An approach to slip boundary conditions in the half-space. Applications to inviscid limits and to non-Newtonian fluids.

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Abstract

The resolution of a very large class of linear and non-linear, stationary and evolutive partial differential problems in the half-space (or similar) under the slip boundary condition is reduced here to that of the corresponding results for the same problem in the whole space. The approach is particularly suitable for proving new results in strong norms. To determine whether this extension is available, turns out to be a simple exercise. The verification depends on a few general features of the functional space X related to the space variables. Hence, we present an approach as much as possible independent of the particular space X. We appeal to a reflection technique. Hence a crucial assumption is to be in the presence of flat boundaries (see below).

Instead of stating "general theorems" we rather prefer to illustrate how to apply our results by considering a couple of interesting problems. As a main example, we show that the resolution of a class of problems for the evolution Navier-Stokes equations under a slip boundary condition can be reduced to that of the corresponding results for the Cauchy problem. In particular, we show that sharp vanishing viscosity limit results that hold for the evolution Navier-Stokes equations in the whole space can be extended to the boundary value problem in the half-space. We also show some applications to non-Newtonian fluid problems.

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1 Introduction

In the following we consider the slip boundary condition

$$\begin{cases} u \cdot \underline{n} = 0, \\ \underline{t} \times \underline{n} = 0\,, \end{cases} \tag{1.1}$$

where $\underline{n}$ is the outward unit normal to the boundary Γ, $\underline{t} = \mathcal{T} \cdot \underline{n}$ is the stress vector, and

$$\mathcal{T} = -\pi\, I + \frac{\nu}{2}\,(\nabla u + \nabla u^T)$$

is the stress tensor. On flat portions of the boundary (1.1) simply reads

$$\begin{cases} u \cdot \underline{n} = 0, \\ \omega \times \underline{n} = 0\,, \end{cases} \tag{1.2}$$

where $\omega = \nabla \times u$. In the general case they differ only by lower order terms. In the sequel we appeal to (1.1), since our results are proved in the case of flat boundaries.

The literature on slip boundary conditions is particularly vast. The boundary conditions (1.1) were proposed by Navier, see [30]. A first mathematical study is due to Solonnikov and Ščadilov in the pioneering paper [32]. In [3], a quite general and self-contained presentation is given. In these two references regularity results up to the boundary are considered. See also [4] and [1], where the regularity problem is considered in the half-space. We also refer to [16], [17], and [34].

As already announced in the abstract, as a main example, we turn the resolution and properties of the evolution Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u(0) = a(x), \end{cases} \tag{1.3}$$

under the slip boundary condition into the corresponding results for the Cauchy problem, all at once. Often, sharp results for the Cauchy problem are known, but counterparts under boundary conditions are known only in weaker forms. This is the situation concerning the convergence of the solutions $\widetilde{u}^\nu$ of the Navier-Stokes equations under the slip boundary condition to the solution of the Euler equations, as the viscosity ν goes to zero. As an application we will consider this problem. We show that if the initial data $\widetilde{a}$ is given in a suitable functional space X then the solutions $\widetilde{u}^\nu$ to the Navier-Stokes problem belong to $C([0, T]; X)$ (T.Kato's persistence property). Further, as the viscosity ν goes to zero, $\widetilde{u}^\nu$ converges in the strong, and uniform in time, $C([0, T]; X)$ norm to the solution of the Euler equations under the zero-flux boundary condition. We may also assume that the initial data $\widetilde{a}^\nu$ to the Navier-Stokes problem depend on the viscosity, and converge in X to some $\widetilde{a}$. See the Theorem 7.2 below, where $X = H^{l,2}_\sigma(\mathbb{R}^3_+)$. By using the same ideas one can also approach stationary problems. Clearly in this case a force term must be included. See the Theorem 8.1 (stationary problem, for shear-thickening fluids), and the Theorem 9.1 (evolution problem for shear-thinning fluids).

It is worth noting that, either in the evolution or in the stationary case, one can solve several problems besides the Navier-Stokes equations, or get different properties for the solutions.

Since we are mainly interested in applications to incompressible fluids, we assume from the very beginning that spaces X consist of divergence free vector fields. Further, as a rule, functions labeled by a tilde are defined in the half-space $\mathbb{R}^3_+$.

The reflection technique, followed here, applies in the presence of flat boundaries, for instance cubic domains (like that used in references [7] and [8]), 3-D strips, or the half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$. To fix ideas, we mainly refer to the half space case and set

$$\Gamma = \{x \in \mathbb{R}^3 : x_3 = 0\}.$$

Note that Γ is called "boundary", even when we consider functions defined in the whole space. The unit normal to Γ (outward, with respect to $\mathbb{R}^3_+$) is denoted by $\underline{n}$. Traces on Γ "from above" and "from below" mean, respectively, from the $x_3 > 0$ side, and from the $x_3 < 0$ side.

2 Functional framework and some basic results.

In the following, by "boundary value problem" we always refer to (1.2).

Roughly speaking, our main interest is showing that results hold for the boundary value problem in the framework of a given functional space $X(\mathbb{R}^3_+)$, *if they hold for the corresponding problem in* $X(\mathbb{R}^3)$. Due to the general nature of this last hypothesis, it would be restrictive to dwell upon the spaces X. This is not difficult, since the main lines depend only on some general properties of X. Some basic assumptions on the functional spaces X are obvious from the context, for example, elements are locally integrable functions, together with a certain number of partial derivatives. To fix ideas, we suggest that the reader thinks of $X(\mathbb{R}^3)$ as being a Sobolev space $W^{s,\,p}_\sigma(\mathbb{R}^3)$ (σ stands for "divergence free").

Concerning the link between the spaces $X(\mathbb{R}^3)$ and $X(\mathbb{R}^3_+)$, we assume the following, standard, properties: Restrictions u_+ to $\mathbb{R}^3_+$ (resp. u_- to $\mathbb{R}^3_-$) of elements $u \in X(\mathbb{R}^3)$ belong to $X(\mathbb{R}^3_+)$ (resp. $X(\mathbb{R}^3_-)$). Moreover, if $u \in X(\mathbb{R}^3)$, the norm of u in this space is equivalent to the sum of the norms of the restrictions u_+ and u_- in the above corresponding spaces. Norms in $X(\mathbb{R}^3_+)$ and in $X(\mathbb{R}^3_-)$, are defined "symmetrically".

