On the global regularity for nonlinear systems of the p-Laplacian type

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Abstract

1 Introduction

We are concerned with the regularity issue for solutions of nonlinear systems of partial differential equations with p-structure, under Dirichlet boundary conditions . In order to emphasize the main ideas we confine ourselves to the following, typical, representative cases, where $p > 1$ and $\mu \geq 0$ are fixed constants:

The "full gradient case'

$$-\nabla \cdot S(\nabla u) = f\,, \tag{1.1}$$

', where

$$S(\nabla u) = (\mu + |\nabla u|)^{p-2}\,\nabla u\,. \tag{1.2}$$

And the "symmetric gradient case"

$$-\nabla \cdot S(\mathcal{D}\,u) = f\,, \tag{1.3}$$

where

$$S(\mathcal{D}\,u) = (\mu + |\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\,, \tag{1.4}$$

and

$$\mathcal{D}\,u = \frac{1}{2}(\nabla u + \nabla u^T)$$

is the symmetric part of the gradient of u.

When $\mu = 0$ in (1.2), the system (1.1) is the well-known p-Laplacian system.

Our main interest is proving global regularity results, up to the boundary, for the second derivatives of the solutions of the previous systems, with Dirichlet boundary conditions. The regularity issue for systems like (1.1) has received many efforts. Actually, these are mostly concerned with an equation in place of a system, and with the $C^{1,\alpha}_{loc}$-regularity. In the scalar case, the existence and interior integrability of the second derivatives are given in [23], for any $p > 1$; in [20] the regularity up to the boundary is obtained for any $p \in (1,2)$. For systems ($N > 1$), which is the case considered in the following, we recall [1] for $p \in (1,2)$, [15] and [24] for $p > 2$, and [17] for any $p > 1$. These papers, however, deal with homogeneous systems and the techniques, quite involved, seem to be not directly applicable to the non-homogeneous setting. In particular, [1] seems to be the only one where L^2-regularity of second derivatives for systems is obtained, but only in the interior.

Another main difference with the above papers is that we do not require differentiability of S, just Lipschitz continuity.

On the other hand, we have not found papers dealing with the equations arising from the choice (1.4) for S. This kind of model is used in various branches of Mathematical Physics as, for instance, in non-linear elasticity or in non-linear diffusion. However, our interest mainly arise from Fluid Dynamics. Indeed, we recall that a good model for non-Newtonian fluids with shear dependent viscosity is the following one

$$-\nabla\cdot\big[\,(\,\mu+|\,\mathcal{D}\,u\,|)^{p-2}\,\mathcal{D}\,u\big]+(u\cdot\nabla)u+\nabla\pi=\,f\,,\quad\nabla\cdot u=0\,,\tag{1.5}$$

which can be obtained from (1.3) by adding the contribution of the pressure field π, the convective term $(u\cdot\nabla)u$ and the divergence free constraint. For this system, the up to the boundary regularity problem has been considered in both the cases $p<2$ and $p>2$. The case $p=2$ corresponding to the well known Navier-Stokes system for Newtonian fluids. For the more general regularity results and a wide bibliography on this topic, we refer the reader to [6] [8] for $p>2$, and to [7] for $p<2$. Despite the very many contributions to the regularity issue, the up to the boundary $W^{2,2}$-regularity of solutions to (1.5) is still an open question, even for the simplified setting of "generalized" Stokes system obtained by dropping the convective term in (1.5). We mention the papers [11] and [12], which, as far as we know, are the only papers where the $W^{2,2}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$-regularity is obtained, under the additional assumption of a small force. The regularity proved below suggests that the main obstacle to the $W^{2,2}$-regularity of solutions of (1.5) is actually the presence of the pressure term.

In the sequel we cover both the cases $p<2$ and $p>2$, however with some differences, and some restrictions on the exponent p.

Case $p<2$: For $p<2$ we consider the "full gradient case' (1.1). In this case, all results hold also in the degenerate case $\mu=0$. For any bounded and sufficiently smooth domain Ω, we prove $W^{2,q}(\Omega)$ regularity, for any $q\geq 2$. Therefore, we get, as by product, the Hölder continuity, up to the boundary, of the gradient of the solution. Results are obtained for p belonging to a suitable interval $[C,2)$, where $C<2$. $W^{2,2}(\Omega)$ regularity is proved under a similar, but less restrictive, assumption on the constant $C<2$.

Case $p>2$: We prove the $W^{2,2}$-regularity in both cases, (1.1) and (1.3), provided that $\mu>0$. We restrict our proofs to the "cubic domain case" (see the next section), where the interesting boundary condition (Dirichlet) is imposed on two opposite sides, and periodicity in the other two directions. This choice, introduced in reference [4] and used in a series of other papers (see for instance [3, 5, 9, 10]), is convenient in order to work with a flat boundary and, at the same time, with a bounded domain. The main reason is that, in proving the regularity theorem for $p>2$ (see Theorem 2.1), we apply the difference quotients method: we appeal to translations parallel to the flat boundary, and then restore the normal derivatives from the equations. Then, the simplified framework of a cubic domain avoids the need of localization techniques and changes of variables.

The results can be extended to smooth domains, by following [6], [7], and [8], where the extension is done for the more involved system of non-Newtonian fluids (see also [21]). See also the Remark 5.1.

2 Notations and statement of the main results

Throughout this paper we will consider problem (1.1) in a *arbitrary* domain Ω or in a *cubic* domain Q. We denote by Ω a bounded three-dimensional domain with smooth boundary, which we assume of class C^2, and we consider the usual homogeneous Dirichlet boundary conditions

$$u_{|\partial\Omega} = 0. \tag{2.1}$$

We denote y Q the cube $Q = (\,]0,1[\,)^3$, and by Γ the two opposite faces of Q in the x_3-direction, i.e.

$$\Gamma = \{\, x : |x_1| < 1,\, |x_2| < 1,\, x_3 = 0\,\} \cup \{\, x : |x_1| < 1,\, |x_2| < 1,\, x_3 = 1\,\}.$$

We impose the Dirichlet boundary conditions on Γ

$$u_{|\,\Gamma} = 0, \tag{2.2}$$

and periodicity, with period equal to 1, in both the x_1, x_2 directions.

By $L^p(\Omega)$ and $W^{m,p}(\Omega)$, m nonnegative integer and $p \in (1, +\infty)$, we denote the usual Lebesgue and Sobolev spaces, with the standard norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{m,p}(\Omega)}$, respectively. We usually denote the above norms by $\|\cdot\|_p$ and $\|\cdot\|_{m,p}$, when the the domain is clear. Further, we set $\|\cdot\| = \|\cdot\|_2$. We denote by $W^{1,p}_0(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C^\infty_0(\Omega)$ and by $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, the strong dual of $W^{1,p}_0(\Omega)$ with norm $\|\cdot\|_{-1,p'}$. In notation concerning duality pairings, norms and functional spaces, we do not distinguish between scalar and vector fields.

We set

$$V_p(\Omega) = \left\{\, v \in W^{1,p}\,(\Omega) : \, v_{|\partial\Omega} = 0\,\right\},$$

and

$$V_p(Q) = \left\{\, v \in W^{1,p}\,(Q) : \, v_{|\Gamma} = 0,\, v \text{ is } x' - \text{periodic}\,\right\}.$$

By $V'_p(\Omega)$ and $V'_p(Q)$ we denote the dual spaces of $V_p(\Omega)$ and $V_p(Q)$, respectively.

