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Direction of Vorticity and Regularity up to the Boundary: On the Lipschitz-Continuous Case

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Abstract. In their famous 1993 paper, Constantin and Fefferman consider the evolution Navier–Stokes equations in the whole space R^3 and prove, essentially, that if the direction of the vorticity is Lipschitz continuous in the space variables, during a given time-interval, then the corresponding solution is regular. Since Lipschitz-continuity is a very natural, basic, property, it looks interesting to go further in this particular direction. In this paper, we consider the initial-boundary value problem for the Navier–Stokes equations in a regular, bounded, domain under a slip boundary condition, and prove regularity of the solution, *up to the boundary*, under a *weakened* Lipschitz-continuity assumption on the direction of the vorticity. The interest of our result highly relies on the fact that the Lipschitz-continuity coefficient $g(x, t)$ is *sharp*. This means, in a sense, that our finding possesses the same level of accuracy as that of the classical "Prodi-Serrin" type conditions; see the introductory section. It should be remarked that a similar result was already obtained in the 2009 paper by Beirão da Veiga and Berselli. In the latter, the proof of an analogous sharp result was shown under the assumption of $\frac{1}{2}$-Hőlder continuity on the direction of vorticity. The authors also claimed, correctly, that by the same ideas the proof of such a result could be extended to Hőlder exponents $\beta \in]\,0,\,1\,]$. However the proofs would be extremely involved. On the contrary, the proof followed in this paper treat the Lipschitz case is definitely more elementary than any other proof, even if restricted to the whole space case.

1. Introduction and Statement of the Main Result

In this paper we consider the initial value problem for the 3D Navier–Stokes equations

$$\begin{cases} u_t + (u\cdot\nabla)\,u - \triangle u + \nabla p = 0 & \text{in } \Omega\times]0,T], \\ \nabla\cdot u = 0 & \text{in } \Omega\times]0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where the unknowns are the velocity u and the pressure p. For simplicity, we assume that the external force vanishes and set the kinematic viscosity equal to 1. The open, bounded, set $\Omega \subset R^3$ has a smooth boundary $\partial\Omega$, say of class $C^{2,\alpha}$, for some $\alpha > 0$.

We supplement the initial value problem with the following "stress-free" boundary conditions

$$\begin{cases} u\cdot n = 0 & \text{on } \partial\Omega\times]0,T], \\ \omega\times n = 0 & \text{on } \partial\Omega\times]0,T], \end{cases} \tag{2}$$

where $\omega = \operatorname{curl} u$ is the vorticity field, while n denotes the exterior unit normal vector. In the case of flat boundaries, the above conditions coincide with the classical Navier boundary conditions without friction; see, e.g., the classical reference Serrin [17].

In the present paper we consider the problem of the regularity, up to the boundary, of weak solutions as a consequence of a (particularly meaningful non-uniform) Lipschitz continuity assumption on the vorticity-direction.

By $\theta(x,y,t)$ we shall denote the angle between the vorticity ω at two distinct points x and y, at the same time t:

$$\theta(x,y,t) \stackrel{def}{=} \angle(\omega(x,t),\omega(y,t)). \tag{3}$$

Clearly

$$\sin\theta(x,y,t) = \frac{|\,\omega(x,t)\times\,\omega(y,t)\,|}{|\omega(x,t)|\,|\omega(y,t)|}. \tag{4}$$

$L^p := L^p(\Omega)$, $1 \le p \le \infty$ denotes the usual Lebesgue spaces equipped with norm $\|.\|_p$. Further, $H^k := H^k(\Omega)$, are the classical Sobolev spaces. We use the same symbol for both scalar and vector function spaces, and set $\partial_i = \frac{\partial}{\partial x_i}$. Moreover

$$L^p_T(X) \stackrel{def}{=} L^p(0,\,T;\,X(\Omega)\,),$$

where $X = X(\Omega)$ is a generic Banach space, and $1 \le p \le \infty$. Arbitrary positive constants are denoted simply by c. We say that a Leray-Hopf weak solution u is strong if $u \in L^\infty(0,\,T;\,H^1(\Omega) \cap L^2(0,\,T;\,H^2(\Omega)\,)$. It is well known that strong solutions are regular.

