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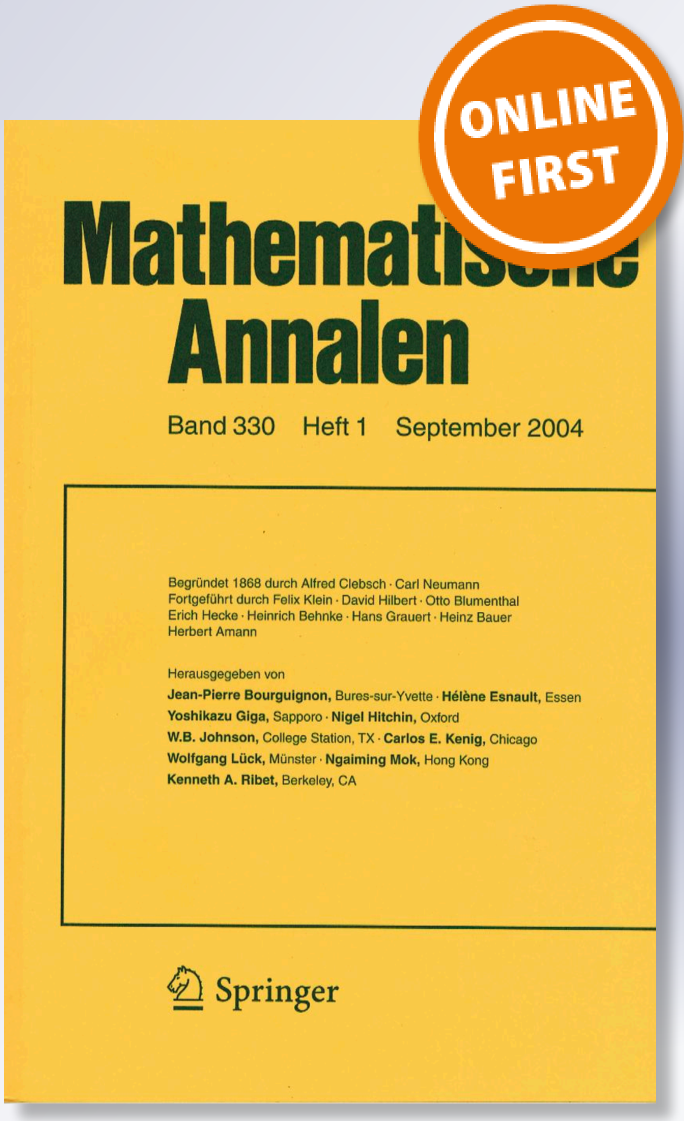

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On a two components condition for regularity of the 3D Navier–Stokes equations under physical slip boundary conditions on non-flat boundaries

Hugo Beirão da Veiga[1] · Josef Bemelmans[2] · Johannes Brand[3]

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Abstract
This work concerns the sufficient condition for the regularity of solutions to the evolution Navier–Stokes equations known in the literature as Prodi–Serrin condition. H.-O. Bae and H. J. Choe proved in 1997 that, in the whole space $\mathbb{R}^3$, it is sufficient that two components of the velocity satisfy the above condition in order to guarantee the regularity of solutions. In 2017, H. Beirão da Veiga extended this result (Beirão da Veiga, J Math Anal Appl 453:212–220, 2017) to the half-space case $\mathbb{R}^n_+$ under slip boundary conditions by assuming that the velocity components *parallel* to the boundary enjoy the above condition. It remained open whether the flat boundary geometry is essential. Below, we prove that, under physical slip boundary conditions imposed in cylindrical boundaries, the result still holds.

Keywords Navier–Stokes Equations · Slip Boundary Conditions · Prodi–Serrin Condition · Two Components Condition · Regularity

Mathematics Subject Classification 35Q30

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✉ Josef Bemelmans
bemelmans@instmath.rwth-aachen.de

Hugo Beirão da Veiga
bveiga@dma.unipi.it

Johannes Brand
J.Brand@wzl.rwth-aachen.de

1 Department of Mathematics, Pisa University, Pisa, Italy

2 Institute for Mathematics, RWTH Aachen University, Aachen, Germany

3 WZL, RWTH Aachen University, Aachen, Germany

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1 Introduction. Related results. The main problem

To explain motivation, setting, and interest of the problem studied below, we start by recalling some well-known results. A sketch is sufficient to this purpose, since we assume that readers are acquainted with the main lines of the subject. Some results that are referred to below also hold for dimensions $n > 3$. For simplicity, since we are interested in the case $n = 3$ below, we do not refer to extensions to larger dimensions, except if strongly connected to our specific problem.

Consider the Navier–Stokes equations described in Cartesian coordinates

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla \pi = 0, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T], \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is an open, smooth set. Below, weak solutions are considered in the so-called Leray–Hopf sense, see Leray [24], Hopf [18], and Kiselev and Ladyzhenskaya [19], and also Lions [25]. Solutions are called *strong* if

$$\mathbf{u} \in L^\infty(0, T; W^{1,2}(\Omega)) \cap L^2(0, T; W^{2,2}(\Omega)). \tag{1.2}$$

A main point in the theory of the $3D$ Navier–Stokes equations is that strong solutions are unique and smooth if data and domain are smooth as well. The result holds in a very large class of domains Ω if suitable boundary conditions, or behavior at infinity, are prescribed. To prove, or disprove, that weak solutions are necessarily strong (or unique) under reasonable but general assumptions, is one of the most challenging open mathematical problems.

In this context, a remarkable and classical sufficient condition for uniqueness and regularity is the so-called strict Prodi–Serrin (P–S) condition, namely

$$\mathbf{u} \in L^q(0, T; L^p(\Omega)), \quad \frac{2}{q} + \frac{3}{p} = 1, \quad p > 3. \tag{1.3}$$

Weak solutions satisfying the P–S condition (1.3) are known to be strong and unique.

Assumption (1.3) was firstly considered by Prodi in his paper [28] of 1959. He proved uniqueness under assumption (1.3), see also Foias [13]. Furthermore, Serrin, see [30,31], particularly proved interior spatial regularity under the stronger (non-strict) assumption

$$\mathbf{u} \in L^q(0, T; L^p(\Omega)), \quad \frac{2}{q} + \frac{3}{p} < 1, \quad p > 3. \tag{1.4}$$

Concerning the above problems, see also Ladyzenskaya's contributions [22,23]. The above setup led to the nomenclature Prodi–Serrin condition.

Complete proofs of the strict regularity result (i.e. under assumption (1.3)) were given by Sohr in [32], von Wahl in [34], and Giga in [16]. A simplified version of the proof was given in [15]. We additionally recommend the references in the bibliography of this last paper. For a quite complete overview on the initial-boundary value problem see contribution [14].

More recently, Escauriaza et al., see [12], extended the regularity result to the case $(q, p) = (\infty, 3)$.

We strongly recommend [29,31] as sources for information on the historical context of the P–S condition by the initiators themselves.

A significant improvement of the P–S condition was obtained by Bae and Choe. This is the main subject of our paper. These authors succeeded in proving that regularity also holds under the weaker assumption

$$\overline{\mathbf{u}} \in L^q(0, T; L^p(\mathbb{R}^3)), \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p > 3, \tag{1.5}$$

where $\overline{\mathbf{u}}$ is a vector consisting of two arbitrary components of $\mathbf{u}$. A complete proof of this result was shown in a preprint from 1997 by Bae and Choe, see also [1].

Furthermore, in contribution [8], this result was extended to the half-space $\mathbb{R}^n_+$ under slip boundary conditions. In this case, the truncated $(n-1)$-dimensional vector field $\overline{\mathbf{u}}$ cannot be chosen arbitrarily. The omitted component has to be normal to the boundary.

$$\overline{\mathbf{u}} = (u_1, u_2, ..., u_{n-1}, 0).$$

The challenging question whether the assumption of a flat boundary was a crucial element for the proof remained open. In order to study this problem, we will consider, below, a cylindrical three-dimensional domain, periodic in the axial direction, see Sect. 2. Equations are studied in cylindrical coordinates $(\xi_1, \xi_2, \xi_3) = (r, \vartheta, z)$ with obvious notation. Hence, the velocity's component normal to the lateral boundary $\partial^l \Omega$ of the cylinder is represented by u_1, and $\overline{\mathbf{u}} = (0, u_2, u_3)$ consists of the angular and the axial components of the velocity field. In order to better highlight the common features in the two approaches, Cartesian and cylindrical, we keep the same notation in both systems of coordinates. For instance, $\mathbf{u}$ and π denote velocity and pressure, respectively, in both systems.

Our main result is Theorem 2.2 below. For definitions and notation, see Sect. 2.

After the contribution by Bae and Choe, related papers appeared that particularly concerned assumptions on two components of velocity or vorticity, see [2,5,8,9,11]. There are also many papers dedicated to sufficient conditions for regularity which depend merely on one component, see, for instance [10,17,21,27,35,36].

Next, we briefly consider the Prodi–Serrin condition for $(q, p) = (\infty, n)$. It deserves a separate treatment. Consult [12,26] for full results, and [4,20] for previous results. Concerning contributions in which the restricted P–S condition

$$\overline{\mathbf{u}} \in L^\infty(0, T; L^n(\Omega)) \tag{1.6}$$

is assumed, we refer to [5,9]. In both cases, $\Omega = \mathbb{R}^n$.

In contribution [5], it was shown that solutions are regular even under the condition that the norm $\|\overline{\mathbf{u}}(t)\|_n$ admits a sufficiently small discontinuity from the left. In other words, they cannot exist. More precisely, it was proved that there is a positive constant $C(n)$ such that the solution is smooth in $(0, T]$ if

$$\sup_{\tau\in(0,T]}\left(\left(\limsup_{t\to\tau-0}\|\overline{\mathbf{u}}(t)\|_n^n\right)-\|\overline{\mathbf{u}}(\tau)\|_n^n\right)\le C(n)\,\nu^n\,. \tag{1.7}$$

In particular, by setting $\tau = 0$, it follows that $\|\overline{\mathbf{u}}\|_{L^\infty(0,T;L^n(\mathbb{R}^n))} \le C(n)\,\nu$ implies regularity. In contribution [9], the author replaced the space $L^n(\mathbb{R}^n)$ by the weak L^n-Marcinkiewicz space $L^n_w(\mathbb{R}^n)$, endowed with the canonical quasi-norm $[v]_n$, and essentially proved that there is a positive constant C such that a weak solution $\mathbf{u}$ is smooth in $(0, T]$ if it satisfies $\|\overline{\mathbf{u}}\|_{L^\infty(0,T;L^n_w(\mathbb{R}^n))} \le C$.

For the reader's convenience, we briefly describe the main points of the classical proof of the sufficiency of the P–S condition for regularity in Sect. 3. The aim of this sketch is merely to provide additional assistance in comprehensively reading the more complicated situation that involves cylindrical coordinates. In this sense, it may be skipped by the reader.