We use the summation convention on repeated indexes, except for the index s. For any given couple of second order tensors B and C, we write $B \cdot C \equiv B_{ij}\, C_{ij}$.

We denote by the symbols c, c_1, c_2, etc., positive constants that may depend on μ; by capital letters, C, C_1, C_2, etc., we denote positive constants independent of $\mu \geq 0$ (eventually, bounded from above). The same symbol c or C may denote different constants, even in the same equation.

We set $\partial_i\, u = \frac{\partial\, u}{\partial\, x_i}$, $\partial^2_{ij}\, u = \frac{\partial^2\, u}{\partial\, x_i \partial\, x_j}$. Moreover we set $(\nabla u)_{ij} = \partial_j\, u_i$ and $(\mathcal{D}\, u)_{ij} = \frac{1}{2}\,((\nabla u)_{ij} + (\nabla u)_{ji})$. We denote by $D^2 u$ the set of all the second partial derivatives of u. The symbol $D^2_* u$ may denote any second-order partial derivative $\partial^2_{hk}\, u$ except for the derivatives $\partial^2_{33}\, u$. Moreover we set

$$|\, D^2 u\,|^2 := \sum_{i,j,k=1}^{3} \left|\, \partial^2_{jk}\, u_i\,\right|^2 \quad \text{and} \quad |\, D^2_* u\,|^2 := \sum_{\substack{i,j,k=1\\(j,k)\neq(3,3)}}^{3} \left|\, \partial^2_{jk}\, u_i\,\right|^2. \tag{2.3}$$

We define the tensor $S(A)$ as

$$S(A) = (\mu + |A|)^{p-2} A, \tag{2.4}$$

with $\mu \geq 0$ fixed constant, $p > 1$, and A an arbitrary tensor field. It is easily seen that $S(A)$ satisfies the following property: there exists a positive constant C_1 such that

$$\frac{\partial S_{ij}(A)}{\partial A_{kl}} B_{ij} B_{kl} \geq C_1 (\mu + |A|)^{p-2} |B|^2, \tag{2.5}$$

for any tensor B. Further

$$(S(A) - S(B)) \cdot (A - B) \geq C_2 \frac{|A - B|^2}{(\mu + |A| + |B|)^{2-p}}, \tag{2.6}$$

and

$$|S(A) - S(B)| \leq C_3 \frac{|A - B|}{(\mu + |A| + |B|)^{2-p}}, \tag{2.7}$$

for any pair of tensors A and B, with C_2 and C_3 positive constants. The proof of the above estimates are essentially contained in [16]. We also refer to [13] for a detailed proof.

Our aim is to prove the regularity results up to the boundary given in the theorems below. We start from the case $p > 2$.

Theorem 2.1. *Assume that $p > 2$ and $\mu > 0$. Let $f \in L^2(Q)$, and let $u \in V_p(Q)$ be a weak solution of problem* (1.1)–(2.2) *or of problem* (1.3)–(2.2). *Then $u \in W^{2,2}(Q)$, moreover there exists a constant c such that*

$$\|D^2 u\| \leq c \|f\|. \tag{2.8}$$

This theorem will be proved in the next section.

In the case $p < 2$ the parameter μ can be equal to zero, covering in this way the case of p-Laplacian systems. Here we consider a general smooth bounded domain. On the other hand, we restrict our considerations to the full gradient case. Before stating the regularity theorems for $p < 2$, let us recall two well known inequalities for the Laplace operator. The first, namely

$$\|D^2 v\| \leq C_4 \|\Delta v\|, \tag{2.9}$$

holds for any function $v \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Here $C_4 = C_4(\Omega)$. Note that if Ω is a convex domain, then $C_4 = 1$. For details we refer to [18] (Chapter I, estimate 20). The second estimate is

$$\|v\|_{2,q} \leq C_5 \|\Delta v\|_q, \tag{2.10}$$

where $C_5 = C_5(q, \Omega)$. It will be used for functions $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, and $q \geq 2$. It derives by standard estimates for solution of the Dirichlet problem for the Poisson equation.

In the sequel we set

$$C_6 = 2 - \frac{1}{C_4}, \tag{2.11}$$

where C_4 is given by (2.9), we set

$$C_7 = 2 - \frac{1}{C_5}\,, \tag{2.12}$$

where C_5 is given by (2.10), and finally we set

$$C_8 = \max\{C_6, C_7\}\,. \tag{2.13}$$

For $p < 2$ we prove the following results.

Theorem 2.2. *Assume that $\mu \geq 0$ and that $C_6 < p \leq 2$. Let $f \in L^{\frac{6}{p+1}}(\Omega)$. Then, the unique weak solution u of problem (1.1)-(2.1) belongs to $W^{2,2}(\Omega)$. Moreover, the following estimate holds*

$$\| u \|_{2,2} \leq C \left(1 + \|f\|_{\frac{6}{p+1}}^{\frac{1}{p-1}} \right) . \tag{2.14}$$

Theorem 2.3. *Let $\mu \geq 0$, $q > 2$, $C_8 < p \leq 2$, with C_8 as in (2.13). Let $f \in L^{\frac{6q}{6-2q+pq}}(\Omega)$ if $q \in (2, \frac{6}{2-p})$, $f \in L^q(\Omega)$ if $q \geq \frac{6}{2-p}$ and let u be the unique weak solution of problem (1.1)–(2.1). Then u belongs to $W^{2,q}(\Omega)$. Moreover, the following estimate holds*

$$\|u\|_{2,q} \leq C \begin{cases} \left(1 + \|f\|_{\frac{6q}{6-2q+pq}}^{\frac{1}{p-1}} \right), & \text{if } q \in \left(2, \dfrac{6}{2-p} \right), \\ \left(1 + \|f\|_{q}^{\frac{1}{p-1}} \right), & \text{if } q \geq \dfrac{6}{2-p}. \end{cases} \tag{2.15}$$

If Ω is convex the result holds for any $1 < p \leq 2$.

Corollary 2.1. *Let p, μ and f be as in Theorem 2.3. Then, if $q > 3$, the weak solution of problem (1.1)–(2.1) belongs to $C^{1,\alpha}(\overline{\Omega})$, for any $\alpha \leq 1 - \frac{3}{q}$.*

Remark 2.1. *When $p < 2$ we could extend to system (1.3) the up to the boundary regularity results obtained for system (1.1), by requiring a smallness condition on a suitable norm of f. Actually, following the arguments already used in [11] and [12] for non-Newtonian fluids, the idea is to study the regularity of suitable approximating linear problems and then prove the regularity for solutions of the nonlinear problem, by employing the method of successive approximations. For brevity, we avoid here this further development.*

3 The $W^{2,2}(Q)$-regularity: $p > 2$ and $\mu > 0$

In this section we prove Theorem 2.1. Therefore, throughout the section we work in the cubic domain Q. Let us introduce the definition of weak solutions of both the problems (1.1) and (1.3).

Definition 3.1. *Assume that $f \in V_p'(Q)$. We say that u is a* weak solution *of problem (1.1)–(2.2), if $u \in V_p(Q)$ satisfies*

$$\int_Q S(\nabla u) \cdot \nabla \varphi \, dx = \int_Q f \cdot \varphi \, dx\,, \tag{3.1}$$

for all $\varphi \in V_p(Q)$.