It is in general false (for spaces used in PDE's) that $u_+ \in X(\mathbb{R}^3_+)$, and $u_- \in X(\mathbb{R}^3_-)$ implies $u \in X(\mathbb{R}^3)$. In the sequel we assume the following typical situation.

Assumption 2.1. *The functional spaces X consist of divergence free vector fields. Further, in correspondence to a given $X(\mathbb{R}^3)$ space, an integer $l_0 = l_0(X)$ exists so that partial derivatives of elements of $X(\mathbb{R}^3)$ have traces on Γ, in the usual sense, if and only if its order is less or equal to l_0. Moreover, u belongs to $X(\mathbb{R}^3)$ if and only if its restrictions satisfy $u_+ \in X(\mathbb{R}^3_+)$, $u_- \in X(\mathbb{R}^3_-)$, and traces of homonymous derivatives coincide, from above and from below, up to the order l_0.*

Assumption 2.1 avoids some singular cases like Sobolev spaces $W^{s,\,p}$, if $s - \frac{1}{p}$ is an integer. In this case, and in similar ones, the theory presented in the sequel needs some adaptation, not considered here.

For convenience, we often use the symbol X^l to specify the largest order of the derivatives that appear in the definition of X. Clearly, $l_0 \leq l$. For instance, if $X^l = H^l = H^{l,\,2}$, l integer, then $l_0 = l - 1$.

The link between $\widetilde{u}$ and u shows that (6.1) holds, in general, for any other X_1-norm satisfying the assumption 2.1, provided that $l_0(X_1) \leq l_0(X)$. Actually, it is worth noting that the simple relation between $\widetilde{u}$ and u shows that from any result in $\mathbb{R}^3$ a similar result in $\mathbb{R}^3_+$ follows. We set

$$\overline{x} = (x_1,\, x_2,\, -x_3)$$

and start this section by introducing the following definition.

Definition 2.1. *Let p and v be an arbitrary scalar field and an arbitrary vector field in $\mathbb{R}^3$. We set, for each $x \in \mathbb{R}^3$, $(T\,p)(x) = p(\overline{x})$, and*

$$(T\,v)(x) = (\,T\,v_1(x), T\,v_2(x), -T\,v_3(x)\,) = (\,v_1(\overline{x}), v_2(\overline{x}), -v_3(\overline{x})\,)\,.$$

We also define $(T\,v)(x)$, for $x \in \mathbb{R}^3_-$, if v is defined in $\mathbb{R}^3_+$.

Proposition 2.1. *One has*

$$\left\{ \begin{array}{l} \nabla\,(T\,p) = T\,(\nabla\,p)\,, \\ \nabla\cdot\,(T\,v) = T\,(\nabla\cdot\,v\,)\,, \\ \nabla\times\,(T\,v) = -T\,(\nabla\times\,v\,)\,, \\ \Delta\,(T\,v) = T\,(\Delta\,v)\,, \\ (\,(T\,v)\cdot\,\nabla\,)(T\,v) = T\,(\,(v\,\cdot\,\nabla)\,v\,)\,, \\ \partial_t\,(T\,v) = T\,(\,\partial_t\,v)\,, \end{array} \right. \tag{2.1}$$

where the terms in the left hand side are taken in x and the corresponding terms in the right hand side in $\overline{x}$. Note that $T^2 = I$.

The proof is left to the reader.

Lemma 2.1. *Let v be a vector field in $\mathbb{R}^3$. If $T\,v = v$ then $v_3(x_1, x_2, 0) = 0$. If $T\,v = -v$ then $v_j(x_1, x_2, 0) = 0$, for $j = 1, 2$. In particular, if $T\,v = v$ then $(\nabla\times\,v\,)_j(x_1, x_2, 0) = 0$, for $j = 1, 2$.*

In fact, from $T\,v = v$ it follows that

$$(T\,v)_3(x_1, x_2, x_3) = v_3(x_1, x_2, x_3)\,.$$

On the other hand,

$$(T\,v)_3(x_1, x_2, x_3) = -\,(T\,v_3)(x_1, x_2, x_3) = -\,v_3(\overline{x})\,.$$

Consequently, $v_3 = 0$ for $x_3 = 0$. The second statement follows by a similar argument. The last statement follows from the relation $(2.1)_3$ for $T\,v = v$

$$\nabla\times\,v = -T\,(\nabla\times\,v\,)\,.$$

Corollary 2.1. *If $v = T\,v$ then v satisfies the boundary conditions* (1.2).

3 An explanatory interlude

Before going to the next section it looks useful to justify it itself by appealing to an example. To this end we consider the solution to the Navier-Stokes equations (1.3). One has the following result.

Proposition 3.1. *If u is a solution to the Cauchy problem* (1.3) *with initial data a, then $T\,u$ is a solution to the Cauchy problem with initial data $T\,a$.*

If $a \in X(\mathbb{R}^3)$, $a = T\,a$, and u is the unique solution to the Cauchy problem (1.3)*, then $u = T\,u$ for each $t \in [\,0, T\,]$.*

Moreover, the restriction of $u(t)$ to $\mathbb{R}^3_+$ solves the initial boundary-value problem in the half space, under the boundary condition (1.2).

Proof. The proof of the first part follows by applying the linear operator T to each of the single equations (1.3), and by appealing to Proposition 2.1. The proof of the second assertion follows from the first part and from the uniqueness of the strong solutions to the Cauchy problem. The third claim follows by appealing to Corollary 2.1. $\square$

The above result essentially shows that if $a \in X(\mathbb{R}^3)$ satisfies $a = T\,a$, then statements that hold for the Cauchy problem (like inviscid limit results) also hold in the half-space, for the boundary value problem (1.2) with initial data $\widetilde{a} = a_{|\mathbb{R}^3_+}$. However, in considering directly the boundary value problem, the initial data $\widetilde{a}$ is defined in the half-space. Hence, we have to study how general $\widetilde{a} \in X(\mathbb{R}^3_+)$ may be, so that it is the restriction to $\mathbb{R}^3_+$ of some $a \in X(\mathbb{R}^3)$ for which $a = Ta$. In other words, we must express the implicit constraint $a = T\,a$ in terms of explicit assumptions on $\widetilde{a}$. This leads to the following problem.

