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NAVIER-STOKES EQUATIONS: SOME QUESTIONS RELATED TO THE DIRECTION OF THE VORTICITY

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To Professor Vicentiu Rădulescu on the occasion of his 60th birthday

Abstract. We consider solutions u to the Navier-Stokes equations in the whole space. We set $\omega = \nabla \times u$, the vorticity of u. Our study concerns relations between $\beta-$Hölder continuity assumptions on the direction of the vorticity and induced integrability regularity results, a significant research field starting from a pioneering 1993 paper by P. Constantin and Ch. Fefferman. Nowadays it is know that if $\beta = \frac{1}{2}$ then $\omega \in L^\infty(L^2)$, a 2002 result by L.C. Berselli and the author. This conclusion implies smoothness of solutions. Assume now that one is able to prove that a strictly decreasing perturbation of β near $\frac{1}{2}$ induces a strictly decreasing perturbation for r near 2. Since regularity holds if merely $\omega \in L^\infty(L^r)$, for some $r \geq \frac{3}{2}$, the above assumption would imply regularity for values $\beta < \frac{1}{2}$. The aim of the present note is to go deeper into this study and related open problems. The approach developed below reinforces the conjecture on the particular significance of the value $\beta = \frac{1}{2}$.

1. **Introduction.** In the following we consider solutions u to the Navier-Stokes equations

$$\begin{cases} u_t + (u \cdot \nabla)\, u - \triangle u + \nabla p = 0\,, & \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^3 \times (0, T], \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^3\,. \end{cases} \tag{1}$$

We will not repeat well know notation as, for instance, Sobolev spaces notation, and so on. For brevity, we set $L^s(\,L^r\,) = L^s(0,\,T;\,L^r(\mathbb{R}^3)\,)$, and similar. Solutions $u \in\, L^2(0,\,T;\,H^1(\mathbb{R}^3)\,) \cap L^\infty(0,\,T;\,L^2(\mathbb{R}^3)\,)$ are defined in the well known Leray-Hopf weak sense. We set $\omega = \nabla \times u$, the vorticity of u.

We start by recalling some know results. The classical Ladyzhenskaya-Prodi-Serrin sufficient condition for regularity (see for instance [24] and references therein) states that if

$$u \in L^s(0,\,T; L^q(\Omega)\,)\,, \tag{2}$$

for some exponents s and q, $2 \leq\, s <\, \infty$ (so, $3 < q \leq \infty$) satisfying

$$\lambda(s,\,q) \equiv\, \frac{2}{s} + \frac{3}{q} =\, 1\,, \tag{3}$$

then u is regular.

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Next we consider sufficient conditions for regularity, again of integral type, but concerning the vorticity $\omega = \nabla \times u$. Solutions u of (1) are regular if

$$\omega \in L^s(0,\, T; L^r(\Omega)\,) \tag{4}$$

for exponents $s,\, r$ satisfying

$$\mu(s,\, r) \equiv \frac{2}{s} + \frac{3}{r} = 2\,, \quad \text{for some} \quad 1 \leq s \leq \infty\,. \tag{5}$$

If $s \geq 2$, this follows from the L-P-S condition by appealing to a Sobolev's embedding theorem. If $1 < s \leq 2$, regularity follows from [2].

Next we define

$$\theta(x,y,t) \stackrel{def}{=} \angle(\omega(x,t),\, \omega(y,t))\,,$$

where the symbol "$\angle$" denotes the amplitude of the angle between two vectors. We are interested in studying possible regularization effects of $\beta-$Hölder continuity assumptions on the direction of the vorticity, namely

$$\sin\theta(x,y,t) \leq \; c\,|x-y|^\beta \tag{6}$$

in $\mathbb{R}^3 \times \mathbb{R}^3 \times (0,T)$, for some $\beta \in (0, 1/2]$. For brevity, in the following, $\beta-$Hölder continuity assumptions on the direction of the vorticity will be simply called $\beta-$Hölder assumptions, or even $\beta-$assumptions.

