Volume 75 Number 11 July 2012 ISSN 0362-546X

ELSEVIER

Nonlinear Analysis

Theory, Methods & Applications

An International Multidisciplinary Journal

Series A: Theory and Methods

Available online at www.sciencedirect.com
SciVerse ScienceDirect

Nonlinear Analysis 75 (2012) 4346–4354

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the global $W^{2,q}$ regularity for nonlinear N-systems of the p-Laplacian type in n space variables

H. Beirão da Veiga[a,*], F. Crispo[b]

[a] Dipartimento di Matematica Applicata "U. Dini", Università di Pisa, Via Buonarroti 1/C, 56127 Pisa, Italy
[b] Dipartimento di Matematica, Seconda Università degli Studi di Napoli, Via Vivaldi 43, 81100 Caserta, Italy

ARTICLE INFO

Article history:
Received 18 October 2011
Accepted 16 March 2012
Communicated by S. Carl

Keywords:
p-Laplacian systems
Regularity up to the boundary
Full regularity

ABSTRACT

We consider the Dirichlet boundary value problem for nonlinear N-systems of partial differential equations with p-growth, $1 < p \leq 2$, in the n-dimensional case. For clearness, we confine ourselves to a particularly representative case, the well known p-Laplacian system.

We are interested in regularity results, *up to the boundary*, for the second order derivatives of the solution. We prove $W^{2,q}$-global regularity results, for arbitrarily large values of q. In turn, the regularity achieved implies the Hölder continuity of the gradient of the solution. It is worth noting that we cover the singular case $\mu = 0$. See Theorem 2.1.

1. Introduction

We are concerned with the regularity problem for solutions of nonlinear systems of partial differential equations with p-structure, $p \in (1, 2]$, under Dirichlet boundary conditions. In order to emphasize the main ideas we confine ourselves to the following representative case, where $\mu \geq 0$ is a fixed constant:

$$\begin{cases} -\nabla \cdot \left((\mu + |\nabla u|)^{p-2} \nabla u \right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The vector field $u = (u_1(x), \ldots, u_N(x))$, $N \geq 1$, is defined on a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. When $\mu = 0$, the system (1.1) is the well-known p-Laplacian system. To avoid meaningless specifications, we fix an arbitrarily large positive constant μ_0, and assume that $\mu \leq \mu_0$ everywhere.

It is worth noting that our interests concern global (up to the boundary), full regularity for the second derivatives of the solutions. Our results also hold in the singular case $\mu = 0$. For any bounded and sufficiently smooth domain Ω, we prove $W^{2,q}(\Omega)$ regularity, for any arbitrarily large q. Therefore, we get, as a by product, the α-Hölder continuity, up to the boundary, of the gradient of the solution, for any $\alpha < 1$. The results are obtained for p belonging to intervals $[C, 2)$, where C are suitable constants, whose expression may be explicitly calculated. In particular, if Ω is convex, solutions belong to $W^{2,2}(\Omega)$ for any $1 < p \leq 2$.

As usual, weak solutions of problem (1.1) are defined as follows (for notation and more precise statements see the sequel).

* Corresponding author. Tel.: +39 050 2213837; fax: +39 050 2213802.
E-mail addresses: bveiga@dma.unipi.it (H. Beirão da Veiga), francesca.crispo@unina2.it (F. Crispo).

0362-546X/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2012.03.021

Definition 1.1. Assume that $f \in W^{-1,p'}(\Omega)$. We say that u is a weak solution of problem (1.1) if $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega (\mu + |\nabla u|)^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \varphi \, dx, \tag{1.2}$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

We recall that the existence and uniqueness of a weak solution can be obtained by appealing to the theory of monotone operators, following [1].

It is immediate to verify that, if $\mu > 0$, sufficiently regular weak solutions to the problem (1.1) satisfy

$$-\Delta u - (p-2)\frac{\nabla u \cdot \nabla\nabla u \cdot \nabla u}{(\mu + |\nabla u|)|\nabla u|} = f\,(\mu + |\nabla u|)^{2-p}. \tag{1.3}$$

Here, and in the following, we use the notation $\nabla u \cdot \nabla\nabla u \cdot \nabla u$ to indicate the vector whose ith component is $\nabla u \cdot \big(\partial_j \nabla u\big)\, \partial_j u_i = (\partial_l u_k)(\partial^2_{jl} u_k)(\partial_j u_i)$.

In the sequel we start by proving the existence of a (unique) strong solution

$$u \in W_0^{1,p}(\Omega) \cap W^{2,q}(\Omega),$$

of problem (1.3), under homogeneous Dirichlet boundary conditions, in the case $\mu > 0$ (see Theorem 3.1). Clearly, u solves (1.2). Furthermore, we prove that the $W^{2,q}(\Omega)$ norms of the above strong solutions are uniformly bounded with respect to μ. This allows us, by passing to the limit as $\mu \to 0$, to extend the $W^{2,q}(\Omega)$ regularity result to weak solutions of problem (1.1) in the case $\mu = 0$ (see Theorem 2.1).