Problem 3.1. *Given an arbitrary $\widetilde{a} \in X(\mathbb{R}^3_+)$, which satisfies the slip boundary condition, look for necessary and sufficient conditions so that $\widetilde{a}$ is the restriction to $\mathbb{R}^3_+$ of some $a \in X(\mathbb{R}^3)$ for which $T\,a = a$.*

This is the subject of the next section. For instance, we will see that, if $l_0 < 3$, the answer is always positive (as, for instance: for H^3_σ ; for $W^{3,\,p}_\sigma$ and arbitrarily large p). On the contrary, H^4_σ and $W^{4,\,p}_\sigma$ require compatibility conditions.

4 The compatibility conditions, and "fitting" on the boundary.

The deduction of the following *compatibility conditions* is the main subject of this section.

Assumption 4.1. *Let $\widetilde{a} = (\widetilde{a}_1, \widetilde{a}_2, \widetilde{a}_3) \in X^l(\mathbb{R}^3_+)$. If $l_0 = l_0(X^l) \geq 3$, then for each* odd *integer $k \in [3, l_0\,]$, the partial derivatives $\partial^k_3\, \widetilde{a}_j$, $j = 1, 2$, vanish on Γ :*

$$\partial^k_3\, \widetilde{a}_1(x_1, x_2, 0) = \partial^k_3\, \widetilde{a}_2(x_1, x_2, 0) = 0\,. \tag{4.1}$$

In the next section we show that conditions (4.1) are independent.

It seems convenient to introduce the following convention. If g is defined in $\mathbb{R}^3$ (or in $\mathbb{R}^3_+ \cup \mathbb{R}^3_-$), we say that g *fits on* Γ, or just *fits*, if the traces on Γ, from above and from below, coincide. Furthermore, the expression *fits by zero* means that both traces vanish on Γ. Clearly, these definitions are meaningful only when the traces from both sides exist. If not, "fitting" is not defined.

The following theorem is the main result in this section.

Theorem 4.1. *Let $\widetilde{a} \in X^l(\mathbb{R}^3_+)$ satisfy the slip boundary conditions* (4.4). *Moreover, if $l_0(X^l(\mathbb{R}^3) \geq 3$, assume that $\widetilde{a}$ verifies the compatibility conditions 4.1. Define in $\mathbb{R}^3$ the vector field a by equation* (4.2) *below (hence $T\,a = a$ in $\mathbb{R}^3$).*

Under the above hypotheses, any partial derivative of the vector field a, of order less or equal to l_0, fit on Γ. Hence

$$a \in X^l(\mathbb{R}^3)\,.$$

For instance, if $l_0 = 3$ or $l_0 = 4$, the assumption

$$\partial_3^3\, \widetilde{a}_1 = \partial_3^3\, \widetilde{a}_2 = 0 \quad \text{on} \quad \Gamma\,,$$

guaranties that the traces, from above and from below, of any derivative of a of order less or equal to l_0 (a defined by (4.3)), fit on Γ.

In order to prove Theorem 4.1, we define in $\mathbb{R}^3$ the *mirror-extension* $\widetilde{\mathbf{T}}\,\widetilde{a}$ of $\widetilde{a}$, defined on $\mathbb{R}^3_+$, by the equation

$$a = \widetilde{\mathbf{T}}\,\widetilde{a} := \begin{cases} \widetilde{a} & \text{in } \mathbb{R}^3_+\,, \\ T\,\widetilde{a} & \text{in } \mathbb{R}^3_-\,. \end{cases} \tag{4.2}$$

It is obvious that $a = \widetilde{\mathbf{T}}\,\widetilde{a}$ is the unique extension of $\widetilde{a}$ for which $a = T\,a$. The point here is whether or not $\widetilde{a} \in X(\mathbb{R}^3_+)$ implies that $a = \widetilde{\mathbf{T}}\,\widetilde{a}$, belongs to $X(\mathbb{R}^3)$. By assumption 2.1, this is equivalent to proving the coincidence of the traces on Γ "from above and from below" of all partial derivatives of a up to order l_0. We will show that this is equivalent to the conditions on the traces (obviously from above) of the derivatives of $\widetilde{a}$ referred to in assumption 4.1.

For convenience we write (4.2) in the more explicit form

$$a_j(x_1,\, x_2,\, x_3) = \widetilde{a}_j(x_1,\, x_2,\, x_3) \quad \text{in} \quad \mathbb{R}^3_+\,, \quad \text{if } j = 1,\, 2,\, 3\,;$$

$$a_j(x_1,\, x_2,\, x_3) = \begin{cases} \widetilde{a}_j(x_1,\, x_2,\, -x_3) & \text{in } \mathbb{R}^3_-\,, \text{ if } j = 1,\, 2\,, \\ -\,\widetilde{a}_3(x_1,\, x_2,\, -x_3) & \text{in } \mathbb{R}^3_-\,, \text{ if } j = 3\,, \end{cases} \tag{4.3}$$

and also the slip boundary condition (1.2) in the explicit form

$$\begin{cases} \widetilde{a}_3(x_1,\, x_2,\, 0) = 0\,, \\ \partial_3\, \widetilde{a}_j(x_1,\, x_2,\, 0) = \partial_j\, \widetilde{a}_3(x_1,\, x_2,\, 0)\,, \qquad \text{for} \quad j = 1,\, 2\,. \end{cases} \tag{4.4}$$

It is worth noting that if two functions fit (respectively, fit by zero), then their tangential derivatives of any order also fit (respectively, fit by zero). Hence, the "fitting problem" for a partial derivative $\partial_\tau^k\, \partial_3^m\, a_j$ is reduced to the same problem for the pure normal derivative $\partial_3^m\, a_j$. For convenience, we put in evidence this result.

Lemma 4.2. *If a partial derivative $\partial_3^m\, a_j$ fits (resp. fits by zero) on Γ, then any partial derivative $\partial_\tau^k\, \partial_3^m\, a_j$ fits (respectively, fits by zero).*

The next result follows easily from the definitions.

Proposition 4.1. *Let $\widetilde{a}$ be given in $\mathbb{R}^3_+$, and let $a = \widetilde{\boldsymbol{T}}\widetilde{a}$ be the mirror-extension of $\widetilde{a}$ to $\mathbb{R}^3$. Then:*

a) Partial derivatives of a_3, of odd order in the normal direction, fit on Γ. Partial derivatives of a_j, for $j = 1,\, 2$, of even order in the normal direction, fit on Γ.

b) Partial derivatives of a_3, of even order in the normal direction, and partial derivatives of a_j, for $j = 1,\, 2$, of odd order in the normal direction, fit on Γ if and only if they fit by zero.

