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Regularity Criteria for Navier-Stokes Equations with Slip Boundary Conditions on Non-flat Boundaries via Two Velocity Components

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Abstract: H.-O. Bae and H.J. Choe, in a 1997 paper, established a regularity criteria for the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^3$ based on two velocity components. Recently, one of the present authors extended this result to the half-space case $\mathbb{R}^3_+$. Further, this author in collaboration with J. Bemelmans and J. Brand extended the result to cylindrical domains under physical slip boundary conditions. In this note we obtain a similar result in the case of smooth arbitrary boundaries, but under a distinct, apparently very similar, slip boundary condition. They coincide just on flat portions of the boundary. Otherwise, a reciprocal reduction between the two results looks not obvious, as shown in the last section below.

Keywords: Navier-Stokes equations; Slip boundary conditions; No flat boundaries; Two components regularity criterium

MSC: 35Q30, 35B65, 76D05.

1 Introduction

The starting point of the present paper is the well known Prodi-Serrin (P-S) sufficient condition for regularity of the solutions to the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = 0\,, & \text{in } \Omega \times (0, T]\,, \\ \nabla \cdot \mathbf{u} = 0\,, & \text{in } \Omega \times (0, T]\,. \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ denotes the unknown velocity of the fluid and p the pressure. To immediately set limits to the circle of our interests, assume for now on that $\Omega \subset \mathbb{R}^3$ is a bounded, smooth domain, even if many results quoted below hold for larger space dimensions. For the time being, assume that suitable boundary conditions are imposed to the velocity $\mathbf{u}$.

The global existence of the so called weak solutions to system (1.1) goes back to J. Leray [1] and E. Hopf [2] classical references. See also A.A. Kiselev and O.A. Ladyzhenskaya [3], and J.L. Lions [4]. Below, solutions of (1.1) are intended in this sense.

A main classical open mathematical problem is to prove, or disprove, that weak solutions are necessarily strong under reasonable but general assumptions, where strong means that

$$\mathbf{u} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))\,. \tag{1.2}$$

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In this context, a remarkable and classical sufficient condition for uniqueness and regularity is the so-called Prodi-Serrin condition, P-S in the sequel, namely

$$\mathbf{u} \in L^q(0, T; L^p(\Omega)), \quad \frac{2}{q} + \frac{3}{p} = 1, \quad p > 3. \tag{1.3}$$

Concerning condition (1.3), we transcribe from [5], Section 1, the following considerations: Assumption (1.3) was firstly considered by G. Prodi in his paper [6] of 1959. He proved uniqueness under this last assumption. See also C. Foias, [7]. Furthermore, J. Serrin, see [8, 9], particularly proved interior spatial regularity under the stronger (non-strict) assumption

$$\mathbf{u} \in L^q(0, T; L^p(\Omega)), \quad \frac{2}{q} + \frac{3}{p} < 1, \quad p > 3. \tag{1.4}$$

Concerning the above problems, see also O.A. Ladyzhenskaya's contributions [10, 11]. The above setup led to the nomenclature Prodi-Serrin condition.

Complete proofs of the strict regularity result (i.e. under assumption (1.3)) were given by H. Sohr in [12], W. von Wahl in [13], and Y. Giga in [14]. A simplified version of the proof was given in reference [15], to which we refer also for bibliography. For a quite complete overview on the main points, and references, on the initial-boundary value problem for Navier-Stokes equations we strongly recommend Galdi's contribution [16]. Further, we refer to [9, 17], as sources for information on the historical context of the P-S condition by the initiators themselves.

Finally, we recall that L. Escauriaza, G. Seregin, and V. Šverák, see [18], extended the regularity result to the case $(q, p) = (\infty, 3)$.

A significant improvement of the P-S condition was obtained by H.-O. Bae and H.J. Choe [19], see also [20]. They proved, in the whole space case, that it is sufficient for regularity of solutions that two components of the velocity satisfy the above condition (1.3). For convenience we call here this situation as being the restricted P-S condition. In 2017, one of the authors, see [21], extended this result to the half-space $\mathbb{R}^3_+$ under slip boundary conditions. In this case, the truncated 2-dimensional vector field $\bar{\mathbf{u}}$ cannot be chosen arbitrarily. The omitted component has to be the normal to the boundary.

Very recently, in reference [5], the result was extended to a cylindrical type three-dimensional domain, consisting on the complement set between two co-axial circular cylinders, with radius ρ_0 and ρ_1, $0 < \rho_0 < \rho_1$, periodic in the axial direction, under the physical slip boundary condition

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad [D(\mathbf{u})\mathbf{n}] \cdot \tau = 0, \quad \text{on } \partial\Omega, \tag{1.5}$$

where $D(\mathbf{u}) = \frac{\nabla \mathbf{u} + (\nabla \mathbf{u})^T}{2}$ is the shear stress. The above exclusion of an interior cylinder was done to avoid the radial coordinate singularities on the symmetry axis, which consideration is out of interest in our context. Below we obtain a similar result, extended to domains with general non-flat boundaries, but under the slip boundary condition (2.1). The two boundary conditions coincide just on flat portions of the boundary. Otherwise, a reciprocal reduction between the two results looks not obvious. This claim is shown in the last section.

Again by following [5] we recall that after the contribution by H.-O. Bae and H.J. Choe, related papers appeared that particularly concerned assumptions on two components of velocity or vorticity, see [21–25]. There are also many papers dedicated to sufficient conditions for regularity which depend merely on one component, see, for instance, [26–31].