It is known, see [9], that assumption (6) for $\beta = \frac{1}{2}$ implies $\omega \in L^\infty(\,L^2\,)$. In particular, $u \in L^\infty(\,L^6\,)$ follows. This shows that $\frac{1}{2}-$Hölder continuity implies regularity for u, by the L-P-S condition. Now assume that a strictly decreasing perturbation of β near $\frac{1}{2}$ induces a strictly decreasing perturbation for r near 2. Since, by the L-P-S condition, regularity holds if merely $\omega \in L^\infty(\,L^r\,)$, for some $r \geq \frac{3}{2}$, the above assumption would imply regularity for values $\beta < \frac{1}{2}$. However this important consequence would be in contrast with a previous conjecture supported by us which suggests that $\beta = \frac{1}{2}$ is the smallest value enjoying (in some non rigorous sense) the above strong regularization property. See section 6. The aim of these notes is to go deeper in the study of the above problem by appealing to a significant generalization of the main lines of proofs shown in previous references. We argue as trying to prove that

$$\omega \in \; L^\infty(0,\, T;\; L^r(\Omega)\,) \tag{7}$$

for some couple r, β satisfying $r < 2$, and $\beta < \frac{1}{2}$. Our final conclusion will be that, *for any value* $r \in (1,\, 2]$, the smallest value β which guarantees (7) is *in all cases* $\beta = \frac{1}{2}$. This conclusion supports our conjecture about the main role played by the $\frac{1}{2}-$Hölder assumption.

Two words about the strategy followed in the sequel. In our proof model, namely the proof of the Hilbertian case $(\beta,\, r) = (\frac{1}{2},\, 2)$, one started by proving a $L^\infty(\,L^2\,)$ estimate for ω, by leaving *completely* free the parameter β. The value $\beta = \frac{1}{2}$ appears at the end of the proof as being the smallest β consistent with the proof of the $L^\infty(\,L^2\,)$ estimate previously obtained (i.e., for which the proof still works). Below we follow this same line, by replacing 2 by r. We start by proving a suitable $L^\infty(\,L^r\,)$ estimate for ω, for all $r \leq 2$ in a neighborhood of 2, instead of merely for $r = 2$, as in the classical case. Then we look for the smallest $\beta = \beta(r)$ consistent with the proof of the $L^\infty(\,L^r\,)$ estimate. It is worth noting that each single manipulation in the proof is formally independent of the particular value of the parameter r, and coincides with the "classical" proof for $r = 2$. This guarantees that a perturbation argument, near $r = 2$, does not present a "discontinuity" at

$r = 2$. However, as for the classical case $r = 2$, at the end of the proof only the value $\beta = \frac{1}{2}$ appears to be admissible.

Summing up, our attempt to feel out if the regularity result (38) may hold for $r < 2$ under a $\beta-$Hölder continuity assumption on the direction of the vorticity, for some $\beta < \frac{1}{2}$, has had here a negative reply. This conclusion supports the argument, still defended in the appendix of [6], that if for some value $\beta < \frac{1}{2}$ the $\beta-$Hölder assumption implies regularity, than a proof seems not obtainable by appealing to classical devices. A negative conclusion, not to be disregard.

Mostly to present a "positive" result we prove, in section 5, the following result.

Proposition 1. *Assume that (6) holds in $\Omega \times (0,T)$, for some $\beta \in [0, 1/2]$, and that*

$$\omega \in L^{\frac{4}{1+2\beta}}(0,T; L^2)\,. \tag{8}$$

Then the solution u of the Navier-Stokes equations (1) is strong in $(0,T)$ and, consequently, is regular.

In reference [3], Theorem 1.3, it was stated that if (6) holds in $\Omega \times (0,T)$, for some $\beta \in [0, 1/2]$, and if

$$\omega \in L^2(0,T; L^{\frac{3}{\beta+1}})\,, \tag{9}$$

then the solution u is strong in $(0,T)$ and, consequently, is regular. We remark that assumptions (8) and (9) have the same strength since

$$\frac{2}{2} + \frac{3}{\frac{3}{\beta+1}} = \frac{2}{\frac{4}{1+2\beta}} + \frac{3}{2} = 2 + \beta\,.$$

However assumption (8) is of greater interest since the spatial norm of the vorticity is taken in the more significant energy space L^2.