Proposition 4.1 shows that a necessary and sufficient condition for the resolution of problem 3.1 is the fitting by zero of the partial derivatives considered in part b). However these conditions are not independent. We start by proving the following result.

Proposition 4.2. *Compatibility conditions are not required for derivatives of order less than or equal to two. In particular if* $l_0(X) \leq 2$.

Proof. We have to show that

$$a_3(x_1, x_2, 0) = \partial_3 a_j(x_1, x_2, 0) = \partial_3^2 a_3(x_1, x_2, 0) = 0, \quad \text{for} \quad j = 1, 2. \tag{4.5}$$

The two first assertions follow easily from (4.4). On the other hand, due to the divergence free property, one has

$$-\partial_3^2 a_3 = \sum_{j=1,2} \partial_j \partial_3 a_j. \tag{4.6}$$

By results already shown, both terms on the right hand side of the above equation have zero trace on Γ.

□

Proposition 4.3. *Let* $\widetilde{a}$ *be a divergence free vector field defined in* $\mathbb{R}^3_+$ *and let* $a = \widetilde{\boldsymbol{T}}\widetilde{a}$ *be the mirror-extension of* $\widetilde{a}$ *to* $\mathbb{R}^3$. *If, for some odd integer* k,

$$\partial_3^k \widetilde{a}_j = 0 \quad on \quad \Gamma, \quad for \quad j = 1, 2, \tag{4.7}$$

then

$$\partial_3^{k+1} a_3 = 0, \tag{4.8}$$

in Γ. *In other words if* $\partial_3^k \widetilde{a}_j$ *fits by zero, for* $j = 1, 2$, *then* $\partial_3^{k+1} a_3$ *fits by zero.*

Proof. By appealing to the divergence free property we get

$$-\partial_3^{k+1} a_3(x) = \begin{cases} \sum_{j=1}^2 (\partial_j \partial_3^k \widetilde{a}_j)(x) & \text{if } x_3 > 0, \\ (-1)^{k+1} \sum_{j=1}^2 (\partial_j \partial_3^k \widetilde{a}_j)(\overline{x}) & \text{if } x_3 < 0. \end{cases} \tag{4.9}$$

This leads to the thesis. □

Note that the result also holds for even values of k. Proposition 4.3, together with results already established, prove Theorem 4.1.

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Remark 4.1. *Thanks to this section and to the main result of the next section, we will see that one can appeal to Proposition 3.1 and Theorem 4.1 for extending to the boundary value problem the properties which hold for the Cauchy problem. The same arguments hold for a stationary problem, where a force field* $\widetilde{f}$ *appears in place of* $\widetilde{a}$. *More precisely, if we want to extend some properties from a problem* $P = P(\mathbb{R}^3)$, *in the whole space, to the corresponding boundary value problem* $\widetilde{P} = \widetilde{P}(\mathbb{R}^3_+)$, *in the half-space, we merely have to check that: 1)* $\widetilde{a}$ *(respectively* $\widetilde{f}$*) satisfies the assumptions 4.1, and* $\widetilde{a}$ *satisfies the boundary conditions* (1.2); *2) if* u *is a solution of problem* P *then* Tu *is solution of the same problem; 3) there exists a unique solution of problem* P.

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5 Independence of the compatibility conditions

We already have shown that (4.7) and (4.8) are necessary conditions for fitting. Proposition 4.3 shows that these conditions are not independent. We wonder whether the set of compatibility conditions in assumption 4.1 is minimal. We show here that this is the case, by constructing a divergence free vector field $v \in C^\infty(\mathbb{R}^3)$, with compact support contained in a sphere centered in the origin, and with radius arbitrarily small, which satisfies the boundary conditions and all the compatibility conditions up to an arbitrary odd order $n-1$, but which do not satisfy the compatibility condition of order n. This shows not only that the last compatibility condition is needed (this was already known), but also that it does not follow from the set of all the previous (lower order) compatibility conditions, together with the boundary conditions and the divergence free property.

Let $n \geq 2$ be an integer, fix a scalar field $\rho \in C_0^\infty(B(0,1))$, and define the vector field $w = x_3^{n+1}(1,1,0)$ in $\mathbb{R}^3$. It is easy to check that the divergence free vector field $v = \nabla \times (\rho\, w)$ vanishes on the boundary, together with any partial derivative of order less or equal to $n-1$. In particular, v satisfies our boundary conditions, and

$$\partial_3^k v_1(x_1, x_2, 0) = \partial_3^k v_2(x_1, x_2, 0) = 0\,, \tag{5.1}$$

on Γ, for each $k < n$. However

$$\partial_3^n v(x_1, x_2, 0) = \rho\,(n+1)!\,(-1, 1, 0)\,, \tag{5.2}$$

shows that $\partial_3^n v_1$ and $\partial_3^n v_2$ do not vanish on Γ. Consider an odd value $n \geq 3$. Then, by (5.1), the compatibility condition (4.1) is satisfied up to order $n-1$. In particular it holds for all odd k, up to order $n-2$ included. But the last compatibility condition does not hold, as follows from (5.2).

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6 On a class of solutions to the Navier-Stokes equations in $\mathbb{R}^3$. The "abstract" theorem 6.1.

The following theorem is the foundation of many possible applications, in particular those considered by us in the sequel.

Note that the preliminary hypotheses below consist in assuming that a very basic result holds in $\mathbb{R}^3$.

We suppose that the functional space $X^l(\mathbb{R}^3)$ satisfies the assumption 2.1.

Theorem 6.1. $\mathbb{R}^3-$ *preliminary hypotheses: for each* $a \in X^l(\mathbb{R}^3)$ *there is a positive* $T = T(a)$ *such that the Cauchy problem* (1.3) *admits a unique solution* $u \in C([0,T]; X^l(\mathbb{R}^3))$.

$\mathbb{R}^3_+$ *result: Assume that the initial data* $\widetilde{a} \in X^l(\mathbb{R}^3_+)$ *satisfy the boundary conditions* (1.2) *and the compatibility conditions described in assumption 4.1.*

Then the initial-boundary value problem (1.3), (1.2) *admits a (unique) solution* $\widetilde{u} \in C([0, T]; X^l(\mathbb{R}^3_+))$ *in* $[0, T]$.