Before going on we want to motivate the particular choice of the domain made below. It takes into account that the real significance of the result has essentially a local character. First of all, a global regular (i.e., without singularities) system of coordinates, two of them parallel and the third orthogonal to the boundary, does not exist in general, even in an arbitrarily thin neighbourhood of the full boundary, as in the case of a sphere and even in the case of a spherical corona. In fact, singularities typically appear, like on the above two cases, and even in full cylinders (due to the symmetry axis). The cylindrical case considered in reference [5] is an exception (see below) due to the removal of a neighbourhood of the symmetry axis.

Luckily, the above type of coordinates' system exists in sufficient small neighbourhoods of any regular boundary point. Hence, to illustrate the full significance of our thesis in a simple, but still convincing way, it looks sufficient to prove it near any “small” piece of smooth boundary with an arbitrary geometrical shape. This is our aim below. The restrictions on the domain Ω below are made in accordance with these lines, a choice which covers the very basic situation, in the simplest way.

2 Main Results

In the sequel we assume the slip boundary condition

$$\mathbf{u}\cdot\mathbf{n}=0\,,\quad \boldsymbol{\omega}\times\mathbf{n}=0\quad \text{on } \partial\Omega\,, \tag{2.1}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the vorticity, $\mathbf{n}$ is the outward normal of $\partial\Omega$, and $\Omega\in\mathbb{R}^3$ is a smooth domain satisfying the following condition:

Assumption 2.1. *There exists a curvilinear orthogonal system of coordinates*

$$q(x)=(q_1(x),q_2(x),q_3(x))$$

such that Ω can be transited into

$$\hat{\Omega}\triangleq\{(q_1,q_2,q_3):0\le q_1<1,\ 0\le q_2<1,\ 0<\rho_0\le q_3\le\rho_1\}\,,$$

where the axis q_3 direct to the outward normal on the boundary $\partial\hat{\Omega}^1 := \{(q_1,q_2,q_3) : q_3=\rho_1\}$ (the inward normal on the boundary $\partial\hat{\Omega}^0 := \{(q_1,q_2,q_3) : q_3=\rho_0\}$, respectively), and q_1, q_2 are periodic.

Remark 2.1. The above "small" piece of a generical smooth boundary is here represented by $q_3 = \rho_0$, and $q_3 = \rho_1$.

Remark 2.2. It is worth noting that the slip boundary condition (2.1) is equivalent to

$$\mathbf{u}\cdot\mathbf{n}=0\,,\quad [D(\mathbf{u})\mathbf{n}]\cdot\tau=-\kappa_\tau\,\mathbf{u}\cdot\tau\,, \tag{2.2}$$

where τ stands for any arbitrary unit tangential vector on $\partial\Omega$, and κ_τ is the principal curvature in the τ direction, positive if the center of curvature lies inside Ω.

The above claim follows immediately by appealing to equation (5.2) in [32], namely

$$[D(\mathbf{u})\mathbf{n}]\cdot\tau=\frac{1}{2}(\boldsymbol{\omega}\times\mathbf{n})\cdot\tau-\kappa_\tau\,\mathbf{u}\cdot\tau\,. \tag{2.3}$$

For a mathematical treatment of some aspects related to slip boundary conditions imposed on smooth, but generic, boundaries see also [33], and the pioneering paper [34].

Next we recall some facts on curvilinear coordinates. The Lamé coefficients (scale factors) of the transition to the system of coordinates q are denoted by the letters H_i

$$H_i(q)=\left(\sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial q_i}\right)^2\right)^{\frac{1}{2}},\quad i=1,2,3.$$

Let $\hat{e}_i=\frac{1}{H_i}\frac{\partial x}{\partial q_i}$, $i=1,2,3$. Note that $|\hat{e}_i|=1$ and $\hat{e}_i\cdot\nabla=\frac{1}{H_i}\frac{\partial}{\partial q_i}$. One can write

$$\mathbf{u}(x)=\hat{\mathbf{u}}(q)=\hat{u}_1(q)\hat{e}_1+\hat{u}_2(q)\hat{e}_2+\hat{u}_3(q)\hat{e}_3$$

and

$$\boldsymbol{\omega}(x)=\hat{\boldsymbol{\omega}}(q)=\hat{\omega}_1(q)\hat{e}_1+\hat{\omega}_2(q)\hat{e}_2+\hat{\omega}_3(q)\hat{e}_3.$$

It is well known (see for example [35, 36]) that

$$\nabla \cdot \mathbf{u} = \frac{1}{H_1 H_2 H_3} \left(\frac{\partial(\hat{u}_1 H_2 H_3)}{\partial q_1} + \frac{\partial(\hat{u}_2 H_1 H_3)}{\partial q_2} + \frac{\partial(\hat{u}_3 H_1 H_2)}{\partial q_3} \right) \tag{2.4}$$

and

$$\begin{aligned} \nabla \times \mathbf{u} = & \frac{1}{H_2 H_3} \left(\frac{\partial(\hat{u}_3 H_3)}{\partial q_2} - \frac{\partial(\hat{u}_2 H_2)}{\partial q_3} \right) \hat{e}_1 \\ & + \frac{1}{H_1 H_3} \left(\frac{\partial(\hat{u}_1 H_1)}{\partial q_3} - \frac{\partial(\hat{u}_3 H_3)}{\partial q_1} \right) \hat{e}_2 \\ & + \frac{1}{H_1 H_2} \left(\frac{\partial(\hat{u}_2 H_2)}{\partial q_1} - \frac{\partial(\hat{u}_1 H_1)}{\partial q_2} \right) \hat{e}_3 . \end{aligned} \tag{2.5}$$

We state our main result as follows.