2 The Navier–Stokes equations in cylindrical coordinates. The restricted P–S condition. The main result

In the sequel, we are interested in the evolution Navier–Stokes equations in the open bounded cylinder $\Omega \subset \mathbb{R}^3$, defined by

$$\Omega := (\rho_0, \rho_1) \times [0,\ 2\pi) \times (0, 1),$$

under the classical Navier slip boundary condition without friction, see below. It is convenient to study these equations in cylindrical coordinates (ξ_1, ξ_2, ξ_3), where the radial coordinate ξ_1 has range

$$0 < \rho_0 < \xi_1 < \rho_1, \tag{2.1}$$

the angular coordinate ξ_2 is 2π-periodic, and the component in axial direction ξ_3 is 1-periodic. We write

$$\mathbf{u} = u_1 \cdot \mathbf{e}_1 + u_2 \cdot \mathbf{e}_2 + u_3 \cdot \mathbf{e}_3,$$

where $\mathbf{e}_k,\ k = 1, 2, 3$, are the unit vectors in radial, angular and axial (orthogonal) directions, respectively. We use the ∇-symbol in the following manner, where $\mathbf{v} : \Omega \to \mathbb{R}^3$ is a vector field and $g : \Omega \to \mathbb{R}$ is a scalar field:

$$\begin{cases} \nabla \cdot \mathbf{v} := \dfrac{1}{\xi_1}\partial_1\left(\xi_1 v_1\right) + \dfrac{1}{\xi_1}\left(\partial_2 v_2\right) + \partial_3 v_3, & \text{(2.2a)} \\ \nabla g := \left(\partial_1 g\right)\cdot \mathbf{e}_1 + \dfrac{1}{\xi_1}\left(\partial_2 g\right)\cdot \mathbf{e}_2 + \left(\partial_3 g\right)\cdot \mathbf{e}_3, & \text{(2.2b)} \\ \nabla^2 g := \dfrac{1}{\xi_1}\partial_1\left(\xi_1 \partial_1 g\right) + \dfrac{1}{\xi_1^2}\left(\partial_2^2 g\right) + \left(\partial_3^2 g\right), & \text{(2.2c)} \\ \nabla^2 \mathbf{v} := \left(\nabla^2 v_1 - \dfrac{2}{\xi_1^2}\partial_2 v_2 - \dfrac{v_1}{\xi_1^2}\right)\cdot \mathbf{e}_1 & \\ \qquad + \left(\nabla^2 v_2 + \dfrac{2}{\xi_1^2}\partial_2 v_1 - \dfrac{v_2}{\xi_1^2}\right)\cdot \mathbf{e}_2 & \text{(2.2d)} \\ \qquad + \left(\nabla^2 v_3\right)\cdot \mathbf{e}_3, & \\ \mathbf{v}\cdot\nabla g := v_1(\partial_1 g) + \dfrac{v_2}{\xi_1}(\partial_2 g) + v_3(\partial_3 g). & \text{(2.2e)} \end{cases}$$

Note that

$$\mathbf{v}\cdot(\nabla g) = (\mathbf{v}\cdot\nabla)g = v_1(\partial_1 g) + \frac{v_2}{\xi_1}(\partial_2 g) + v_3(\partial_3 g). \tag{2.3}$$

The three-dimensional evolution Navier–Stokes equations in cylindrical coordinates, see [3, p. 602], are given by:

$$\begin{cases} \mathcal{E}_1 := \partial_t u_1 + N_1 - \nu\left(\nabla^2 u_1 - \dfrac{2}{\xi_1^2}\partial_2 u_2 - \dfrac{u_1}{\xi_1^2}\right) + \partial_1 \pi = 0, & (2.4)_1 \\ \mathcal{E}_2 := \partial_t u_2 + N_2 - \nu\left(\nabla^2 u_2 + \dfrac{2}{\xi_1^2}\partial_2 u_1 - \dfrac{u_2}{\xi_1^2}\right) + \dfrac{1}{\xi_1}\partial_2 \pi = 0, & (2.4)_2 \\ \mathcal{E}_3 := \partial_t u_3 + N_3 - \nu\left(\nabla^2 u_3\right) + \partial_3 \pi = 0. & (2.4)_3 \end{cases}$$

The fluid's incompressibility is expressed by

$$\nabla\cdot\mathbf{u} = \frac{1}{\xi_1}\partial_1\left(\xi_1 u_1\right) + \frac{1}{\xi_1}\left(\partial_2 u_2\right) + \left(\partial_3 u_3\right) = 0. \tag{2.5}$$

N_1, N_2 and N_3 denote the three components of the non-linear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ in cylindrical coordinates, namely

$$\begin{cases} N_1 := \mathbf{u} \cdot \nabla u_1 - \frac{u_2^2}{\xi_1} = u_1(\partial_1 u_1) + \frac{u_2}{\xi_1}(\partial_2 u_1) + u_3(\partial_3 u_1) - \frac{u_2^2}{\xi_1}, & (2.6)_1 \\ N_2 := \mathbf{u} \cdot \nabla u_2 + \frac{u_1 u_2}{\xi_1} = u_1(\partial_1 u_2) + \frac{u_2}{\xi_1}(\partial_2 u_2) + u_3(\partial_3 u_2) + \frac{u_1 u_2}{\xi_1}, & (2.6)_2 \\ N_3 := \mathbf{u} \cdot \nabla u_3 = u_1(\partial_1 u_3) + \frac{u_2}{\xi_1}(\partial_2 u_3) + u_3(\partial_3 u_3). & (2.6)_3 \end{cases}$$

On the lateral boundary of the cylinder,

$$\partial^l \Omega := \{(\xi_1, \xi_2, \xi_3) \, : \, \xi_1 = \rho_0, \rho_1; \; \xi_2 \in [0, 2\pi); \; \xi_3 \in (0, 1)\}, \tag{2.7}$$

we impose slip boundary conditions defined by requiring that the normal component of $\mathbf{u}$ vanishes, i.e. $u_1 \equiv 0$, and that the tangential components of the stress vector vanish, too. By appealing to the tangent vector fields $\mathbf{e}_2$ and $\mathbf{e}_3$ on $\partial^l \Omega$ and to the stress vector

$$[-\pi + 2\,(\partial_1 u_1)] \cdot \mathbf{e}_1 + \left[\frac{(\partial_2 u_1)}{\xi_1} + \xi_1 \left(\partial_1 \frac{u_2}{\xi_1} \right) \right] \cdot \mathbf{e}_2 + [(\partial_3 u_1) + (\partial_1 u_3)] \cdot \mathbf{e}_3, \tag{2.8}$$

we get

$$\begin{cases} u_1 = 0, & (2.9)_1 \\ \partial_1 \frac{u_2}{\xi_1} = 0, & (2.9)_2 \\ \partial_1 u_3 = 0 & (2.9)_3 \end{cases}$$

on $\partial^l \Omega$, because of $\partial_2 u_1 \equiv \partial_3 u_1 \equiv 0$ on $\partial^l \Omega$.

Note that we may assume Ω being a ξ_3-periodic cylinder, and so do not consider its base and top as parts of the boundary.

For a mathematical treatment of quite general physical slip boundary conditions imposed on smooth, but generic, boundaries, with applications to stationary (classical and generalized) Stokes systems, see reference [6]. See also [33]. Further, in reference [7], applications to evolution problems of the results shown in [6] are illustrated by some significant examples.

Definition 2.1 Let $\mathbf{u}$ be a weak solution of the Navier–Stokes equations given by (2.4)–(2.6). Set

$$\overline{\mathbf{u}} = (0, u_2, u_3).$$

We say that $\mathbf{u}$ satisfies the *restricted* Prodi–Serrin condition if

$$\overline{\mathbf{u}} \in L^q(0, T; L^p(\Omega)), \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p > 3, \tag{2.10}$$

holds.

In the sequel, we prove the following result.

Theorem 2.2 *Let* $\mathbf{u}$ *be a weak solution of the Navier–Stokes equations given by* (2.4)–(2.6) *in the cylinder* Ω*, subject to the slip boundary conditions* (2.9)*. Furthermore, assume that* $\mathbf{u}$ *satisfies the restricted P–S condition* (2.10)*. Then,* $\mathbf{u}$ *is a strong solution*

$$\mathbf{u} \in L^{\infty}(0, T; W^{1,2}(\Omega)) \cap L^{2}(0, T; W^{2,2}(\Omega)). \tag{2.11}$$

Strong solutions are smooth provided that data and domain are smooth as well.

3 Remarks on the whole- and half-space cases

The proof of Theorem 2.2 is quite intricate, particularly due to the appearance of many "lower order terms". We believe that an anticipatory knowledge of the main lines of the proof, in a simpler case, could help readers to follow the complete proof of the Theorem shown in the next sections. We try to accomplish this purpose by briefly describing the main points in the classical proof of the P–S condition's sufficiency for regularity, in the simplest case, namely $\Omega = \mathbb{R}^n$, in Cartesian coordinates. Our aim is merely to assist in the understanding of the more complicated situation involving cylindrical coordinates. In this sense, this section may be fully skipped by the reader.

To better highlight the common features in the two approaches, Cartesian and cylindrical, we stick to the same notation $(\mathbf{u}, \pi)$. Hence, we write

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla \pi = 0, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T]. \end{cases} \tag{3.1}$$

In this simplified case, the proof of (2.11) has the following structure. By differentiating both sides of the first equation in (3.1) with respect to x_k, $k = 1, 2, 3$, by taking the scalar product with $\partial_k \mathbf{u}$, and by summing up over k, one shows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla \mathbf{u}|^2 \, \mathrm{d}x + \nu \int \left|\nabla^2 \mathbf{u}\right|^2 \mathrm{d}x = -\int \nabla[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \nabla \mathbf{u} \, \mathrm{d}x, \tag{3.2}$$

where obvious integrations by parts have been done, and $\nabla \cdot \mathbf{u} = 0$ was taken into account. On the other hand, an integration by parts yields

$$\left| \int \nabla[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \nabla \mathbf{u} \, \mathrm{d}x \right| \le c(n) \int |\mathbf{u}| \, |\nabla \mathbf{u}| \left|\nabla^2 \mathbf{u}\right| \mathrm{d}x. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla \mathbf{u}|^2 \, \mathrm{d}x + \nu \int \left|\nabla^2 \mathbf{u}\right|^2 \mathrm{d}x \le c(n) \int |\mathbf{u}| \, |\nabla \mathbf{u}| \left|\nabla^2 \mathbf{u}\right| \mathrm{d}x. \tag{3.4}$$

Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla \mathbf{u}|^2 \, \mathrm{d}x + \nu \int \left|\nabla^2 \mathbf{u}\right|^2 \mathrm{d}x \le c(n) \, \||\mathbf{u}| \, \nabla \mathbf{u}\|_2 \left\|\nabla^2 \mathbf{u}\right\|_2. \tag{3.5}$$

By Hölder's inequality, one has

$$\| |\mathbf{u}| \, \nabla \mathbf{u} \|_2 \leq \| \mathbf{u} \|_p \, \| \nabla \mathbf{u} \|_{\frac{2p}{p-2}} \, .$$

Furthermore, by interpolation and by Sobolev's embedding theorem,

$$\| \nabla \mathbf{u} \|_{\frac{2p}{p-2}} \leq \| \nabla \mathbf{u} \|_2^{1-\frac{n}{p}} \, \| \nabla \mathbf{u} \|_{2^*}^{\frac{n}{p}} \leq c \, \| \nabla \mathbf{u} \|_2^{1-\frac{n}{p}} \, \left\| \nabla^2 \mathbf{u} \right\|_2^{\frac{n}{p}} ,$$

since $(p-2)/(2p) = (1-n/p)/2 + (n/p)/2^*$. Here, $2^* = 2n/(n-2)$ is the well-known exponent in Sobolev's embedding theorem. Consequently,

$$\| |\mathbf{u}| \, \nabla \mathbf{u} \|_2 \left\| \nabla^2 \mathbf{u} \right\|_2 \leq c \, \| \mathbf{u} \|_p \, \| \nabla \mathbf{u} \|_2^{1-\frac{n}{p}} \, \left\| \nabla^2 \mathbf{u} \right\|_2^{1+\frac{n}{p}} .$$

Hence, by Young's inequality,

$$\| |\mathbf{u}| \, \nabla \mathbf{u} \|_2 \left\| \nabla^2 \mathbf{u} \right\|_2 \leq C(\varepsilon) \, \| \mathbf{u} \|_p^q \, \| \nabla \mathbf{u} \|_2^2 + \varepsilon \left\| \nabla^2 \mathbf{u} \right\|_2^2 . \tag{3.6}$$

From (3.5) and (3.6), we get, for $t \in (0, T]$,

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \nabla \mathbf{u} \|_2^2 + \frac{\nu}{2} \left\| \nabla^2 \mathbf{u} \right\|_2^2 \leq C(\varepsilon) \, \| \mathbf{u} \|_p^q \, \| \nabla \mathbf{u} \|_2^2 + \varepsilon \left\| \nabla^2 \mathbf{u} \right\|_2^2 . \tag{3.7}$$