Let us end this section by proposing the following problem.

Problem. *Assume that a $\beta-$Hölder continuity assumption on the direction of the vorticity holds for some value $\beta < \frac{1}{2}$. Show that (4) holds for a couple of exponents s and r satisfying $\mu(s,r) = \frac{2}{s} + \frac{3}{r} < \frac{5}{2}$.*

Since $\mu(s,r) = \frac{5}{2}$ corresponds, formally, to the maximum regularity known for generical weak solutions (proved for $s = r = 2$), the above result would show additional regularity to weak solutions, in terms of integrability. The possibility of this "transfer" of regularity from direction of vorticity to integrability looks quite natural.

Remark. A first version of this paper was previously presented in reference [7]. We are pleased that, in the meantime, some authors appealed to our techniques to obtain quite interesting new results.

2. **A new estimate.** In the following $f(s)$ denotes a real continuous differentiable function $f : \mathbb{R}^+ \to \mathbb{R}^+$. We set

$$F(s) = \int_0^s f(\tau)\, d\tau\,.$$

Hence $F'(s) = f(s)$. By applying the curl operator to equation (1) we get the well-known equation

$$\omega_t + (u \cdot \nabla)\,\omega - \nu\,\Delta\omega = (\omega \cdot \nabla)\,u\,.$$

Scalar multiplication by $f(|\omega|^2)\,\omega$, integration in $\mathbb{R}^3$, and integrations by parts easily show that

$$\frac{1}{2}\,\frac{d}{dt}\int F(|\omega|^2)\,dx - \int f(|\omega|^2)\,\Delta\,\omega\cdot\,\omega\,dx = \int f(|\omega|^2)\,(\omega\cdot\nabla)\,u\cdot\,\omega\,dx\,. \tag{10}$$

Non-labeled integrals are over $\mathbb{R}^3$. Straightforward calculations show that

$$-\int f(|\omega|^2)\,\Delta\,\omega\cdot\,\omega\,dx = \int f(|\omega|^2)\,|\nabla\,\omega|^2\,dx + 2\int f'(|\omega|^2)\,|\omega|^2\,|\nabla\,\omega|^2\,dx\,.$$

Hence

$$\begin{aligned} &\frac{1}{2}\,\frac{d}{dt}\int F(|\omega|^2)\,dx + \int f(|\omega|^2)\,|\nabla\,\omega|^2\,dx \\ &\leq 2\int f'(|\omega|^2)\,|\omega|^2\,|\nabla\,\omega|^2\,dx + \int f(|\omega|^2)\,(\omega\cdot\nabla)\,u\cdot\,\omega\,dx\,. \end{aligned} \tag{11}$$

In these notes we are interested in the particular case $f(s) = s^{-\,\alpha}$. In the Hilbertian case, in which $\alpha = 0$, many of the devices used in the sequel are superfluous.

It is useful to start by considering the approximation functions

$$f_\epsilon(s) = (\epsilon + s)^{-\alpha}, \tag{12}$$

where $\varepsilon > 0$, and $0 \leq \alpha \leq \frac{1}{2}$. Note that $f'_\varepsilon(s) < 0$. Straightforward calculations show that the absolute value of the first integral on the right hand side of equation (11) is bounded by α times the second integral on the left hand side of the same equation. So, one has

$$\begin{aligned} &\frac{1}{2(1-\alpha)}\,\frac{d}{dt}\int (\varepsilon + |\omega|^2)^{1-\,\alpha}\,dx + (1-2\,\alpha)\int (\varepsilon + |\omega|^2)^{-\,\alpha}|\nabla\,\omega|^2\,dx \\ &\qquad\leq \int (\varepsilon + |\omega|^2)^{-\,\alpha}\,|\mathcal{K}(x)|\,dx \end{aligned} \tag{13}$$

where

$$\mathcal{K}(x) \equiv (\,(\omega\cdot\nabla)\,u\cdot\,\omega)\,(x)\;. \tag{14}$$