We start with an overview on some results particularly related to the one we are going to prove. We assume everywhere that the initial data $u_0 \in H^1(\Omega)$ is divergence free. We denote by u a weak solution to (1)–(2) in $[0,T]$. The meaning of Ω (open bounded set, whole space, or half-space)will be clear from the particular statement under consideration.

The study of conditions involving the direction of vorticity, and its physical-geometric interpretation, started with Constantin and Fefferman celebrated paper [11], who first derived some exact formulas and employed them in order to prove regularity in the whole of R^3. In [11] the authors show that if

$$\sin\theta(x,y,t) \le\, g(t)\,|x-y|, \qquad \text{a.e. } x,y\in R^3,\ \text{a.e. } t\in]0,T[, \tag{5}$$

for some $g(t) \in L^{12}(0,\,T;\,L^\infty(\Omega)\,)$, then the solution u is strong. The above result has been improved in reference [5], by replacing the Lipschitz condition by a 1/2-Hölder condition. If

$$\sin\theta(x,y,t) \le c\,|x-y|^{1/2}, \qquad \text{a.e. } x,y\in R^3,\ \text{a.e. } t\in]0,T[, \tag{6}$$

then the solution u is strong. The proof employs a number of fundamental ideas introduced in [11]. Actually, the authors of [5] consider a family of sufficient conditions which contain (6) as a particular case. A weak solution u is a strong solution if there exists $\beta\in[1/2,1]$ and $g\in L^a(0,T;L^b(R^3))$,

$$\frac{2}{a}+\frac{3}{b} = \beta-\frac{1}{2} \quad \text{with} \quad a\in\left[\frac{4}{2\beta-1},\infty\right], \tag{7}$$

such that

$$\sin\theta(x,y,t) \le g(x,\,t)|x-y|^\beta, \qquad \text{a.e. } x,y\in R^3,\ \text{a.e. } t\in]0,T[. \tag{8}$$

More recently, in [2], we extended the 1/2-Hölder condition in the whole of R^3 to solutions to the boundary value problem (2) in the half-space case by showing that if for some $\beta\in]0,1/2]$

$$\sin\theta(x,y,t) \le c|x-y|^\beta, \qquad \text{a.e. } x,y\in \Omega,\ \text{a.e. } t\in]0,T[,$$

and, in addition,

$$\omega\in L^2(0,T;L^s(\Omega)), \qquad \text{with } s=\frac{3}{\beta+1}, \tag{9}$$

then u is a strong solution. This result is proved by means of a different method with respect to that used in all previously referenced articles. In fact, to treat the presence of the boundary we exploit directly the classical Dirichlet and Neumann Green's functions in the half space. This can be done since, for flat boundaries, conditions (2) may be split as follows:

$$\omega_1 = \omega_2 = 0\,; \quad \frac{\partial\,\omega_3}{\partial\,x_3} = 0. \tag{10}$$

Note that in [5] the advantage of assuming $\beta > \frac{1}{2}$ is counterbalanced by replacing in (8) the constant c by a function $g\in L^a(0,T;L^b(\Omega))$. On the other hand, in [2] we mitigate the penalizing situation $\beta < \frac{1}{2}$ by assuming (9).

The fact that condition (6) alone implies regularity may be viewed as a particular case of each of the two families of results formerly recalled. In fact, if we consider $\beta = \frac{1}{2}$ as a particular case of $\beta \in [\frac{1}{2}, 1]$, if follows by (8) that $a = b = \infty$, which corresponds to the constant c in Eq. (6). On the other hand, if we consider $\beta = \frac{1}{2}$ as a particular case of $\beta \in [0, \frac{1}{2}]$, regularity holds since, by (9) it follows $s = 2$, and weak solutions necessarily satisfy $\omega \in L^2(0,T;L^2(\Omega))$. This argument shows that the two families of results, from above and from below, match perfectly at $\beta = \frac{1}{2}$. On the other hand, going through the proofs of the above two families of results, given in [6] and [2], respectively, one finds that the single results obtained for the distinct values $\beta \in]0, 1]$ have the same formal level of "strength". Furthermore, in the borderline case $\beta = 0$, the Hölder's condition disappears, and (9) simply reads $\omega \in L^2(0,T;L^3(\Omega))$. As shown in [1], this result corresponds exactly to the classical, so called, "Prodi-Serrin" conditions. These facts lead us to call the above family of β dependent results, $\beta \in [0, 1]$ *sharp results*. In particular the borderline Lipschitz-continuous case, which states that regularity holds if