Finally, (2.11) is proved by appealing to Gronwall's Lemma, since, by the classical version of the P–S condition,

$$\| \mathbf{u} \|_p^q \in L^1(0, T).$$

The crucial contribution of Bae and Choe was to succeed in replacing, in the right hand side of (3.3), the term $|\mathbf{u}|$ simply by $|\overline{\mathbf{u}}|$, where $\overline{\mathbf{u}} = (u_1, u_2, ..., u_{n-1}, 0)$. So

$$\left| \int \nabla [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \nabla \mathbf{u} \, \mathrm{d}x \right| \leq c(n) \int |\overline{\mathbf{u}}| \, |\nabla \mathbf{u}| \, \left| \nabla^2 \mathbf{u} \right| \, \mathrm{d}x \tag{3.8}$$

holds instead of the weaker estimate (3.3). The reader immediately verifies that all the above calculations hold simply by replacing $\mathbf{u}$ by $\overline{\mathbf{u}}$ in the appropriate places. In particular, the inequality (3.7) holds with $\| \mathbf{u} \|_p^q$ replaced by $\| \overline{\mathbf{u}} \|_p^q$. This leads to the generalized P–S condition

$$\| \overline{\mathbf{u}} \|_p^q \in L^1(0, T). \tag{3.9}$$

4 Structure and method of proof of Theorem 2.2

In order to prove Theorem 2.2, we start from the integral identities

$$\begin{cases} I_{1,1}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\xi_1\mathcal{E}_1)] \cdot \left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{1,1} \\ I_{1,2}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\xi_1\mathcal{E}_2)] \cdot \left[\partial_1 \frac{u_2}{\xi_1}\right] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{1,2} \\ I_{1,3}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1\mathcal{E}_3] \cdot [\partial_1 u_3] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{1,3} \\ I_{2,1}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_2\mathcal{E}_1] \cdot [\partial_2 u_1] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{2,1} \\ I_{2,2}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_2\mathcal{E}_2] \cdot [\partial_2 u_2] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{2,2} \\ I_{2,3}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_2\mathcal{E}_3] \cdot [\partial_2 u_3] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{2,3} \\ I_{3,1}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_3\mathcal{E}_1] \cdot [\partial_3 u_1] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{3,1} \\ I_{3,2}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_3\mathcal{E}_2] \cdot [\partial_3 u_2] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0, & (4.1)_{3,2} \\ I_{3,3}(\mathcal{E}) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_3\mathcal{E}_3] \cdot [\partial_3 u_3] \cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3 = 0 & (4.1)_{3,3} \end{cases}$$

which follow immediately from Eq. (2.4). To exploit incompressibility, we have to combine the above equations in the following manner:

$$I_{j,1}(\mathcal{E}) + I_{j,2}(\mathcal{E}) + I_{j,3}(\mathcal{E}) = 0, \quad j = 1, 2, 3. \tag{4.2}$$

Note that $\mathcal{E}_k,\ k = 1, 2, 3$, consist of four distinct terms, time, non-linear, viscous, and pressure, respectively. This leads to the following decomposition of the integrals appearing in Eq. (4.1).

$$I_{j,k}(\mathcal{E}) = I_{j,k}(N) + I_{j,k}(\pi) + I_{j,k}(\nu) + I_{j,k}(\partial t)\,. \tag{4.3}$$

The integrals on the right hand side will be studied separately. Just at the end of this paper, we will put all together by appealing to the core identity

$$\sum_{j,k=1,2,3} I_{j,k}(\mathcal{E}) = 0\,. \tag{4.4}$$

Roughly, we will prove in the next sections that the time terms give rise to the first term in the left hand side of (3.5), the viscous terms generate the second term, the pressure terms vanish, and the non-linear terms give rise to the right hand side of (3.5), obviously with $|\mathbf{u}|$ replaced by $|\overline{\mathbf{u}}|$. This leads to (3.7), with $\|\mathbf{u}\|_p^q$ replaced by $\|\overline{\mathbf{u}}\|_p^q$. However, in our cylindrical setting, this identification is possible only up to the appearance of a large number of negligible terms, see below.

Convention 4.1 *In the sequel, claiming that some quantity $H(t)$ is* negligible *means that one can show,* without *appealing to the restricted P–S condition* (2.10)*, that, given an arbitrary $\varepsilon > 0$, there is a real function $b_\varepsilon(t) \in L^1(0,T)$, such that*

$$|H(t)| \le b_\varepsilon(t)\left(\|\mathbf{u}\|_2^2 + \|\mathrm{D}\mathbf{u}\|_2^2\right) + \varepsilon\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2, \tag{4.5}$$

a.e. in (0,T). We will also call a quantity $h(t)$ negligible, *if*

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} |h(t)|\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$

is negligible, *and such quantities may, thus, be eliminated from equations.*

If an equality or an estimate holds up to negligible terms, *we will write $\simeq$ or $\preceq$, respectively.*

Due to the integrability of the function $b_\varepsilon(t)$, negligible terms $H(t)$ are trivially controlled by our main left hand side by appealing to Gronwall's Lemma. Equation (3.7) shows the typical situation where now the above term $|H(t)|$ appears on the left hand side.

The above convention is useful, since it allows us to avoid many similar calculations and unnecessarily long equations as the verification of the negligibility of many quantities becomes routine and may be left to the reader.

We could give simple expressions, case by case, for the above functions $b_\varepsilon(t)$, see Lemma 4.2 below for examples. However, by appealing to a generic b_ε, we invite the reader to retrace these simple calculations on his own.

Note that the above convention is quite significant in the context of the P–S condition as it separates terms requiring this extra assumption from terms that can be treated without appealing to it.

Lemma 4.2 *Terms of the following forms are negligible:*

$$\begin{cases} u^3, & (4.6a) \\ u^2(\partial u), & (4.6b) \\ u^2(\partial^2 u), & (4.6c) \\ u(\partial u)^2, \text{ and} & (4.6d) \\ (\partial u)(\partial^2 u) & (4.6e) \end{cases}$$

Furthermore,

$$\|\mathbf{u}\cdot \mathrm{D}\mathbf{u}\|_2 \le c\,(\|\mathbf{u}\|_2 + \|\mathrm{D}\mathbf{u}\|_2)^{3/2}\left(\|\mathrm{D}\mathbf{u}\|_2^{1/2} + \left\|\mathrm{D}^2\mathbf{u}\right\|_2^{1/2}\right). \tag{4.7}$$

Products of functions that are bounded by terms in (4.6) *are still negligible.*

Proof The integral of the absolute value of the term (4.6e) is clearly bounded by the right hand side of (4.5) with $b_\varepsilon(t) = \varepsilon^{-1}$. The integral of the absolute value of the term (4.6d) can be estimated by appealing to Hölder's inequality with exponents 3, 2, and 6, and to Sobolev's embedding theorem $W^{1,2} \subset L^6$ applied to D**u**. It follows that

$$\begin{aligned}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} |u|\cdot|\partial u|\cdot|\partial u|\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 &\le c\,\|\mathbf{u}\|_3\,\|\mathrm{D}\mathbf{u}\|_2\,\|\mathrm{D}\mathbf{u}\|_6 \\ &\le c\,\|\mathbf{u}\|_3\,\|\mathrm{D}\mathbf{u}\|_2\left(\|\mathrm{D}\mathbf{u}\|_2 + \left\|\mathrm{D}^2\mathbf{u}\right\|_2\right) \\ &\le c\,\|\mathbf{u}\|_3\,\|\mathrm{D}\mathbf{u}\|_2^2 + \frac{c}{\varepsilon}\,\|\mathbf{u}\|_3^2\,\|\mathrm{D}\mathbf{u}\|_2^2 + \varepsilon\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2.\end{aligned}$$

The coefficient $\|\mathbf{u}\|_3^2 \le c\,\|\mathbf{u}\|_6^2 \in L^1(0,T)$ satisfies condition (4.5) without appealing to the restricted P–S condition (2.10) as Leray-Hopf solutions belong to $L^2(0,T; W^{1,2}(\Omega))$. Similarly, the integral of the absolute value of the term (4.6c) may be bounded as follows.

$$\begin{aligned}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} |u|\cdot|u|\cdot\left|\partial^2 u\right|\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 &\le c\,\|\mathbf{u}\|_3\,\|\mathbf{u}\|_6\left\|\mathrm{D}^2\mathbf{u}\right\|_2 \\ &\le c\,\|\mathbf{u}\|_3\,(\|\mathbf{u}\|_2 + \|\mathrm{D}\mathbf{u}\|_2)\left\|\mathrm{D}^2\mathbf{u}\right\|_2 \\ &\le \frac{c}{\varepsilon}\,\|\mathbf{u}\|_3^2\left(\|\mathbf{u}\|_2^2 + \|\mathrm{D}\mathbf{u}\|_2^2\right) + \varepsilon\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2.\end{aligned}$$

The integrals of the absolute values of the terms (4.6a) and (4.6b) are bounded by $c\,\|\mathbf{u}\|_3\left(\|\mathbf{u}\|_2^2 + \|\mathrm{D}\mathbf{u}\|_2^2\right)$.

Equation (4.7) will be used much later only. Since the proof follows the same ideas, it seems appropriate to state it right away for the reader's convenience. By Hölder's inequality with exponents 3 and 3/2, one shows that $\|\mathbf{u}\cdot \mathrm{D}\mathbf{u}\|_2 \le \|\mathbf{u}\|_6\, \|\mathrm{D}\mathbf{u}\|_3$. Furthermore, by interpolation, we obtain the relation $\|\mathrm{D}\mathbf{u}\|_3^2 \le \|\mathrm{D}\mathbf{u}\|_2\, \|\mathrm{D}\mathbf{u}\|_6$. On the other hand, $\|\mathbf{u}\|_6 \le c(\|\mathbf{u}\|_2 + \|\mathrm{D}\mathbf{u}\|_2)$, similarly for $\|\mathrm{D}\mathbf{u}\|_6$. The estimate (4.7) now follows easily. The last claim in the Lemma is obvious. □

Note that, with regard to the boundary condition for u_2 that we consider, the quantity $\|\mathrm{D}\mathbf{u}\|_2$ is merely a semi-norm. This led to the addition of $\|\mathbf{u}\|_2$.

It is worth noting that Hölder and Sobolev theorems, due to (2.1), hold in Ω in the context of cylindrical coordinates, formally as for Cartesian coordinates, at most with an obvious adaptation.

5 Contribution of the non-linear terms

We start by remarking that the role of the non-linear terms is central here, since the P–S condition is necessary especially because of these terms.

In this section, we study the integrals obtained by restricting the terms $\mathcal{E}_1,\ \mathcal{E}_2,\ \mathcal{E}_3$ in (4.1) to their non-linear parts, i.e. $N_1,\ N_2$, and N_3, respectively. Thus, we consider

$$
\begin{cases}
I_{1,1}(N) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\xi_1 N_1)]\cdot\left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right]\cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3, & (5.1)_{1,1}\\
I_{1,2}(N) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\xi_1 N_2)]\cdot\left[\partial_1\frac{u_2}{\xi_1}\right]\cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3, & (5.1)_{1,2}\\
I_{1,3}(N) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1 N_3]\cdot[\partial_1 u_3]\cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3, & (5.1)_{1,3}\\
I_{j,k}(N) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left[\partial_j N_k\right]\cdot\left[\partial_j u_k\right]\cdot \xi_1\, d\xi_1 d\xi_2 d\xi_3,\ j=2,3,\ k=1,2,3. & (5.1)_{j,k}
\end{cases}
$$

We start by investigating the integrands

$$
\begin{cases}
N_{1,1} := [\partial_1(\xi_1 N_1)]\cdot\left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right]\cdot \xi_1, & (5.2)_{1,1}\\
N_{1,2} := [\partial_1(\xi_1 N_2)]\cdot\left[\partial_1\frac{u_2}{\xi_1}\right]\cdot \xi_1, & (5.2)_{1,2}\\
N_{1,3} := [\partial_1 N_3]\cdot[\partial_1 u_3]\cdot \xi_1, & (5.2)_{1,3}\\
N_{j,k} := \left[\partial_j N_k\right]\cdot\left[\partial_j u_k\right]\cdot \xi_1,\ j=2,3,\ k=1,2,3, & (5.2)_{j,k}
\end{cases}
$$

and by replacing the quantities $N_k,\ k = 1, 2, 3$, by their definitions from (2.6). In this way, each $N_{j,k}$ appears as a sum of single terms which are trilinear in u, possibly with coefficients consisting of powers of ξ_1. This claim is obvious. Hence, we may decompose each $N_{j,k}$ in the following manner:

$$N_{j,k} = B_{j,k} + K_{j,k} + R_{j,k}, \quad j, k = 1, 2, 3, \tag{5.3}$$

where, up to negligible terms (cf. Remark 5.1), $B_{j,k}$ denotes the summation of all terms having a factor of the form $u(\partial u)(\partial\partial u)$, and $K_{j,k}$ consists of all terms containing a factor of the form $(\partial u)(\partial u)(\partial u)$. It is worth noting that the really significant terms are the $B_{j,k}$- and the $K_{j,k}$-terms, since they are characterized by three differentiations. The sums of all other terms which have, at most, two differentiations, are denoted by $R_{j,k},\ j, k = 1, 2, 3$.

Remark 5.1 In the sequel, some negligible terms will be dropped from the expressions of the $B_{j,k}$- and the $K_{j,k}$-terms without changing notation. However, the definition (5.7) is strict due to the equality required in (5.9). On the contrary, the definition of the $K_{j,k}$-terms shown in (5.8) is neither strict nor particularly significant. To this extent, note that, in (5.14), one has a $\preceq$-sign.

We now proceed to prove the negligibility of $R_{j,k}$-terms.