Definition 3.2. *Assume that* $f \in V_p'(\Omega)$. *We say that* u *is a* weak solution *of problem* (1.3)–(2.2) *if* $u \in V_p(\Omega)$ *satisfies*

$$\int_Q S(\mathcal{D}\,u) \cdot \mathcal{D}\,\varphi\,dx \;=\; \int_Q f\,\cdot\,\varphi\,dx\,, \tag{3.2}$$

for all $\varphi \in V_p(Q)$.

We recall that the existence and uniqueness of a weak solution can be obtained by appealing to the theory of monotone operators, following J.-L. Lions [19].

In proving Theorem 2.1 we focus on the symmetric gradient case, since the full gradient case is, in some respects, easier to handle. Hence we assume that S is given by

$$S(\mathcal{D}\,u) = \;(\,\mu + |\,\mathcal{D}\,u\,|)^{p-2}\,\mathcal{D}\,u\,,$$

with $\mu > 0$ and $p > 2$.

We highly follow the arguments used in [5], in the context of non-Newtonian fluids. Therefore, we will try to preserve the notations. However in [5] (due to the divergence free constraint) the symbol $D_*^2 u$ has a slightly different meaning from that introduced in definition (2.3) below, since it also includes the derivatives $\partial_{33}^2\,u_3$ (see (2.8) in [5]).

As in in [5], in order to avoid arguments already developed in other papers by the authors, we replace the use of difference quotients simply by differentiation.

It is an easy matter to obtain the following Korn's type inequality, proceeding, for instance, as in the proof given in [22].

Lemma 3.1. *There exists a constant* c *such that*

$$\|\,v\,\|_p \;+\; \|\,\nabla v\,\|_p \;\leq\; c\,\|\,\mathcal{D}\,v\,\|_p\,,$$

for all $v \in V_p(Q)$.

Lemma 3.2. *There exists a constant* c *such that*

$$\|\,D_*^2\,u\,\|_p \leq c\,\|\,\nabla_*\mathcal{D}\,u\|_p\,,$$

for all $u \in V_p(Q)$ *.*

This result reproduces Lemma 3.1 in [5], adapted to the new definition of $D_*^2\,u$. Note that $\partial_s\,u = 0$ on Γ, $s = 1, 2$.

Actually, the above two lemmas hold for each $p > 1$.

Define, for $s = 1, 2$,

$$J_s(u) \;:=\; \int_Q \nabla \cdot \Big[(\mu + |\mathcal{D}\,u\,|)^{p-2}\,\mathcal{D}\,u\Big] \cdot \partial_{ss}^2 u\,dx\,, \tag{3.3}$$

and

$$I_s(u) \;:=\; \int_Q (\mu + |\mathcal{D}\,u\,|)^{p-2}\;|\partial_s \mathcal{D}\,u\,|^2\,dx\,. \tag{3.4}$$

Lemma 3.3. *For any smooth function $u \in V_p(Q)$ the following inequality holds true*

$$J_s(u) \geq C_1\, I_s(u)\,, \tag{3.5}$$

with the constant C_1 given by (2.5)*.*

Proof. Integrating twice by parts in (3.3) one gets

$$J_s(u) = \int_Q \partial_s \left[(\mu + |\mathcal{D}\, u\,|)^{p-2}\, \mathcal{D}\, u \right] \cdot \partial_s \nabla\, u\, dx\,.$$

Note that, due to symmetry, we replace $\partial_s\, \nabla\, u$ by $\partial_s\, \mathcal{D}\, u$. From the above expression, one has

$$J_s(u) = \int_Q \frac{\partial}{\partial D_{kl}}\, \left[(\mu + \,|D|)^{p-2}\, D_{ij} \right] \frac{\partial(\mathcal{D}u)_{kl}}{\partial x_s} \frac{\partial(\mathcal{D}u)_{ij}}{\partial x_s}\, dx\,,$$

where the derivatives with respect to $D_{k\,l}$ are evaluated at the point $D = \mathcal{D}\, u$. Note that here we merely appeal to the chain rule. Then the result follows by using estimate (2.5). □

next we prove the following result which, roughly speaking, shows that the second tangential derivatives of u are square integrable.

Lemma 3.4. *Assume that $f \in L^2(Q)$. Then $D^2_* u \in L^2(Q)$ and*

$$\|D^2_* u\,\| \leq \frac{c}{\mu^{\,p-2}}\, \|\, f\, \|\,. \tag{3.6}$$

Proof. Multiply both sides of the equations (1.1) by $\partial^2_{ss}\, u$, $s = 1, 2$, and integrate over Q. By appealing to (3.3) and Lemma 3.3 it readily follows that

$$I_s(u) \leq\; c\, \|\, f\, \|\, \|\, \partial^2_{ss}\, u\, \| \leq c\, \|\, f\, \|\, \|\, \nabla\, \partial_s\, u\, \|\,,$$

hence, from Lemma 3.1 applied to $\partial_s\, u$,

$$I_s(u) \leq c\, \|\, f\, \|\, \|\, \partial_s\, \mathcal{D}\, u\, \|\,.$$

Finally, observing that

$$\mu^{p-2}\, \|\, \partial_s\, \mathcal{D}\, u\, \|^2 \leq I_s(u)\,,$$

one gets

$$\|\, \partial_s\, \mathcal{D}\, u\, \| \leq\; \frac{c}{\mu^{\;p-2}}\, \|\, f\, \|\,.$$

Application of Lemma 3.2, gives the result. □

In order to complete the proof of theorem 2.1 we have to show the integrability of the remaining second derivatives, namely the normal derivatives $\partial^2_{33}\, u$. In doing this we follow the argument used in the paper [2]; we express these derivatives, pointwisely, in terms of the derivatives of u already estimated, and solve the corresponding system in the unknowns $\partial^2_{33}\, u_i$, $i = 1, 2, 3$. Note that the main differences between this situation reference [2], are the following: in [2] the L^2-integrability of $\partial^2_{33}\, u_3$ is known from the integrability of the second tangential derivatives, thanks to the divergence free constraint, $\partial^2_{33}\, u_3 = -\partial^2_{11}\, u_1 - \partial^2_{22}\, u_2$. Hence the 3×3 linear system considered below is replaced, in [2], by a 2×2 linear system in the unknowns $\partial^2_{33}\, u_i$, $i = 1, 2$. On the other hand, in reference [2], the presence of the pressure prevents the full $W^{2,2}$-regularity.

For the missing derivatives we prove the following lemma.