Next we estimate from below the second integral on the left hand side of equation (13) (see [1] and [2] for similar manipulations). One has

$$(\varepsilon + |\omega|^2)^{-\,\alpha}\,|\nabla\,|\omega|\,|^2 \;=\; \frac{1}{(1-\alpha)^2}\,\frac{|\omega|^{2\,\alpha}}{(\varepsilon + |\omega|^2)^\alpha}\,\big|\,\nabla(\,|\omega|^{1-\,\alpha})\,\big|^2\,. \tag{15}$$

Since $|\,\nabla\,\omega|\,\geq |\,\nabla\,|\omega|\,|$, it follows from equation (13) that

$$\begin{aligned} &\tfrac{1}{2(1-\alpha)}\,\tfrac{d}{dt}\int (\varepsilon + |\omega|^2)^{1-\,\alpha}\,dx + \tfrac{(1-2\,\alpha)}{(1-\alpha)^2}\int \tfrac{|\omega|^{2\,\alpha}}{(\varepsilon + |\omega|^2)^\alpha}\,\big|\,\nabla(\,|\omega|^{1-\,\alpha})\,\big|^2\,dx \\ &\leq \int (\varepsilon + |\omega|^2)^{-\,\alpha}\,|\mathcal{K}(x)|\,dx\,. \end{aligned}$$

By letting $\varepsilon \to 0$ one gets

$$\begin{aligned} &\tfrac{1}{2(1-\alpha)}\,\tfrac{d}{dt}\,\|\omega\|_{2(1-\alpha)}^{2(1-\alpha)} + \tfrac{(1-2\,\alpha)}{(1-\alpha)^2}\,\|\nabla(\,|\omega|^{1-\,\alpha})\|_2^2 \\ &\leq \int |\omega|^{-2\,\alpha}\,|\mathcal{K}(x)|\,dx \end{aligned} \tag{16}$$

which, for $\alpha = 0$, is precisely the estimate obtained in the Hilbertian case. Now we apply the Sobolev's embedding theorem $\|g\|_6 \leq c_0 \|\nabla g\|_2$ to the function $g = |\omega|^{1-\alpha}$. After this device, equation (16) reads

$$\frac{d}{dt} \|\omega\|^{2(1-\alpha)}_{2(1-\alpha)} + c_1 \|\omega\|^{2(1-\alpha)}_{6(1-\alpha)} \leq \int |\omega|^{-2\alpha} |\mathcal{K}(x)| \, dx \, . \tag{17}$$

The symbol c, and similar, may denote distinct positive constants.

3. **Estimating the nonlinear term by a Riesz potential.** In this section we estimate the right hand side of equation (17) by means of a related Riesz potential. This is one of the main ideas introduced by Constantin and Fefferman in [18], and took again in [8]. We follow here the presentation given in reference [6] (where bounded domains are considered). See also [9].

Since $-\Delta u = \nabla \times (\nabla \times u) - \nabla (\nabla \cdot u)$, it follows that

$$-\Delta u = \nabla \times \omega \quad \text{in} \quad \mathbb{R}^3 , \tag{18}$$

for each t. So, by Biot-Savart law, one has

$$u(x) = \int G(x,y) \, (\nabla \times \omega)(y) \, dy \, , \tag{19}$$

where

$$G(x,y) = \frac{1}{4\pi |x-y|} \, .$$

In particular

$$\left| \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \right| \leq \frac{c}{|x-y|^3} \, . \tag{20}$$

Set, for each triad (j,k,l), $j,k,l \in \{1,2,3\}$,

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation} \, , \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation} \, , \\ 0 & \text{if two indexes are equal} \, . \end{cases}$$

These are the components of the totally anti-symmetric Ricci tensor. One has

$$(a \times b)_j = \epsilon_{jkl} \, a_k \, b_l \, , \qquad (\nabla \times v)_j = \epsilon_{jkl} \, \partial_k v_l \, . \tag{21}$$

The usual convention about summation of repeated indexes is assumed.