Proposition 5.2 *The $R_{j,k}$-terms, $j, k = 1, 2, 3$, are negligible.*

Proof Since every term in $N_{j,k},\ j, k = 1, 2, 3$, is trilinear in u, the residual terms $R_{j,k}$ must fall into one of the five categories of terms given in (4.6), possibly multiplied by an integer power of ξ_1. Due to this particular form, these coefficients remain in the very same class after differentiation. Furthermore, coefficients in this class are bounded, cf. (2.1). The Proposition becomes immediate by appealing to (4.6a)–(4.6d): The negligibility of these expressions has been shown in Lemma 4.2. □

Clearly, in order to eliminate the ε-term from the right hand side of estimates like (3.6), we need a suitable estimate of the term $\|\mathrm{D}^2\mathbf{u}\|_2^2$, present on the left hand side of (3.4). This crucial estimate will be obtained from the viscous ν-terms in Sect. 7.

Next, note that, in Eq. (2.6), the terms u_2^2/ξ_1 and $(u_1u_2)/\xi_1$ give rise to negligible terms. Thus, we drop these terms from the expression of N_1 and N_2 :

$$N_k \simeq (\mathbf{u}\cdot\nabla)u_k, \quad k = 1, 2, 3. \tag{5.4}$$

Suitable expressions for the $B_{j,k}$- and the $K_{j,k}$-terms can easily be obtained as follows. One starts by noting that, in Eq. (5.2), each time we differentiate a coefficient with respect to ξ_1, we obtain a negligible term. Thus,

$$N_{j,k} \simeq \left(\partial_j N_k\right)\left(\partial_j u_k\right)\xi_1 \simeq \left[\partial_j\left(\mathbf{u}\cdot\nabla u_k\right)\right]\left(\partial_j u_k\right)\xi_1, \tag{5.5}$$

where we also have appealed to the equivalence (5.4). Hence,

$$\begin{aligned} N_{j,k} \simeq& \left[\mathbf{u}\cdot\nabla\left(\partial_j u_k\right)\right]\left(\partial_j u_k\right)\xi_1 \\ &+\left[\left(\partial_j u_1\right)\left(\partial_1 u_k\right)+\left(\partial_j \frac{u_2}{\xi_1}\right)\left(\partial_2 u_k\right)+\left(\partial_j u_3\right)\left(\partial_3 u_k\right)\right]\left(\partial_j u_k\right)\xi_1 \\ \simeq& \frac{\xi_1}{2}\mathbf{u}\cdot\nabla\left[\left(\partial_j u_k\right)^2\right] \\ &+\left[\left(\partial_j u_1\right)\left(\partial_1 u_k\right)+\frac{1}{\xi_1}\left(\partial_j u_2\right)\left(\partial_2 u_k\right)+\left(\partial_j u_3\right)\left(\partial_3 u_k\right)\right]\left(\partial_j u_k\right)\xi_1, \end{aligned} \tag{5.6}$$

where we have appealed to (2.3) and to the fact that $\partial_j\xi_1^{-1}$ gives rise to a negligible term (which vanishes if $j \neq 1$).

The first term on the right hand side of (5.6) denotes the explicit form of the $B_{j,k}$-terms:

$$B_{j,k} = \frac{\xi_1}{2}\mathbf{u}\cdot\nabla\left[\left(\partial_j u_k\right)^2\right]. \tag{5.7}$$

To fix ideas, we choose the second term in (5.6) as being the explicit form of the $K_{j,k}$-terms,

$$K_{j,k} = \left[\xi_1\left(\partial_j u_1\right)\left(\partial_1 u_k\right)+\left(\partial_j u_2\right)\left(\partial_2 u_k\right)+\xi_1\left(\partial_j u_3\right)\left(\partial_3 u_k\right)\right]\left(\partial_j u_k\right). \tag{5.8}$$

We now prove that the $B_{j,k}$-terms do not contribute to the integrals (5.1). The following identity holds.

Proposition 5.3 *One has*

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} B_{j,k}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = 0, \quad j,k = 1,2,3. \tag{5.9}$$

The result follows from the following statement.

Lemma 5.4 *Let g be a scalar field that is 2π-periodic with respect to ξ_2 and 1-periodic with respect to ξ_3. Then, there holds*

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\mathbf{u}\cdot\nabla g)\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = 0. \tag{5.10}$$

Proof Integration by parts yields

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} u_1(\partial_1 g)\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\xi_1 u_1)(\partial_1 g)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$
$$= -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_1(\xi_1 u_1)g\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,$$

since the corresponding boundary integral vanishes due to the boundary condition $u_1 = 0$ on the lateral boundary.

Similarly,

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_2}{\xi_1}(\partial_2 g)\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\partial_2 u_2)g\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,$$

and

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} u_3(\partial_3 g)\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \xi_1(\partial_3 u_3)g\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,$$

since the boundary integrals vanish due to periodicity in ξ_2 or ξ_3, respectively.

Adding up the three above equations, it follows that

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\mathbf{u}\cdot\nabla)\,g\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\nabla\cdot\mathbf{u})\cdot g\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = 0,$$

and Eq. (5.10) is proved. □

The reader should take note that the main ingredient for the estimate of the $B_{j,k}$-terms was the incompressibility of the velocity $\mathbf{u}$. The weak P–S condition was not used. It will be used, though, while considering the $K_{j,k}$-terms in order to prove the following result.

Proposition 5.5 *One has*

$$\left|\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} K_{j,k}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right| \preceq c\cdot\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} |\overline{u}|\;|\mathrm{D}\mathbf{u}|\;\left|\mathrm{D}^2\mathbf{u}\right|\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,\quad j,k=1,2,3,\qquad (5.11)$$

where $\overline{u}$ may denote the angular component u_2 or the axial component u_3 of the velocity.

Proof For arbitrary but fixed $j, k = 1, 2, 3$, the three parts of $K_{j,k}$ have the particular form

$$a(\xi_1)\,(\partial_j u_i)\,(\partial_i u_k)\,(\partial_j u_k),\quad i = 1, 2, 3, \qquad (5.12)$$

where $a(\xi_1) = 1$ or $a(\xi_1) = \xi_1$. Hence, in order to prove Proposition 5.5, it is sufficient to show that

$$\left| \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} a(\xi_1)\,(\partial_j u_i)\,(\partial_i u_k)\,(\partial_j u_k)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right| \preceq c \cdot \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} |\overline{u}|\;|\mathbf{Du}|\;\left|\mathrm{D}^2\mathbf{u}\right| \cdot \xi_1\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \tag{5.13}$$

for each triad of indices i, j, k.

Assume that the term $\partial_2 u_2$ is present in the left hand side of (5.13). Then, after integrating by parts with respect to the angular variable ξ_2, one gets

$$\int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} a(\xi_1)\,(\partial_j u_i)\,(\partial_i u_k)\,(\partial_j u_k)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = -\int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} a(\xi_1)\,\partial_2\,[(\partial_* u_*)(\partial_* u_*)]\,u_2\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3,$$

since the corresponding boundary integral vanishes due to ξ_2-periodicity. Take note that the factor $(\partial_* u_*)(\partial_* u_*)$ must be of the the form $(\partial_2 u_k)(\partial_2 u_k)$, $(\partial_j u_2)(\partial_j u_2)$, or $(\partial_2 u_i)(\partial_i u_2)$. This already proves (5.13) with $\overline{u} = u_2$ (and $c = \rho_1$ or $c = a(\rho_1)$).

A similar proof applies if we assume that the term $\partial_3 u_3$ is present in the left hand side of (5.13). In this case, we appeal to the ξ_3-periodicity.

Next, assume that the term $\partial_1 u_1$ is present in the left hand side of (5.13). As $\mathbf{u}$ is incompressible, we may now replace $\partial_1 u_1$ by

$$-\frac{u_1}{\xi_1} - \frac{\partial_2 u_2}{\xi_1} - \partial_3 u_3.$$

The expression coming from u_1/ξ_1 is negligible. The other two are treated as above.

If the left hand side of Eq. (5.13) does not fall into one of the above three cases, then, necessarily, the three indices i, j, k are pairwise distinct. One easily verifies that, in this case, at least one of the two terms $\partial_2 u_3$ or $\partial_3 u_2$ must be present. In the first case, we integrate by parts with respect to ξ_2, and we end up with $\overline{u} = u_3$ in Eq. (5.13). The boundary integral vanishes due to ξ_2-periodicity. The second case is similar and the argumentation reads as above if we interchange the indices 2 and 3. □

Equation (5.3) and Propositions 5.2, 5.3 and 5.5 lead to the following result.

Proposition 5.6 *One has*

$$\begin{aligned}\left|I_{j,k}(N)\right| &= \left|\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} N_{j,k}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right| \\ &\preceq c\cdot\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} |\overline{u}|\;|\mathrm{D}\mathbf{u}|\;\left|\mathrm{D}^2\mathbf{u}\right|\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,\quad j,k=1,2,3,\end{aligned} \tag{5.14}$$

where $\overline{u}$ may denote the angular component u_2 or the axial component u_3 of the velocity.

By arguing as in the proof of (3.6) with $|\mathbf{u}|$ replaced by $|\overline{\mathbf{u}}|$, we prove the following result.

Theorem 5.7 *One has*

$$\left|I_{j,k}(N)\right| \preceq C(\varepsilon)\,\|\overline{\mathbf{u}}\|_p^q\,\|\mathrm{D}\mathbf{u}\|_2^2+\varepsilon\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2,\quad j,k=1,2,3. \tag{5.15}$$

Recall that, due to Convention 4.1, we may replace in (5.15) the $\preceq$-sign by the $\leq$-sign, and add $b_\varepsilon(t)\left(\|\mathbf{u}\|_2^2+\|\mathrm{D}\mathbf{u}\|_2^2\right)$ to the right hand side of (5.15).

6 Contribution of the pressure terms

In this section, we study the integrals obtained by restricting the terms $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ in (4.1) to their pressure parts, i.e. $\partial_1\pi$, $\frac{1}{\xi_1}\partial_2\pi$, and $\partial_3\pi$, respectively. Thus, we consider

$$
\begin{cases}
I_{1,1}(\pi) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\xi_1\partial_1\pi)] \cdot \left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right] \cdot \xi_1 \, d\xi_1 d\xi_2 d\xi_3, & (6.1)_{1,1} \\
I_{1,2}(\pi) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\partial_2\pi)] \cdot \left[\partial_1 \frac{u_2}{\xi_1}\right] \cdot \xi_1 \, d\xi_1 d\xi_2 d\xi_3, & (6.1)_{1,2} \\
I_{1,3}(\pi) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1(\partial_3\pi)] \cdot [\partial_1 u_3] \cdot \xi_1 \, d\xi_1 d\xi_2 d\xi_3, & (6.1)_{1,3} \\
I_{j,2}(\pi) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left[\partial_j\left(\frac{1}{\xi_1}\partial_2\pi\right)\right] \cdot \left[\partial_j u_2\right] \cdot \xi_1 \, d\xi_1 d\xi_2 d\xi_3, \; j=2,3, & (6.1)_{j,2} \\
I_{j,k}(\pi) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left[\partial_j(\partial_k\pi)\right] \cdot \left[\partial_j u_k\right] \cdot \xi_1 \, d\xi_1 d\xi_2 d\xi_3, \; j=2,3, \; k=1,3. & (6.1)_{j,k}
\end{cases}
$$

In order to handle the pressure terms, we consider the three sums

$$
I_j(\pi) := I_{j,1}(\pi) + I_{j,2}(\pi) + I_{j,3}(\pi), \;\; j = 1, 2, 3.
$$

This crucial device allows us to exploit the incompressibility of the velocity field $\mathbf{u}$.

With the help of straightforward calculations, by appealing to the boundary conditions, and with suitable integrations by parts, we show that

$$
I_1(\pi) = -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1\pi] \cdot \partial_1 [\nabla \cdot \mathbf{u}] \cdot \xi_1 \, d\xi_1 d\xi_2 d\xi_3 + \mathring{I}_{1,1}(\pi), \tag{6.2}
$$

where

$$
\mathring{I}_{1,1}(\pi) := \int_0^1 \int_0^{2\pi} [\xi_1\partial_1\pi] \cdot \left[\frac{1}{\xi_1}\partial_1(\xi_1 u_1)\right]\Bigg|_{\xi_1=\rho_0}^{\xi_1=\rho_1} d\xi_2 d\xi_3.
$$

The volume integral on the right hand side of (6.2) vanishes due to $\nabla \cdot \mathbf{u} \equiv 0$. Hence,

$$
I_1(\pi) = \mathring{I}_{1,1}(\pi),
$$

and we are left to study the boundary integral $\mathring{I}_{1,1}(\pi)$.

Similar calculations show that

$$
I_2(\pi) = I_3(\pi) = 0.
$$

We now turn to treating the remaining boundary integral $\mathring{I}_{1,1}(\pi)$.