Theorem 2.2. *Let Ω satisfy Assumption 2.1, and suppose that there exist two positive constants c and C such that*

$$c \le H_i \le C \quad \text{and} \quad \left| \frac{\partial^2 x_i}{\partial q_i \partial q_j} \right| , \left| \frac{\partial^3 x_i}{\partial q_i \partial q_j \partial q_k} \right| \le C \,, \tag{2.6}$$

for any $i, j, k = 1, 2, 3$. Let $\mathbf{u}$ be a weak solution of the system (1.1) *under the boundary condition* (2.1), *and set $\bar{\mathbf{u}} = \hat{u}_1 \hat{e}_1 + \hat{u}_2 \hat{e}_2$. If $\bar{\mathbf{u}}$ satisfies*

$$\bar{\mathbf{u}} \in L^q(0, T; L^p(\Omega)) \,, \quad \frac{2}{q} + \frac{3}{p} \le 1, \quad p > 3 \,, \tag{2.7}$$

then the solution $\mathbf{u}$ is strong, namely,

$$\mathbf{u} \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \,.$$

Note that assumption (2.6) implies $|\partial_i H_j| , |\partial_{ij} H_k| \le C$.

It is worth noting that our proof applies to a more general set of geometrical situations. let's just give some hint in this direction.

Remark 2.3. The above statement does not contain the result proved in reference [5], due to the distinct boundary conditions, see Section 4. On the other hand, we may replace the two circular, vertical, cylinders by more general vertical cylinders where the external circle $q_3 = \rho_1$ is replaced by a smooth Jordan curve y_1, and the internal circle $q_3 = \rho_0$ by a parallel Jordan curve y_0, at a sufficient small distance $\delta > 0$ from y_1. The coordinate θ is now an arc length coordinate on y_1. All points in the same orthogonal segment to y_0 and y_1 enjoy the same θ coordinate. The coordinate $r \in (0, \delta)$ is given by the distance to y_1. The "vertical" coordinate z preserves his periodic character. Clearly, the role played by the above Jordan curve may be immediately extended to much more general situations.

Another significant application is obtained by replacing the above two cylindrical boundaries by two torus of revolution, generated by revolving two concentric circles y_0 and y_1 about an axis coplanar with the circles, which does not touch the circles (roughly, we obtain the complement set between two closed tubes). Now $z \in [0, 2\pi)$ is an angular periodic coordinate, the toroidal coordinate. The result still applies by replacing the two circles by two parallel Jordan curves.

Let's propose the following benchmark problem:

Problem 2.3. *Consider two concentric spheres Ω_R and Ω_ρ, of radius respectively ρ and R, $0 < \rho < R$. Let u be a weak solution in $\Omega_R \times (0, T]$ of* (1.1) *under one of the above slip boundary conditions. Further, assume that the restricted P-S condition holds in $(\Omega_R - \Omega_\rho) \times (0, T]$ with respect to the tangential components of the velocity, and holds in $\Omega_\rho \times (0, T]$ with respect to two arbitrary components of the velocity. Problem: To prove that u is a strong solution in $\Omega_R \times (0, T]$.*

3 Proof of Theorem 2.2

Proof. We start by reducing the system (1.1) under the boundary condition (2.1) into the classical vorticity form

$$\begin{cases} \partial_t \boldsymbol{\omega} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} - \Delta \boldsymbol{\omega} = 0 \,, & \text{in } \Omega \times (0, T] \,, \\ \nabla \cdot \mathbf{u} = 0 \,, & \text{in } \Omega \times (0, T] \,, \\ \mathbf{u} \cdot \mathbf{n} = 0 \,, \quad \boldsymbol{\omega} \times \mathbf{n} = 0 \,, & \text{on } \partial\Omega \,. \end{cases}$$

Then we take the scalar product with $\boldsymbol{\omega}$, and integrate by parts. One easily gets

$$\frac{1}{2} \partial_t \int_\Omega |\boldsymbol{\omega}|^2 dx + \int_\Omega |\nabla \boldsymbol{\omega}|^2 dx = \int_{\partial\Omega} \mathbf{n} \cdot \nabla \boldsymbol{\omega} \cdot \boldsymbol{\omega} dS + \int_\Omega \boldsymbol{\omega} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega} dx := I_1 + I_2 \,. \tag{3.1}$$

Next, we focus on the estimates of I_1 and I_2.

Control of I_1: First, it follows from (2.1) that

$$\hat{u}_3 = 0 \,, \quad \hat{\omega}_1 = \hat{\omega}_2 = 0 \,, \quad \text{as } \ q_3 = \rho_0 \,, \rho_1 \,. \tag{3.2}$$