More precisely, the solution $\widetilde{u}$ *is constructed as follows. Given* $\widetilde{a} \in X^l(\mathbb{R}^3_+)$ *as above, we define* $a = \widetilde{\boldsymbol{T}}\widetilde{a} \in X^l(\mathbb{R}^3)$ *as being the mirror-extension of* $\widetilde{a}$ *to* $\mathbb{R}^3$ *(see* (4.2) *below). Furthermore, let* $u \in C([0, T]; X^l(\mathbb{R}^3))$ *be the solution to the Cauchy problem with the initial data* a. *Then, for each* $t \in [0, T]$, *the above solution* $\widetilde{u}(t)$ *is simply the restriction of* $u(t)$ *to the half-space* $\mathbb{R}^3_+$. *In particular,*

$$c\,\|u(t)\|_{X(\mathbb{R}^3)} \leq \|\widetilde{u}(t)\|_{X(\mathbb{R}^3_+)} \leq \|u(t)\|_{X(\mathbb{R}^3)}\,, \tag{6.1}$$

where $c = c(l)$, *is a positive constant.*

Proof. The Theorem 6.1 follows immediately from the Theorem 4.1 together with Proposition 3.1. Indeed the hypotheses on the data $\widetilde{a}$ in Theorem 6.1 are the same as in Theorem 4.1. From this last theorem we get $a \in X^l(\mathbb{R}^3)$ and $a = T\,a$. Therefore the assumptions on a in Proposition 3.1 are satisfied and we get that the restriction of $u(t)$ to $\mathbb{R}^3_+$ solves the initial boundary-value problem in the half space. □

Concerning time instant $T(a)$, in typical situations there is a lower bound for the values T, which depends (decreasingly) only on the X^l-norm of a, and not on a itself. Often, a weaker norm is sufficient to determine T.

Note that Theorem 6.1 is not the more general result that one can get. We have preferred to avoid the full generality, since our interest is mainly concerned with the inviscid limit result in strong topologies. This means that if the initial data is given in a Banach space X, the convergence result should be established in $C([0, T]; X)$. Actually, one can deduce that the initial-boundary value problem (1.3), (1.2) has a unique solution in some class (such as $L^p(0, T; W^{l,q}(\mathbb{R}^3_+))$) if the the Cauchy problem has a unique solution in the corresponding class, provided that the initial data (prescribed in $\mathbb{R}^3_+$) satisfies the boundary conditions and the compatibility conditions given in assumption 4.1.

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7 The inviscid limit

Vanishing viscosity limit results in $3-D$ domains, without boundary conditions, have been studied by many authors. See, for instance, [13], [20], [21], [23], [25], [33], and the more recent papers [6], [28]. In [6], [21] and [28] results are proved in the *strong topology*. For results concerning inviscid limits in non-smooth situations we refer to [14].

Concerning the vanishing viscosity problem in bounded domains, under slip boundary conditions, we refer to [7],[8], [11], [18], [35] and references therein.

In the particular 2-D case the assumption $\omega \times n = 0$ on Γ is simply replaced by $\omega = 0$. For specific 2-D vanishing viscosity results under slip-type boundary conditions we refer to the classical papers, [2], [19], [29], [31]. See also the more recent papers [12], [26], and [34].

In the following we consider the Navier-Stokes equations

$$\begin{cases} \partial_t \widetilde{u}^\nu + (\widetilde{u}^\nu \cdot \nabla)\, \widetilde{u}^\nu - \nu\, \Delta\, \widetilde{u}^\nu + \nabla\, \widetilde{p}^\nu = 0, \\ \nabla \cdot \widetilde{u}^\nu = 0\,, \\ \widetilde{u}^\nu(0) = \widetilde{a}_\nu(x)\,, \end{cases} \tag{7.1}$$

in $\mathbb{R}^3_+$, under the boundary condition

$$\begin{cases} \widetilde{u}^\nu \cdot \underline{n} = 0, \\ \\ \widetilde{\omega}^\nu \times \underline{n} = 0\,. \end{cases} \tag{7.2}$$

We assume that the positive viscosities ν are bounded from above by an arbitrary, but fixed, constant.

As already remarked, on flat portions of the boundary, (1.2) coincides with the well known boundary condition (1.1). In the 3-D problem, if the boundary is not flat, it is not clear how to prove *strong* inviscid limit results. In fact, a substantial obstacle appears. See [7] for some details on this point.

Denote by $X^l(\mathbb{R}^3_+)$ the initial data's space. We want to prove the convergence in $C([0,\,T\,];\,X^l(\mathbb{R}^3_+)\,)$ of the solutions $\widetilde{u}^\nu$, as the viscosity ν goes to zero (and, possibly, $\widetilde{a}_\nu$ converging to some $\widetilde{a}$) to the solution $\widetilde{u}^0$ of the Euler equations

$$\begin{cases} \partial_t \widetilde{u}^0 + (\widetilde{u}^0 \cdot \nabla)\, \widetilde{u}^0 + \nabla\, \widetilde{p}^0 = 0, \\ \nabla \cdot \widetilde{u}^0 = 0\,, \\ \widetilde{u}^0(0) = \widetilde{a}(x)\,, \end{cases} \tag{7.3}$$

in $\mathbb{R}^3_+$, under the zero-flux boundary condition

$$\widetilde{u}^0 \cdot \underline{n} = 0\,. \tag{7.4}$$

Previous results, particularly related to ours, were proved in [35], in spaces $W^{3,\,2}$; in [7], in spaces $W^{2,\,p}$ and $W^{3,\,p}$, for any arbitrarily large p; and in reference [8], in arbitrary $W^{k,\,p}$ spaces. However, uniform convergence in time with values in the initial data space X is not proved. On the contrary, for the Cauchy problem, some sharp vanishing viscosity limit results are known. Theorem 6.1 allows immediate extension of these results to the initial-boundary value problem. The following assumption is, in fact, a condition on $X^l(\mathbb{R}^3)$. It requires that the vanishing viscosity limit result holds in $X^l(\mathbb{R}^3)$ for the Cauchy problem

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla)\, u^\nu - \nu\, \Delta\, u^\nu + \nabla\, p^\nu = 0, \\ \nabla \cdot u^\nu = 0\,, \\ u^\nu(0) = a_\nu(x)\,. \end{cases} \tag{7.5}$$