Lemma 6.1 *The boundary integral*

$$\mathring{I}_{1,1}(\pi) = \int_0^1 \int_0^{2\pi} [\partial_1 \pi] \cdot [\partial_1 (\xi_1 u_1)] \Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} \mathrm{d}\xi_2 \mathrm{d}\xi_3$$

is negligible in the sense that there holds

$$\left| \mathring{I}_{1,1}(\pi) \right| \leq b_\varepsilon(t) \cdot \|\mathbf{Du}\|_2^2 + \varepsilon \cdot \left\| \mathrm{D}^2 \mathbf{u} \right\|_2^2.$$

Proof According to Eqs. (2.4)$_1$ and (2.6)$_1$, we have

$$\partial_1 \pi = -\partial_t u_1 - \mathbf{u} \cdot \nabla u_1 + \frac{u_2^2}{\xi_1} + \nu \left(\nabla^2 u_1 - \frac{2}{\xi_1^2} \partial_2 u_2 - \frac{u_1}{\xi_1^2} \right).$$

First, we evaluate this equation for $\xi_1 = \rho_0, \rho_1$, using the boundary condition $u_1 = 0$, and, thus, as a consequence, that also the tangential derivatives $\partial_i u_1$, $\partial_i \partial_j u_1$, $i, j = 2, 3$, vanish for $\xi_1 = \rho_0, \rho_1$. Hence, we get

$$\partial_t u_1 = 0, \qquad \mathbf{u} \cdot \nabla u_1 = 0, \quad \text{and} \quad \nabla^2 u_1 = \frac{1}{\xi_1} \partial_1 (\xi_1 \partial_1 u_1),$$

and that leads to

$$\partial_1 \pi = \nu \cdot \frac{1}{\xi_1} \partial_1 (\xi_1 \partial_1 u_1) - \nu \frac{2}{\xi_1^2} \partial_2 u_2 + \frac{u_2^2}{\xi_1} \quad \text{on } \partial^l \Omega.$$

Since $\xi_1 \partial_1 u_1 = \partial_1 (\xi_1 u_1) - u_1$ and $\nabla \cdot \mathbf{u} = 0$, we get

$$\begin{aligned} \frac{1}{\xi_1} \partial_1 (\xi_1 \partial_1 u_1) &= \frac{1}{\xi_1} \partial_1 (-\partial_2 u_2 - \xi_1 \partial_3 u_3 - u_1) \\ &= -\frac{1}{\xi_1} \cdot \partial_2 (\partial_1 u_2) - \frac{1}{\xi_1} \cdot \partial_3 u_3 - \frac{1}{\xi_1} \cdot \xi_1 \cdot \partial_3 \partial_1 u_3 - \frac{1}{\xi_1} \cdot \partial_1 u_1. \end{aligned}$$

Now, we use the boundary conditions for u_2 and u_3 and get

$$\partial_3 \partial_1 u_3 = 0, \quad \text{and} \quad \partial_2 \partial_1 \left(\frac{u_2}{\xi_1} \right) = 0.$$

This leads to

$$\frac{1}{\xi_1} \partial_1 (\xi_1 \partial_1 u_1) = -\frac{1}{\xi_1^2} (\partial_2 u_2) - \frac{1}{\xi_1} (\partial_3 u_3) - \frac{1}{\xi_1} (\partial_1 u_1), \tag{6.3}$$

and, because $\partial_1 u_1 = \frac{1}{\xi_1}\partial_1(\xi_1 u_1)$ with $u_1 = 0$, the right-hand side of (6.3) equals $-\frac{1}{\xi_1}\nabla\cdot u$ and, therefore, vanishes. So we finally arrive at

$$\partial_1\pi = -\frac{2\nu}{\xi_1^2}\partial_2 u_2 + \frac{u_2^2}{\xi_1}.$$

The second factor in the integrand is

$$\partial_1(\xi_1 u_1) = -\partial_2 u_2 - \xi_1\partial_3 u_3,$$

hence, we have

$$\mathring{I}_{1,1}(\pi) = \int_0^1\int_0^{2\pi}\left[-\frac{2\nu}{\xi_1^2}\partial_2 u_2 + \frac{u_2^2}{\xi_1}\right]\cdot\left[-\partial_2 u_2 - \xi_1\partial_3 u_3\right]\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1}\mathrm{d}\xi_2\mathrm{d}\xi_3.$$

We now expand the product and consider the four appearing summands.

For the integrals over $|\partial_2 u_2|^2$ and $\partial_2 u_2\cdot\partial_3 u_3$, we use Gagliardo's trace theorem and get

$$\begin{aligned}\int_0^1\int_0^{2\pi}|\mathbf{Du}|^2\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1}\mathrm{d}\xi_2\mathrm{d}\xi_3 &\le c\,\left\|\,|\mathbf{Du}|^2\,\right\|_{1,1}\\ &\le c\left(\|\mathbf{Du}\|_2^2 + \left\|\,|\mathbf{Du}|\,\left|\mathrm{D}^2\mathbf{u}\right|\,\right\|_1\right)\\ &\le c\,\|\mathbf{Du}\|_2^2 + C(\varepsilon)\cdot\|\mathbf{Du}\|_2^2 + \varepsilon\cdot\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2.\end{aligned} \tag{6.4}$$

Here, $\|\cdot\|_{1,1}$ denotes the $W^{1,1}(\Omega)$-norm.

The integral with the integrand

$$\frac{1}{\xi_1}u_2^2(\partial_2 u_2)$$

vanishes because

$$u_2^2(\partial_2 u_2) = \frac{1}{3}\partial_2\left(u_2^3\right),$$

and we can integrate by parts with respect to ξ_2.

Finally, we consider

$$
\begin{aligned}
&-\int_0^1\int_0^{2\pi}\left(\frac{1}{\xi_1}\cdot u_2^2\right)\cdot(\xi_1\partial_3 u_3)\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1}\mathrm{d}\xi_2\mathrm{d}\xi_3\\
&=-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[u_2^2\cdot(\partial_3 u_3)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\
&=-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}2u_2(\partial_1 u_2)(\partial_3 u_3)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}u_2^2\partial_3\partial_1 u_3\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3.
\end{aligned}
$$

The first integral is of type (4.6d), and the second integral is of type (4.6c). Integrals of these types have been treated in Lemma 4.2. □

The above Lemma and the fact that all volume integrals, if summed up suitably, vanish identically lead to the following result.

Theorem 6.2 *All pressure terms $I_{j,k}(\pi)$ are negligible.*

7 Contribution of the viscous terms

To estimate the contribution of the viscous terms in equations (4.1), we consider the nine integrals obtained by restricting the terms $\mathcal{E}_1,\ \mathcal{E}_2,\ \mathcal{E}_3$ to their respective viscous parts. For instance,

$$
I_{1,1}(\nu):=-\nu\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[\xi_1\left(\nabla^2 u_1-\frac{2}{\xi_1^2}\partial_2 u_2-\frac{u_1}{\xi_1^2}\right)\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3. \quad (7.1)_{1,1}
$$

The two "lower order terms" in the expression

$$
\nabla^2 u_1-\frac{2}{\xi_1^2}\partial_2 u_2-\frac{u_1}{\xi_1^2}
$$

clearly generate negligible quantities. Thus, we drop these two terms in Eq. $(7.1)_{1,1}$ and instead investigate the integral

$$
\begin{aligned}
I_{1,1}\left(\nabla^2\right)&:=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[\xi_1\cdot\nabla^2 u_1\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\
&=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[\partial_1(\xi_1\partial_1 u_1)+\frac{1}{\xi_1}\partial_2^2 u_1+\xi_1\partial_3^2 u_1\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1(\xi_1 u_1)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3. \quad (7.2)_{1,1}
\end{aligned}
$$

In the same way, we obtain integrals $I_{j,k}\left(\nabla^2\right),\ \ j,k=1,2,3,$ that we will refer to with equation numbers $(7.2)_{j,k},\ \ j,k=1,2,3.$

In order to obtain integrands of the form $\left|\partial_i\partial_j u_1\right|^2,\ i,j=1,2,3,$ we separate the three terms which make up the right hand side of Eq. $(7.2)_{1,1}$. Hence, we write

$$I_{1,1}\left(\nabla^2\right)=I^1_{1,1}\left(\nabla^2\right)+I^2_{1,1}\left(\nabla^2\right)+I^3_{1,1}\left(\nabla^2\right),$$

where the upper index $l=1,2,3$ indicates that, in the right hand side of $(7.2)_{1,1}$, we have only considered the l-th term of the decomposition of the expression $\xi_1(\nabla^2 u_1)$. Subsequently, we integrate by parts: the first term with respect to ξ_1, the second one with respect to ξ_2, and the third one with respect to ξ_3. We start with the ξ_1-term:

$$\begin{aligned}I^1_{1,1}\left(\nabla^2\right)&:=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[\partial_1\left(\xi_1\partial_1 u_1\right)\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1\left(\xi_1 u_1\right)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\&=-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[\partial_1\left(\xi_1\partial_1 u_1\right)\right]\cdot\partial_1\left[\frac{1}{\xi_1}\partial_1\left(\xi_1 u_1\right)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\qquad(7.3)_1\end{aligned}$$

because the boundary integral

$$\int_0^1\int_0^{2\pi}\left[\partial_1\left(\xi_1\partial_1 u_1\right)\right]\cdot\left[\frac{1}{\xi_1}\partial_1\left(\xi_1 u_1\right)\right]\Bigg|_{\xi_1=\rho_0}^{\xi_1=\rho_1}\mathrm{d}\xi_2\mathrm{d}\xi_3$$

disappears, since the first factor vanishes identically. We have already established this fact whilst deriving Eq. (6.3).

For the second part of the Laplacian, we get

$$\begin{aligned}I^2_{1,1}\left(\nabla^2\right)&:=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[\frac{1}{\xi_1}\partial_2^2 u_1\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1\left(\xi_1 u_1\right)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\&=-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\partial_1\left[\frac{1}{\xi_1}\partial_2 u_1\right]\cdot\left[\frac{1}{\xi_1}\partial_2\left(\partial_1\left(\xi_1 u_1\right)\right)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\qquad(7.3)_2\end{aligned}$$

because the boundary integral vanishes due to periodicity in ξ_2.

Integration by parts with respect to ξ_3 yields

$$I^3_{1,1}\left(\nabla^2\right) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_1\left[\xi_1\partial_3^2 u_1\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1\left(\xi_1 u_1\right)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$

$$= -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_1\left[\xi_1\partial_3 u_1\right]\cdot\left[\frac{1}{\xi_1}\partial_3\left(\partial_1\left(\xi_1 u_1\right)\right)\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \qquad (7.3)_3$$

because $\mathbf{u}$ is periodic in ξ_3.

From (7.3), it looks clear that

$$I^l_{1,1}\left(\nabla^2\right) \simeq -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_l\partial_1 u_1\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \simeq -c\left\|\partial_l\partial_1 u_1\right\|_2^2, \quad l = 1, 2, 3.$$

We can argue similarly to show the following result.

Proposition 7.1 *One has*

$$I^l_{j,k}\left(\nabla^2\right) \simeq -c\left\|\partial_l\partial_j u_k\right\|_2^2, \quad j, k, l = 1, 2, 3.$$

Proof The integral $(7.2)_{1,1}$ has already been considered. The other eight integrals can be handled in the same manner. After a decomposition of each integral in three summands, the respective first summands should be integrated by parts with respect to ξ_1, the respective second summands with respect to ξ_2, and the respective third summands with respect to ξ_3. Integrating by part with respect to ξ_2 or ξ_3 does not lead to boundary integrals due to the periodicity in the corresponding variables. Therefore, we only check the remaining eight first summands that contain an integration by parts with respect to ξ_1. One has

$$I^1_{1,2}\left(\nabla^2\right) := \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_1\left[\partial_1\left(\xi_1\partial_1 u_2\right)\right]\cdot\left[\partial_1\frac{u_2}{\xi_1}\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$

$$= -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left[\partial_1\left(\xi_1\partial_1 u_2\right)\right]\cdot\partial_1\left[\xi_1\partial_1\frac{u_2}{\xi_1}\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$

because the boundary integral

$$\int_0^1\int_0^{2\pi} \left[\partial_1\left(\xi_1\partial_1 u_2\right)\right]\cdot\left[\xi_1\partial_1\frac{u_2}{\xi_1}\right]\Bigg|_{\xi_1=\rho_0}^{\xi_1=\rho_1}\mathrm{d}\xi_2\mathrm{d}\xi_3$$

vanishes, since $\partial_1\dfrac{u_2}{\xi_1} \equiv 0$ on $\partial^l\Omega$.