Let $\partial\Omega^l = \partial\hat{\Omega}^l := \{(q_1, q_2, q_3) \in \hat{\Omega} : q_3 = \rho_l\}$, where $l = 0\,, 1$. One can deduce from (3.2) that

$$\begin{aligned} (2l-1) \int_{\partial\Omega^l} \mathbf{n} \cdot \nabla \boldsymbol{\omega} \cdot \boldsymbol{\omega} dS =& (2l-1) \int_{\partial\Omega^l} \mathbf{n} \cdot \nabla \left(\frac{|\boldsymbol{\omega}|^2}{2} \right) dS \\ =& \int_0^1 \int_0^1 \left[\partial_{q_3} \left(\frac{|\hat{\boldsymbol{\omega}}|^2}{2} \right) H_1 H_2 H_3^{-1} \right] \Big|_{q_3 = \rho_l} dq_1 dq_2 \\ =& \int_0^1 \int_0^1 \left[(\partial_{q_3} \hat{\omega}_3) \, \hat{\omega}_3 H_1 H_2 H_3^{-1} \right] |_{q_3 = \rho_l} dq_1 dq_2 \\ =& \int_0^1 \int_0^1 \left[\partial_{q_3} \left(H_1 H_2 \hat{\omega}_3 \right) H_3^{-1} \hat{\omega}_3 \right] |_{q_3 = \rho_l} dq_1 dq_2 \\ & - \int_0^1 \int_0^1 \left[\partial_{q_3} \left(H_1 H_2 \right) H_3^{-1} \hat{\omega}_3^2 \right] |_{q_3 = \rho_l} dq_1 dq_2 \,. \end{aligned} \tag{3.3}$$

Since $\nabla \cdot \boldsymbol{\omega} = 0$, from (2.4) one gets

$$\frac{\partial(\hat{\omega}_1 H_2 H_3)}{\partial q_1} + \frac{\partial(\hat{\omega}_2 H_1 H_3)}{\partial q_2} + \frac{\partial(\hat{\omega}_3 H_1 H_2)}{\partial q_3} = 0 \,,$$

which gives

$$\begin{aligned} & \int_0^1 \int_0^1 \left[\partial_{q_3} \left(H_1 H_2 \hat{\omega}_3 \right) H_3^{-1} \hat{\omega}_3 \right] |_{q_3 = \rho_l} dq_1 dq_2 \\ = & - \int_0^1 \int_0^1 \left[\partial_{q_1} \left(H_2 H_3 \, \hat{\omega}_1 \right) H_3^{-1} \hat{\omega}_3 \right] |_{q_3 = \rho_l} dq_1 dq_2 - \int_0^1 \int_0^1 \left[\partial_{q_2} \left(H_1 H_3 \hat{\omega}_2 \right) H_3^{-1} \hat{\omega}_3 \right] |_{q_3 = \rho_l} dq_1 dq_2 \\ = & \, 0 \,, \end{aligned}$$

since $\hat{\omega}_1 = \hat{\omega}_2 = \partial_{q_1} \hat{\omega}_1 = \partial_{q_2} \hat{\omega}_2 = 0$ on $\partial\hat{\Omega}^l$. Hence, one obtains

$$(2l-1) \int_{\partial\Omega^l} \mathbf{n} \cdot \nabla \boldsymbol{\omega} \cdot \boldsymbol{\omega} dS = - \int_0^1 \int_0^1 \left[\partial_{q_3} \left(H_1 H_2 \right) H_3^{-1} \hat{\omega}_3^2 \right] |_{q_3 = \rho_l} dq_1 dq_2 .$$

By appealing to (2.6) one shows that

$$\left| \int_{\partial\Omega} \mathbf{n} \cdot \nabla \boldsymbol{\omega} \cdot \boldsymbol{\omega} dS \right| \leq C \int_{\partial\Omega} |\boldsymbol{\omega}|^2 dS \leq \| |\boldsymbol{\omega}|^2 \|_{W^{1,1}(\Omega)} ,$$

where we have used Gagliardo's trace theorem, see [37]. See also [38], Theorem 4.2 (for an English recent text see, for example, the Theorem III.2.21 in [39]). It follows that

$$\left| \int_{\partial\Omega} \mathbf{n} \cdot \nabla \boldsymbol{\omega} \cdot \boldsymbol{\omega} dS \right| \leq C(\epsilon) \|\boldsymbol{\omega}\|^2_{L^2(\Omega)} + \epsilon \|\nabla \boldsymbol{\omega}\|^2_{L^2(\Omega)} , \tag{3.4}$$

for all $0 < \epsilon < 1$.

Control of I_2: First, one has

$$\begin{aligned} \int_{\Omega} \boldsymbol{\omega} \cdot \nabla \mathbf{u} \cdot \boldsymbol{\omega} dx &= \sum_{i,j,k} \int_{\hat{\Omega}} \hat{\omega}_i H_i^{-1} \partial_{q_i} \left(\hat{u}_j \hat{e}_j \right) \cdot (\hat{\omega}_k \hat{e}_k) H_1 H_2 H_3 dq_1 dq_2 dq_3 \\ &= \sum_{i,j,k} \int_{\hat{\Omega}} \hat{u}_j \hat{\omega}_i \hat{\omega}_k (\partial_{q_i} \hat{e}_j \cdot \hat{e}_k) H_i^{-1} H_1 H_2 H_3 dq_1 dq_2 dq_3 \\ &\quad + \sum_{i,j} \int_{\hat{\Omega}} \hat{\omega}_i (\partial_{q_i} \hat{u}_j) \hat{\omega}_j H_i^{-1} H_1 H_2 H_3 dq_1 dq_2 dq_3 \\ &:= I_{21} + I_{22} . \end{aligned}$$