In particular

$$\left| \frac{\partial^2 G(x,y)}{\partial y_k \partial x_i} \right| \leq \frac{c}{|x-y|^3} \, . \tag{22}$$

By considering in equation (19) a single component u_j, and by appealing to (21), an integration by parts yields

$$u_j(x) = \int G(x,y) \, \epsilon_{jkl} \, \partial_k \, \omega_l(y) \, dy = -\int \epsilon_{jkl} \, \frac{\partial \, G(x,y)}{\partial \, y_k} \, \omega_l(y) \, dy \, .$$

Hence

$$\frac{\partial \, u_j(x)}{\partial \, x_i} = -P.V. \int \epsilon_{jkl} \, \frac{\partial^2 \, G(x,y)}{\partial \, x_i \partial \, y_k} \, \omega_l(y) \, dy \, .$$

It readily follows that

$$\mathcal{K}(x) = -\int \epsilon_{jkl} \, \tfrac{\partial^2 \, G(x,y)}{\partial \, y_k \partial \, x_i} \, \omega_i(x) \, \omega_j(x) \, \omega_l(y) \, dy \, .$$

Since $-\epsilon_{jkl}\,\omega_j(x)\,\omega_l(y) = \big(\omega_j(x) \times \omega_l(y)\big)_k$, it follows that

$$\mathcal{K}(x) = P.V. \int \frac{\partial^2 G(x,y)}{\partial y_k\,\partial x_i}\,\omega_i(x)\,\big(\omega_j(x) \times \omega_l(y)\big)_k\,dy\,.$$

By appealing to (22) one shows that

$$|\,\mathcal{K}(x)\,| \leq \int \frac{c}{|x-y|^3}\,|\omega(x)|^2\,|\omega(y)|\,\sin\theta(x,y,t)\,dy\,. \tag{23}$$

Now we appeal to the main assumption (6) where, for now, $\beta \in [0, 1/2]$ is left free. By appealing to (23) one gets

$$|\mathcal{K}(x)| \leq c\,|\omega(x)|^2\,I(x)\,, \tag{24}$$

where

$$I(x) = \int_\Omega |\omega(y)|\,\frac{dy}{|x-y|^{3-\beta}}$$

is the Riesz potential in $\mathbb{R}^3$. Recall that (see [33]) if $0 < \beta < 3$, and if $\omega \in L^{\widehat{r}}(\Omega)$ for some exponent $\widehat{r}$ satisfying

$$1 < \widehat{r} < 3\,, \tag{25}$$

then $I \in L^q(\mathbb{R}^3)$, where

$$1/q = 1/\widehat{r} - \beta/3\,, \tag{26}$$

and

$$\|I\|_q \leq c\,\|\omega\|_{\widehat{r}}\,. \tag{27}$$

In particular, by (24), the right hand side of equation (17) satisfies the estimate

$$\int |\omega|^{-2\alpha}\,\mathcal{K}(x)\,dx \leq c \int |\omega|^r\,I(x)\,dx\,, \tag{28}$$

where $r = 2(1-\alpha)$. So,

$$\frac{d}{dt}\|\omega\|_r^r + \|\omega\|_{3r}^r \leq c \int |\omega|^r\,I(x)\,dx\,, \tag{29}$$

for

$$1 < r \leq 2\,. \tag{30}$$

From now on we eliminate the above parameter α by appealing to the new exponent r. By appealing to (27), we write the basic estimate

$$\frac{d}{dt}\|\omega\|_r^r + \|\omega\|_{3r}^r \leq c\,\|\omega\|_{\widehat{r}}\,\|\omega\|_{q'r}^r\;, \tag{31}$$

where

$$\frac{1}{q'} = 1 - \frac{1}{\widehat{r}} + \frac{\beta}{3}\,. \tag{32}$$

More precisely, by (16), we could write (not used in the sequel)

$$\frac{d}{dt}\|\omega\|_r^r + \|\,\nabla\,|\omega|^{\frac{r}{2}}\,\|_2^2 \leq c\,\|\omega\|_{\widehat{r}}\,\|\omega\|_{q'r}^r\;. \tag{33}$$

Note that from (31) it immediately follows that the exponents $\widehat{r}$ and $q'\,r$ should be chosen less or equal to the exponent $3\,r$. The first condition holds by assumption (25). The second one easily leads to the restriction

$$\frac{3}{2-\beta} \leq \widehat{r} < 3\,. \tag{34}$$

In the classical case $\beta = \frac{1}{2}$ one has $\widehat{r} \geq 2$. Fortunately, if β decreases, the value on the above left hand side also decreases, so the range of $\widehat{r}$ expands to the left of the significant value 2.