$$\sin\theta(x,y,t) \le g(x,t)|x-y|, \qquad \text{a.e. } x,y \in R^3, \text{ a.e. } t \in]0,T[, \tag{11}$$

for some $g \in L^a(0,T;L^b(R^3))$, where

$$\frac{2}{a} + \frac{3}{b} = \frac{1}{2} \quad \text{with} \quad a \in [4, \infty], \tag{12}$$

is a sharp result. In the sequel we show a very elementary proof of this last result, up to the boundary, under the slip boundary condition (see the Theorem 1.2 below). In this respect, it is worth noting that in reference [6] the following result was claimed.

Theorem 1.1 ([6]). *Let $\Omega \subset R^3$ be an open, bounded set with a smooth boundary $\partial\Omega$, say of class $C^{3,\alpha}$, for some $\alpha > 0$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla \cdot u_0 = 0$, and u is a weak solution to (1)–(2) in $[0,T]$. In addition, suppose either that there exist $\beta \in [1/2, 1]$ and $g \in L^a(0,T;L^b(\Omega))$, with a, b given in (7), such that*

$$\sin\theta(x,y,t) \le g(x,t)|x-y|^\beta, \qquad a.e.\ x,y \in \Omega,\ a.e.\ t \in]0,T[,$$

or that there exists $\beta \in]0, 1/2]$ such that

$$\sin\theta(x,y,t) \le c|x-y|^\beta, \qquad a.e.\ x,y \in \Omega,\ a.e.\ t \in]0,T[,$$

and that (9) holds. Then, the solution u is strong in $[0,T]$, and hence smooth. In particular, this result holds if

$$\sin\theta(x,y,t) \le c\,|x-y|^{1/2}, \qquad a.e.\ x,y \in \Omega,\ a.e.\ t \in]0,T[. \tag{13}$$

However, in [6], the proof is presented only for the main case $\beta = \frac{1}{2}$. Since the boundary is not flat, one has to localize the problem, a not trivial and quite technical matter. Moreover, one can not use separately the classical Dirichlet and Neumann Green's functions, as in the half space case, since the boundary conditions do not split as in (10). So, in their proof, the authors employ the representation formulas for Green's matrices derived in Solonnikov's fundamental work [18,19]. With the aid of these explicit formulas, original local representation formulas for the velocity (in terms of the vorticity) were introduced and, as a result, useful estimates for the vortex stretching terms were proved. The proof is particularly involved. Its extension to all the values of the parameter β considered in references [5] and [2], namely $\beta \in (0, 1]$, is feasible, but hard. This led us to look for a much simpler proof under the original, and mathematically quite natural, Constantin and Fefferman's Lipschitz condition $\beta = 1$. Actually, our proof furnishes regularity up to the boundary under the slip boundary condition, and allow a sharp coefficients $g(x,t)$ as in (8). More precisely, we present a really straightforward proof of the following result.

Theorem 1.2. *Let $\Omega \subset R^3$ be an open, bounded set with a smooth boundary $\partial\Omega$, say, of class $C^{2,\alpha}$, for some $\alpha > 0$. Suppose that $u_0 \in H^1(\Omega)$, $\nabla \cdot u_0 = 0$, and u is a weak solution to (1)–(2) in $[0,T]$. In addition, suppose that there exists $g \in L^a(0,T;L^b(\Omega))$, where*

$$\frac{2}{a} + \frac{3}{b} = \frac{1}{2}, \quad \textit{with} \quad a \in [4, \infty], \tag{14}$$

and a positive $\delta(x,\,t)$*, such that, for almost all* $t\in]0,T[$*, one has*

$$\sin\theta(x,y,t)\le g(x,\,t)\,|y-\,x|, \tag{15}$$

for a.a. $x,y\in\Omega$*, satisfying* $|y-\,x|<\,\delta(x,\,t)$*. Then, the solution* u *is a strong solution, and hence it is smooth, in* $[0,T]$*.*

For instance, $g\in L^4(0,T;L^\infty(\Omega))$, or $g\in L^\infty(0,T;L^6\Omega))$, are sufficient for regularity.