Lemma 3.5. *The vector field $\partial^2_{33}\, u$ satisfies the pointwise estimate*

$$|\,\partial^2_{33} u\,| \leq c\, \left(\frac{1}{\mu^{\,p-2}}\, |f\,| + |\,D^2_* u\,| \right), \quad a.e.\ in\ Q\,. \tag{3.7}$$

Proof. Straightforward calculations show that

$$\begin{aligned} \partial_s\, [(\mu + |\mathcal{D}\, u|)^{p-2}\, \mathcal{D}\, u] &= (\mu + |\mathcal{D}\, u|)^{p-2}\, \partial_s\, \mathcal{D}\, u \\ &\quad + (p-2)(\mu + |\mathcal{D}\, u|)^{p-3}\, |\mathcal{D}\, u|^{-1}\, (\mathcal{D}\, u \cdot \partial_s\, \mathcal{D}\, u)\, \mathcal{D}\, u\,. \end{aligned} \tag{3.8}$$

For convenience, we set $\mathcal{D}_{jk} = (\mathcal{D}\, u)_{jk}$ and $B := (\mu + |\mathcal{D}\, u|)$. By using (3.8), the j^{th} equation (1.1), for any $j = 1, 2, 3$, takes the following form

$$B^{p-2} \left(\partial^2_{kk} u_j + \partial^2_{jk} u_k\right) + (p-2) B^{p-3} |\mathcal{D}\, u|^{-1} \mathcal{D}_{lm} \mathcal{D}_{jk} \left(\partial^2_{km} u_l + \partial^2_{kl} u_m\right) = -2 f_j. \tag{3.9}$$

Let us write the previous three equations as a system in the unknowns $\partial^2_{33}\, u_j$. For $j = 1, 2$ we have

$$B^{p-2}\, \partial^2_{33}\, u_j + 2(p-2)\, B^{p-3}\, |\mathcal{D}\, u|^{-1}\, \mathcal{D}_{j3} \sum_{l=1}^{3} \mathcal{D}_{l3}\, \partial^2_{33}\, u_l = F_j\ - 2\, f_j\,, \tag{3.10}$$

where

$$\begin{aligned} F_j := &\ - B^{p-2} \sum_{k=1}^{2} \partial^2_{kk}\, u_j - B^{p-2} \sum_{k=1}^{3} \partial^2_{jk}\, u_k \\ & -2(p-2)\, B^{p-3}\, |\mathcal{D}\, u|^{-1} \sum_{\substack{l,m,k=1 \\ (m,k)\neq(3,3)}}^{3} \partial^2_{km}\, u_l\, \mathcal{D}_{jk}\, \mathcal{D}_{lm}\,. \end{aligned} \tag{3.11}$$

For $j = 3$ we have

$$2B^{p-2}\, \partial^2_{33}\, u_j + 2(p-2)\, B^{p-3}\, |\mathcal{D}\, u|^{-1}\, \mathcal{D}_{j3} \sum_{l=1}^{3} \mathcal{D}_{l3}\, \partial^2_{33}\, u_l = F_j\ - 2\, f_j\,, \tag{3.12}$$

where, for $j = 3$,

$$\begin{aligned} F_j := &\ -B^{p-2} \sum_{k=1}^{2} \partial^2_{kk}\, u_j -\ B^{p-2} \sum_{k=1}^{2} \partial^2_{jk}\, u_k \\ & -2(p-2)\, B^{p-3}\, |\mathcal{D}\, u|^{-1} \sum_{\substack{l,m,k=1 \\ (m,k)\neq(3,3)}}^{3} \partial^2_{km}\, u_l\, \mathcal{D}_{jk}\, \mathcal{D}_{lm}\,. \end{aligned} \tag{3.13}$$

The equations (3.10), for $j = 1, 2$, together with the equation (3.13) for $j = 3$ can be treated as a 3×3 linear system in the unknowns $\partial^2_{33} u_j$, $j = 1, 2, 3$. Multiply all the three equations by B^{2-p}. We denote the elements of the matrix $A = A(x)$ associated with this system as a_{jl}, where $j, l = 1, 2, 3$. Then, we can write the system in a compact form as

$$a_{jl}\, \partial^2_{33}\, u_l = G_j\,, \tag{3.14}$$

where the elements of the matrix of the system are given by

$$a_{jl} := \delta_{jl} + 2(p-2)\, \left(B\, |\mathcal{D}\, u|\right)^{-1} \mathcal{D}_{j3} \mathcal{D}_{l3}\,,$$

for $j = 1, 2$, by

$$a_{jl} := 2\delta_{jl} + 2(p-2)\, \left(B\, |\mathcal{D}\, u|\right)^{-1} \mathcal{D}_{j3} \mathcal{D}_{l3}\,,$$

and for $j = 3$, and

$$G_j := B^{2-p} \left(F_j \; - 2\, f_j \right)\,. \tag{3.15}$$

Note that $a_{jl} = a_{lj}$; moreover, if ξ denotes any vector field then

$$a_{jl} \xi_j \xi_l = |\, \xi\, |^2 + \xi_3^2 + 2\, (p-2)\, \left(\, B\, |\mathcal{D}\, u|\, \right)^{-1} [\, \mathcal{D}\, u \cdot \xi\,]^2_3\,.$$

Hence, the matrix $A = (a_{jl})$ is also definite positive, a.e. in $x \in Q$, and the previous identity shows that

$$a_{jl}\, \xi_j\, \xi_l \geq \, |\, \xi\, |^2\,.$$

By setting $\xi = \partial^2_{33}\, u$, we have obtained

$$|\, \partial^2_{33}\, u\, |^2 \leq \, |\, G\, |\, |\, \partial^2_{33}\, u\, |, \text{ a.e. in } Q\,, \tag{3.16}$$

where, obviously, by G we mean the vector (G_1, G_2, G_3). Noting that, from (3.15), (3.11) and (3.13), there holds

$$|\, G_j\, | \leq \frac{2}{\mu^{\,p-2}}\, |f_j\, | + c\, |\, D^2_* u\, |\,, \text{ a.e. in } Q\,, \tag{3.17}$$

from this estimate and (3.16) we get (3.5). □

Finally, by combining (3.6) and (3.5) we readily obtain

$$\|D^2\, u\, \| \leq \frac{c}{\mu^{\,p-2}}\, \|\, f\, \|$$

which is just (2.8). The proof of theorem 2.1 is accomplished.

4 A regularity result for an approximating system: $p < 2$.

In the sequel we introduce an auxiliary positive parameter η and study the regularity of solutions of the following approximating problem

$$\begin{cases} -\eta\, \Delta v - \nabla \cdot S\, (\nabla\, v) = \, f, \quad \text{in } \Omega\,, \\ v = \, 0, \quad \text{on } \partial\Omega\,, \end{cases} \tag{4.1}$$

with S defined by (2.4), $\eta > 0$, $\mu > 0$ and $p \in (1,2)$. The solutions v_η satisfy the estimate (4.15) below, with the constant C independent of η. This allows us to show that, as $\eta \to 0$, v_η tends, in a suitable sense, to the solution v of problem (4.1) with $\eta = 0$. A similar situation occurs, with respect to μ, as $\mu \to 0$.

We explicitly note that we introduce the above model to approximate our nonlinear problem (1.1) when μ vanishes.

Let us introduce the definition of weak solution of both the problems (4.1) and (1.1)–(2.1).

Definition 4.1. *Assume that $f \in V_2'(\Omega)$. We say that v is a* weak solution *of problem* (4.1) *if $v \in V_2(\Omega)$ and satisfies*

$$\eta \int_\Omega \nabla v \cdot \nabla \varphi \, dx + \int_\Omega S(\nabla v) \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \varphi \, dx \,, \tag{4.2}$$

for all $\varphi \in V_2(\Omega)$.

Definition 4.2. *Assume that $f \in V_p'(\Omega)$. We say that u is a* weak solution *of problem* (1.1)–(2.1)*, if $u \in V_p(\Omega)$ satisfies*

$$\int_\Omega S(\nabla u) \cdot \nabla \varphi \, dx = \int_\Omega f \cdot \varphi \, dx \,, \tag{4.3}$$

for all $\varphi \in V_p(\Omega)$.

