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Sharp Inviscid Limit Results under Navier Type Boundary Conditions. An L^p Theory

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Abstract. We consider the evolutionary Navier–Stokes equations with a Navier slip-type boundary condition, and study the convergence of the solutions, as the viscosity goes to zero, to the solution of the Euler equations under the zero-flux boundary condition. We obtain quite sharp results in the 2-D and 3-D cases. However, in the 3-D case, we need to assume that the boundary is flat.

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1. Introduction and results

Throughout this paper, Ω is an open bounded set in $\mathbb{R}^3$ (or $\mathbb{R}^2$) locally situated on one side of its boundary Γ. The unit outward normal to Γ is denoted by $\underline{n} = (n_1, n_2, n_3)$. We consider, in a suitable time-interval $[0, T]$, the Navier–Stokes equations

$$\begin{cases} \partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla \pi = 0, \\ \operatorname{div} u^\nu = 0, \\ u(0) = u_0, \end{cases} \tag{1.1}$$

and the corresponding Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla \pi = 0, \\ \operatorname{div} u = 0, \\ u(0) = u_0. \end{cases} \tag{1.2}$$

Our aim is to investigate strong convergence, up to the boundary, of the solutions u^ν of the first system to the solution u of the second system, as $\nu \to 0$, under physical meaningful boundary conditions. Concerning the Euler equations, we assume the classical zero-flux boundary condition

$$u \cdot \underline{n} = 0. \tag{1.3}$$

Existence of local in time classical solutions to the Euler equations in $\mathbb{R}^3$ was proved by L. Lichtenstein, see [29]. W. Wolibner, [40], proves the existence of global in time solutions in $\mathbb{R}^2$. For other 2-D results see also [2], [22], [23], [27] and [39]. Concerning 3-D existence results under the boundary condition (1.3) we refer the reader to the classical papers [13], [19], [26] and [36]. For a very interesting overview of problems relating to the Euler equations, we refer to [15].

Concerning the Navier–Stokes equations, the first thought goes to the classical non-slip boundary condition $u = 0$ on Γ. However, it is well known that in this case "strong" convergence does not hold up to the boundary, since boundary layers appear; see the thorough analysis of [15], in particular Section 3.

The other boundary condition that comes to mind is the slip boundary condition:

$$\begin{cases} u \cdot \underline{n} = 0, \\ \underline{t} \cdot \underline{\tau} = 0, \end{cases} \tag{1.4}$$

for any tangential vector $\underline{\tau}$. The stress vector $\underline{t}$ is defined by $\underline{t} = \mathcal{T} \cdot \underline{n}$, where

$$\mathcal{T} = -\pi I + \frac{\nu}{2}(\nabla u + \nabla u^T),$$

is the stress tensor. The literature on this type of boundary conditions is endless. We quote here Solonnikov and Ščadilov's pioneering paper [34], and also [9], where a more general self-contained presentation is given. In these two references regularity results up to the boundary are considered (see also [8] and [1], where the regularity problem is considered in the half-space). Further, in [10], it is shown that the linear Stokes operator, under slip boundary conditions, generates an analytical semi-group. This may be applied to study evolution Stokes and Navier–Stokes equations.

It is worth noting that, on flat portions of the boundary, the boundary conditions (1.4) and

$$\begin{cases} (u \cdot \underline{n})_{|\Gamma} = 0, \\ \omega \times \underline{n} = 0 \end{cases} \tag{1.5}$$

coincide, where $\omega = \operatorname{curl} u$. This leads us to consider (1.5), even when the boundary is not flat. Note that the boundary condition (1.5) is strongly related to the slip boundary condition (1.4). In fact,

$$\underline{t} \cdot \underline{\tau} = \frac{\nu}{2}(\omega \times \underline{n}) \cdot \underline{\tau} - \nu u \cdot \frac{\partial \underline{n}}{\partial \underline{\tau}}.$$

Note, also, that the last term on the right-hand side is a lower order term, and that $\omega \times \underline{n}$ and $\frac{\partial \underline{n}}{\partial \underline{\tau}}$ are tangential to Γ, while $\left|\frac{\partial \underline{n}}{\partial \underline{\tau}}\right|$ is the normal curvature in the $\underline{\tau}$ direction. For $n = 2$, the second boundary condition in equation (1.5) is simply replaced by $\omega = 0$. Furthermore,

$$\underline{t} \cdot \underline{\tau} = \frac{\nu}{2}\omega - \nu u \cdot (k\underline{\tau}),$$

where k is the curvature of Γ. For a discussion on Navier–Stokes equations under some non-standard boundary conditions related to (1.5), we refer to [3] and [21].

Vanishing viscosity limit results in $\mathbb{R}^3$ without boundary conditions, can be found, for instance, in [16], [24], [25], [28], [31], [35], and in the recent paper [12]. Note that here we are interested only in "smooth solution" situations. For some main results concerning inviscid limits in non-smooth situations we refer to [17] (L^2 theory) and [18] (L^p theory).