The regularity issue for systems like (1.1) has received substantial attention, mostly concerned with the scalar case ($N = 1$), and with $C^{1,\alpha}_{\rm loc}$-regularity. Here and in the following, by local regularity we mean interior regularity. The pioneering result dates back to Ural'tseva [2], where, for $p > 2$ and $N = 1$, the author proves $C^{1,\beta}_{\rm loc}$-regularity for a suitable exponent β. Still in the case $N = 1$ we recall the following contributions. In [3] the author proves $W^{2,p}_{\rm loc}$-regularity for any $p < 2$, and also $W^{2,2}_{\rm loc}$-regularity, for $p > 2$. In [4], for $p > 2$, the author proves $C^{1,\beta}$-regularity up to the boundary, in $\Omega \subset \mathbb{R}^n$. In [5] the authors show, for any $p \in (1,2)$, $W^{2,2} \cap C^{1,\alpha}$-regularity up to the boundary, in $\Omega \subset \mathbb{R}^2$.

For systems (solutions are N-dimensional vector fields, $N > 1$), we recall [6] for $p \in (1,2)$, [7,8] for $p > 2$, and [9] for any $p > 1$. The results proved in papers [6–8] are local. Moreover all these papers deal only with homogeneous systems and the techniques, sometimes quite involved, seem not to be directly applicable to the non-homogeneous setting. In particular, [6] is the only paper in which the $L^2_{\rm loc}$-regularity of second derivatives is considered. The results below are, in the non-scalar case, the first regularity results up to the boundary, for the second derivatives of solutions.

For related results and for an extensive bibliography we also refer to papers [10–18] and references therein.

Last but not least, we recall the classical Ladyzhenskaya and Ural'tseva famous treatise [19], where related results and deep methods are shown.

We observe that we do not consider a more general dependence on ∇u, as for instance $\varphi(|\nabla u|)\,\nabla u$, under suitable assumptions on the scalar function φ, just to emphasize the core aspects of the results and to avoid additional technicalities. For the same reason we avoid the introduction of lower order terms. Note that another, very similar, representative case can be obtained with the regular term $(\mu + |\nabla u|^2)^{\frac{p-2}{2}}$ in place of $(\mu + |\nabla u|)^{p-2}$ in (1.1). This latter function is only Lipschitz continuous, hence in this case it seems not possible to get stronger regularity results. Finally one could also extend the results to non-homogeneous Dirichlet boundary conditions, if the boundary data belongs to a suitable $W^{2,q}$-space.

Remark 1.1. Different, more intricate, proofs of Theorem 2.1 and its corollaries were given in [11], in the particular case $N = n = 3$. In [11] we also consider the case where ∇u is replaced by $\mathcal{D} u = \frac{1}{2}(\nabla u + \nabla u^T)$, and $p \in (1, +\infty)$. Results and proofs where also presented at the conference Vorticity, Rotation and Symmetry (II)—Regularity of Fluid Motion, held at the CIRM, (Luminy, Marseille) from May 23–27, 2011.

2. Notation and statement of the main results

Throughout this paper we denote by Ω a bounded n-dimensional domain, $n \geq 3$, with smooth boundary, which we assume of class C^2, and we consider the usual homogeneous Dirichlet boundary conditions

$$u_{|\partial\Omega} = 0. \tag{2.1}$$

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, m nonnegative integer and $p \in (1, +\infty)$, we denote the usual Lebesgue and Sobolev spaces, with the standard norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$, respectively. We usually denote the above norms by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, when the domain is clear. Further, we set $\|\cdot\| = \|\cdot\|_2$. We denote by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ and by $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, the strong dual of $W_0^{1,p}(\Omega)$ with norm $\|\cdot\|_{-1,p'}$.

In notation concerning norms and functional spaces, we do not distinguish between scalar and vector fields. For instance $L^p(\Omega;\mathbb{R}^N) = [L^p(\Omega)]^N$, $N > 1$, is simply $L^p(\Omega)$.

We use the summation convention on repeated indexes. For any given pair of matrices B and C in R^{Nn} (linear space of $N \times n$-matrices), we write $B \cdot C \equiv B_{ij}\, C_{ij}$.

We denote by the symbols c, c_1, c_2, etc., positive constants that may depend on μ; by capital letters, C, C_1, C_2, etc., we denote positive constants independent of $\mu \geq 0$ (eventually, μ bounded from above). The same symbol c or C may denote different constants, even in the same equation.

We set $\partial_i\, u = \frac{\partial u}{\partial x_i}$, $\partial^2_{ij}\, u = \frac{\partial^2 u}{\partial x_i \partial x_j}$. Moreover we set $(\nabla u)_{ij} = \partial_j\, u_i$. We denote by $D^2 u$ the set of all the second partial derivatives of u. Moreover we set

$$|D^2 u|^2 := \sum_{i=1}^{N} \sum_{j,h=1}^{n} \left| \partial^2_{jh}\, u_i \right|^2 . \tag{2.2}$$

Before stating our main results, let us recall two well known inequalities for the Laplace operator. The first, namely

$$\| D^2\, v \| \leq C_1 \| \Delta v \|, \tag{2.3}$$

holds for any function $v \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$, with $C_1 = C_1(\Omega)$. Note that $C_1 = 1$ if Ω is convex. For details we refer to [20, Chapter I, estimate (20)]. The second kind of estimates which we are going to use says that

$$\|D^2\, v\|_q \leq C_2(q)\, \| \Delta v\|_q, \tag{2.4}$$

for $v \in W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega)$, $q > 1$, where the constant C_2 depends on q and Ω. It relies on standard estimates for solution of the Dirichlet problem for the Poisson equation. Actually, there is a constant K, independent of q, such that

$$C_2(q) \leq K\, q, \tag{2.5}$$

for $q > \frac{2\,n}{n+2}$. Similarly, one has

$$\|v\|_{\,2,q} \leq C \|\Delta v\|_q, \tag{2.6}$$

where the constant C depends on q and Ω. For further details we refer to [21,22]. For convenience, since we are interested in large values of q, we assume from now on that $q \geq 2$.