As recalled in the previous section, the existence and uniqueness of a weak solution is known by the theory of monotone operators.

We start by proving the $W^{2,2}$-regularity result stated in proposition 4.1 below. In (4.4), the dependence of the constant c on Ω_0, η and μ is omitted since the aim of the proposition is just to ensure that second derivatives are well defined a.e. in Ω. Following the notations introduced in section 2, by capital letters, C, C_1, C_2, etc., we denote positive constants independent of μ and of η also.

Proposition 4.1. *Let $p \in (1,2)$, $f \in L^2(\Omega)$, and v be a weak solution of problem* (4.1)*. Then $v \in W^{2,2}_{loc}(\Omega)$ and, for any fixed open set $\Omega_0 \subset\subset \Omega$, there exists a constant c such that*

$$\| D^2 v \|_{L^2(\Omega_0)} \leq c \, \|f\| \,. \tag{4.4}$$

Proof. As in the previous section, we formally use derivatives instead of difference quotients, to make the computation simpler. Fix an open set $\Omega_0 \subset\subset \Omega$. Let ζ be a $C^2_0(\Omega)$-function, such that $0 \leq \zeta(x) \leq 1$ in Ω, and $\zeta(x) = 1$ in Ω_0. Multiplying the first three equations in (4.1) by $-\nabla \cdot (\zeta^2 \nabla v)$ and integrating over Ω we get

(4.5)
$$\eta \int_\Omega \partial^2_{jj} v_i \, \partial_h \left(\zeta^2 \partial_h v_i\right) dx + \int_\Omega \partial_j \left[(\mu + |\nabla v|)^{p-2} (\nabla v)_{ij} \right] \partial_h \left(\zeta^2 \partial_h v_i\right) dx$$
$$= - \int_\Omega f_i \, \partial_h \left(\zeta^2 \partial_h v_i\right) dx \,.$$

By integration by parts, with respect to x_j and x_h, on the left-hand side one has

(4.6)
$$\begin{aligned}\eta \int_\Omega (\partial^2_{jh}\, v_i)^2\, \zeta^2\, dx + \int_\Omega \partial_h\, \left[(\mu + |\nabla v|)^{p-2}\, (\nabla v)_{i\,j}\right]\, \partial_h (\nabla v)_{i\,j}\, \zeta^2\, dx \\ = -\eta \int_\Omega \left(\partial^2_{jh}\, v_i\right)\, R_{i\,j\,h}(x)\, dx - \int_\Omega \partial_h\, \left[(\mu + |\nabla v|)^{p-2}\, (\nabla v)_{i\,j}\right]\, R_{i\,j\,h}(x)\, dx \\ - \int_\Omega f_i \left(\partial^2_{hh}\, v_i\right) \zeta^2\, dx - 2 \int_\Omega f_i\, (\partial_h\, v_i)\, \zeta\, (\partial_h\, \zeta)\, dx = \sum_{i=1}^4 I_i\,,\end{aligned}$$

where, with obvious notation, $R_{i\,j\,h}$ are lower order terms satisfying estimates

$$|R_{i\,j\,h}(x)| \leq c\, |\zeta|\, |\nabla\, \zeta|\, |\nabla v|\,. \tag{4.7}$$

As in the proof of Lemma 3.3, it is easy to verify, by appealing to (2.5), that

(4.8)
$$\int_\Omega \partial_h\, \left[(\mu + |\nabla v|)^{p-2}\, (\nabla v)_{i\,j}\right]\, \partial_h (\nabla v)_{i\,j}\, \zeta^2\, dx \geq c \int_\Omega (\mu + |\nabla v|)^{p-2}\, |D^2 v|^2\, \zeta^2\, dx\,.$$

On the other hand, by Hölder's and Cauchy-Schwartz inequalities,

$$|I_1| \leq \epsilon\, \|\, |D^2 v|\, \zeta\|^2 + c(\epsilon)\, \|\nabla\, \zeta\|^2_\infty\, \|\nabla\, v\|^2\,, \tag{4.9}$$

$$|I_3| \leq \epsilon\, \|\, |D^2 v|\, \zeta\|^2 + c(\epsilon)\, \|f\|^2\,, \tag{4.10}$$

and

$$|I_4| \leq c\, \|\nabla\, \zeta\|_\infty \|f\|\, \|\nabla\, v\|\,. \tag{4.11}$$

Further, by using the estimate

$$\frac{\partial S_{i\,j}(A)}{\partial A_{k\,l}} \leq c\, (\mu + |\, A\,|)^{p-2},$$

we have

$$|I_2| \leq c \int_\Omega (\mu + |\nabla v|)^{p-2}\, |D^2\, v|\, |\zeta|\, |\nabla\, \zeta|\, |\nabla v|\, dx\,, \tag{4.12}$$

and, by the Cauchy-Schwartz inequality,

$$|I_2| \leq \epsilon\, \|\, |D^2 v|\, \zeta\|^2 + c(\epsilon)\, \|\nabla\, \zeta\|^2_\infty\, \|\nabla\, v\|^2_2\,. \tag{4.13}$$

Further, from (4.5) together with $\|\nabla\, v\| \leq c\, \|f\|$, it readily follows that

$$\|\, |D^2 v|\, \zeta\| \leq c\, \|f\|\,. \tag{4.14}$$

Hence (4.4) follows. □

Our next step is to get a global estimate for the L^2-norm of the second derivatives, uniform in η. This is the aim of the following proposition.

Proposition 4.2. *Let $C_6 < p < 2$, with C_6 given by* (2.11). *Let $f \in L^{\frac{6}{p+1}}(\Omega)$, and let v be a weak solution of problem* (4.1). *Then v belongs to $W^{2,2}(\Omega)$. Moreover, there exists a constant C such that*

$$\| v \|_{2,2} \leq C \left(1 + \|f\|_{\frac{6}{p+1}}^{\frac{1}{p-1}} \right) . \tag{4.15}$$

Proof. In order to avoid an useless dependence on μ, we assume, without loss of generality, $\mu \in (0, 1]$. At first note that, by replacing φ by v in (4.2) it is easy to get the following estimate for $\|\nabla v\|_p$, uniformly in η,

$$\|\nabla v\|_p^p \leq \mu^p\, |\Omega| + 2^{2-p} \int_\Omega f \cdot v\, dx \leq C \left(1 + \int_\Omega f \cdot v\, dx \right) .$$

Since, by Proposition 4.1, $v \in W^{2,2}_{loc}(\Omega)$, the i^{th} equation (4.1) can be written almost everywhere in Ω as

$$\begin{aligned} &\eta \Delta v_i + (\mu + |\nabla v|)^{p-2}\, \Delta v_i \\ &\qquad\qquad + (p-2)(\mu + |\nabla v|)^{p-3}\, |\nabla v|^{-1} \nabla v \cdot (\partial_j \nabla v)\, \partial_j\, v_i = - f_i\,. \end{aligned}$$

By multiplying both sides by Δv_i and summing over $i = 1, 2, 3$, we have

$$\begin{aligned} &\eta\, |\Delta v|^2 + (\mu + |\nabla v|)^{p-2}\, |\Delta v|^2 \\ &= (2-p)(\mu + |\nabla v|)^{p-3} |\nabla v|^{-1} \nabla v \cdot (\partial_j \nabla v)\, \partial_j\, v_i\, \Delta v_i - f_i\, \Delta v_i\,, \text{ a.e. in } \Omega\,. \end{aligned}$$