For I_{21}, from (2.6), one has

$$\begin{aligned} |I_{21}| &\leq C \int_{\Omega} |\mathbf{u}| \, |\boldsymbol{\omega}|^2 \, dx \\ &\leq C \|\mathbf{u}\|_{L^2(\Omega)} \|\boldsymbol{\omega}\|^2_{L^4(\Omega)} \\ &\leq C \|\mathbf{u}\|_{L^2(\Omega)} \|\boldsymbol{\omega}\|^{\frac{1}{2}}_{L^2(\Omega)} (\|\boldsymbol{\omega}\|_{L^2(\Omega)} + \|\nabla \boldsymbol{\omega}\|_{L^2(\Omega)})^{\frac{3}{2}} \\ &\leq C \|\mathbf{u}\|_{L^2(\Omega)} \|\boldsymbol{\omega}\|^2_{L^2(\Omega)} + C \|\mathbf{u}\|_{L^2(\Omega)} \|\boldsymbol{\omega}\|^{\frac{1}{2}}_{L^2(\Omega)} \|\nabla \boldsymbol{\omega}\|^{\frac{3}{2}}_{L^2(\Omega)} \\ &\leq C \|\mathbf{u}\|_{L^2(\Omega)} \|\boldsymbol{\omega}\|^2_{L^2(\Omega)} + C(\epsilon) \|\mathbf{u}\|^4_{L^2(\Omega)} \|\boldsymbol{\omega}\|^2_{L^2(\Omega)} + \epsilon \|\nabla \boldsymbol{\omega}\|^2_{L^2(\Omega)} . \end{aligned} \tag{3.5}$$

For I_{22}, we consider separately the three cases $j \neq 3$; $j = 3$ and $i \neq 3$; $i = j = 3$.

Case I: $j \neq 3$. By integration by parts, one has

$$\begin{aligned} &\int_{\hat{\Omega}} \hat{\omega}_i (\partial_{q_i} \hat{u}_j) \hat{\omega}_j H_i^{-1} H_1 H_2 H_3 dq_1 dq_2 dq_3 \\ &= - \int_{\hat{\Omega}} \hat{u}_j (\partial_{q_i} \hat{\omega}_i) \hat{\omega}_j H_i^{-1} H_1 H_2 H_3 dq_1 dq_2 dq_3 - \int_{\hat{\Omega}} \hat{u}_j \hat{\omega}_i (\partial_{q_i} \hat{\omega}_j) H_i^{-1} H_1 H_2 H_3 dq_1 dq_2 dq_3 \\ &\quad - \int_{\hat{\Omega}} \hat{u}_j \hat{\omega}_i \hat{\omega}_j \partial_{q_i} \left(H_i^{-1} H_1 H_2 H_3 \right) dq_1 dq_2 dq_3 . \end{aligned} \tag{3.6}$$

Case II: $j = 3$ **and** $i \neq 3$. From (2.5) one has

$$\hat{\omega}_3 = \frac{1}{H_1 H_2} \left(\frac{\partial(\hat{u}_2 H_2)}{\partial q_1} - \frac{\partial(\hat{u}_1 H_1)}{\partial q_2} \right) .$$

Hence, by integration by parts, it follows that

$$
\begin{aligned}
&\int_{\hat{\Omega}} \hat{\omega}_i\,(\partial_{q_i}\hat{u}_3)\,\hat{\omega}_3 H_i^{-1} H_1 H_2 H_3\, dq_1\, dq_2\, dq_3 \\
&= \int_{\hat{\Omega}} \hat{\omega}_i\,(\partial_{q_i}\hat{u}_3)\,\big(\partial_{q_1}(H_2\hat{u}_2) - \partial_{q_2}(H_1\hat{u}_1)\big) H_i^{-1} H_3\, dq_1\, dq_2\, dq_3 \\
&= -\int_{\hat{\Omega}} \hat{u}_2\, \partial_{q_1}\big(\,\hat{\omega}_i\,(\partial_{q_i}\hat{u}_3)\, H_i^{-1} H_3\,\big) H_2\, dq_1\, dq_2\, dq_3 \\
&\quad + \int_{\hat{\Omega}} \hat{u}_1 \partial_{q_2}\,\big(\hat{\omega}_i\,(\partial_{q_i}\hat{u}_3)\, H_i^{-1} H_3\big)\, H_1\, dq_1\, dq_2\, dq_3\,.
\end{aligned}
\tag{3.7}
$$

Case III: $i = j = 3$. Note that, due to $\nabla \cdot \mathbf{u} = 0$, it follows

$$
\frac{\partial(\hat{u}_1 H_2 H_3)}{\partial q_1} + \frac{\partial(\hat{u}_2 H_1 H_3)}{\partial q_2} + \frac{\partial(\hat{u}_3 H_1 H_2)}{\partial q_3} = 0\,. \tag{3.8}
$$

One has

$$
\begin{aligned}
&\int_{\hat{\Omega}} \hat{\omega}_3\,(\partial_{q_3}\hat{u}_3)\,\hat{\omega}_3 H_1 H_2\, dq_1\, dq_2\, dq_3 \\
&= \int_{\hat{\Omega}} \hat{\omega}_3 \partial_{q_3}\,\big(H_1 H_2 \hat{u}_3\big)\,\hat{\omega}_3\, dq_1\, dq_2\, dq_3 - \int_{\hat{\Omega}} \hat{\omega}_3 \hat{u}_3 \hat{\omega}_3 \partial_{q_3}(H_1 H_2)\, dq_1\, dq_2\, dq_3 \\
&= -\int_{\hat{\Omega}} \hat{\omega}_3 \partial_{q_1}\,\big(H_2 H_3 \hat{u}_1\big)\,\hat{\omega}_3\, dq_1\, dq_2\, dq_3 - \int_{\hat{\Omega}} \hat{\omega}_3 \partial_{q_2}\,\big(H_1 H_3 \hat{u}_2\big)\,\hat{\omega}_3\, dq_1\, dq_2\, dq_3 \\
&\quad - \int_{\hat{\Omega}} \hat{\omega}_3 \hat{u}_3 \hat{\omega}_3 \partial_{q_3}(H_1 H_2)\, dq_1\, dq_2\, dq_3 \\
&= \int_{\hat{\Omega}} \hat{u}_1 \partial_{q_1}\,\big(\hat{\omega}_3^2\big)\, H_2 H_3\, dq_1\, dq_2\, dq_3 + \int_{\hat{\Omega}} \hat{u}_1 \partial_{q_2}\,\big(\hat{\omega}_3^2\big)\, H_1 H_3\, dq_1\, dq_2\, dq_3 \\
&\quad - \int_{\hat{\Omega}} \hat{u}_3 \hat{\omega}_3^2 \partial_{q_3}(H_1 H_2)\, dq_1\, dq_2\, dq_3\,,
\end{aligned}
\tag{3.9}
$$

