.$$

$$(1.12)^* \quad \left\|\frac{d^2 v}{dt^2}\right\|_{k,T} \leqslant c\left\|\frac{d^2\theta}{dt^2}\right\|_{k-1,T} + c\|v\|_{k+2,T}\left\|\frac{dv}{dt}\right\|_{k+1,T} + c\left\|\frac{df}{dt}\right\|_{k,T}\,.$$

$$(2.2)_5^* \quad \frac{d}{dt}(v, u^{(l)}) + ((v\cdot\nabla)v - f, u^{(l)}) = 0 \text{ classically in } [0, T]\,, \qquad l = 1, 2, \dots, N\,.$$

$$(2.8)^* \quad \|v\|_{k+2,T} \leqslant c(\bar{B} + D + \|a\| + [f]_{0,T}) \exp[c(\bar{B} + D)T]\,.$$

$$(2.26)^* \quad \begin{cases} \|\zeta\|_{k+1,T} \leqslant (\|\alpha\|_{k+1} + c[f]_{k+2,T}) \exp[cT(\|v\|_{k+2,T} + \|\theta\|_{k+1,T})]\,, \\ \left\|\dfrac{d\zeta}{dt}\right\|_{k,T} \leqslant c(\|v\|_{k+1,T} + \|\theta\|_{k,T})\|\zeta\|_{k+1,T} + c\|f\|_{k+1,T}\,. \end{cases}$$

$$(2.27)^* \quad \begin{cases} \|\zeta\|_{k+1,T} \leqslant (\|\alpha\|_{k+1} + c_1[f]_{k+2,T}) \exp[cT(\bar{B} + D + \|a\| + [f]_{0,T})\,e^{c(\bar{B}+D)T}]\,, \\ \left\|\dfrac{d\zeta}{dt}\right\|_{k,T} \leqslant c(\bar{B} + D + \|a\| + [f]_{k+2,T}) \exp[c(\bar{B} + D)T]\|\zeta\|_{k+1,T} + c\|f\|_{k+1,T}\,. \end{cases}$$

(3.1)* $$D = 2(\|\alpha\|_{k+1} + c_1[f]_{k+2,T_0})\,.$$

(3.3)* $$D_1 = c(\bar{B} + D + \|a\| + [f]_{0,T_0'})D + c\|f\|_{k+1,T_0}\,.$$

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