Therefore we can drop the term $\eta|\Delta v|^2$, bounding the left-hand side from below by $(\,\mu + |\nabla v|\,)^{p-2}\, |\Delta v|^2$. Multiplying the estimate thus obtained by $(\mu + |\nabla v|)^{2-p}$ and then integrating over Ω we get

$$\int_\Omega |\, \Delta v\,|^2\, dx \leq (2-p) \int_\Omega |\, D^2 v\,|\,|\, \Delta v\,|\, dx + \int_\Omega (\,\mu + |\nabla v|\,)^{2-p}\, |\, f\,|\,|\, \Delta v\,|\, dx\,,$$

where we have used the following estimate (for details see the Appendix)

$$|\, \nabla v \cdot (\partial_j\, \nabla v)\, (\partial_j\, v_i\,)\, \Delta v_i\,| \leq |\, \nabla v\,|^2\, |\, D^2\, v\,|\,|\, \Delta v\,|\,.$$

Observing that $(\mu + |\nabla v|)^{2-p} \leq \mu^{2-p} + |\nabla v|^{2-p}$, using Hölder's inequality and, finally, by dividing both sides by $\|\, \Delta v\,\|$, we get

$$\|\, \Delta v\,\| \leq (2-p)\, \|\, D^2 v\,\| + \|\, |\nabla v|^{2-p}\, f\,\| + \|\, f\,\|\,. \tag{4.16}$$

Let us estimate the first two terms on the right-hand side. For the first term we employ estimate (2.9). As far as the second term in (4.16) is concerned, by applying Hölder's inequality with exponents $3/(2-p)$ and $3/(p+1)$, the Sobolev embedding of $W^{2,2}(\Omega)$ in $W^{1,6}(\Omega)$, and by appealing to the estimate (2.10) with $q = 2$, we get

$$\|\, |\nabla v|^{2-p}\, f\| \leq \|\, \nabla v\,\|_6^{2-p}\, \|\, f\|_{\frac{6}{p+1}} \leq C\, \|\, \Delta v\,\|^{2-p}\, \|\, f\|_{\frac{6}{p+1}}\,.$$

By using the above estimates in (4.16), we get

$$\|\, \Delta v\,\| \leq (2-p)\, C_4 \|\, \Delta v\,\| + C\, \|\, \Delta v\,\|^{2-p}\, \|\, f\|_{\frac{6}{p+1}} + \|\, f\,\|\,.$$

Recalling that, by assumption, $p > 2 - \frac{1}{C_4} = C_6$ and $2 - p < 1$, it is easy to recognize that the estimate

$$\| \Delta v \| \leq C \left(1 + \|f\|_{\frac{6}{p+1}}^{\frac{1}{p-1}} \right) \tag{4.17}$$

holds. Note that the constant C may blow up as p tends to C_6. By using once again (2.10) we prove (4.15). $\square$

5 The $W^{2,2}$-regularity result: $p < 2$.

Proof of Theorem 2.2. We deal separately with the case $\mu > 0$ and the degenerate case $\mu = 0$.

The case $\mu > 0$ - Consider the "sequence" (v_η) consisting of the solutions to problem (4.1), for $\eta > 0$. By the above proposition the sequence (v_η) is uniformly bounded in $W^{2,2}(\Omega)$. Therefore, by Rellich's theorem, there exists a field $u \in W^{2,2}(\Omega)$ and a subsequence, which we continue to denote by (v_η), such that $v_\eta \rightharpoonup u$ weakly in $W^{2,2}(\Omega)$, and strongly in $W^{1,q}(\Omega)$ for any $q < 6$. Let us prove that

$$\int_\Omega S(\nabla u) \cdot \nabla \varphi \, dx = \lim_{\eta \to 0^+} \left\{ \int_\Omega S(\nabla v_\eta) \cdot \nabla \varphi \, dx + \eta \int_\Omega \nabla v_\eta \cdot \nabla \varphi \, dx \right\}, \tag{5.1}$$

for any $\varphi \in C_0^\infty(\Omega)$. By applying (2.7) and then Hölder's inequality, we get

$$\begin{aligned} &\left| \int_\Omega S(\nabla u) \cdot \nabla \varphi \, dx - \int_\Omega S(\nabla v_\eta) \cdot \nabla \varphi \, dx \right| \\ &\leq c \int_\Omega (\mu + |\nabla u| + |\nabla v_\eta|)^{p-2} \, |\nabla u - \nabla v_\eta| \, |\nabla \varphi| \, dx \\ &\leq c \int_\Omega |\nabla u - \nabla v_\eta|^{p-1} \, |\nabla \varphi| \, dx \leq c \, \| \nabla v_\eta - \nabla u \|_p^{p-1} \, \| \nabla \varphi \|_p \, . \end{aligned}$$

The right-hand side of the last inequality tends to zero, as η goes to zero, thanks to the strong convergence of v_η to u in $W^{1,p}(\Omega)$. Further

$$\left| \eta \int_\Omega \nabla v_\eta \cdot \nabla \varphi \, dx \right| \leq \eta \, \|\nabla v_\eta\| \, \|\nabla \varphi\| \, ,$$

where the right-hand side tends to zero as η goes to zero. Finally, observing that for any $\eta > 0$ and any $\varphi \in C_0^\infty(\Omega)$ the right-hand side of (5.1) is equal to $\int_\Omega f \cdot \varphi \, dx$, we show that u satisfies the integral identity (4.3) for any $\varphi \in C_0^\infty(\Omega)$. By a standard argument we show that u satisfies the integral equation (4.3), for any $\varphi \in V_p(\Omega)$. Hence u is a weak solution of (1.1), and belongs to $W^{2,2}(\Omega)$. Moreover, (2.14) follows from the relation $\| u \|_{2,2} \leq \liminf_{\eta \to 0^+} \| v_\eta \|_{2,2}$, together with (4.15). From the uniqueness of weak solutions we obtain the desired result.

The case $\mu = 0$ - Let us denote by u_μ the sequence of solutions of (1.1) for the different values of $\mu > 0$. We have shown that the sequence (u_μ) is uniformly bounded in $W^{2,2}(\Omega)$. Therefore, exactly as above, we can prove the weak convergence of a suitable subsequence in $W^{2,2}(\Omega)$, and the strong convergence in $W^{1,q}(\Omega)$ for any $q < 6$, to the solution $u \in W^{2,2}(\Omega)$ of the problem (1.1) with $\mu = 0$. In this regard note that estimate (2.7) also holds with $\mu = 0$.

Convex domains - Finally we prove the last assertion in the Theorem 2.2. If Ω is convex (or, for instance, the cubic domain in Theorem 2.1), then the Theorem holds for any $p > 1$. Indeed, as previously observed, for a smooth convex domain the estimate (2.9) holds with $C_4 = 1$, hence the lower bound $p > C_6 = 1 - \frac{1}{C_4}$ is merely $p > 1$. $\square$

Remark 5.1. *We can adapt the above arguments to the case $p > 2$. Via a result similar to Proposition 4.1, one shows that the solution v of the approximated system 4.1 belongs to $W^{2,2}_{loc}(\Omega)$ and then, reasoning as in the proof of Proposition 4.2, one obtains a global estimate for v in $W^{2,2}(\Omega)$, uniformly in η, with a restriction on the range of p, $p \in (2, 2 + \frac{1}{C_5})$, C_5 as in (2.9). Hence, as in the above Theorem 2.2, one proves that the solution of (1.1), with $\mu > 0$, belongs to $W^{2,2}(\Omega)$. This result has the advantage to hold in a general smooth domain, without need of localization techniques. However, such estimate is not uniform in μ, hence one cannot cover the case $\mu = 0$.*

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6 The $W^{2,q}$-regularity result: $q \geq 2$ and $p < 2$.