4. **Towards the final conclusion.** The main task in this section is looking for pairs r and β such that the estimate

$$\|\omega\|_{\widehat{r}}\, \|\omega\|^r_{q'r} \leq C_\varepsilon\, \|\omega\|^2_2\, \|\omega\|^r_r + \varepsilon\, \|\,\omega\|^r_{3r} \tag{35}$$

holds. By (31), this would lead to

$$\frac{d}{dt}\|\omega\|^r_r + \|\omega\|^r_{3r} \leq C_\varepsilon\, \|\omega\|^2_2\, \|\omega\|^r_r\,, \tag{36}$$

for a sufficient large value of C_ε. As usual, the meaning of the equation (35) is that ε may be any positive, arbitrarily small real number, at the price of having corresponding large values of C_ε. The motivation for this requirement is standard. Assume that (48) holds for some pair of values r and β. Then, by appealing to

$$\|\omega(t)\|^2_2 \in L^1(0,\, T) \tag{37}$$

and to Gronwall's lemma, we show that

$$\omega \in L^\infty(0,\, T; L^r(\Omega)\,) \cap L^r(0,\, T; L^{3r}(\Omega)\,)\,. \tag{38}$$

Even though $r \neq 2$, a central role in the right hand side of equation (35) is still required to the integrability exponent 2. The reason for this choice is that (37) is the strongest known estimate for the vorticity of weak solutions.

Our aim is now to find pairs $(r,\, \beta) \in (1,\, 2] \times (0,\, 1/2\,]$ such that (35) holds. We decompose both norms $\|\omega\|_{\widehat{r}}$ and $\|\omega\|_{q'r}$, present in the right hand side of (31), by appealing to interpolation. To keep again the maximum generality, we give the largest width to the range of the exponent $\widehat{r}$, by interpolating $\widehat{r}$ between the values r and $3r$ as suggested by the left hand side of (31), and by the right hand side of (35).

Note that the restriction (34) does not prevent interpolation between spaces outside the above range, for instance r and $3\,r$, as done below. The same motivation and remarks apply to the exponent $q'\,r$.

We start by considering parameters $\alpha,\, \theta,\, \gamma$ and $\alpha',\, \theta',\, \gamma'$, in the interval $[0,\, 1]$, satisfying $\alpha + \theta + \gamma = \alpha' + \theta' + \gamma' = 1$, and related to the exponents $q'\,r$ and $\widehat{r}$ by the following equations:

$$\begin{cases} \dfrac{1}{q'r} = \dfrac{\alpha}{r} + \dfrac{\theta}{2} + \dfrac{\gamma}{3r}\,, \\[2mm] \dfrac{1}{\widehat{r}} = \dfrac{\alpha'}{r} + \dfrac{\theta'}{2} + \dfrac{\gamma'}{3r}\,. \end{cases} \tag{39}$$

It follows, by interpolation, that

$$\begin{cases} \|\omega\|_{q'r} \leq \|\omega\|^\alpha_r\, \|\omega\|^\theta_2\, \|\omega\|^\gamma_{3r}\,, \\ \|\omega\|_{\widehat{r}} \leq \|\omega\|^{\alpha'}_r\, \|\omega\|^{\theta'}_2\, \|\omega\|^{\gamma'}_{3r}\,. \end{cases} \tag{40}$$

The values of the above parameters will be fixed in the sequel. One has

$$B \equiv \|\omega\|^r_{q'r}\, \|\omega\|_{\widehat{r}} \leq \|\omega\|^{\alpha' + \alpha\, r}_r\, \|\omega\|^{\theta' + \theta\, r}_2\, \|\omega\|^{\gamma' + \gamma\, r}_{3r}\,. \tag{41}$$