We set

$$r(q) = \begin{cases} \dfrac{nq}{n(p-1) + q(2-p)} & \text{if } q \in [\,2,\ n], \\ q & \text{if } q \geq n. \end{cases} \tag{2.7}$$

Note that $r(q)$ is a strictly increasing function. Moreover, $r(q) > q$ for any $q < n$. Clearly, in (2.7), $r(n) = n$ in booth cases. Our main results is the following.

Theorem 2.1. *Let be $\mu \geq 0$. Further, let $p \in (1, 2]$ and $q \geq 2,\ q \neq n$, be such that the couple $(p,\ q)$ satisfies the condition*

$$(2-p)\, C_2(q) < 1, \tag{2.8}$$

where $C_2(q)$ is given by (2.4). In particular (2.8) holds if

$$2 - \frac{1}{K\,q} < p \leq 2, \tag{2.9}$$

where K is independent of q (see (2.5)).

Assume that $f \in L^{r(q)}(\Omega)$, and let u be the unique weak solution of problem (1.1). Then u belongs to $W^{2,q}(\Omega)$. Moreover, the following estimate holds

$$\|u\|_{2,q} \leq C \left(\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}} \right). \tag{2.10}$$

Corollary 2.1. *Let $p, \mu,\ q$, and f be as in Theorem 2.1. Then, if $q > n$, the weak solution of problem (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$, for $\alpha = 1 - \frac{n}{q}$.*

In particular, when $q = 2$, one has the following corollary.

Corollary 2.2. *Let $p \in (1, 2]$ satisfy $(2-p)\, C_1 < 1$, where C_1 is defined by (2.3). Assume that $\mu \geq 0$. Let $f \in L^{r(2)}(\Omega)$, where*

$$r(2) = \frac{2n}{(n-2)\,p + (4-n)}, \tag{2.11}$$

and let u be the unique weak solution u of problem (1.1). Then u belongs to $W^{2,2}(\Omega)$. Moreover, there is a constant C such that

$$\|u\|_{2,2} \leq C\left(\|f\| + \|f\|_{r(2)}^{\frac{1}{p-1}}\right). \tag{2.12}$$

If Ω is convex the result holds for any $1 < p \leq 2$.

It is worth noting that in the limit case $p = 2$, when system (1.1) reduces to the Poisson equations, we recover exactly the well known result

$$\|u\|_{2,q} \leq C\|f\|_q,$$

since $r(q) = q$ for $p = 2$.

Note that in estimates (2.10) and (2.12), the terms $\|f\|_q$ and $\|f\|$ can be replaced by 1.

Remark 2.1. One could also consider the case where $f \in L^n(\Omega)$. We omit this further case and leave it to the interested reader. In this regard we stress that our interest mostly concerns the maximal integrability of the second derivatives of the solution.

We end this section by some remarks:

It could be more natural to state the results by inverting the (strictly increasing) function $r = r(q)$ since, in practice, r is given and we look for q. This means to consider the inverse function $q = q(r)$, to assume $f \in L^r(\Omega)$, and conclude that $u \in W^{2,\,q(r)}(\Omega)$.

Let us make some comment on practical application of Theorem 2.1. The p-growth exponent is fixed, as soon as we consider the problem (1.1). Next, by assuming that right hand sides f are given in some $L^s(\Omega)$ space, we look for the solution q to the equation $r(q) = s$ (note that this equation depends on the given value p). If the pair $(p,\ q)$ enjoys (2.8), the solution u belongs to $W^{2,\,q}(\Omega)$. Otherwise, we look for the largest value $\widehat{q} \geq 2$ for which (2.8) holds. Since $r(\widehat{q}) < r(q) = s$, it follows that $f \in L^{r(\widehat{q})}(\Omega)$. Hence Theorem 2.1 shows that $u \in W^{2,\,\widehat{q}}(\Omega)$. Roughly speaking, in this last case, f is “unnecessarily” regular.

An interesting question is to look for the minimal L^s-regularity needed by f in order to prove that the second derivatives of solutions are integrable with some exponent q. In our framework the values s are artificially restricted from below, due to the assumption $q \geq 2$. However, it is easily checked that the proof of Theorem 2.1 also hold without this restriction. Some restriction is necessary if we want to appeal to (2.5). For instance $q > \frac{2n}{n+2}$.

Finally we note that, in the presence of the same right hand side f, the regularity proved for the solutions tends to increase with the growth's exponent p, since the value $r(q)$, for fixed q, is a decreasing function of p (strictly decreasing if $q < n$).