where the first equality is an identity, the second is obtained by appealing to (3.8), and the third one follows by integration by parts. From (3.6), (3.7), (3.9) and the assumption (2.6), one can obtain

$$
|I_{22}| \le C \int_{\Omega} |\bar{\mathbf{u}}||\nabla \mathbf{u}||\nabla^2 \mathbf{u}|\, dx + C \int_{\Omega} |\mathbf{u}||\nabla \mathbf{u}|^2\, dx + C \int_{\Omega} |\mathbf{u}|^2 |\nabla \mathbf{u}|\, dx\,.
$$

It is easy to get that

$$
\begin{aligned}
\int_{\Omega} |\mathbf{u}|^2 |\nabla \mathbf{u}|\, dx &\le \|\mathbf{u}\|_{L^2(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} \|\nabla \mathbf{u}\|_{L^4(\Omega)} \\
&\le \|\mathbf{u}\|_{L^2(\Omega)}^2 \|\mathbf{u}\|_{L^4(\Omega)}^2 + \|\nabla \mathbf{u}\|_{L^4(\Omega)}^2 \\
&\le \|\mathbf{u}\|_{L^2(\Omega)}^2 \|\mathbf{u}\|_{L^4(\Omega)}^2 + \|\nabla \mathbf{u}\|_{L^2(\Omega)}^{\frac{1}{2}} (\|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|\nabla^2 \mathbf{u}\|_{L^2(\Omega)})^{\frac{3}{2}} \\
&\le \|\mathbf{u}\|_{L^2(\Omega)}^2 \|\mathbf{u}\|_{L^4(\Omega)}^2 + C(\epsilon) \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \epsilon \|\nabla^2 \mathbf{u}\|_{L^2(\Omega)}^2\,,
\end{aligned}
$$

and similarly to the proof of (3.5)

$$
\int_{\Omega} |\mathbf{u}||\nabla \mathbf{u}|^2\, dx \le C\, \|\mathbf{u}\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + C(\epsilon) \|\mathbf{u}\|_{L^2(\Omega)}^4 \|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 + \epsilon\, \|\nabla^2 \mathbf{u}\|_{L^2(\Omega)}^2\,.
$$

Hence, one has

$$\begin{aligned}|I_{22}| \leq & C\int_\Omega |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}|dx + C\|\mathbf{u}\|^2_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^4(\Omega)} + C(\epsilon)\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \\ & + C\,\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + C(\epsilon)\|\mathbf{u}\|^4_{L^2(\Omega)}\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + C\epsilon\,\|\nabla^2\mathbf{u}\|^2_{L^2(\Omega)}\,.\end{aligned} \tag{3.10}$$

By Hölder's inequality, interpolation, and a Sobolev's embedding theorem, one can easily show that

$$\begin{aligned}&\int_\Omega |\bar{\mathbf{u}}||\nabla\mathbf{u}||\nabla^2\mathbf{u}|dx \\ \leq & \||\bar{\mathbf{u}}|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega)} \\ \leq & \|\bar{\mathbf{u}}\|_{L^p(\Omega)}\|\nabla\mathbf{u}\|_{L^{\frac{2p}{p-2}}(\Omega)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega)} \\ \leq & \|\bar{\mathbf{u}}\|_{L^p(\Omega)}\|\nabla\mathbf{u}\|^{1-\frac{3}{p}}_{L^2(\Omega)}\|\nabla\mathbf{u}\|^{\frac{3}{p}}_{L^6(\Omega)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega)} \\ \leq & C\|\bar{\mathbf{u}}\|_{L^p(\Omega)}\|\nabla\mathbf{u}\|^{1-\frac{3}{p}}_{L^2(\Omega)}(\|\nabla\mathbf{u}\|_{L^2(\Omega)} + \|\nabla^2\mathbf{u}\|_{L^2(\Omega)})^{\frac{3}{p}}\|\nabla^2\mathbf{u}\|_{L^2(\Omega)} \\ \leq & C\left(\|\bar{\mathbf{u}}\|_{L^p(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega)} + \|\bar{\mathbf{u}}\|_{L^p(\Omega)}\|\nabla\mathbf{u}\|^{1-\frac{3}{p}}_{L^2(\Omega)}\|\nabla^2\mathbf{u}\|^{1+\frac{3}{p}}_{L^2(\Omega)}\right) \\ \leq & C(\epsilon)\left(\|\bar{\mathbf{u}}\|^{\frac{2p}{p-3}}_{L^p(\Omega)} + \|\bar{\mathbf{u}}\|^2_{L^p(\Omega)}\right)\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} + \epsilon\,\|\nabla^2\mathbf{u}\|^2_{L^2(\Omega)}\,.\end{aligned} \tag{3.11}$$