Actually, the Theorem 1.2 is a little more more general then that stated for $\beta=1$ in Theorem 1.1, since it does not require a positive lower bound for $\delta(x,\,t)$). The present result also extends, and simplifies, the classical proof given in [11].

We conclude this section with a bibliographical remark. Since the 1993 Constantin and Fefferman pioneering paper [11], many other interesting papers on the relation between direction of vorticity and regularity for 3-D Navier–Stokes equations appeared. After the 2002 improvement [5], and besides the works already referred above, see, for instance, [3,7,9,10,12–16,20], and [21]. In [4] and [8] suitable conditions on the angle between velocity and vorticity are studied.

2. A Dimensional Analysis

In this section, by simple dimensional analysis techniques, we show that the sufficient condition for regularity described in the Theorem 1.2 (where $\beta=1$) and the main sufficient condition (6) (where $\beta=\frac{1}{2}$) enjoy the same "strength". These two conditions are included in condition (8), the first for $\beta=1$, the last for $\beta=\frac{1}{2}$. Roughly speaking they are continuously connected by condition (8), as β moves from $\frac{1}{2}$ to 1. In the following, we show that all this family of results enjoy the same "strength".

Assume that $(\,(u(x,\,t),\,p(x,\,t)\,)$ is a solution to the Navier–Stokes equations in $]\,0,\,+\infty\,[\times R^3$. Then

$$(\,(u_\lambda(x,\,t),\,p_\lambda(x,\,t)\,)\equiv(\,(\,\lambda\,u(\,\lambda x,\,\lambda^2 t),\,\lambda^2 p(\,\lambda x,\,\lambda^2 t)\,)$$

is a solution in the same domain. In particular

$$\omega_\lambda(x,\,t)\equiv\,\mathrm{curl}\;u_\lambda(x,\,t)=\,\lambda^2\,\omega(\,\lambda x,\,\lambda^2 t).$$

If

$$\theta_\lambda(x,y,t)\overset{def}{=}\angle(\omega_\lambda(x,t),\omega_\lambda(y,t))$$

then, by (4),

$$\sin\theta_\lambda(x,y,t)=\,\sin\theta(\lambda\,x,\lambda\,y,\lambda^2\,t).$$

Assume now that the solution $u(x,t)$ satisfies (8) (here $T=+\infty$) for some $\beta=\frac{1}{2}$, where $g\in L^a(0,\,+\infty\,;L^b(R^3))$, and (7) holds. Then (as the reader will check with no pain)

$$\sin\theta_\lambda(x,y,t)\le g_\lambda(x,\,t)|x-y|^\beta,$$

where the function g_λ, given by

$$g_\lambda(x,\,y)\overset{def}{=}\lambda^\beta\,g(\,\lambda x,\,\lambda^2 t),$$

belong to the functional space $L^a(0,\,+\infty\,;L^b(R^3))$, moreover

$$\|\,g_\lambda\,\|_{L^a(\,0,\,+\infty;\,L^b(R^3)\,)}=\,\lambda^{\frac{1}{2}}\,\|\,g\,\|_{L^a(\,0,\,+\infty;\,L^b(R^3)\,)}. \tag{16}$$

Our claim of "equivalent strength" follows here from the fact that the exponent $\frac{1}{2}$ in Eq. (16) does not depend on the particular value $\beta\in\,[\frac{1}{2},\,1]$. Weaker (resp. stronger) sufficient conditions lead to larger (resp. smaller) exponents.

3. Preliminaries

The following argument is well know, so that we leave the details to the reader. The assumption $u(0) = u_0 \in H^1(\Omega)$, the existence of a corresponding unique, smooth, solution on a positive time interval $]\,t_0, t_0 + \epsilon\,[$, if $u(t_0) \in H^1(\Omega)$, and the higher regularity in $]\,0, t_0\,] \times \Omega$ of weak solutions $u \in L^\infty(0, t_0; H^1(\Omega))$, allow us prove Theorem 1.2 merely by showing that smooth solutions in $]\,0, t_0\,[$, which enjoy the hypotheses of the theorem in the interval $]\,0, t_0\,[$, necessarily satisfy $\omega \in L^\infty(0, t_0; L^2(\Omega))$. This shows that $u \in L^\infty(0, t_0; H^1(\Omega))$ (see, for instance, the Lemma 2.7 in [6]). In the following we replace, without loosing generality, t_0 by T.