3. Proof of Theorem 2.1. The case $\mu > 0$

In this section we assume that $\mu > 0$. Let us consider the system (1.3). In the left hand side of (1.3) the fraction is continuously extended by zero to $\nabla u = 0$. Formally the system (1.3) can be obtained from system (1.1) by computing the divergence on the left-hand side and then multiplying the equation by $(\mu + |\nabla u|)^{2-p}$.

It is immediate to verify that if u is a sufficiently regular solution of (1.3), say $u \in W^{2,2}(\Omega)$, then u is a weak solutions of (1.1). So, from the uniqueness of weak solutions of (1.1), it follows that to prove Theorem 2.1 under the assumption $\mu > 0$ it is sufficient to prove the following result for strong solutions.

Theorem 3.1. *Let be $\mu > 0$. Further, let $p \in (1, 2]$ and $q \geq 2,\ q \neq n$, be such that the couple $(p,\ q)$ satisfies the condition (2.8). Let $f \in L^{r(q)}(\Omega)$ for some $q \geq 2$ and $q \neq n$. Then, there is a strong solution $u \in W^{2,q}(\Omega)$ of problem (2.1), (1.3). Moreover, the following estimate holds*

$$\|u\|_{2,q} \leq C\left(\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}}\right). \tag{3.1}$$

In the sequel we appeal to the following fixed point theorem in order to prove Theorem 3.1. In order to avoid misunderstanding between indexes concerning sequences and lower indexes concerning components of a vector, the first ones will be denoted by upper-indexes.

Theorem 3.2. *Let X be a reflexive Banach space and $\mathbb{K}$ a non-empty, convex, bounded, closed subset of X. Let F be a map defined in $\mathbb{K}$, such that $F(\mathbb{K}) \subset \mathbb{K}$.*

Assume that there is a Banach space Y such that:

(i) *$X \subset Y$, with compact (completely continuous) immersion.*
(ii) *If $v^n \in \mathbb{K}$ converges weakly in X to some $v \in \mathbb{K}$ then there is a subsequence v^m such that $F(v^m) \to F(v)$ in Y.*

Under the above hypotheses the map F has a fixed point in $\mathbb{K}$.

For the proof and some comments see Section 5.

In the sequel we appeal to the above theorem with $X = W^{2,q}$ and $Y = L^q$. Clearly, point (i) in Theorem 3.2 holds.

Proof of Theorem 3.1. For any $v \in W^{2,q} \cap W_0^{1,q}$ we define $C_3 = C_3(q)$ by

$$\begin{aligned} &\|\nabla v\|_{q^*} \le C_3 \|\Delta v\|_q, \quad \text{if } q \in (1, n), \\ &\|\nabla v\|_\infty \le C_3 \|\Delta v\|_q, \quad \text{if } q \in (n, +\infty). \end{aligned} \tag{3.2}$$

These estimates can be easily obtained by applying the Sobolev embeddings and then using estimate (2.6).

Define δ by

$$\delta = 1 - (2 - p)\, C_2,$$

where C_2 is given by (2.4), and fix a positive real a by

$$1 + 2\, C_3^{2-p}\, a^{2-p} \le a\, \delta.$$

Note that, under our assumptions, $\delta > 0$. It is worth noting that $\delta,\ a$ and b are constants of type C.

Define

$$\mathbb{K} = \{v \in W^{2,q}(\Omega) : \|\Delta v\|_q \le R,\ v = 0 \text{ on } \partial\Omega\},$$

where

$$R = a \left(\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}} \right).$$

Let $f \in L^{r(q)}$ be given. For each $v \in \mathbb{K}$ define $u = F(v)$ as being the solution to the linear problem

$$\begin{cases} -\Delta u = (p-2) \dfrac{\nabla v \cdot \nabla\nabla v \cdot \nabla v}{(\mu + |\nabla v|)|\nabla v|} + f\, (\mu + |\nabla v|)^{2-p}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

To apply Theorem 3.2 we start by showing that $F(\mathbb{K}) \subset \mathbb{K}$. Note that if the right-hand side of (3.3) belongs to $L^q(\Omega)$, from well known results on the Poisson equation, there exists a unique $u \in W^{2,q}(\Omega)$ solving the Dirichlet problem (3.3). For $q \in (1, n)$, by using $(3.2)_1$ we have

$$\| |\nabla v|^{2-p} f \|_q \le \|\nabla v\|_{q^*}^{2-p} \|f\|_{r(q)} \le C_3^{2-p} \|\Delta v\|_q^{2-p} \|f\|_{r(q)}.$$

For $q > n$, by using $(3.2)_2$, and by recalling that $r(q) = q$ if $q > n$, we have

$$\| |\nabla v|^{2-p} f \|_q \le \|\nabla v\|_\infty^{2-p} \|f\|_q \le C_3^{2-p} \|\Delta v\|_q^{2-p} \|f\|_q.$$

So, in both the cases,

$$\| |\nabla v|^{2-p} f \|_q \le C_3^{2-p} \|\Delta v\|_q^{2-p} \|f\|_{r(q)}. \tag{3.4}$$

Therefore, since the first term on the right-hand side of (3.3) obviously belongs to $L^q(\Omega)$, there exists a unique $u \in W^{2,q}(\Omega)$ solving the Dirichlet problem (3.3).