Proof of Theorem 2.3. Since $C_6 < p < 2$, From Theorem 2.2, we already know that the solution u of problem (1.1) belongs to $W^{2,2}(\Omega)$. Therefore, by multiplying the equations (1.1) by $(\mu + |\nabla u|)^{2-p}$, we can write the system, a.e. in Ω, as

$$-\Delta u - (p-2)\frac{\nabla u \cdot \nabla\nabla u \cdot \nabla u}{(\mu + |\nabla u|)\,|\nabla u|} = f\,(\mu + |\nabla u|)^{2-p}\ , \tag{6.1}$$

where we have used the notation $\nabla u \cdot \nabla\nabla u \cdot \nabla u$ to denote the vector whose i^{th} component is $\nabla u \cdot (\partial_j \nabla u)\,\partial_j u_i$.

Let us assume, for the moment, that $u \in W^{2,q}(\Omega)$ and let us find a L^q-estimate of its second derivatives. We follow an argument similar to that used for proving the $W^{2,2}$-estimates of u. We take equations (6.1), multiply both sides by $\Delta u\,|\Delta u|^{q-2}$ and integrate over Ω. We get (for details see the Appendix)

$$\int_\Omega |\,\Delta u\,|^q\,dx \leq (2-p)\int_\Omega |\,D^2 u\,|\,|\,\Delta u\,|^{q-1}\,dx + \int_\Omega (\,\mu + |\nabla u|\,)^{2-p}\,|\,f\,|\,|\,\Delta u\,|^{q-1}\,dx\,.$$

By using Hölder's inequality, since $(\mu + |\nabla u|)^{2-p} \leq 1 + |\nabla u|^{2-p}$, we have

$$\begin{aligned} \|\,\Delta u\,\|_q^q \leq\ & (2-p)\,\|\,D^2 u\,\|_q \|\,\Delta u\,\|_q^{q-1} \\ & + \|\,f\,\|_q \|\,\Delta u\,\|_q^{q-1} + \|\,|\nabla u|^{2-p} f\,\|_q \|\,\Delta u\,\|_q^{q-1}\,. \end{aligned} \tag{6.2}$$

By dividing both sides by $\|\,\Delta u\,\|_q^{q-1}$, one gets

$$\|\,\Delta u\,\|_q \leq (2-p)\,\|\,D^2 u\,\|_q + \|\,f\,\|_q + \|\,|\nabla u|^{2-p} f\,\|_q. \tag{6.3}$$

The first term on the right-hand side of (6.3) can be estimated via inequality (2.10).

Let us assume that $q \in \left(2, \frac{6}{2-p}\right)$. Then, the term $(\,\mu + |\nabla u|\,)^{2-p}\,|\,f\,|$ belongs to $L^q(\Omega)$, since, by applying Hölder's inequality, with exponents $6/[q(2-p)]$ and $6/(6-2q+pq)$, and the Sobolev embedding theorem, we get

$$\|\,|\nabla u|^{2-p}\,f\|_q \leq \|\,\nabla u\,\|_6^{2-p}\,\|\,f\|_{\frac{6q}{6-2q+pq}} \leq c\,\|\,u\,\|_{2,2}^{2-p}\,\|\,f\|_{\frac{6q}{6-2q+pq}}\,. \tag{6.4}$$

By using (6.4) and then the estimate (2.14) for the second derivatives of u, from (6.3) one easily gets that, for $p > 2 - \frac{1}{C_5} = C_7$,

$$\|u\|_{\,2,q} \leq C\left(1 + \|f\|_{\frac{6q}{6-2q+pq}}^{\frac{1}{p-1}}\right)\,, \tag{6.5}$$

where C is independent of μ.

With this result in hand, it is easy to prove the $W^{2,q}$-regularity of the solution for any $q \geq \frac{6}{2-p}$. Indeed, assume that $f \in L^q(\Omega)$, for such an exponent q. Then $f \in L^{\frac{6s}{6-2s+ps}}(\Omega)$, for any $s \in \left(3, \frac{3}{2-p}\right)$. Then, from the first part of the proof, the solution u belongs to $W^{2,s}(\Omega)$, and, by embedding, $\nabla u \in L^\infty(\Omega)$. We can now estimate the L^q-norm of the force term as follows:

$$\|\,|\nabla u|^{2-p}\,f\|_q \leq \|\,\nabla u\,\|_\infty^{2-p}\,\|\,f\|_q \leq c\,\|\,u\,\|_{2,s}^{2-p}\,\|\,f\|_q\,. \tag{6.6}$$

Then, by replacing estimate (6.4) with (6.6), we can repeat verbatim the arguments used above and show that u is bounded in $W^{2,q}(\Omega)$, uniformly in μ, and the following estimate holds

$$\|\,u\,\|_{2,q} \leq C\left(1 + \|f\|_q^{\frac{1}{p-1}}\right). \tag{6.7}$$

The same arguments used in the proof of the previous Theorem 2.2 show that the result also holds in the degenerate case $\mu = 0$.

The previous arguments are formal, since we do not know that the solution belongs to $W^{2,q}(\Omega)$ yet. However the following arguments make everything rigorous. Let us consider the following problem

$$\begin{cases} -\Delta w^\varepsilon - (p-2)\dfrac{\nabla J_\varepsilon(u)\cdot\nabla\nabla w^\varepsilon\cdot\nabla J_\varepsilon(u)}{(\mu +\ J_\varepsilon(|\nabla u|))\ J_\varepsilon(|\nabla u|)} = f\,(\mu + |\nabla\, u|)^{2-p}\,, & \text{in } \Omega\,, \\ w^\varepsilon = 0\,, \quad \text{on } \partial\Omega\,, \end{cases} \tag{6.8}$$

in the unknown w^ε, where J_ε denotes the Friedrichs mollifier. The coefficients of the previous system belong to $C^\infty(\mathbb{R}^n)$. We can also write the system in divergence form as follows:

$$-\partial_h\left[\,m_{ijhk}(x)\,\partial_k\,w_j^\varepsilon\,\right] + (p-2)\,\partial_h\left[\,c_{ijhk}^\varepsilon(x)\,\right]\,\partial_k\,w_j^\varepsilon = f\,(\mu + |\nabla\,u|)^{2-p}\,, \tag{6.9}$$

where

$$m_{ijhk}(x) = \delta_{ij}\delta_{hk} + (p-2)\,c_{ijhk}^\varepsilon(x)$$

and

$$c_{ijhk}^\varepsilon(x) = \partial_h J_\varepsilon(u_i)\,\partial_k J_\varepsilon(u_j)\frac{1}{(\mu +\ J_\varepsilon(|\nabla u|))\ J_\varepsilon(|\nabla u|)}.$$