Collecting (3.1) and the estimates (3.4), (3.5), (3.10) and (3.11), one obtains

$$\begin{aligned}\frac{1}{2}\partial_t\int_\Omega |\boldsymbol{\omega}|^2dx + \int_\Omega |\nabla\boldsymbol{\omega}|^2dx \leq & C(\epsilon)\left(1 + \|\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\|^4_{L^2(\Omega)} + \|\bar{\mathbf{u}}\|^{\frac{2p}{p-3}}_{L^p(\Omega)} + \|\bar{\mathbf{u}}\|^2_{L^p(\Omega)}\right)\|\nabla\mathbf{u}\|^2_{L^2(\Omega)} \\ & + C\|\mathbf{u}\|^2_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^4(\Omega)} + C\epsilon\,\|\nabla^2\mathbf{u}\|^2_{L^2(\Omega)}\,.\end{aligned}$$

On the other hand, the following well known estimates (see for instance Theorem IV.4.8 and Theorem IV.4.9 in [39]), hold:

$$\|\nabla\mathbf{u}\|_{L^2(\Omega)} \leq C\|\boldsymbol{\omega}\|_{L^2(\Omega)}\,, \quad \|\nabla^2\mathbf{u}\|_{L^2(\Omega)} \leq C\left(\|\mathbf{u}\|_{L^2(\Omega)} + \|\boldsymbol{\omega}\|_{H^1(\Omega)}\right)\,. \tag{3.12}$$

Therefore, from equation (3.12), by letting ϵ be sufficiently small, one has

$$\begin{aligned}\partial_t\int_\Omega |\boldsymbol{\omega}|^2dx + \int_\Omega |\nabla\boldsymbol{\omega}|^2dx \leq & C\left(1 + \|\mathbf{u}\|_{L^2(\Omega)} + \|\mathbf{u}\|^4_{L^2(\Omega)} + \|\bar{\mathbf{u}}\|^{\frac{2p}{p-3}}_{L^p(\Omega)} + \|\bar{\mathbf{u}}\|^2_{L^p(\Omega)}\right)\|\boldsymbol{\omega}\|^2_{L^2(\Omega)} \\ & + C\|\mathbf{u}\|^2_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^4(\Omega)} + C\|\mathbf{u}\|^2_{L^2(\Omega)}\,.\end{aligned}$$

Finally (1.2) follows by taking into account equations (2.7) ($q \geq \frac{2p}{p-3} > 2$) and (3.12), and by appealing to a well known argument, which is based on Gronwall's inequality. Recall that weak solutions verify $\|\mathbf{u}\|_{L^2(\Omega)} \in L^\infty(0,\,T)$ and $\|\mathbf{u}\|^2_{L^6(\Omega)} \in L^1(0,\,T)$. Hence we have proved that $\mathbf{u}$ is a strong solution. □

4 On related slip boundary conditions.

In this section we present a first attempt to prove the statement of Theorem 2.2 with the slip boundary condition (2.1) replaced by the slip boundary condition (1.5) (assumed in reference [5]) by means of a simple modification of our proof. This attempt fails. Hence this significant problem remains open to further investigation. This leads us to briefly show our calculations.

Let's start by explaining our guess. As still shown in Remark 2.2 condition (1.5) is equivalent to

$$\mathbf{u}\cdot\mathbf{n} = 0\,, \quad (\boldsymbol{\omega}\times\mathbf{n})\cdot\tau = 2\,\kappa_\tau\,\mathbf{u}\cdot\tau\,, \quad \text{on} \quad \partial\Omega\,. \tag{4.1}$$

We may replace the arbitrary tangent vector τ simply by a couple of independent vectors like, for instance, the principal direction's vectors τ_1 and τ_2. In this case $\kappa_1 = \kappa_{\tau_1}$ and $\kappa_2 = \kappa_{\tau_2}$ are the maximum and the minimum principal curvatures.

A more natural choice here is to consider the couple of tangent, orthogonal, vectors $\hat{e}_1$ and $\hat{e}_2$. In this case κ_1 and κ_2 are the related curvatures. This second choice easily leads to the couple of linear equations

$$\begin{cases} \hat{\omega}_2 = 2\,\kappa_1\,\hat{u}_1\,, \\ \hat{\omega}_1 = -2\,\kappa_2\,\hat{u}_2\,. \end{cases} \tag{4.2}$$

Hence to replace the slip boundary condition (2.1) by $[D(\mathbf{u})\mathbf{n}]\cdot\tau = 0$ means to replace assumption (3.2) by

$$\hat{u}_3 = 0\,, \quad \hat{\omega}_1 = -2\,\kappa_2\hat{u}_2\,, \quad \hat{\omega}_2 = 2\,\kappa_1\,\hat{u}_1\,, \quad \text{as } q_3 = \rho_0\,, \rho_1\,. \tag{4.3}$$

To prove our main statement with the boundary condition (2.1) replaced by the boundary condition (4.1) we have to control some new boundary integrals, which no longer vanish since now $\hat{\omega}_1$ and $\hat{\omega}_2$ do not vanish. However, by (4.2), $\hat{\omega}_1$ and $\hat{\omega}_2$ can be expressed in terms of the (lower order) velocity components $\hat{u}_2$ and $\hat{u}_1$. Well known inverse trace theorems allow us to control boundary-norms of these two components by suitable internal norms. Since our P-S assumption guarantees additional regularity just for these two velocity components, one could expect that the above internal norms could be estimated in a convenient way. Unfortunately this device seems not sufficient to prove our goal. So this interesting problem remains open.