We are mainly interested in 3-D problems. However, if the boundary is not flat, it is not clear how to prove the inviscid limit result. In fact, a substantial obstacle appears. In Section 4 we discuss this point in some detail. This situation leads us to consider, together with the 3-D flat boundary case, also the 2-D general problem, since in this case we are able to prove the convergence results for non-flat boundaries (Section 5).

The 3-D inviscid limit for solutions u^ν to the boundary value problem (1.1), (1.5) has been considered by Xiao and Xin in smooth domains, by means of a new and very interesting approach to the problem. In [41] these authors state the following result (see [41], Theorem 8.1). Assume that $\operatorname{div} u_0 = 0$, and that $u_0 \in H^3$ satisfies the boundary conditions (1.5). Then, as $\nu \to 0$,

$$u^\nu \to u \quad \text{in} \quad L^p(0,T;H^3(\Omega)) \cap C([0,T];H^2(\Omega)),$$

for some $T > 0$ and any $p \in [1,+\infty)$, where u is the solution to the Euler equations (1.2), (1.3). However, if the boundary is not flat, the proof seems to contain an oversight (see Remark 4.1 below).

We develop here an L^p theory, for arbitrarily large values of p, which is a novelty in the context of the above vanishing viscosity limit problems. In particular, our convergence results are substantially stronger than the previous ones. Since the problem has a local character, in studying the 3-D flat boundary case we consider a cubic domain Q (for details, see the following section), and prove the following result.

Theorem 1.1. *Assume $p > \frac{3}{2}$. For each $\nu > 0$ denote by u^ν the solution to the initial boundary value problem* (1.1), (1.5)*, in the "cubic domain" Q (flat boundary case). Assume also that the initial data u_0 belongs to $W^{3,p}(Q)$, is divergence free in Ω and satisfies the boundary conditions* (1.5)*. Then*

$$\begin{cases} u^\nu \rightharpoonup u \quad \textit{in} \quad L^\infty\big(0,T_0;W^{3,p}(\Omega)\big) \quad \textit{weak-}*, \\ u^\nu \to u \quad \textit{in} \quad C\big([0,T_0];W^{s,p}(\Omega)\big) \;\; \textit{for each } s<3, \end{cases} \tag{1.6}$$

where u is the unique solution to the Euler equations (1.2), (1.3)*. Further,*

$$\partial_t u^\nu \to \partial_t u \quad \textit{in} \quad L^\infty\big(0,T_0;W^{1,p}(\Omega)\big) \tag{1.7}$$

and, if $p \geq 2$,

$$\partial_t u^\nu \to \partial_t u \quad \textit{in} \quad L^p\big(0,T_0;W^{1,3p}(\Omega)\big). \tag{1.8}$$

In the 2-D case the assumption $\omega \times n = 0$ on Γ is simply replaced by $\omega = 0$. Moreover equation (3.4) is replaced by (5.1). For 2-D vanishing viscosity results under slip-type boundary conditions we refer to the classical papers, [2], [22], [33]. See also the more recent papers [14] and [38]. In [14] the authors consider the slip boundary condition

$$u \cdot \underline{n} = 0, \quad \underline{\tau}(u) + \alpha(x) u_\tau = 0, \tag{1.9}$$

where α is a given positive, twice continuous differentiable function on Γ. They assume that $u_0 \in H^2$ satisfies the boundary conditions (1.9), that $\operatorname{div} u_0 = 0$, and that $\operatorname{curl} u_0 \in L^\infty$, and prove that

$$u^\nu \to u \quad \text{in} \quad L^q\big(0, T; W^{\alpha, q'}(\Omega)\big),$$

for any $\alpha \in (0,1)$ and $q, q' \in (1, \infty)$. Further, $u^\nu \rightharpoonup u$ in $L^2\big(0, T; H^1(\Omega)\big)$ weakly, and $\omega^\nu \rightharpoonup \omega$ in $L^\infty\big(0, T; L^\infty(\Omega)\big)$ weak-*.

In fact, we show that in the 2-D case the results stated in Theorem 1.1 hold without the flat-boundary assumption. More precisely, we prove the following result.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected, open set, locally situated on on side of its boundary Γ that we assume to be a C^3 manifold. Then, the results stated in Theorem 1.1 hold with Q replaced by Ω. More precisely, the estimate* (1.8) *holds with the exponent* $3p$ *replaced by any finite* q*. Moreover, the results are global in time, in the sense that they hold for any arbitrarily large T_0.*

The effective construction of the solution to the problem (1.1), (1.5) follows easily from our a priori estimates. We may appeal to [10] or to well-known approximation methods like, for instance, Faedo–Galerkin procedure (see, for instance, [16], [20], [30], [37]).

In studying problems like inviscid limits, incompressible limits, well-posedness, etc., the relevance of the results strictly depends on the topology in which convergence is proved. In the framework of smooth solutions, if the initial data are given in a Banach space X, ultimate results should establish convergence in $C([0,T]; X)$. In the present context, this means to replace (1.6) by

$$u^\nu \to u \quad \text{in} \quad C\big([0, T_0]; W^{3,p}(\Omega)\big). \tag{1.10}$$

We believe that (1.10) can be proved by following ideas developed in previous papers, see [11] for references, even though the proofs become much more involved. Moreover it seems that there is not sufficient awareness of the mathematical importance of this kind of achievement.