Further, let

$$c_{ijhk}(x) = (\,\partial_h\, u_i\,)\,(\,\partial_k\, u_j\,)\frac{1}{(\mu + |\nabla u|)\;|\nabla u|}\,.$$

We recall the following well known estimate

$$|\nabla J_\varepsilon(u)| = |J_\varepsilon(\nabla u)| \leq J^\varepsilon(|\nabla u|)\,. \tag{6.10}$$

From (6.10) we get

$$|c^\varepsilon_{ijhk}(x)| \leq 1, \quad \text{uniformly in } x\,, \varepsilon\,, \text{ and } \mu\,. \tag{6.11}$$

As a consequence of the above estimate, we get that system (6.8) (or, equivalently, (6.9)) is a linear elliptic system with regular coefficients. For such kind of system it is well known that if the prescribed force term, say F, belongs to $L^q(\Omega),\ q \geq 2$, then the solution belongs to $W^{2,q}(\Omega)$ (see, for instance, [14]). Following the arguments above with u replaced by w^ε, by using (6.10) and (6.11), it is straightforward to obtain the estimates (6.5) and (6.7) for w^ε. Note that such estimates are uniform both in μ and ε. Then there exists a subsequence, still denoted by w^ε, and $w \in W^{2,q}(\Omega)$ such that, as ε goes to zero, w^ε converges to w, weakly in $W^{2,q}$ and strongly in $W^{1,r}(\Omega)$, for any r if $q \geq 3$ and for any $r \in \left(1, \frac{3q}{3-q}\right)$ if $q < 3$. Let us show that w is a solution of the following system

$$-\Delta w - (p-2)\frac{\nabla u \cdot \nabla\nabla w \cdot \nabla u}{(\mu + |\nabla u|)\;|\nabla u|} = f\,(\mu + |\nabla\, u|)^{2-p}\,. \tag{6.12}$$

For this purpose we write equations (6.8) and (6.12) in the weak form and take their difference, side by side. This leads to the expression

$$\begin{aligned}&\int_\Omega (\partial_h w^\varepsilon_i - \partial_h w_i)\,\partial_h \varphi_i\,dx + (2-p)\int_\Omega \left(c^\varepsilon_{ijhk} - c_{ijhk}\right)\partial^2_{hk} w^\varepsilon_j\,\varphi_i\,dx\\ &+ (2-p)\int_\Omega c_{ijhk}\left(\partial^2_{hk} w^\varepsilon_j - \partial^2_{hk} w_j\right)\,\varphi_i\,dx,\end{aligned} \tag{6.13}$$

for any $\varphi \in C^\infty_0(\Omega)$. The first integral goes to zero, thanks to the strong convergence of w^ε to w in $W^{1,2}(\Omega)$. Concerning the second integral, from the L^p-convergence of the mollified functions it follows the almost everywhere convergence of a subsequence. Therefore, c^ε_{ijhk} converges a.e. to c_{ijhk}. From (6.11), recalling that Ω is bounded and by using the dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0}\int_\Omega |c^\varepsilon_{ijhk} - c_{ijhk}|^2\,dx = 0\,. \tag{6.14}$$

Hence the second integral in (6.13) goes to zero. The last integral in (6.13) tends to zero thanks to (6.11) and the weak convergence of w^ε to w in $W^{2,q}(\Omega)$.

Finally, it is easy to verify that w coincides with u. Indeed, by taking the difference of (6.1) and (6.12), side by side, and by setting $V = u - w$, we have

$$\begin{cases} -\Delta V - (p-2)\dfrac{\nabla u \cdot \nabla\nabla V \cdot \nabla u}{(\mu + |\nabla u|)\;|\nabla u|} = 0, & \text{in } \Omega\,,\\ V = 0\,, \quad \text{on } \partial\Omega\,. \end{cases}$$

Then, multiplying by ΔV, integrating over Ω and then applying arguments already used, one readily recognizes that, under our assumptions on p, the vector V satisfies $\| \Delta V \| = 0$, hence $V = 0$, by uniqueness.

$\square$

The result in Corollary 2.1 is an immediate consequence of Theorem 2.3. Hence we omit any detail of the proof.

7 Appendix

Our aim is to show the estimate

$$|\, I \,| := |\, \nabla v \cdot (\partial_j \nabla v) \, (\partial_j \, v_i) \, \Delta v_i \,| \le \, |\nabla v|^2 \, |D^2 \, v| \, |\Delta v| \, .$$

For convenience, in the sequel we sometimes avoid the summation convention and explicitly write the sums, even if repeated indexes appear. We recall that

$$(\, D^2 v_k)^2 \, := \sum_{j,h=1}^{3} \big| \, \partial^2_{jh} \, v_k \, \big|^2 \qquad \text{and} \qquad |\, D^2 v|^2 := \sum_{k=1}^{3} (\, D^2 v_k)^2 \, := \sum_{k,j,h=1}^{3} \big| \, \partial^2_{jh} \, v_k \, \big|^2 \, .$$

We introduce the vectors b and w, whose components are defined as follows

$$b_j := (\partial_j \, v) \cdot \, \Delta v \, , \quad w_k^2 := \sum_{j,h=1}^{3} \big(\, (\partial_h \, v_k) \, b_j \big)^2 \, .$$

The moduls of vector b satisfies the following estimate:

$$|\, b \,| = \sum_{j=1}^{3} \, b_j^2 \, \le \sum_{j=1}^{3} |\, \partial_j \, v|^2 |\Delta v|^2 = |\Delta v|^2 \sum_{j=1}^{3} \sum_{i=1}^{3} (\, \partial_j \, v_i)^2 = |\Delta v|^2 |\nabla v|^2.$$

Hence

$$w_k^2 = \sum_{h=1}^{3} \, (\partial_h \, v_k)^2 \sum_{j=1}^{3} \, b_j^2 = |\nabla v_k|^2 |\Delta v|^2 |\nabla v|^2 \, . \tag{7.1}$$

Moreover

$$\begin{aligned} |\, I \,| = \left| \sum_{j,h,k=1}^{3} \, (\partial_h \, v_k) \left(\partial^2_{hj} \, v_k \right) \, b_j \right| &\le \sum_{k=1}^{3} \left| \sum_{j,h=1}^{3} \, \left(\partial^2_{hj} \, v_k \right) \, (\partial_h \, v_k) \, b_j \right| \\ &\le \sum_{k=1}^{3} \sqrt{\sum_{j,h=1}^{3} \, \left(\partial^2_{hj} \, v_k \right)^2} \, \sqrt{\sum_{j,h=1}^{3} \, \big(\, (\partial_h \, v_k) \, b_j \big)^2} \, , \end{aligned}$$

where, in the last step, we have used that, for any pair of tensors A and B, there holds $|A \cdot B| \le |A| \, |B|$. Hence, by the above notations and estimate (7.1), we get

$$\begin{aligned} |\, I \,| &\le \sum_{k=1}^{3} |D^2 v_k| \, |w_k| \le \, |\Delta v| \, |\nabla v| \, \sum_{k=1}^{3} |D^2 v_k| \, |\nabla v_k| \\ &\le \, |\Delta v| \, |\nabla v| \sqrt{\sum_{k=1}^{3} |D^2 v_k|^2} \, \sqrt{\sum_{k=1}^{3} |\nabla v_k|^2} = \, |\Delta v| \, |\nabla v|^2 \, |D^2 v| \, , \end{aligned}$$

that is what we wanted to prove.

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