We shall often drop in the notation the time variable t, since time is assumed to be frozen. We start by recalling some results proved in [6].

Denote by ϵ_{ijk} the components of the totally anti-symmetric Ricci tensor.

The following result was proved in [6], Lemma 2.2. We repeat here the proof.[1]

Lemma 3.1. *Assume that u is divergence-free, that $u \cdot n = 0$, and that $\omega \times n = 0$ on $\partial\Omega$. Then*

$$- \frac{\partial \omega}{\partial n} \cdot \omega = (\epsilon_{1jk}\, \epsilon_{1\beta\gamma} + \epsilon_{2jk}\, \epsilon_{2\beta\gamma} + \epsilon_{3jk}\, \epsilon_{3\beta\gamma})\, \omega_j\, \omega_\beta\, \partial_k n_\gamma. \tag{17}$$

In particular,

$$- \int_\Omega \triangle\omega \cdot \omega\, dx \geq \int_\Omega |\nabla \omega|^2\, dx - c \int_{\partial\Omega} |\omega|^2\, dS. \tag{18}$$

Proof. The vorticity ω is parallel to the normal unit vector on $\partial\Omega$. Hence $\partial_\tau(\omega \times n) = 0$, for each vector field τ tangential to the boundary. Since on the boundary ω is orthogonal to tangent vectors, it follows that $\omega \times \nabla[\,(\omega \times n)_i\,] \equiv 0$ for $i = 1, 2, 3$, on $\partial\Omega$. In more explicit coordinates we can write, for i, $\alpha = 1, 2, 3$,

$$\epsilon_{ijk}\, \epsilon_{\alpha\beta\gamma}\, \omega_j\, \partial_k(\omega_\beta\, n_\gamma) = 0, \qquad \text{on } \partial\Omega. \tag{19}$$

Hence, by considering Eq. (19) for (i, α) equal to $(1, 1)$, $(2, 2)$, and $(3, 3)$ we get, respectively:

$$\begin{cases} n_3\, \omega_2\, \partial_3\omega_2 + n_2\, \omega_3\, \partial_2\omega_3 - n_2\, \omega_2\, \partial_3\omega_3 - n_3\, \omega_3\, \partial_2\omega_2 + \epsilon_{1jk}\, \epsilon_{1\beta\gamma}\, \omega_j\, \omega_\beta\, \partial_k n_\gamma = 0, \\ n_1\, \omega_3\, \partial_1\omega_3 + n_3\, \omega_1\, \partial_3\omega_1 - n_3\, \omega_3\, \partial_1\omega_1 - n_1\, \omega_1\, \partial_3\omega_3 + \epsilon_{2jk}\, \epsilon_{2\beta\gamma}\, \omega_j\, \omega_\beta\, \partial_k n_\gamma = 0, \\ n_2\, \omega_1\, \partial_2\omega_1 + n_1\, \omega_2\, \partial_1\omega_2 - n_1\, \omega_1\, \partial_2\omega_2 - n_2\, \omega_2\, \partial_1\omega_1 + \epsilon_{3jk}\, \epsilon_{3\beta\gamma}\, \omega_j\, \omega_\beta\, \partial_k n_\gamma = 0. \end{cases} \tag{20}$$