Assumption 7.1. *a) For each $\nu > 0$ and each $a^\nu \in X^l(\mathbb{R}^3)$ the Cauchy problem (7.5) admits a unique solution $u^\nu \in C(\,[0,\,T];\,X^l(\mathbb{R}^3)\,)$, where $T > 0$ is independent of ν and of the particular $a^\nu \in X^l(\mathbb{R}^3)$, provided that their norms are bounded from above by a given constant. Furthermore, if the parameter ν tends to zero and a^ν tends to a in $X^l(\mathbb{R}^3)\,)$, then u^ν converges in $C(\,[0,\,T];\,X^l(\mathbb{R}^3)\,)$ to the unique solution u^0 of the Euler equations in $\mathbb{R}^3$.*

Theorem 7.1. *Under the assumption 7.1 one has the following result:*

Let the vector fields $\widetilde{a}^\nu \in X^l(\mathbb{R}^3_+)$ *satisfy the boundary conditions* (7.2)*, and the compatibility conditions described in the assumption 4.1. Then, for each* $\nu > 0$*, the initial-boundary value problem* (7.5)*,* (7.2) *admits a unique solution* $\widetilde{u}^\nu \in C([0, T]; X^l(\mathbb{R}^3_+))$*. Furthermore, if the parameter* ν *tends to zero (vanishing viscosity limit) and the initial data* $\widetilde{a}^\nu$ *converge to some* a *in* $X^l(\mathbb{R}^3_+)$*, then* $\widetilde{u}^\nu$ *converges in* $C([0, T]; X^l(\mathbb{R}^3_+))$*, to the unique solution* $\widetilde{u}^0$ *of the Euler equations* (7.3) *under the boundary condition* (7.4)*.*

The result follows from Theorem 6.1 together with the assumption 7.1. We define $a^\nu = \widetilde{\mathbf{T}}\,\widetilde{a}^\nu$ as being the mirror-images of the $\widetilde{a}^\nu$'s, and $u^\nu \in C([0, T]; X^l(\mathbb{R}^3))$ as being the solutions to the Cauchy problems (7.5) with viscosity ν and initial data a^ν. Then, the solutions $\widetilde{u}^\nu$ of the boundary value problems are the restrictions to the half-space $\mathbb{R}^3_+$ of the solutions u^ν, and $\widetilde{u}^0$ is the restriction to the half-space $\mathbb{R}^3_+$ of u^0.

Additional regularity and convergence results for the solutions u^ν, their time-derivatives, and pressure follow immediately from corresponding results proved for the Cauchy problem.

To apply the above theorem to a specific problem, we simply replace the assumption 7.1 by the known, desired, vanishing viscosity result for the Cauchy problem. For instance, let us show an application of the above theorem, in the case $X^l(\mathbb{R}^3) = H^{l,\,2}_\sigma(\mathbb{R}^3)$.

Theorem 7.2. *Assume that the initial data* $\widetilde{a}_\nu \in H^{l,\,2}_\sigma(\mathbb{R}^3_+)$ *satisfy the boundary condition* (7.2)*. Further, if* $l \geq 4$*, assume that for each* odd *integer* $k \in [3, l-1]$*, the compatibility condition*

$$\partial_3^k \widetilde{a}_j = 0\,, \quad on \quad \Gamma\,, \quad for \quad j = 1,\, 2 \tag{7.6}$$

holds. Then the initial-boundary value problem (7.5)*,* (7.2) *admits a unique solution* $u^\nu \in C([0, T]; H^{l,\,2}_\sigma(\mathbb{R}^3_+))$*. Furthermore, if*

$$\widetilde{a}_\nu \to a\,, \quad in \quad H^{l,\,2}_\sigma(\mathbb{R}^3_+)\,,$$

as $\nu \to 0$*, then, as* $\nu \to 0$*,*

$$u_\nu \to u\,, \quad in \quad C([0, T]; H^{l,\,2}_\sigma(\mathbb{R}^3_+))\,, \tag{7.7}$$

where u *is the solution to the Euler equations* (7.3) *under the boundary condition* (7.4)*.*

For instance, if $l = 3$ one has $l_0 = 2$. Hence the vanishing viscosity limit holds in the space $H^3_\sigma(\mathbb{R}^3_+)$ *without* assuming compatibility conditions on the initial data. If $X^5(\mathbb{R}^3_+) = H^5_\sigma(\mathbb{R}^3_+)$, one has $l_0 = 4$. Hence we have to assume the compatibility condition $\partial_3^3 \widetilde{a}_j = 0$, for $j = 1, 2$.

In Theorem 7.2 the assumption 7.1 holds, as follows essentially from results due to T. Kato, see [20] and [21]. A simpler proof is shown by N. Masmoudi, see Theorem 2.1 in reference [28]. We may also appeal to [6], to prove the above result in the cubic domain case.

As another application, we may extend to the boundary value problem the results proved by T.Kato and G. Ponce in [23], where convergence of Navier-Stokes to Euler follows in Lebesgue spaces $L^p_s(\mathbb{R}^n)$. These spaces are similar to $W^{s,\,p}$ spaces. See also [22]. It would be redundant to state here other specific results. Checking this possibility case by case is, on the whole, an easy task.

Remark 7.1. *In this section we have appealed to Proposition 3.1 and Theorem 4.1 (or, directly, to Theorem 6.1) for extending to the boundary value problem the properties described in assumption 7.1 for the Cauchy problem. If we want to extend different properties, we merely have to replace the assumption 7.1 by an assumption describing the corresponding properties for the Cauchy problem (see Remark 4.1).*

8 Regularity for shear-thickening stationary flows

In this section we consider the following stationary system describing the motion of a non-Newtonian fluid:

$$\begin{cases} -\nabla \cdot S\,(\mathcal{D}\,u) + \nabla \pi = \, f\,, \\ \nabla \,\cdot\, u = \, 0\,. \end{cases} \tag{8.1}$$

We assume that the "extra stress" S is given by

$$S(\,\mathcal{D}\,u) = \,(\,\nu_0 + \nu_1 |\mathcal{D}u|^{p-2}\,)\,\mathcal{D}u\,, \tag{8.2}$$

where $\mathcal{D}u$ is the symmetric gradient of u, i.e.

$$\mathcal{D}\,u = \,\frac{1}{2}\,(\,\nabla\,u + \,\nabla\,u^T\,)\,,$$

ν_0, ν_1 are positive constants, and $p > 2$. Here, and in the next section, we sacrifice a greater generality to emphasizing the main ideas. Thus (8.2) and (9.5) (see below) are just the canonical representative of a wider class of extra stress tensors to which our proof applies.