The terms $I^1_{1,3}\left(\nabla^2\right)$, $I^1_{2,1}\left(\nabla^2\right)$, and $I^1_{3,1}\left(\nabla^2\right)$ lead to boundary integrals that vanish for the same reason, namely $\partial_1 u_3 \equiv \partial_2 u_1 \equiv \partial_3 u_1 \equiv 0$ on $\partial^l \Omega$.

$I^1_{2,2}\left(\nabla^2\right)$ and $I^1_{3,2}\left(\nabla^2\right)$ lead to boundary integrals that contain first order derivatives only. In fact,

$$I^1_{2,2}\left(\nabla^2\right) := -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_2\left[\partial_1 u_2\right]\cdot\partial_1\left[\partial_2 u_2\right]\cdot\xi_1\, d\xi_1 d\xi_2 d\xi_3 + \int_0^1\int_0^{2\pi} \partial_2\left[\partial_1 u_2\right]\cdot\left[\partial_2 u_2\right]\cdot\xi_1\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} d\xi_2 d\xi_3.$$

However, due to the fact that

$$\partial_1 u_2 = \xi_1\cdot\left(\partial_1\frac{u_2}{\xi_1}\right) + \frac{u_2}{\xi_1} \tag{7.4}$$

and $\partial_1\frac{u_2}{\xi_1} \equiv 0$ on $\partial^l\Omega$, we obtain the identity $\partial_2\left(\partial_1 u_2\right) = \dfrac{\partial_2 u_2}{\xi_1}$ on $\partial^l\Omega$ and arrive at a boundary integral that is quadratic in a first order derivative.

More precisely, we get

$$I^1_{2,2}\left(\nabla^2\right) = -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_2\left[\partial_1 u_2\right]\cdot\partial_1\left[\partial_2 u_2\right]\cdot\xi_1\, d\xi_1 d\xi_2 d\xi_3 + \int_0^1\int_0^{2\pi} \left|\partial_2 u_2\right|^2\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} d\xi_2 d\xi_3. \tag{7.5}$$

The boundary integral in (7.5) is negligible; this was shown in (6.4).

The term $I^1_{3,2}\left(\nabla^2\right)$ can be treated in the exact same manner, simply by replacing ∂_2 by ∂_3 in each step.

Regarding $I^1_{2,3}\left(\nabla^2\right)$, after an integration by parts, we arrive at

$$I^1_{2,3}\left(\nabla^2\right) := -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_2\left[\xi_1\partial_1 u_3\right]\cdot\partial_1\left[\partial_2 u_3\right] d\xi_1 d\xi_2 d\xi_3,$$

since the boundary integral, namely

$$\int_0^1\int_0^{2\pi} \partial_2\left[\xi_1\partial_1 u_3\right]\cdot\left[\partial_2 u_3\right]\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} d\xi_2 d\xi_3,$$

vanishes due to the condition $\partial_2\left(\partial_1 u_3\right) \equiv 0$ on $\partial^l\Omega$ in the integrand's first factor.

Considering $I^1_{3,3}\left(\nabla^2\right)$, we can argue in the very same manner if we appeal to the condition $\partial_3\left(\partial_1 u_3\right) \equiv 0$ on $\partial^l\Omega$. Hence,

$$I^1_{3,3}\left(\nabla^2\right) := -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_3\left[\xi_1\partial_1 u_3\right]\cdot\partial_1\left[\partial_3 u_3\right] \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$

since, as mentioned before, the boundary integral,

$$\int_0^1\int_0^{2\pi} \partial_3\left[\xi_1\partial_1 u_3\right]\cdot\left[\partial_3 u_3\right]\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} \mathrm{d}\xi_2\mathrm{d}\xi_3,$$

vanishes. □

The proof of Proposition 7.1 shows that the following result holds.

Theorem 7.2 *One has*

$$\sum_{j,k} I_{j,k}\left(\nu\right) \simeq \nu\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2, \tag{7.6}$$

uniformly in t *for almost all* $t \in (0, T)$.

8 Contribution of the time derivatives

In this section, we study the integrals obtained by restricting the terms $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ in (4.1) to the time derivatives, $\partial_t u_k$, $k = 1, 2, 3$, of the velocity.

Hence, we consider

$$\begin{cases} I_{1,1}\left(\partial_t\right):=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[\partial_1\left(\xi_1\partial_t u_1\right)\right]\cdot\left[\frac{1}{\xi_1^2}\partial_1\left(\xi_1 u_1\right)\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, & (8.1)_{1,1}\\ I_{1,2}\left(\partial_t\right):=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[\partial_1\left(\xi_1\partial_t u_2\right)\right]\cdot\left[\partial_1\frac{u_2}{\xi_1}\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, & (8.1)_{1,2}\\ I_{1,3}\left(\partial_t\right):=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[\partial_1\left(\partial_t u_3\right)\right]\cdot\left[\partial_1 u_3\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, & (8.1)_{1,3}\\ I_{j,k}\left(\partial_t\right):=\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[\partial_j\left(\partial_t u_k\right)\right]\cdot\left[\partial_j u_k\right]\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3,\ j=2,3,\ k=1,2,3. & (8.1)_{j,k}\end{cases}$$

Except for the consideration of $(8.1)_{1,1}$ and $(8.1)_{1,2}$, this leads to integrals of the form

$$I_{j,k}(\partial_t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_j u_k\right|^2 \cdot \xi_1 \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3, \qquad (8.2)_{j,k}$$

and these are the quantities that we need in the main inequality (3.7) (with $\|\mathbf{u}\|_p^q$ replaced by $\|\overline{\mathbf{u}}\|_p^q$).

Straightforward calculations show that the integrand of $(8.1)_{1,1}$ reads

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\xi_1^2(\partial_1 u_1)^2 + 2\xi_1 u_1(\partial_1 u_1) + u_1^2\right]\frac{1}{\xi_1}\\ &\quad = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(\partial_1 u_1)^2\right]\xi_1 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[(\partial_1(u_1^2))\right] + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\xi_1}u_1^2. \end{aligned} \qquad (8.3)$$

The first term on the right hand side of (8.3) gives the integral

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} |\partial_1 u_1|^2 \cdot \xi_1 \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3,$$

and this is of the form that we need in (3.7) (with $\|\mathbf{u}\|_p^q$ replaced by $\|\overline{\mathbf{u}}\|_p^q$).

The integral of the second term in (8.3),

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left[\partial_1(u_1^2)\right] \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3,$$

vanishes as we can integrate by parts with respect to ξ_1 and use the boundary condition $u_1 = 0$ on $\partial^l \Omega$ for all $t \in (0, T)$.

We have proved that

$$I_{1,1}(\partial_t) \simeq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}|\partial_1 u_1|^2 \cdot \xi_1 \,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_1^2}{\xi_1}\,\mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3, \qquad (8.4)$$

where the second term on the right hand side is the integral of the third term in (8.3). We note that, in the sequel, this term will appear on the left hand side of our equation of type (3.7). Further, in Sect. 9, the application of Gronwall's Lemma in the proof of the main theorem will give the additional conclusion $u_1 \in L^\infty(0, T; L^2(\Omega))$.

The integral $(8.1)_{1,2}$ must be treated differently because, now, the integrand differs from $(1/2)\partial_t(\partial_1 u_2)^2$ by terms that cannot be handled in the way above. Therefore, we proceed in the following way:

$$\partial_1\left(\xi_1\partial_t u_2\right) = \partial_t\left[\partial_1\left(\xi_1^2\cdot\frac{u_2}{\xi_1}\right)\right] = \partial_t\left[2\xi_1\cdot\frac{u_2}{\xi_1} + \xi_1^2\left(\partial_1\frac{u_2}{\xi_1}\right)\right],$$

and the integrand of (8.1)$_{1,2}$ can be rewritten in the form

$$\left[\partial_1\left(\xi_1\partial_t u_2\right)\right]\cdot\left[\partial_1\frac{u_2}{\xi_1}\right]\cdot\xi_1 = \frac{1}{2}\partial_t\left(\left|\partial_1\frac{u_2}{\xi_1}\right|^2\right)\cdot\xi_1^3 + 2(\partial_t u_2)(\partial_1 u_2) - \frac{2}{\xi_1}(\partial_t u_2)u_2.$$

Thus, we have

$$\begin{aligned} I_{1,2}\left(\partial_t\right) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left|\partial_1\frac{u_2}{\xi_1}\right|^2\cdot\xi_1^3\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &\quad + 2\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}(\partial_t u_2)(\partial_1 u_2)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} u_2^2\cdot\frac{1}{\xi_1}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 =: I_{1,2}^1\left(\partial_t\right) + 2I_{1,2}^2\left(\partial_t\right) - I_{1,2}^3\left(\partial_t\right). \end{aligned} \tag{8.5}$$

The term $I_{1,2}^3\left(\partial_t\right)$ will be easily estimated, since it is integrable on $(0, T)$ because the weak solution belongs to $L^2(0, T; L^2(\Omega))$.

Proposition 8.1 *The term $I_{1,2}^2\left(\partial_t\right)$ is negligible.*

Proof In order to estimate $I_{1,2}^2\left(\partial_t\right)$, we replace $\partial_t u_2$ according to the equation of motion (2.4)$_2$:

$$\partial_t u_2 = -\left(\mathbf{u}\cdot\nabla\right)u_2 - \frac{u_1 u_2}{\xi_1} + \nu\left(\nabla^2 u_2 + \frac{2}{\xi_1^2}\partial_2 u_1 - \frac{u_2}{\xi_1^2}\right) - \frac{1}{\xi_1}\partial_2\pi. \tag{8.6}$$

An integration of $(\partial_t u_2)(\partial_1 u_2)$ then leads to integrals of types which we already treated in Lemma 4.2, except for the integral that contains the pressure:

$$-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[\frac{1}{\xi_1}\partial_2\pi\right]\cdot\left[\partial_1\frac{u_2}{\xi_1}\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \le c\left\|\nabla\pi\right\|_2\left\|\mathbf{D}\mathbf{u}\right\|_2. \tag{8.7}$$

On the other hand, by scalar multiplication of $\mathcal{E}_1\cdot\mathbf{e}_1 + \mathcal{E}_2\cdot\mathbf{e}_2 + \mathcal{E}_3\cdot\mathbf{e}_3 = 0$ by $\nabla\pi$, we get

$$\langle\nabla\pi, \nabla\pi\rangle = -\langle\partial_t\mathbf{u}, \nabla\pi\rangle - \langle N(\mathbf{u}), \nabla\pi\rangle + \langle\nu(\mathbf{u}), \nabla\pi\rangle. \tag{8.8}$$

When, subsequently, integrating (8.8) over Ω, we note that the first summand on the right hand side vanishes:

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \langle \partial_t \mathbf{u}, \nabla\pi\rangle \cdot \xi_1 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$= \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left\{ [(\partial_t u_1)\cdot \partial_1 \pi] + \left[(\partial_t u_2)\cdot \frac{\partial_2 \pi}{\xi_1}\right] + [(\partial_t u_3)\cdot \partial_3\pi] \right\} \cdot \xi_1 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$= -\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} [\partial_1 (\xi_1 \partial_t u_1) + \partial_2 (\partial_t u_2) + \xi_1 \partial_3 (\partial_t u_3)]\cdot \pi \cdot \xi_1 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = 0$$

because $\nabla \cdot (\partial_t \mathbf{u}) = 0$. Note that the boundary integrals vanish, again, since, when integrating with respect to ξ_1, we can exploit $\partial_t u_1 = 0$ on $\partial^l \Omega$ and, when integrating with respect to ξ_2 and ξ_3, we can draw on the periodicity in these variables.

Therefore, we have

$$\|\nabla\pi\|_2 \le c\,(\|N(\mathbf{u})\|_2 + \|\nu(\mathbf{u})\|_2)\,. \tag{8.9}$$

So,

$$\|\nabla\pi\|_2 \,\|\mathrm{D}\mathbf{u}\|_2 \le c\,(\|N(\mathbf{u})\|_2 + \|\nu(\mathbf{u})\|_2)\,\|\mathrm{D}\mathbf{u}\|_2 \preceq c\,\|N(\mathbf{u})\|_2\,\|\mathrm{D}\mathbf{u}\|_2$$

as $\|\nu(\mathbf{u})\|_2 \,\|\mathrm{D}\mathbf{u}\|_2 \preceq c \left\|\mathrm{D}^2\mathbf{u}\right\|_2 \|\mathrm{D}\mathbf{u}\|_2$ is negligible. By appealing to (4.7), it follows that

$$\|\nabla\pi\|_2 \,\|\mathrm{D}\mathbf{u}\|_2 \preceq c\,\|\mathrm{D}\mathbf{u}\|_2^{5/2} \left\|\mathrm{D}^2\mathbf{u}\right\|_2^{1/2}.$$

By Young's equality with exponents $4/3$ and 4, we obtain

$$\|\nabla\pi\|_2 \,\|\mathrm{D}\mathbf{u}\|_2 \preceq C(\varepsilon)\,\|\mathrm{D}\mathbf{u}\|_2^{4/3}\,\|\mathrm{D}\mathbf{u}\|_2^2 + \varepsilon \left\|\mathrm{D}^2\mathbf{u}\right\|_2^2.$$

The desired result follows as $\mathrm{D}\mathbf{u} \in L^2(0,T;L^2(\Omega)) \subset L^{4/3}(0,T;L^2(\Omega))$. □

We have proved that

$$I_{1,2}(\partial_t) \simeq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left(\partial_1 \frac{u_2}{\xi_1}\right)^2 \cdot \xi_1^3 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} u_2^2 \cdot \frac{1}{\xi_1} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3. \tag{8.10}$$

From $(8.2)_{j,k}$, (8.4), and (8.10), we get the following result.