Next, by adding term-by-term, Eq. (20) together with

$$\begin{aligned} &(n_2\, \omega_2\, \partial_2\, \omega_2 - n_2\, \omega_2\, \partial_2\, \omega_2) + (n_3\, \omega_3\, \partial_3\, \omega_3 - n_3\, \omega_3\, \partial_3\, \omega_3) \\ &\quad + (n_1\, \omega_1\, \partial_1\, \omega_1 - n_1\, \omega_1\, \partial_1\, \omega_1) = 0, \end{aligned}$$

we show that

$$n_i\, \omega_k\, \partial_i\omega_k - (\omega_i\, n_i)(\partial_k\omega_k) + (\epsilon_{1jk}\, \epsilon_{1\beta\gamma} + \epsilon_{2jk}\, \epsilon_{2\beta\gamma} + \epsilon_{3jk}\, \epsilon_{3\beta\gamma})\, \omega_j\, \omega_\beta\, \partial_k n_\gamma = 0$$

on $\partial\,\Omega$. Finally, since $\nabla \cdot \omega = 0$ we show that (17) holds. In particular, Eq. (18) follows as a consequence of the well known Green's formula

$$- \int_\Omega \triangle\omega \cdot \omega\, dx = \int_\Omega |\nabla \omega|^2\, dx - \int_{\partial\Omega} \frac{\partial \omega}{\partial n} \cdot \omega\, dS, \tag{21}$$

since (17) shows that there exists $c = c(\Omega) > 0$ such that

$$\left| \frac{\partial \omega(x)}{\partial n} \cdot \omega(x) \right| \leq c\, |\omega(x)|^2, \qquad \forall\, x \in \partial\Omega. \tag{22}$$

□

We next observe that (17) immediately leads to the following result.

[1] We assume summation over repeated indices.

Lemma 3.2. *Under the assumptions of Lemma 3.1, it follows that*

$$- \frac{\partial \omega}{\partial n} \cdot \omega = \kappa_2 \,|\, \omega_1 \,|^2 + \kappa_1 \,|\, \omega_2 \,|^2 \tag{23}$$

on Γ. *Here* $\kappa_j, j = 1, 2$, *denote the principal curvatures, and the* ω_j *are the coordinates of* ω *with respect to the* τ_j, *the unit tangent vectors to the principal directions. In particular, if* Ω *is convex, the boundary integral in Eq. (21) is less or equal to zero.*

Proof. Note that, with respect to the orthogonal system of coordinates $\{\tau_1,\, \tau_2,\, n\}$, the third component ω_3 vanishes.

The three "ϵ-terms" on the right-hand side of (17) have the same common value $\delta_{j\,\beta}\, \delta_{k\,\gamma} - \delta_{j\,\gamma}\, \delta_{k\,\beta}$. So

$$- \frac{\partial \omega}{\partial n} \cdot \omega = 3 \left(|\, \omega \,|^2 \, \partial_k n_k - (\, \partial_k n_j \,)\, \omega_k \, \omega_j \,\right). \tag{24}$$

Denote by ∇n the 3×3, self-adjoint, matrix with entries $\partial_k n_\gamma$. With respect to the system of coordinates $\{\tau_1,\, \tau_2,\, n\}$, defined in a suitable neighborhood of Γ, the matrix ∇n is diagonal, with diagonal elements given by $\kappa_1,\, \kappa_2,\, 0$. So (23) follows from (24). □

Next, by applying the curl operatoron both sides of (1) we get the well-known equation

$$\begin{cases} \omega_t + (u \cdot \nabla)\, \omega - \Delta \omega = (\omega \cdot \nabla)\, u & \text{in } \Omega \times]0, T], \\ \nabla \cdot \omega = 0 & \text{in } \Omega \times]0, T]. \end{cases} \tag{25}$$

By taking the scalar product of both sides of the first Eq. (25) with ω, by integrating in Ω, and by appealing to Eq. (18) one gets (see [6], Lemma 2.6)

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\omega|^2 dx + \int_\Omega |\nabla \omega|^2 \, dx \leq c \int_{\partial \Omega} |\omega|^2 \, dS + \left| \int_\Omega (\omega \cdot \nabla) u \cdot \omega \, dx \right|. \tag{26}$$

This easily leads to the main estimate (see [6], equation (23))

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\omega|^2 dx + \frac{1}{2} \int_\Omega |\nabla \omega|^2 \, dx \leq c(\Omega) \int_\Omega |\omega|^2 dx + \left| \int_\Omega (\omega \cdot \nabla) u \cdot \omega \, dx \right|. \tag{27}$$

For an alternative proof see the Appendix.

The next section is dedicated to estimate the last integral on the right-hand side of Eq. (27).