Next, by appealing to Hölder's inequality with dual exponents

$$\frac{r}{\gamma' + \gamma\, r}\,, \quad \frac{r}{(1-\gamma)r - \gamma'}\,,$$

we get

$$B \leq C_\varepsilon \, \|\omega\|_r^{\frac{(\alpha'+\alpha\, r)\, r}{(1-\gamma) r-\gamma'}} \, \|\omega\|_2^{\frac{(\theta'+\theta\, r)\, r}{(1-\gamma) r-\gamma'}} + \varepsilon \, \|\omega\|_{3r}^r \,, \tag{42}$$

where the meaning of ε and C_ε is clear. We want

$$\frac{\alpha' + \alpha\, r}{(1-\gamma)\, r - \gamma'} = 1 \,, \qquad \frac{\theta' + \theta\, r}{(1-\gamma)\, r - \gamma'} = \frac{2}{r} \,, \tag{43}$$

since this immediately leads to (48). By setting $\gamma = 1-(\alpha+\theta)$ and $\gamma' = 1-(\alpha'+\theta')$ in the first equation (43), one easily shows that (43) is equivalent to

$$\begin{cases} \theta' + \theta\, r = 1 \,, \\ \alpha' + \alpha\, r = \dfrac{r}{2} \,. \end{cases} \tag{44}$$

In addition, the exponents q' and $\widehat{r}$ must also verify equation (32). This last constraint will be satisfied since the parameter β is still free. In other words, (32) determines the set of values of the Hölder exponent $\beta = \beta(r)$ which lead to the regularity result (38). Then we choose the minimal one. Let's calculate these values. By appealing to the equation (32) and to the second equation (39), one shows that the first equation (39) can be written in the equivalent form

$$1 - \frac{\alpha'}{r} - \frac{\theta'}{2} - \frac{\gamma'}{3r} + \frac{\beta}{3} = \alpha + \frac{r}{2}\,\theta + \frac{\gamma}{3} \,.$$

Further, by replacing γ and γ' respectively by $1-(\alpha+\theta)$ and $1-(\alpha'+\theta')$, straightforward calculations lead to the desired expression of $\beta(r)$, namely

$$\beta(r) = \frac{2}{r}\,(\alpha' + \alpha\, r) + \Big(\frac{3}{2} - \frac{1}{r}\Big)(\theta' + \theta\, r) - 2 + \frac{1}{r} \,. \tag{45}$$

Lastly, by appealing to (44) and (45), we realize, in agrement to our prediction but also with some disappointment, that the exponent β obtained here does not depend on r. In fact one gets

$$\beta = \frac{1}{2} \,. \tag{46}$$

5. **Proof of Proposition 1.**

Proof. By setting $\widehat{r} = 2$ in (32) it follows that $q' = \frac{6}{3+2\,\beta}$. Hence, from (31),

$$\frac{d}{dt}\|\omega\|_r^r + \|\omega\|_{3\,r}^r \leq c\, \|\omega\|_2 \, \|\omega\|^r_{\frac{6}{3+2\,\beta}\, r} \,. \tag{47}$$

Further, by interpolation, one gets

$$\|\omega\|_{\frac{6}{3+2\,\beta}\, r} \leq \|\omega\|_r^\nu \,, \|\omega\|_{3\,r}^{1-\nu} \,,$$

where $\nu = \frac{1+2\,\beta}{4}$. Hence

$$\frac{d}{dt}\|\omega\|_r^r + \|\omega\|_{3r}^r \leq c\,\Big(\|\omega\|_2 \, \|\omega\|_r^{\frac{1+2\,\beta}{4}\, r}\Big)\, \|\omega\|_{3\,r}^{\frac{3-2\,\beta}{4}\, r} \,.$$