It remains to show that u satisfies the estimate $\|\Delta u\|_q \le R$. We multiply both sides of Eq. (3.3) by $-\Delta u |\Delta u|^{q-2}$, and integrate in Ω. Note that (for details see the Appendix)

$$|I| := |(\nabla v \cdot \nabla\nabla v \cdot \nabla v) \cdot \Delta u| \le |\nabla v|^2\, |D^2 v|\, |\Delta u|. \tag{3.5}$$

We get

$$\int_\Omega |\Delta u|^q\, dx \le (2-p) \int_\Omega |D^2 v|\, |\Delta u|^{q-1}\, dx + \int_\Omega (\mu + |\nabla v|)^{2-p} |f|\, |\Delta u|^{q-1}\, dx.$$

The Hölder inequality and the inequality $(\mu + |\nabla v|)^{2-p} \le 1 + |\nabla v|^{2-p}$ yield

$$\|\Delta u\|_q^q \le (2-p) \|D^2 v\|_q \|\Delta u\|_q^{q-1} + \|f\|_q \|\Delta u\|_q^{q-1} + \| |\nabla v|^{2-p} f \|_q \|\Delta u\|_q^{q-1}, \tag{3.6}$$

and, by dividing both sides by $\|\Delta u\|_q^{q-1}$, one has

$$\|\Delta u\|_q \le (2-p) \|D^2 v\|_q + \|f\|_q + \| |\nabla v|^{2-p} f \|_q. \tag{3.7}$$

Let us estimate the last term on the right-hand side of (3.7). Since $v \in K$, one has

$$\|\Delta v\|_q^{2-p} \le a^{2-p} \left(\|f\|_q^{2-p} + \|f\|_{r(q)}^{\frac{2-p}{p-1}} \right). \tag{3.8}$$

Hence, from (3.4), by using (3.8) and

$$\|f\|_q^{2-p}\|f\|_{r(q)} \le \|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}},$$

one gets

$$\||\nabla v|^{2-p} f\|_q \le C_3^{2-p}\, a^{2-p}\left(\|f\|_q + 2\|f\|_{r(q)}^{\frac{1}{p-1}}\right).$$

Therefore (3.7) becomes

$$\|\Delta u\|_q \le \left((2-p)\,C_2\; a\; + 1 + 2\,C_3^{2-p}\,a^{2-p}\right)\left(\|f\|_q + \|f\|_{r(q)}^{\frac{1}{p-1}}\right), \tag{3.9}$$

where we have appealed to (2.4). Finally from the definition of δ, it readily follows that $u \in \mathbb{K}$. So, $F(\mathbb{K}) \subset \mathbb{K}$.

To end the proof of Theorem 3.1 it is sufficient to show the following result (which corresponds to point (ii) in Theorem 3.2).

Proposition 3.1. *Let $v^n \rightharpoonup v$ weakly in $W^{2,q}$, where $v^n \in \mathbb{K}$. If $u^n = F(v^n)$ are the solutions to the problem*

$$-\,\Delta u^n = (2-p)\frac{\nabla v^n \cdot \nabla\nabla v^n \cdot \nabla v^n}{(\mu + |\nabla v^n|)|\nabla v^n|} + f\,\left(\mu + |\nabla\, v^n|\right)^{2-p}, \tag{3.10}$$

then there is a subsequence v^m of v^n such that $u^m = F(v^m) \to u$ in $Y = L^q$, where $u = F(v)$.

In the sequel we use the label $(3.10)_l$ to mean that the sequences u^n and v^n are replaced by subsequences u^l and v^l respectively. For instance we can denote identity (3.10) also by $(3.10)_n$.

Since $u^n \in \mathbb{K}$, there is a subsequence u^k and an element $u \in \mathbb{K}$ such that $u^k \rightharpoonup u$ weakly in $W^{2,q}$ (since this space is reflexive). In particular $-\Delta\, u^k \rightharpoonup -\Delta\, u$ weakly in L^q. Moreover, $u^k \to u$ strongly in $W^{s,q}$, for each $s < 2$, hence $u^k \to u$ in $Y = L^q$.

The proof is accomplished by showing that one can pass to the limit in $(3.10)_m$, along subsequences u^m and v^m, to obtain

$$-\,\Delta u = (p-2)\frac{\nabla v \cdot \nabla\nabla v \cdot \nabla v}{(\mu + |\nabla v|)|\nabla v|} + f\,(\mu + |\nabla\, v|)^{2-p}. \tag{3.11}$$

To prove the proposition it is sufficient to consider the Eq. $(3.10)_k$ and to show that there is a subsequence v^m of v^k such that each of the two terms in the right hand side of $(3.10)_m$ converge, in the distributional sense, to the corresponding terms in Eq. (3.11). This verification would be quite immediate. However, we rather prefer to prove the convergence in a topology stronger than the distributional one.

It is worth noting that below, on passing to the limit, the particular structure of the fractional term in the right hand side of Eq. (3.10) has not a particular role. So, for clearness and also to simplify notation, we denote similar single terms by the same notation, by setting

$$A(w) = \frac{(\partial_l\, w_k)\,(\partial_j\, w_i)}{(\mu + |\nabla w|)|\nabla w|},$$

and also $\partial^2\, w = \partial^2_{jl}\, w_k$, for arbitrary vector fields w and indexes $l,\ j,\ k,\ i$. Clearly, we may appeal to successive extraction of subsequences.