4. Proof of Theorem 1.2

Lemma 4.1. *Under the assumption (15), for almost all* $x \in \Omega$ *one has*

$$|\, \omega_l \, \partial_j \, \omega_k - \omega_k \, \partial_j \, \omega_l \,| \leq g(x,\, t) \,|\, \omega \,|^2, \tag{28}$$

for each triplet of indexes $\{i,\, j,\, k\}$.

Proof. By setting $y = x + h$, where $h < \delta(t, x)$, from assumption (15) one shows that for almost all $x \in \Omega$,

$$|\, \omega(x) \times \omega(x + h) \,| \leq g(x,\, t) \,|\, h \,|\, |\, \omega(x) \,|\, |\, \omega(x + h) \,|,$$

for sufficiently small h. So,

$$\left| \omega(x) \times \frac{\omega(x + h) - \omega(x)}{|\, h \,|} \right| \leq g(x,\, t) \,|\, \omega(x) \,|\, |\, \omega(x + h) \,|. \tag{29}$$

In particular, by letting $h \to 0$, one gets

$$|\,\omega(x) \times \partial_j\, \omega(x)\,| \leq g(x,\, t)\,|\,\omega(x)\,|^2, \tag{30}$$

for each $j = 1, 2, 3$. This implies that (28) holds for $k \neq l$. For $k = l$ the result is obvious. □

Lemma 4.1 allows us to get a simple bound for the second integral on the right-hand side of (27). In fact,

$$\int_\Omega (\omega \cdot \nabla)u \cdot \omega\, dx = -\int_\Omega (\partial_j\, \omega_k)\, \omega_j\, u_k\, dx - \int_\Omega (\nabla \cdot \omega)\,(u \cdot \omega)\, dx + \int_{\partial\Omega} (\omega \cdot u)\,(\omega \cdot n)\, d\Gamma.$$

Hence

$$\int_\Omega (\omega \cdot \nabla)u \cdot \omega\, dx = -\int_\Omega \Big((\partial_j\, \omega_k)\, \omega_j - (\partial_j\, \omega_j)\, \omega_k \Big)\, u_k\, dx.$$

So, from (28) one gets

$$\Big|\int_\Omega (\omega \cdot \nabla)u \cdot \omega\, dx\Big| \leq \int_\Omega g(x,\, t)\, |u|\, |\omega|^2\, dx.$$

By Hőlder's inequality

$$\Big|\int_\Omega (\omega \cdot \nabla)u \cdot \omega\, dx\Big| \leq \|\, g\,\|_{\frac{6}{\alpha}}\, \|\, u\,\|_{\frac{6}{2-\alpha}}\, \|\,\omega\,\|_2\, \|\,\omega\,\|_6,$$

for each $\alpha \in [0, 1]$. Hence, by (27), the immersion $H^1(\Omega) \subset L^6(\Omega)$, and the interpolation inequality

$$\|\, u\,\|^2_{\frac{6}{2-\alpha}} \leq \|\, u\,\|_6^{1+\alpha}\, \|\, u\,\|_2^{1-\alpha},$$

one gets

$$\frac{1}{2}\, \frac{d}{dt} \|\,\omega\|_2^2 + \frac{1}{4}\, \|\nabla\omega\|_2^2 \leq c \left(1 + \|\, g\,\|^2_{\frac{6}{\alpha}}\, \|\, u\,\|_6^{1+\alpha}\, \|\, u\,\|_2^{1-\alpha} \right) \|\,\omega\,\|_2^2. \tag{31}$$

Using one more time Hőlder's inequality, now with respect to the time variable, we show that

$$\int_0^T \left(\|\, g(t)\,\|^2_{\frac{6}{\alpha}}\, \|\, u(t)\,\|_6^{1+\alpha}\, \|\, u(t)\,\|_2^{1-\alpha} \right) dt \leq \|\, g\,\|^2_{L^{\frac{4}{1-\alpha}}(0,T;\, L^{\frac{6}{\alpha}}(\Omega))}\, \|\, u\,\|^{1+\alpha}_{L^2(0,T;\, L^6(\Omega))}\, \|\, u\,\|^{1-\alpha}_{L^\infty(0,T;\, L^2(\Omega))}. \tag{32}$$