Theorem 8.2 *For the time terms, one gets, with $I_{j,k}(\partial_t)$ as in $(8.2)_{j,k}$,*

$$\sum_{j,k=1,2,3} I_{j,k}(\partial_t) \simeq \sum_{(j,k)\neq(1,2)} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_j u_k\right|^2 \cdot \xi_1 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_1 \frac{u_2}{\xi_1}\right|^2 \xi_1^3 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3$$
$$+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_1^2}{\xi_1} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 - \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_2^2}{\xi_1} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3.$$

9 The core estimate. Related remarks

The aim of this section is twofold: We give the core estimate, actually, in more than one explicit form, and we also explain how, and why, we will proceed in the sequel. These explanations should be helpful for the readers.

The integrals $I_{j,k}(\mathcal{E})$ of the basic identities (4.1) have been split up according to (4.3) into four distinct parts: time, pressure, non-linear and viscous terms. These quantities have been estimated in Sect. 5–8, cf. Theorems 5.7, 6.2, 7.2, and 8.2. Adding up these inequalities according to (4.4) gives the following main result.

Theorem 9.1 *The estimate*

$$\begin{aligned}
&\sum_{(j,k)\neq(1,2)} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_j u_k\right|^2 \cdot \xi_1 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\
&+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_1 \frac{u_2}{\xi_1}\right|^2 \xi_1^3 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\
&+ \nu \left\|\mathrm{D}^2 \mathbf{u}\right\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_1^2}{\xi_1} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 - \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} u_2^2 \cdot \frac{1}{\xi_1} \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\
&\leq C(\varepsilon) \left\|\overline{\mathbf{u}}\right\|_p^q \left\|\mathrm{D}\mathbf{u}\right\|_2^2 + b_\varepsilon(t) \left(\|\mathbf{u}\|_2^2 + \|\mathrm{D}\mathbf{u}\|_2^2\right) + \varepsilon \left\|\mathrm{D}^2 \mathbf{u}\right\|_2^2
\end{aligned} \tag{9.1}$$

holds for almost all $t \in (0, T)$.

Note that, by inserting, on the right hand side, the b_ε-term, we were allowed to replace the symbol "$\preceq$" by "$\leq$". Equation (9.1), up to secondary terms, enjoys the canonical structure of Eq. (3.7). In view of the application of Gronwall's lemma, an apparent main difference is that, on the left hand side of (9.1), one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left|\partial_1\frac{u_2}{\xi_1}\right|^2\cdot\xi_1^3\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \tag{9.2}$$

instead of

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}(\partial_1 u_2)^2\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3, \tag{9.3}$$

but, on the right hand side of the same inequality, one must have

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left|\partial_1\frac{u_2}{\xi_1}\right|^2\cdot\xi_1^3\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \tag{9.4}$$

instead of

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}(\partial_1 u_2)^2\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\,. \tag{9.5}$$

We overcome this obstacle by appealing to the following result.

Lemma 9.2 *One has the following equivalence up to negligible terms.*

$$\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left(\partial_1\frac{u_2}{\xi_1}\right)^2\cdot\xi_1^3\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \simeq \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}(\partial_1\,u_2)^2\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\,. \tag{9.6}$$

The proof follows immediately from the identity

$$(\partial_1(u_2/\xi_1))^2\,\xi_1^3 = (\partial_1 u_2)^2\xi_1 - 2u_2(\partial_1 u_2) + u_2^2/\xi_1\,. \tag{9.7}$$

Since the two terms in (9.4) and (9.5) are equivalent, and $\|\partial_1 u_2\|_2^2$ still appears on the right hand side of (9.1), we may add $\|\partial_1(u_2/\xi_1)\|_2^2$ to this same right hand side. So we will show, by appealing to Gronwall's Lemma, that

$$\partial_1\frac{u_2}{\xi_1} \in L^\infty(0,T;L^2(\Omega)).$$

Using Lemma 9.2 once more, we will obtain, in particular, that

$$\partial_1 u_2 \in L^\infty(0,T;L^2(\Omega)),$$

which is the desired result.

Therefore, we define, in addition to $\|\mathbf{Du}\|_2^2$, the quite similar quantity

$$\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2 = \sum_{(j,k)\neq(1,2)} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_j u_k\right|^2 \cdot \xi_1 \,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 + \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left|\partial_1 \frac{u_2}{\xi_1}\right|^2 \xi_1^3 \,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \,. \tag{9.8}$$

By appealing to (9.7), one shows that

$$\left| \left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2 - \|\mathrm{D}\mathbf{u}\|_2^2 \right| \le c\left(\|\mathbf{u}\|_2 \left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2 + \|\mathbf{u}\|_2^2\right) \le c\left(\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2 + \|\mathbf{u}\|_2^2\right), \tag{9.9}$$

where we may replace, on the right hand side, $\tilde{\mathrm{D}}\mathbf{u}$ by $\mathrm{D}\mathbf{u}$. The core argument is that (9.9) leads to the crucial estimate

$$\|\overline{\mathbf{u}}\|_p^q \,\|\mathrm{D}\mathbf{u}\|_2^2 \le \|\overline{\mathbf{u}}\|_p^q \left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2 + c\left(\|\overline{\mathbf{u}}\|_p^q \left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2 + \|\mathbf{u}\|_2^2\right). \tag{9.10}$$

It is worth noting that, in the sequel, the equivalence would be not sufficient.

By setting $\varepsilon = \nu/2$ in Eq. (9.1), and by taking into account Eq. (9.10), it readily follows that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2 + \frac{\nu}{2}\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2 \\ &\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_1^2}{\xi_1}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \frac{u_2^2}{\xi_1}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &\le B(t)\left(\|\mathbf{u}\|_2^2 + \left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2\right), \end{aligned} \tag{9.11}$$

where, from now on, $B(t)$ denotes any generical non-negative real function satisfying

$$B(t) \in L^1(0,\, T)\,.$$

Basically, Eq. (9.11) is well prepared to apply Gronwall's Lemma. However, there are two minor obstacles. The first one is the presence of the two last terms on the left hand side of (9.11), especially the one with the negative sign (actually, the other one is even helpful). The second point is that, in some cases of axial symmetry of Ω, see [6], the quantities $\|\mathbf{u}\|_2^2 + \|\mathrm{D}\mathbf{u}\|_2^2$ and $\|\mathrm{D}\mathbf{u}\|_2^2$ are not equivalent. In the present case, this concerns the third component. Hence, in order to control the term $\|\mathbf{u}\|_2^2$ on the right hand side of (9.1) by means of Gronwall's Lemma, we will add its time derivative to the left hand side, which is obtained from an energy type estimate. This additional term also allows us to control the above integral with the minus sign in front of it.

10 The energy inequality

According to (2.4), we define $\mathcal{E}_j(\partial_t)$, $\mathcal{E}_j(N)$, $\mathcal{E}_j(\nu)$, and $\mathcal{E}_j(\pi)$, $j = 1, 2, 3$, through the following identity

$$\mathcal{E}_j = \mathcal{E}_j(\partial_t) + \mathcal{E}_j(N) + \mathcal{E}_j(\nu) + \mathcal{E}_j(\pi), \quad j = 1, 2, 3. \tag{10.1}$$

Note that $\mathcal{E}_j(N) = N_j$, cf. (2.6), and $\mathcal{E}_j(\partial_t) = \partial_t u_j$.

A full energy inequality is obtained by time integration of the main identity:

$$\sum_{j=1}^{3} \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \mathcal{E}_j \cdot (\xi_1 u_j)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = 0. \tag{10.2}$$

Since integrations by parts with respect to ξ_2 and ξ_3 always lead to vanishing boundary integrals due to periodicity, we will not treat these integrals explicitly.

Lemma 10.1 *We have*

$$\int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \mathcal{E}_j(\partial_t) \cdot (\xi_1 u_j)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} |u_j|^2 \cdot \xi_1\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3. \tag{10.3}$$

Proof Obvious. □

Lemma 10.2 *We have*

$$\sum_{j=1}^{3} \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \mathcal{E}_j(\pi) \cdot (\xi_1 u_j)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = 0. \tag{10.4}$$

Proof For $j = 1$, we obtain

$$\int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \mathcal{E}_1(\pi) \cdot (\xi_1 u_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = -\int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \pi \cdot \partial_1(\xi_1 u_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3,$$

since the boundary integral vanishes due to $u_1 = 0$ on $\partial^l \Omega$. For $j = 2, 3$, we proceed in an analogous manner. Now, the boundary integrals vanish due to periodicity. Summing up, we draw on the velocity's divergence-free property to obtain the desired result. □

Lemma 10.3 *We have*

$$\sum_{j=1}^{3} \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \mathcal{E}_j(N) \cdot (\xi_1 u_j)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 = 0. \tag{10.5}$$

Proof One easilly shows that, for each $j = 1, 2, 3$,

$$(\mathbf{u} \cdot \nabla u_j)\,(\xi_1 u_j) = \frac{1}{2}\left(\mathbf{u} \cdot \nabla u_j^2\right) \cdot \xi_1\,.$$

Hence, the integral of each of the above terms vanishes according to Lemma 5.4. It follows that the integral on the left hand side of Eq. (10.5) consists merely of the two "lower order terms" appearing in $(2.6)_1$ and $(2.6)_2$. These terms cancel each other due to their opposite signs. □

Next, we consider the viscous terms. We start by the "higher order terms". From (2.2c), one has

$$\nabla^2 u_j = \frac{1}{\xi_1}\partial_1(\xi_1 \partial_1 u_j) + \frac{1}{\xi_1^2}\left(\partial_2^2 u_j\right) + \left(\partial_3^2 u_j\right). \tag{10.6}$$

Hence,

$$\begin{aligned} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\nabla^2 u_j) \cdot (\xi_1 u_j)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 &= \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \partial_1(\xi_1\partial_1 u_j) \cdot u_j\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ + \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left(\partial_2^2 u_j\right) \cdot \frac{u_j}{\xi_1}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 &+ \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} \left(\partial_3^2 u_j\right) \cdot u_j \cdot \xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3. \end{aligned} \tag{10.7}$$

By suitable integrations by parts, one shows that, for each $j = 1, 2, 3$,

$$\begin{aligned} \int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\nabla^2 u_j) \cdot (\xi_1 u_j)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = &-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\partial_1 u_j)^2 \cdot \xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\partial_2 u_j)^2 \cdot \frac{1}{\xi_1}\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &-\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1} (\partial_3 u_j)^2 \cdot \xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &+\int_0^1\int_0^{2\pi} (\partial_1 u_j) \cdot u_j \cdot \xi_1\Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1}\,\mathrm{d}\xi_2\mathrm{d}\xi_3. \end{aligned}$$

For $j = 1$, the boundary integral vanishes due to $u_1 \equiv 0$ on $\partial^l\Omega$. For $j = 3$, the boundary integral vanishes, since $\partial_1 u_3 \equiv 0$ on $\partial^l\Omega$. Furthermore, for $j = 2$, due to boundary condition (2.9)$_2$, one has $\partial_1 u_2 = u_2/\xi_1$ on $\partial^l \Omega$. Hence,

$$\int_0^1 \int_0^{2\pi} (\partial_1 u_2) \cdot u_2 \cdot \xi_1 \Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} \mathrm{d}\xi_2 \mathrm{d}\xi_3 = \int_0^1 \int_0^{2\pi} u_2^2 \Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} \mathrm{d}\xi_2 \mathrm{d}\xi_3 .$$

It readily follows that

$$\begin{aligned}
&\sum_{j=1}^{3} \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} (\nabla^2 u_j) \cdot (\xi_1 u_j) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\
&= - \sum_{j=1}^{3} \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} \left[(\partial_1 u_j)^2 \cdot \xi_1 + (\partial_2 u_j)^2 \cdot \frac{1}{\xi_1} + (\partial_3 u_j)^2 \cdot \xi_1 \right] \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \qquad (10.8) \\
&\quad + \int_0^1 \int_0^{2\pi} u_2^2 \Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} \mathrm{d}\xi_2 \mathrm{d}\xi_3 ,
\end{aligned}$$

where the first integral is the principal part of the ν-term.