Next, by Young's inequality,

$$\frac{d}{dt}\|\omega\|_r^r + \|\omega\|_{3r}^r \leq C_\varepsilon\, \|\omega\|_2^{\frac{4}{1+2\,\beta}} \, \|\omega\|_r^r + \|\omega\|_{3r}^r \,, \tag{48}$$

for arbitrary $\varepsilon > 0$. Since the exponent $\frac{4}{1+2\,\beta}$ is independent of r, the strongest result is obtained by setting $r = 2$. $\square$

6. **On the "equal strength" of distinct assumptions.** In a previous version of these notes we have introduced a notion of *equal strength*, suitable to compare distinct assumptions (see also the appendix of reference [6]). Roughly, it was a kind of "continuity method with respect to a parameter", where real continuity is replaced by the formal independence of proofs with respect to parameters (a clearly non rigorous concept). For instance, since the proof of the Ladyzhenskaya-Prodi-Serrin sufficient condition for regularity is, formally, totally independent of the particular values of the parameters s and q, we say that all the L-P-S sufficient conditions for regularity have the same strength (clearly, the limit case $(\infty, 3)$, see [23], is out of the above criterium of equal strength). We have applied the above criterium of equal strength, based on independence of proofs with respect to parameters, to other similar situations. We also included other two different situations leading to "equal strength" claims, namely when two distinct sets of conditions, depending on parameters, have an element in common, or when a sharp Sobolev's embedding theorem allows an equal strength claim. With this idea in hands, we have deduced that *all* the regularity results refereed in the present notes have the same strength. In particular, the $\beta = \frac{1}{2}$ regularity assumption has the same strength as any of the L-P-S conditions for regularity. Clearly, we can not expect a "strong" equivalence between the integral assumption and the pointwise assumption. However, the equivalence conclusion claimed by us, is sufficient to alert authors that to prove smoothness of solutions (if true) under a $\beta < \frac{1}{2}$ assumption should be a quite hard matter.

7. **Some particularly related known results.** In this section we limit us to describing some results which are strongly related to the author's approach, and to the present notes, by methods of proof. Other main references are given at the end of the section, without any claim of completness. We begin by recalling the fundamental pioneering paper [18], by P. Constantin and Ch. Fefferman, where the authors prove, in particular, that solutions to the evolution Navier-Stokes equations in the whole space are smooth if the direction of the vorticity is Lipschitz continuous with respect to the space variables, namely assumption (6) for $\beta = 1$. This condition is assumed for almost all x and y in R^3, and almost all t in $(0, T)$. Actually, in [18], the assumption is merely required for points x and y where the vorticity at both x and y is larger than a given, arbitrary constant k. This improvement was, or can be, extended in the same way to many subsequent papers on the subject. It is also easily show that assumption (6) can be restricted to couples of points x and y satisfying $|x - y| < \delta$, for an arbitrary positive constant δ.

In reference [8] L.C. Berselli and the author showed, in particular, that regularity still holds in the whole space by replacing Lipschitz continuity by $\frac{1}{2}-$Hölder continuity. This is, up to now, the strongest result in the literature. Actually, the above reference has been a fundamental basis to the subsequent papers by the present author. In reference [3] the result reported at the end of section 1, see (9), was proved.

Concerning other related papers, we start by recalling reference [4] where the above kind of results is extended to the half-space $\Omega = \mathbb{R}^3_+$, under the "stress-free" slip boundary condition

$$\begin{cases} u \cdot n = 0\,, \\ \omega \times n = 0\,, \end{cases} \tag{49}$$

where n is normal to the boundary. In reference [9], L.C. Berselli and the author succeed in extending this result to the case in which $\Omega \subset \mathbb{R}^3$ is an open, bounded set with a smooth boundary, by appealing to suitable representation formulas for Green's matrices. In reference [10] regularity is proved by replacing continuity requirements on $\sin\theta(x,y,t)$ by a smallness assumption. Essentially, it is proved that there is a sufficiently small constant C_1 such that regularity holds if $\sin\theta(x,y,t) \leq C_1$. Clearly, there are many very interesting papers, even crucial papers, related to the present contribution. We recall here, without any claim of completeness, references [5], [6], [11], [12], [13], [14],[15], [16], [17], [19], [20], [21], [22], [25], [26], [27], [28], [29], [30] [31], [32], [34].

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