Lemma 3.3. *There is a subsequence v^m of v^k such that*

$$A(v^m) \to A(v),$$

strongly in L^t, for each finite $t > 1$.

Proof. Since, in particular, $v^k \to v$ in $W^{1,q}$, it follows, by a classical result, that almost everywhere convergence of the gradient in Ω also holds, for some v^m. So, $A(v^m) \to A(v)$, a.e. in Ω. Further, $|A(w(x))| \le 1$, point-wisely. Hence

$$|A(v^m(x)) - A(v(x))|^t \le 2^t.$$

The desired norm-convergence follows by appealing to Lebesgue's dominated convergence theorem. □

Next, we prove that each of the two terms in the right hand side of $(3.10)_m$ converges to the corresponding term in Eq. (3.11). We start by the first term. Each single addend has the form $A(v^m)\,\partial^2\,v^m$, where $\partial^2\, w$ denotes an arbitrary, fixed, second order derivative. We prove the following result.

Lemma 3.4. *One has*

$$A(v^m)\,\partial^2\,v^m \rightharpoonup A(v)\,\partial^2\,v$$

weakly in L^s, for each $s < q$.

Proof. Set $g = A(v)$, $g^m = A(v^m)$, $h = \partial^2 v$, and $h^m = \partial^2 v^m$. Clearly, $h^m \rightharpoonup h$ weakly in L^q. Moreover, by the previous lemma, $g^m \to g$ strongly in L^t, $t := \frac{qs}{q-s}$.

Write

$$g^m h^m - g\,h = g(h^m - h) + (g^m - g)\,h^m, \tag{3.12}$$

and let $\phi \in L^{q'}$. Since $g(x)$ is bounded it follows that $g\,\phi \in L^{q'}$. So the quantity

$$\langle g(h^m - h),\ \phi\rangle = \langle (h^m - h),\ g\,\phi\rangle$$

goes to zero as $m \to \infty$. This proves the weak convergence to zero, in L^q, of the first term in the right hand side of (3.12).

On the other hand, by the Hölder inequality,

$$\|(g^m - g)\,h^m\|_s^s \le \|g^m - g\|^s_{\frac{qs}{q-s}}\,\|h^m\|_q^s.$$

This proves the strong convergence to zero, in L^s, of the second term in the right hand side of (3.12). In conclusion, the first term in the right hand side of $(3.10)_m$ converges to the first term in the right hand side of (3.11). □

Finally, the convergence of the second term in the right hand side of $(3.10)_m$ to the corresponding term in (3.11) holds, since

$$|(\mu + |\nabla v^m|)^{2-p} - (\mu + |\nabla v|)^{2-p}| \le \frac{2-p}{\mu^{p-1}}|\nabla v^m - \nabla v|.$$

By the Cauchy–Schwarz inequality

$$\|f(\mu + |\nabla v^m|)^{2-p} - f(\mu + |\nabla v|)^{2-p}\|_{\frac{q}{2}} \le \frac{2-p}{\mu^{p-1}}\|f\|_q\,\|\nabla v^m - \nabla v\|_q,$$

and the right-hand side goes to zero thanks to the compact embedding of $W^{2,q}$ in $W^{1,q}$.

The solution u obviously satisfies (3.1), as $u \in \mathbb{K}$. □

4. Proof of Theorem 2.1. The case $\mu = 0$

In the previous step we have obtained estimates on the L^q-norm of the second derivatives, uniformly in μ, $\mu > 0$. Let us denote by u^μ the "sequence" of solutions of (1.1) for the different values of $\mu > 0$. We have shown that the sequence u^μ is uniformly bounded in $W^{2,q}(\Omega)$. Therefore, there exists a vector field $u \in W^{2,q}(\Omega)$ and a subsequence, which we continue to denote by u^μ, such that $u^\mu \rightharpoonup u$ weakly in $W^{2,q}(\Omega)$, and, by Rellich's theorem, strongly in $W^{1,s}(\Omega)$, for any s if $q > n$, and for $s < q^*$ if $q < n$. In particular u^μ converges to u strongly in $W^{1,p}(\Omega)$. Let us prove that

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u\,\cdot\nabla\varphi\,dx = \lim_{\mu\to 0^+}\int_\Omega (\mu + |\nabla u^\mu|)^{p-2}\,\nabla u^\mu\,\cdot\nabla\varphi\,dx, \tag{4.1}$$

for any $\varphi \in W_0^{1,p}(\Omega)$. We write the integral on the right-hand side of (4.1) as

$$\int_\Omega \left[(\mu + |\nabla u^\mu|)^{p-2}\,\nabla u^\mu\ - (\mu + |\nabla u|)^{p-2}\,\nabla u\right]\cdot\nabla\varphi\,dx + \int_\Omega (\mu + |\nabla u|)^{p-2}\,\nabla u\,\cdot\nabla\varphi\,dx, \tag{4.2}$$

and we recall the following well known estimate (see, for instance, [23])