Next we pass to showing our calculations. Let's turn back to equation (3.3), by taking into account that now we can not apply to $\hat{\omega}_1 = \hat{\omega}_2 = 0$. One has

$$\begin{aligned}(2l-1)\int_{\partial\Omega^l} \mathbf{n}\cdot\nabla\boldsymbol{\omega}\cdot\boldsymbol{\omega}\,dS &= (2l-1)\int_{\partial\Omega^l}\mathbf{n}\cdot\nabla\left(\frac{|\boldsymbol{\omega}|^2}{2}\right)dS \\ &= \int_0^1\int_0^1 \left[\partial_{q_3}\left(\frac{|\hat{\boldsymbol{\omega}}|^2}{2}\right)H_1H_2H_3^{-1}\right]\Big|_{q_3=\rho_l} dq_1dq_2 \\ &= \int_0^1\int_0^1 \left[\sum_j (\partial_{q_3}\hat{\omega}_j)\,\hat{\omega}_jH_1H_2H_3^{-1}\right]\Big|_{q_3=\rho_l} dq_1dq_2\,.\end{aligned}$$

We will drop terms which could be easily manipulated, called here "lower order terms". Dropping lower order terms and also cancelling non significant multiplication coefficients, lead us to introduce the symbols " $\simeq$ " and " $\preceq$ ", which have a clear meaning here.

One has

$$(\partial_{q_3}\hat{\omega}_j)\,\hat{\omega}_j\,H_1H_2H_3^{-1} = \partial_{q_3}\left(H_1H_2\hat{\omega}_j\right)H_3^{-1}\hat{\omega}_j - \partial_{q_3}(H_1H_2)\,H_3^{-1}\hat{\omega}_j^2 \preceq \partial_{q_3}\left(H_1H_2\hat{\omega}_j\right)H_3^{-1}\hat{\omega}_j\,. \tag{4.4}$$

Since $\nabla\cdot\boldsymbol{\omega} = 0$, from (2.4) one gets

$$\frac{\partial(\hat{\omega}_1H_2H_3)}{\partial q_1} + \frac{\partial(\hat{\omega}_2H_1H_3)}{\partial q_2} + \frac{\partial(\hat{\omega}_3H_1H_2)}{\partial q_3} = 0\,,$$

which gives, on $\partial\hat{\Omega}$,

$$\partial_{q_3}\left(H_1H_2\hat{\omega}_3\right)H_3^{-1}\hat{\omega}_3 = -\partial_{q_1}\left(H_2H_3\,\hat{\omega}_1\right)H_3^{-1}\hat{\omega}_3 - \partial_{q_2}\left(H_1H_3\hat{\omega}_2\right)H_3^{-1}\hat{\omega}_3\,.$$

Under the new boundary conditions we can not apply to $\hat{\omega}_1 = \hat{\omega}_2 = \partial_{q_1}\hat{\omega}_1 = \partial_{q_2}\hat{\omega}_2 = 0$ on $\partial\hat{\Omega}^l$ to claim the cancellation of the above right hand side. By noting that the two terms on the right hand side are symmetric, with respect to the index 1 and 2, we may consider just the first one.

One has

$$\partial_{q_1}\left(H_2H_3\,\hat{\omega}_1\right)H_3^{-1}\hat{\omega}_3 = (\partial_{q_1}\hat{\omega}_1)\hat{\omega}_3\,H_2 - \hat{\omega}_1\,\hat{\omega}_3\,\partial_{q_1}(H_2H_3)H_3^{-1} \simeq (\partial_{q_1}\hat{\omega}_1)\hat{\omega}_3\,.$$

Note that the smooth coefficients H_j, as their derivatives, are not significant on our estimate below. Further, since ∂_{q_1} is a tangential derivative, we may apply to the second equality (4.2) to assume that $\partial_{q_1}\,\hat{\omega}_1 \simeq -\partial_{q_1}\,\hat{u}_2$ on $\partial\hat{\Omega}$. Hence

$$\int_0^1\int_0^1 \left[\partial_{q_1}\,(H_2 H_3\,\hat{\omega}_1)\,H_3^{-1}\hat{\omega}_3\right]|_{q_3=\rho_l} dq_1 dq_2 \simeq \int_0^1\int_0^1 (\partial_{q_1}\hat{u}_2\,)\,\hat{\omega}_3|_{q_3=\rho_l} dq_1 dq_2\,. \tag{4.5}$$

By appealing to Gagliardo's theorem we show that the above right hand side is bounded by $C(\epsilon)\,\|\nabla\,\mathbf{u}\,\|_2 + \epsilon\,\|\nabla^2\,\mathbf{u}\,\|_2$, which is sufficient to our purposes.

Let's now consider in equation (4.4) the terms $\partial_{q_3}(H_1 H_2\hat{\omega}_j)\,H_3^{-1}\hat{\omega}_j$, for $j = 1,\,2$. Assume, for instance, $j = 1$. One has $\partial_{q_3}(H_1 H_2\hat{\omega}_1)H_3^{-1}\hat{\omega}_1 \simeq (\partial_{q_3}\hat{\omega}_1)\hat{u}_2$. Hence we need to control the integral

$$\int_0^1\int_0^1 (\partial_{q_3}\hat{\omega}_1\,)\,\hat{u}_2|_{q_3=\rho_l} dq_1 dq_2\,. \tag{4.6}$$

Roughly speaking the above integrand has the same order as that on the right hand side of (4.5). However in (4.6) the derivation symbol ∂_{q_3} appears now in the "bad position". A suitable control of the above integral turns out to be not obvious.

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