As done for the initial boundary value problem (1.3), we draw new results for the system (8.1) under the slip-boundary conditions (1.2) from the corresponding known results for the whole space. We are mainly interested in regularity results up to the boundary. This problem has received various contributions in recent years. The main open problem is to prove the L^2-integrability, up to the boundary, of the second derivatives of the solutions, in both the cases $p < 2$ and $p > 2$. In reference [5] the half-space case $\mathbb{R}^3_+$ is considered, under slip (and non-slip) boundary conditions, and $p > 2$. The author shows that the second "tangential" derivatives belong to $L^2(\mathbb{R}^3_+)$, while the second "normal" derivatives belong to some $L^l_{loc}(\overline{\mathbb{R}^3_+})$, for a suitable $l < 2$. See [9] for recent, and more general, related results (under the non-slip boundary condition), and for references.

In the sequel the reflection technique enables us to improve the regularity results, by overcoming the loss of regularity from the tangential to the normal direction. See Theorem 8.1 below.

Following the notation in [5], we define $\widetilde{D}^1(\mathbb{R}^3_+) := D^{1,2}(\mathbb{R}^3_+)$ as the completion of $C^\infty_0(\overline{\mathbb{R}^3_+})$ with respect to the norm $\|\nabla\,v\|$, and set

$$\widetilde{V}_2(\mathbb{R}^3_+) = \Big\{\, v \in\, \widetilde{D}^1(\mathbb{R}^3_+) :\, \nabla \cdot v \,=\, 0\,,\, v_3|_{x_3=0} \,=\, 0 \Big\}\,,$$

endowed with the norm $\|\,\nabla\,v\,\|$. We denote by $(\widetilde{V}_2(\mathbb{R}^3_+))'$ the dual space of $\widetilde{V}_2(\mathbb{R}^3_+)$. Finally we set

$$\widetilde{V}(\mathbb{R}^3_+) = \Big\{\, v \in \widetilde{V}_2(\mathbb{R}^3_+) : \|\,\mathcal{D}\,v\,\|_p < \infty \Big\}\,,$$

endowed with the norm $\| \nabla v \| + \| \mathcal{D} v \|_p$. We use

$$D^1(\mathbb{R}^3)\,,\ V_2(\mathbb{R}^3)\,,\ (V_2(\mathbb{R}^3))'\,,\ V(\mathbb{R}^3)\,,$$

for the corresponding spaces in $\mathbb{R}^3$.

Assume that $f \in (V_2)'$. We say that u is a weak solution of system (8.1) if $u \in V$ satisfies

$$\frac{1}{2} \int_{\mathbb{R}^3} \left(\nu_0 + \nu_1 \, |\mathcal{D}u|^{p-2} \right) \mathcal{D}u \cdot \mathcal{D} v dx = \int_{\mathbb{R}^3} f \cdot v \, dx\,, \tag{8.3}$$

for all $v \in V$. A corresponding definition holds for the boundary value problem.

We start by recalling the following result (for a sketch of the proof, see below).

Proposition 8.1. *For each* $f \in (V_2(\mathbb{R}^3))' \cap L^2(\mathbb{R}^3)$, *the system* (8.1) *in* $\mathbb{R}^3$ *admits a unique weak solution* $u \in V(\mathbb{R}^3)$. *Furthermore, the derivatives* $D^2 u$ *belong to* $L^2(\mathbb{R}^3)$.

By the above proposition, and thanks to the procedure developed in the previous sections, one has the following theorem.

Theorem 8.1. *Let be* $\widetilde{f} \in (\widetilde{V}_2(\mathbb{R}^3_+))' \cap L^2(\mathbb{R}^3_+)$. *Then, the boundary value problem* (8.1), (1.2) *in* $\mathbb{R}^3_+$ *admits a unique weak solution* $\widetilde{u} \in \widetilde{V}(\mathbb{R}^3_+)$. *Furthermore, the derivatives* $D^2 \widetilde{u}$ *belong to* $L^2(\mathbb{R}^3_+)$.

Proof. Actually, we merely have to check that if u is a solution of (8.1) in $\mathbb{R}^3$ so is $T u$, where T is given by Definition 2.1. This last property is immediate, since

$$(\mathcal{D}(T u))_{ij}(x) = \begin{cases} (\mathcal{D} u)_{ij}(\overline{x}), & \text{as } i = j = 3 \ \text{ or } \ i, j \in \{1, 2\}\,, \\[2ex] -(\mathcal{D} u)_{ij}(\overline{x}), & \text{as } i = 3\,, j \in \{1, 2\}\,. \end{cases}$$

It is worth noting that, due to the low regularity of the force term, we do not have to require any extra assumption, like assumption 4.1.

As for Proposition 8.1, the existence and uniqueness of weak solutions in $\mathbb{R}^3$ are well known, and derive from basic theory of monotone operators. See [24]. As far as the L^2 regularity is concerned, we note that the results in [27] immediately show that the derivatives $D^2 u$ belong to $L^2_{loc}(\mathbb{R}^3)$. By appealing to our reflection technique results, this yields $D^2 \widetilde{u} \in L^2_{loc}(\overline{\mathbb{R}^3_+})$. Here the restriction "local" means "at finite distance". This restriction is not substantial, since it is formally due to the fact that in [27] the authors consider a bounded domain. Clearly, one gets "global" regularity in $\mathbb{R}^3$ by appealing to the Nirenberg's translation technique in all the space (as in [5], Lemma 4.1). Note that in [5] translations are allowed only in the tangential directions, since the problem is considered in the half-space. However, in $\mathbb{R}^3$, translations can be done in any direction. Since we turn the problem in the half-space with slip-boundary conditions into the problem in the whole space, the solution $\widetilde{u}$ of the boundary value problem has second derivatives in $L^2(\mathbb{R}^3_+)$, since it is the restriction to the half-space of the solution u, with $D^2 u \in L^2(\mathbb{R}^3)$. $\square$

9 The evolution, shear-thinning problem, in the "periodic cube"

As announced in the introduction, the technique followed for the half-space applies for other domains with flat boundaries. Here we show in which way one can extend the procedure to problems in the so called "periodic cube", with slip boundary conditions on two opposite faces, and periodicity on the remaining two pairs of faces. This is by now a canonical situation, that allows to avoid localization techniques and unbounded domains, hence to focus attention on the main (the boundary value) problem. Consider the half-cube

$$\widetilde{\Omega} = (-1,1)^2 \times \left(-\frac{1}{2}, \frac{1}{2}\right) . \tag{9.1}$$

We assume the boundary conditions (recall (4.4))

$$a_3(x) = \partial_3 \widetilde{a}_1(x) = \partial_3 \widetilde{a}_2(x) = 0 \,, \text{if } x_3 = \pm\frac{1}{2} \,, \tag{9.2}$$

together with periodicity in the x_1 and x_2 directions, where $\widetilde{a} \in X^l(\widetilde{\Omega})$. Notation apes that used in the previous sections for the half-space case, with the obvious adaptations.