$$|(\mu + |A|)^{p-2}A - (\mu + |B|)^{p-2}B| \le C\,\frac{|A - B|}{(\mu + |A| + |B|)^{2-p}}, \tag{4.3}$$

for any pair A and B in $\mathbb{R}^{Nn}$, where C is a positive constant independent of μ. By applying (4.3), followed by the Hölder inequality, we get

$$\begin{aligned}
&\left|\int_\Omega (\mu + |\nabla u|)^{p-2}\,\nabla u\,\cdot\nabla\varphi\,dx - \int_\Omega (\mu + |\nabla u^\mu|)^{p-2}\,\nabla u^\mu\,\cdot\nabla\varphi\,dx\right|\\
&\le C\int_\Omega (\mu + |\nabla u| + |\nabla u^\mu|)^{p-2}\,|\nabla u - \nabla u^\mu|\,|\nabla\varphi|dx\\
&\le C\int_\Omega |\nabla u - \nabla u^\mu|^{p-1}\,|\nabla\varphi|dx \le C\|\nabla u^\mu - \nabla u\|_p^{p-1}\|\nabla\varphi\|_p.
\end{aligned} \tag{4.4}$$

The second inequality in (4.4) easily comes from $|\nabla u - \nabla u^\mu| \le |\nabla u| + |\nabla u^\mu|$. We may also appeal to [24], Lemma 6.3, that can be particularly useful for extension to more general operators.

The right-hand side of the last inequality tends to zero, as μ goes to zero, thanks to the strong convergence of u^μ to u in $W^{1,p}(\Omega)$. On the other hand, since

$$\left|(\mu + |\nabla u|)^{p-2}\, \nabla u \cdot \nabla \varphi\right| \leq |\nabla u|^{p-1} |\nabla \varphi|$$

and

$$(\mu + |\nabla u|)^{p-2}\, \nabla u \to |\nabla u|^{p-2} \nabla u, \quad \text{a.e. on } \Omega,$$

we apply Lebesgue's dominated convergence theorem to obtain

$$\lim_{\mu \to 0^+} \int_\Omega (\mu + |\nabla u|)^{p-2}\, \nabla u\, \cdot \nabla \varphi\, dx = \int_\Omega |\nabla u|^{p-2}\, \nabla u\, \cdot \nabla \varphi .dx.$$

The proof of (4.1) is accomplished.

Finally, for each $\mu > 0$, the right-hand side of (4.1) is equal to $\int_\Omega f \cdot \varphi\, dx$. So, u satisfies the integral identity (1.2). Hence u is a weak solution of (1.1), and belongs to $W^{2,q}(\Omega)$. Eq. (2.12) follows since $\|\, u\|_{2,q} \leq \liminf_{\mu \to 0^+} \|\, u^\mu\|_{2,q}$.

Corollary 2.1 is an immediate consequence of Theorem 2.1, by using the regularity of the domain and the Sobolev embedding.

The results in Corollary 2.2 can be obtained by replacing in the proof of Theorem 2.1, hence in the proof of Theorem 3.1, the constant C_2 with the constant C_1. The last assertion in Corollary 2.2 follows from the validity of (2.3), for a smooth convex domain, with $C_1 = 1$. We omit further details.

5. The fixed point theorem. Proof and remarks

Theorem 3.2 is a simplification of an idea introduced in Ref. [25] to prove existence of strong solutions to initial boundary value problems for non-linear systems of evolution equations, specially in Sobolev spaces. See Section 3, in the above reference. Successively, the method has been applied with success to many other problems, in particular to the compressible Euler equations (see [26]). Main requirements, in applications, are the reflexivity of the Banach space X, and its sufficiently strong topology. Schauder's fixed point theorem is applied with respect to a quite arbitrary "container space" Y. Roughly speaking, the above two properties allow us to trivialize both compactness and continuity requirements, respectively. So, to apply the theorem, the main point is to show that $F(\mathbb{K}) \subset \mathbb{K}$, for some convex, bounded, closed subset $\mathbb{K}$.

Proof of Theorem 3.2. Obviously $\mathbb{K}$ is convex, bounded, and pre-compact in Y.

Let $y_n \in \mathbb{K}$ converge to some y in the Y norm. We start by showing that $\mathbb{K}$ is closed, hence compact, in Y, and that the sequence $y_n \rightharpoonup y$ weakly in X. Since $\mathbb{K}$ is X-bounded, and X is reflexive, there is a subsequence y_m which is X-weakly convergent to some $u \in X$. Since the immersion $X \subset Y$ is continuous, y_m is also weakly convergent to u in Y. Since, by assumption, this sequence is strongly convergent in Y to y, it follows that $u = y$. Further, since convex sets in Banach spaces are weakly closed if and only if they are strongly closed, it follows that $y \in \mathbb{K}$. So, $\mathbb{K}$ is Y-closed. Further, from the uniqueness of the limit y, we deduce that the whole sequence y_n converges weakly in X to y.