If we set

$$a = \frac{4}{1-\alpha}, \qquad b = \frac{6}{\alpha},$$

we then get

$$\int_0^T \left(\|\, g(t)\,\|^2_{\frac{6}{\alpha}}\, \|\, u(t)\,\|_6^{1+\alpha}\, \|\, u(t)\,\|_2^{1-\alpha} \right) dt \leq \|\, g\,\|^2_{L^a(0,T;\, L^b)}\, \|\, u\,\|^{2-\frac{4}{a}}_{L^2(0,T;\, L^6(\Omega))}\, \|\, u\,\|^{\frac{4}{a}}_{L^\infty(0,T;\, L^2(\Omega))}, \tag{33}$$

where now the parameters a and b are related by (14).

From (31) and (33) it follows that

$$\|\,\omega\,\|^2_{L^\infty(0,T;\, L^2(\Omega))} \leq \|\,\omega(0)\|^2 \exp\left\{ c \left(t + \|\, g\,\|^2_{L^a(0,T;\, L^b)}\, \|\, u\,\|^{2-\frac{4}{a}}_{L^2(0,T;\, L^6(\Omega))}\, \|\, u\,\|^{\frac{4}{a}}_{L^\infty(0,T;\, L^2(\Omega))} \right) \right\}. \tag{34}$$

The boundedness of u in $L^\infty(\,0,\,T\,;\,H^1(\Omega)\,)$, hence regularity in $]\,0,\,T\,]$, follows. Theorem 1.2 is proved.

Turning back to (26), and employing to (34), we get a suitable estimate for $\|\,\nabla\,\omega\,\|^2_{L^2(\,0,\,T\,;\,L^2(\Omega)\,)}$. Estimates for $\|\,u\,\|^2_{L^\infty(\,0,\,T\,;\,H^1(\Omega)\,)}$ and $\|\,u\,\|^2_{L^2(\,0,\,T\,;\,H^2(\Omega)\,)}$ follow easily.

5. Appendix

Here we give an alternative proof of (26). Set $f =$ curl ω and $g = \omega$ in the identity

$$\int_\Omega (\mathrm{curl}\; f)\cdot g\,dx = \int_\Omega f\cdot(\mathrm{curl}\; g\,)\,dx + \int_\Gamma (n\times\, f\,)\,\cdot\, g\,d\Gamma. \tag{35}$$

By taking into account that the mixed product $(\underline{n}\times$ curl $\underline{\omega})\,\cdot\,\underline{\omega}$ vanishes on Γ, it readily follows that

$$-\int_\Omega (\Delta\,\omega\,)\cdot\,\omega\,dx = \int_\Omega |\,\mathrm{curl}\;\omega\,|^2\,dx,$$

where we have used the identity

$$\Delta\,\omega = -\mathrm{curl}\;\mathrm{curl}\;\omega.$$

So, by dot-multiplying both sides of (25) by $\underline{\omega}$, and integrating by parts over Ω, we immediately infer that

$$\frac{1}{2}\,\frac{d}{d\,t}\|\,\omega\|_2^2 + \|\,\mathrm{curl}\;\omega\,\|_2^2 = \int_\Omega \big(\,(\,\omega\,\cdot\,\nabla)\,u\,\big)\,\cdot\,\omega\,dx. \tag{36}$$

As $\nabla\cdot\omega = 0$ in Ω and $\omega\times n = 0$ on Γ, one has $\|\,\nabla\,\omega\,\|_2^2 \leq c\,(\|\,\mathrm{curl}\;\omega\,\|_2^2 + \|\,\omega\,\|_2^2\,)$. Equation (26) follows.

An interesting, related, estimate follows by starting from the well-known equation

$$\omega_t - \Delta\omega + \mathrm{curl}\,(\omega\times\,u\,) = 0.$$

Scalar multiplication by ω, followed by integration over Ω plus suitable integration by parts lead to

$$\frac{1}{2}\,\frac{d}{d\,t}\|\,\omega\|_2^2 + \|\,\mathrm{curl}\;\omega\,\|_2^2 = \int_\Omega u\,\times\,\omega\,\cdot\,\mathrm{curl}\;\omega\,dx, \tag{37}$$

since $n\,\times\,(\omega\times\,u\,)\cdot\,\omega = 0$.

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