The boundary integral can be estimated by appealing to Gagliardo's trace theorem. This immediately shows that

$$\begin{aligned}
&\left| \int_0^1 \int_0^{2\pi} u_2^2 \Big|_{\xi_1=\rho_0}^{\xi_1=\rho_1} \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right| \\
&\leq C \cdot \left\| u_2^2 \right\|_{1,1} \\
&\leq C \cdot \left(\int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} u_2^2 \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 + \int_0^1 \int_0^{2\pi} \int_{\rho_0}^{\rho_1} |u_2 \cdot \mathrm{D} u_2| \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right) \qquad (10.9) \\
&\leq C \cdot \left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2 \, \|\mathbf{D}\mathbf{u}\|_2 \right)
\end{aligned}$$

which is clearly a negligible term because

$$C \cdot \left(\|\mathbf{u}\|_2^2 + \|\mathbf{u}\|_2 \, \|\mathbf{D}\mathbf{u}\|_2 \right) \leq C_\varepsilon \cdot \|\mathbf{u}\|_2^2 + \varepsilon \cdot \|\mathbf{D}\mathbf{u}\|_2^2 . \qquad (10.10)$$

Next, we consider the "lower order terms" which are present for $j = 1, 2$, cf. (2.4)$_1$ and (2.4)$_2$. All these terms are clearly negligible. Hence, for the purpose of proving our main result, the reader does not have to take these terms into account. However, it might still by interesting for the reader to study their contribution in order to obtain a stringent energy inequality in the current context. Instead of appealing to negligibility, we might, therefore, note that the contribution of the "lower order terms" that have not been taken into account yet is bounded by the left hand side of (10.10), as can be easily verified by the reader. Hence, with an obvious ε-notation, one has, by appealing to (10.8), (10.9), and (10.10), the following statement.

Lemma 10.4 *We have*

$$\begin{aligned}&\sum_{j=1}^{3}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\mathcal{E}_j(\nu)\cdot(\xi_1 u_j)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\&\geq\nu\sum_{j=1}^{3}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\left[(\partial_1 u_j)^2\cdot\xi_1+(\partial_2 u_j)^2\cdot\frac{1}{\xi_1}+(\partial_3 u_j)^2\cdot\xi_1\right]\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\&\quad-\nu\cdot\left(C_\varepsilon\cdot\|\mathbf{u}\|_2^2+\varepsilon\cdot\|\mathrm{D}\mathbf{u}\|_2^2\right).\end{aligned}\tag{10.11}$$

From the main identity 10.2, by appealing to Lemmata 10.1–10.4, one obtains the following energy inequality.

Theorem 10.5 *One has*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2+\frac{\nu}{2}\|\mathrm{D}\mathbf{u}\|_2^2\leq C\nu\,\|\mathbf{u}\|_2^2.\tag{10.12}$$

In particular,

$$\mathbf{u}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega)).\tag{10.13}$$

11 Proof of Theorem 2.2.

It looks convenient to write the Eq. (10.12) in the more explicit form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}\sum_{j=1}^{3}\left|u_j\right|^2\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3+\frac{\nu}{2}\|\mathrm{D}\mathbf{u}\|_2^2\leq C\nu\,\|\mathbf{u}\|_2^2.\tag{11.1}$$

Addition, side by side, of Eq. (9.11) with Eq. (11.1) multiplied by a suitable positive constant α (to control the previous integral with a minus sign in front of it), leads to the estimate

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2+\frac{\nu}{2}\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2+\alpha\frac{\nu}{2}\left\|\mathrm{D}\mathbf{u}\right\|_2^2\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}u_1^2\left(\alpha\,\xi_1+\frac{1}{\xi_1}\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}u_2^2\left(\alpha\,\xi_1-\frac{2}{\xi_1}\right)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \qquad (11.2)\\
&\quad+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^{2\pi}\int_{\rho_0}^{\rho_1}|u_3|^2\,\alpha\cdot\xi_1\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\\
&\le B(t)\left(\|\mathbf{u}\|_2^2+\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2\right),
\end{aligned}$$

which is clearly suitable for the application of Gronwall's Lemma, up to minor obvious adaptations. Note that the right hand side of (11.1) has been incorporated in the right hand side of (11.2) by replacing $B(t)+C\nu\alpha$ simply by $B(t)$.

Next, fix α such that $\alpha\rho_0 = 1+2/\rho_1$. Since $\rho_0\le\xi_1\le\rho_1$, it follows that,

$$\alpha\,\xi_1-\frac{2}{\xi_1}\ge 1\,.$$

For convenience, let us denote the three explicit space integrals on the left hand side of (11.2) by, respectively, $K_1^2,\ K_2^2$, and K_3^2, and let us introduce $\mathcal{K}^2 = K_1^2+K_2^2+K_3^2$. Due to the above choice of α, one has $K_j^2\simeq\left\|u_j\right\|_2^2$, for $j = 1,\,2,\,3$, which means $\mathcal{K}^2\simeq\|\mathbf{u}\|_2^2$. It follows that,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2+\mathcal{K}^2\right)+\frac{\nu}{2}\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2+\alpha\frac{\nu}{2}\left\|\mathrm{D}\mathbf{u}\right\|_2^2\le B(t)\left(\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2+\mathcal{K}^2\right). \qquad (11.3)$$

A classical argument, based on integration with respect to time of (11.3) and Gronwall's Lemma, shows that

$$\left(\left\|\tilde{\mathrm{D}}\mathbf{u}\right\|_2^2+\mathcal{K}^2\right)\in L^\infty(0,\,T),\quad\text{and}\quad\left\|\mathrm{D}^2\mathbf{u}\right\|_2^2\in L^1(0,\,T)\,.$$

This is obviously equivalent to (2.11), namely

$$\mathbf{u}\in L^\infty(0,T;W^{1,2}(\Omega))\cap L^2(0,T;W^{2,2}(\Omega)).$$

Theorem 2.2 is proved.

References

1. Bae, H.-O., Choe, H.J.: A regularity criterion for the Navier–Stokes equations. Commun. Partial Differ. Equations **32**, 1173–1187 (2007)
2. Bae, H.-O., Wolf, J.: A local regularity condition involving two velocity components of Serrin-type for the Navier–Stokes equations. C. R. Acad. Sci. Paris Ser. **I**(354), 167–174 (2016)
3. Batchelor, G.K.: An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge (1967)
4. Beirão da Veiga, H.: Remarks on the smoothness of the $L^{\infty}(0, T; L^3)$ solutions of the 3-D Navier–Stokes equations. Port. Math. **54**, 381–391 (1997)
5. Beirão da Veiga, H.: On the smoothness of a class of weak solutions to the Navier–Stokes equations. J. Math. Fluid Mech. **2**, 315–323 (2000)
6. Beirão da Veiga, H.: Regularity for Stokes and generalized Stokes systems under nonhomogeneous slip-type boundary conditions. Adv. Differ. Equations **9**, 1079–1114 (2004)
7. Beirão da Veiga, H.: Remarks on the Navier–Stokes equations under slip type boundary conditions with linear friction. Port. Math. **64**, 377–387 (2007)
8. Beirão da Veiga, H.: On the extension to slip boundary conditions of a Bae and Choe regularity criterion for the Navier-Stokes equations. The half space case. J. Math. Anal. Appl. **453**, 212–220 (2017)
9. Berselli, L.C.: A note on regularity of weak solutions of the Navier-Stokes equations in R^n. Jpn. J. Math. **28**, 51–60 (2002)
10. Cao, C., Titi, E.S.: Regularity criteria for the three-dimensional Navier–Stokes equations. Indiana Univ. Math. J. **57**, 2643–2661 (2008)
11. Chae, D., Choe, H.-J.: Regularity of solutions to the Navier–Stokes equation. Electron. J. Differ. Equations **05**, 1–7 (1999)
12. Escauriaza, L., Seregin, G., Šverák, V.: $L_{3,\infty}$-Solutions to the Navier–Stokes equations and backward uniqueness. Russ. Math. Surv. **58**, 211–250 (2003)
13. Foias, C.: Une remarque sur l'unicité des solutions des équations de Navier-Stokes en dimension n.. Bull. Soc. Math. Fr. **89**, 1–8 (1961)
14. Galdi, G.P.: An Introduction to the Navier–Stokes initial-boundary value problems. In: Galdi, G.P., Heywood, M.I., Rannacher, R. (eds.) Fundamental Directions in Mathematical Fluid Mechanics. Advances in Mathematical Fluid Mechanics, pp. 1–70. Birkhäuser, Basel (2000)
15. Galdi, G.P., Maremonti, P.: Sulla regolarità delle soluzioni deboli al sistema di Navier–Stokes in domini arbitrari. Ann. Univ. Ferrara Sez. VII. Sci. Mat. **34**, 59–73 (1988)
16. Giga, Y.: Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system. J. Differ. Equations **62**, 186–212 (1986)
17. He, C.: Regularity for solutions to the Navier–Stokes equations with one velocity component regular. Electron. J. Differ. Equations **29**, 1–13 (2002)
18. Hopf, E.: Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nachr. **4**, 213–231 (1951)
19. Kiselev, A.A., Ladyzhenskaya, O.A.: On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid. Izv. Akad. Nauk SSSR Ser. Mat. **21**, 655–680 (1957)
20. Kozono, H., Sohr, H.: Regularity criterion on weak solutions to the Navier–Stokes equations. Adv. Differ. Equations **2**, 2924–2935 (2007)
21. Kukavica, I., Ziane, M.: Navier–Stokes equations with regularity in one direction. J. Math. Phys. **48**, 2643–2661 (2007)
22. Ladyzhenskaya, O.A.: On uniqueness and smoothness of generalized solutions to the Navier–Stokes equations. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **5**, 169–185 (1967)
23. Ladyzhenskaya, O.A.: La théorie mathématique des fluides visqueux incompressibles. Moscou (1961) [**English edition. 2nd edn. Gordon & Breach, New York (1969)**]
24. Leray, J.: Sur le mouvement d'un liquide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
25. Lions, J.L.: Sur l'existence de solutions des équations de Navier–Stokes. C. R. Acad. Sci. Paris **248**, 2847–2849 (1959)
26. Mikhailov, A.S., Shilkin, T.N.: $L_{3,\infty}$-solutions to the 3D-Navier–Stokes system in a domain with a curved boundary. J. Math. Sci. (N. Y.) **143**, 2924–2935 (2007)
27. Neustupa, J., Penel, P.: Anisotropic and geometric criteria for interior regularity of weak solutions to the 3D Navier–Stokes equations. In: Neustupa, J., Penel, P. (eds.) Mathematical Fluid Mechanics. Advances in Mathematical Fluid Mechanics, pp. 237–265. Birkhäuser, Basel (2001)

28. Prodi, G.: Un teorema di unicità per le equazioni di Navier–Stokes. Ann. Mat. Pura Appl. **48**, 173–182 (1959)
29. Prodi, G.: Résultats récents et problèmes anciens dans la théorie des équations de Navier-Stokes. In: Les Équations aux Dérivées Partielles, pp. 181–196. Éditions du CNRS, Paris (1962)
30. Serrin, J.: On the interior regularity of weak solutions of the Navier–Stokes equations. Arch. Ration. Mech. Anal. **9**, 187–195 (1962)
31. Serrin, J.: The initial value problem for the Navier–Stokes equations. In: Langer, R.E. (ed.) Nonlinear Problems, pp. 69–98. University of Wisconsin Press, Madison (1963)
32. Sohr, H.: Zur Regularitätstheorie der instationären Gleichungen von Navier–Stokes. Math. Z. **184**, 359–375 (1983)
33. Solonnikov, V.A., Ščadilov, V.E.: On a boundary value problem for a stationary system of Navier–Stokes equations. Proc. Steklov Inst. Math. **125**, 186–199 (1973)
34. von Wahl, W.: Regularity of weak solutions of the Navier–Stokes equations. Proc. Symp. Pure Math. **45**, 497–503 (1986)
35. Zhang, Z., Zhong, D., Huang, X.: A refined regularity criterion for the Navier–Stokes equations involving one non-diagonal entry of the velocity gradient. J. Math. Anal. Appl. **453**, 1145–1150 (2017)
36. Zhou, Y., Pokorný, M.: On the regularity of the solutions of the Navier–Stokes equations via one velocity component. Nonlinearity **23**, 1097–1107 (2010)