Finally, to prove that $F(y_n) \to F(y)$ strongly in Y it is sufficient to show, by using standard arguments, that any subsequence y_k contains a subsequence y_m such that $F(y_m) \to F(y)$ strongly in Y. Obviously, $y_k \rightharpoonup y$ weakly in X. By assumption (ii), there is a subsequence y_m such that $F(y_m) \to F(y)$ strongly in Y. This shows that the map F is continuous on $\mathbb{K}$ with respect to the Y topology. So, Schauder's fixed point theorem guarantees the existence of, at least, one fixed point $y_0 \in \mathbb{K}$, $F(y_0) = y_0$. □

Acknowledgments

The authors like to thank Professor M. Fuchs and Professor P. Kaplický for giving some interesting references. The second author thanks Dipartimento di Matematica Applicata "U. Dini", University of Pisa, for the kind hospitality during the preparation of the paper.

Appendix

We prove here the estimate (3.5). One has

$$|I| = \left| \sum_{j,l} \left(\sum_i (\Delta u_i)(\partial_j\, v_i) \right) \left(\sum_k (\partial_l\, v_k)(\partial^2_{jl}\, v_k) \right) \right|.$$

By appealing to the Cauchy–Schwarz inequality one shows that

$$\begin{aligned} |I| &\leq \sum_{j,\,l} (|\Delta u|\,|\partial_j\, v|)(|\partial_l\, v||\partial^2_{jl}\, v|) \\ &\leq |\Delta u| \sum_{j,\,l} (|\partial_j\, v|\,|\partial_l\, v|)|\partial^2_{jl}\, v| \\ &\leq |\Delta u| \left(\sum_{j,\,l} |\partial_j\, v|^2 |\partial_l\, v|^2 \right)^{\frac{1}{2}} |D^2\, v|. \end{aligned}$$

Eq. (3.5) follows.

References

[1] J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Gauthier-Villars, Paris, 1969.
[2] N.N. Ural'tseva, Degenerate quasilinear elliptic systems, Zap. Nauk. Sem., LOMI 7 (1968) 184–222 (in Russian).
[3] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.
[4] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[5] W.B. Liu, J.W. Barrett, A remark on the regularity of the solutions of the p-Laplacian and its application to their finite element approximation, J. Math. Anal. Appl. 178 (1993) 470–487.
[6] E. Acerbi, N. Fusco, Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$, J. Math. Anal. Appl. 140 (1989) 115–135.
[7] M. Giaquinta, G. Modica, Remarks on the regularity of the minimizers of certain degenerate functionals, Manuscripta Math. 57 (1986) 55–99.
[8] K. Uhlenbeck, Regularity for a class of non-linear elliptic systems, Acta Math. 138 (1977) 219–240.
[9] C. Hamburger, Regularity of differential forms minimizing degenerate elliptic functionals, J. Reine Angew. Math. 431 (1992) 7–64.
[10] E. Acerbi, G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, J. Reine Angew. Math. 584 (2005) 117–148.
[11] H. Beirão da Veiga, F. Crispo, On the global regularity for nonlinear systems of the p-Laplacian type, Discrete Contin. Dyn. Syst. Ser. S. Online arXiv:1008.3262v1 [math.AP] (in press).
[12] E. DiBenedetto, $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations, Nonlinear Anal. 7 (1983) 827–850.
[13] E. DiBenedetto, J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, Amer. J. Math. 115 (1993) 1107–1134.
[14] M. Fuchs, G. Mingione, Full $C^{1,\alpha}$-regularity for free and constrained local minimizers of elliptic variational integrals with nearly linear growth, Manuscripta Math. 102 (2000) 227–250.
[15] M. Fuchs, G. Seregin, Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids, in: Lecture Notes in Mathematics, vol. 1749, Springer-Verlag, Berlin, 2000.
[16] G.M. Lieberman, Gradient estimates for a new class of degenerate elliptic and parabolic equations, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) 21 (1994) 497–522.
[17] P. Marcellini, G. Papi, Nonlinear elliptic systems with general growth, J. Differential Equations 221 (2006) 412–443.
[18] G. Mingione, Regularity of minima: an invitation to the dark side of the calculus of variations, Appl. Math. 51 (2006) 355–426.
[19] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, first Russian ed., Izdat. Nauka, Moscow, 1964, English translation: Academic Press, New York, 1968.
[20] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, revised second english ed., Gordon and Breach Science Publishers, New York, London, 1969, Translated from the Russian by Richard A. Silverman.
[21] A.I. Koševlev, On boundedness of L^p of derivatives of solutions of elliptic differential equations, Mat. Sb. NS 38 (1956) 359–372 (in Russian).
[22] V.I. Yudovic, Some estimates connected with integral operators and with solutions of elliptic equations, Dokl. Akad. Nauk SSSR 138 (1961) 805–808; English translation in Soviet Math. Dokl. 2 (1961) 746–749.
[23] L. Diening, C. Ebmeyer, M. Růžička, Optimal convergence for the implicit space–time discretization of parabolic systems with p-structure, SIAM J. Numer. Anal. 45 (2007) 457–472.
[24] M. Růžička, L. Diening, Non-Newtonian fluids and function spaces, in: Proccedings of NAFSA 2006, Prague, in: Nonlinear Analysis, Function Spaces and Applications, vol. 8, 2007, pp. 95–144.
[25] H. Beirão da Veiga, On an Euler type equation in hydrodynamics, Ann. Mat. Pura Appl. 125 (1980) 279–294.
[26] H. Beirão da Veiga, On the barotropic motion of compressible perfect fluids, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) 8 (1981) 317–351.