Now we perform a mirror-extension (recall (4.2) and (4.3)), on the upper and lower faces, from each $\widetilde{a} \in X^l(\widetilde{\Omega})$ to a corresponding a, defined in the cube $\Omega = (-1,1)^3$, in the following way

$$\begin{cases} a(x) = \widetilde{a}(x) \,, & \text{if } x \in \widetilde{\Omega} \,, \\ a_i(x) = \widetilde{a}_i(x_1,\, x_2,\, 1 - x_3) \,, & \text{if } x_3 \in \left[\frac{1}{2}, 1\right] \text{ and } i = 1,2 \,, \\ a_3(x) = -\widetilde{a}_3(x_1,\, x_2,\, 1 - x_3) \,, & \text{if } x_3 \in \left[\frac{1}{2}, 1\right] \,, \\ a_i(x) = \widetilde{a}_i(x_1,\, x_2,\, -1 - x_3) \,, & \text{if } x_3 \in \left[-1, -\frac{1}{2}\right] \text{ and } i = 1,2 \,, \\ a_3(x) = -\widetilde{a}_3(x_1,\, x_2,\, -1 - x_3) \,, & \text{if } x_3 \in \left[-1, -\frac{1}{2}\right] . \end{cases} \tag{9.3}$$

By imposing the compatibility conditions (similar to (4.1))

$$\partial_3^k \, \widetilde{a}_1 \left(x_1,\, x_2,\, \pm\tfrac{1}{2}\right) = \partial_3^k \, \widetilde{a}_2 \left(x_1,\, x_2,\, \pm\tfrac{1}{2}\right) = 0 \,, \quad \forall\, k \in \mathbb{N},\ k \text{ odd},\ k \in [\,3,\, l_0\,] \,,$$

we get $a \in X^l(\Omega)$. A direct computation shows that, for any admissible multi index α,

$$D^\alpha a(x_1, x_2, -1) = D^\alpha a(x_1, x_2, 1).$$

By using this procedure, we have transformed the original problem with mixed boundary conditions in a purely periodic one. Likewise in the half-space case, if we start from a suitable solution to the totally periodic problem, its restriction to the half-cube turns out to be a solution of the slip-periodic problem. Clearly, regularity properties are preserved. As an application we show here an existence and regularity result in the framework of the shear–thinning fluids. The problem considered is the evolution of a non–Newtonian fluid in a periodic cube with slip boundary conditions, namely

$$\begin{cases} \partial_t \, u - \nabla \cdot S\,(\,\mathcal{D}\,u\,) \,+ (\,u\, \cdot \nabla\,)\, u + \nabla\, p = 0 \,, \\ \nabla\, \cdot\, u = \,0 \,, \\ u(0,x) = \,\widetilde{a}(x) \,, \end{cases} \tag{9.4}$$

where, for the sake of simplicity, we assume that the extra stress S is of the following type

$$S(\mathcal{D}u) = (\,1 + \,|\,\mathcal{D}u\,|)^{p-2}\,\mathcal{D}\,u\,. \tag{9.5}$$

We want to solve problem (9.4) in a cube $\widetilde{\Omega}$ with boundary conditions (9.2) and $p < 2$. We require vanishing mean value, as usual in the space periodic case. Let us introduce the spaces

$$V_p = \left\{\, v \in W^{1,p}(\Omega),\, \nabla \cdot v = 0,\; v \text{ is } x - \text{periodic} \right\}\,,$$

$$\widetilde{V}_p = \left\{ v \in W^{1,p}(\widetilde{\Omega}),\, \nabla \cdot v = 0,\; v \text{ is } (x_1, x_2) - \text{periodic, } v_3 = 0 \text{ if } x_3 = \pm\frac{1}{2} \right\}\,.$$

We start by recalling the following result.

Proposition 9.1. *Let be* $p \in \left(\frac{7}{5}, 2\right)$. *For each* $a \in W^{2,2}(\Omega) \cap V_p$ *the initial value problem* (9.4) *in* Ω *admits a unique solution* $u \in L^\infty(0,T;\, V_p\,) \cap L^2(0,T;\, W^{2,2}(\Omega))\,,\; \partial_t u \in L^2(0,T;\, L^2(\Omega)\,)$*, for some positive* $T = T(a)$.

The existence of a solution is ensured by Theorem 17 in [15], while its uniqueness follows by Corollary 18 and Theorem 19 in the same reference. See also [10] for the degenerate case $S(\mathcal{D}u) = \,|\,\mathcal{D}u\,|^{p-2}\,\mathcal{D}\,u\,$.

One has the following theorem.

Theorem 9.1. *Let be* $p \in \left(\frac{7}{5}, 2\right)$. *Let the vector field* $\widetilde{a} \in W^{2,2}(\widetilde{\Omega}) \cap \widetilde{V}_p$ *satisfy the boundary conditions* (9.2). *Then, the initial-boundary value problem* (9.4), (9.2) *in* $\widetilde{\Omega}$ *admits a unique solution* $\widetilde{u} \in L^\infty(0,T;\, \widetilde{V}_p\,) \cap L^2(0,T;\, W^{2,2}(\widetilde{\Omega}))$ *and* $\partial_t \widetilde{u} \in L^2(0,T;\, L^2(\widetilde{\Omega}))$.

The proof follows by extending the initial datum $\widetilde{a}$ by means of equations (9.3), and by observing that if u solves system (9.4) then Tu solves the same system (see the previous section).

We remark that in the present case the compatibility conditions (see assumption 4.1) are not needed, as stated in Proposition 4.2.

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