The vanishing viscosity limit in *strong topology*, without a spatial boundary, was proved in [25], pages 54–56. For a very elementary proof of this same result see the recent paper [12]. See also [28] and, for $n = 2$, [27]. In [25] Kato points out that in his previous paper [24] he was able to prove *only* the weak topology result, and not the strong result. Kato's remark shows how large is the gap between the strong, and any weaker convergence result.

Remark 1.1. Let us present some other remarks concerning the gap between weak and strong topology convergence results. We take the very classical problem of the uniform continuous dependence on the initial data as model problem since some basic technical obstacles are similar. The uniform continuous dependence, in strong topology, on the initial data for solutions to the boundary value problem (1.2), (1.3), was proved in [19] by appealing to infinite dimensional Riemannian geometry techniques. A first analytical proof, well-founded only in 2-D, is given in [4]. For arbitrary dimensions the first analytical proof is given in [26]. The following claim is included in the introduction of reference [26]. The authors wrote: "A remark is in order regarding the continuous dependence in *strong* of the solution on the data. It is the most difficulty part in a theory dealing with nonlinear equations of evolution. As far as we know, [19] is the only place where continuous dependence (in the strong sense) has been proved for the Euler equation in a bounded domain. The general theory developed in [25] by one of the authors for non-linear equations is unfortunately nor applicable, since it is difficult to find the operator S with the required properties in the case of a bounded domain." (Actually, the result was proved later on, even in $W^{k,p}$ spaces, just by appealing to the above Kato's general perturbation theory; see references [6], [7], and [28].)

2. Some main estimates in general 3-D domains

The following result is a main tool in our proofs.

Theorem 2.1. *Let Ω be a regular open, bounded, set in $\mathbb{R}^3$. Then, for each $p > 1$, and sufficiently regular vector fields v,*

$$-\int \Delta v \cdot (|v|^{p-2} v) dx = \frac{1}{2} \int |v|^{p-2} |\nabla v|^2 dx + 4 \frac{p-2}{p^2} \int \left|\nabla |v|^{\frac{p}{2}}\right|^2 dx - \int_\Gamma |v|^{p-2} (\partial_i v_j) n_i v_j d\Gamma. \tag{2.1}$$

See [32], and [5] Lemma 1.1. Note that

$$\int_\Gamma |v|^{p-2} (\partial_i v_j) n_i v_j d\Gamma = \frac{1}{p} \int_\Gamma \partial_n |v|^p d\Gamma. \tag{2.2}$$

As remarked in [5], one has

$$\left|\nabla |v|^{\frac{p}{2}}\right|^2 \leq \left(\frac{p}{2}\right)^2 |v|^{p-2} |\nabla v|^2. \tag{2.3}$$

Note that, for $p \geq \frac{3}{2}$, the absolute value of the second term on the right-hand side of equation (2.1) is bounded by a positive constant times the first term, and that the above term is no-negative if $p \geq 2$.

It is easily shown that if v and n are two arbitrary, sufficiently regular, vector fields then

$$(\partial_i v_j) n_i v_j = (\operatorname{curl} v) \times \underline{n} \cdot v + (\partial_j v_i) n_i v_j. \tag{2.4}$$

Since we mainly work with the solution u^ν of the Navier–Stokes equations (1.1), in the following we denote this solution by u, except when both solutions u^ν and u appear at the same time.

In the sequel we set

$$\omega = \operatorname{curl} u, \quad \zeta = \operatorname{curl} \omega, \quad \chi = \operatorname{curl} \zeta, \tag{2.5}$$

and assume, for convenience, that Ω is simply-connected.

By multiplying both sides of equation (3.4) by $|\omega|^{p-2}\omega$, by integrating in Ω, and by taking into account the Theorem 2.1, one gets the general relation

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p + \frac{\nu}{2}\int |\omega|^{p-2}|\nabla\omega|^2 dx + 4\nu\frac{p-2}{p^2}\int \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 dx \\ &\quad + \frac{1}{p}\int (u\cdot\nabla)|\omega|^p dx - \int |\omega|^{p-2}\big((\omega\cdot\nabla)u\big)\cdot\omega dx \\ &= \nu\int_\Gamma |\omega|^{p-2}(\partial_i\omega_j)n_i\omega_j d\Gamma. \end{aligned} \tag{2.6}$$

Due to $\operatorname{div} u = 0$ in Ω, and to $u \cdot n = 0$ on Γ, the third integral on the left-hand side vanishes. Hence,

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\omega\|_p^p + \nu\frac{2(2p-3)}{p^2}\int \left|\nabla|\omega|^{\frac{p}{2}}\right|^2 dx \\ &\leq \int |\nabla u|\,|\omega|^p dx + \nu\int_\Gamma |\omega|^{p-2}(\partial_i\omega_j)n_i\omega_j d\Gamma, \end{aligned} \tag{2.7}$$

where we have used (2.3).

Next we follow the above argument with ω replaced by ζ. By applying the operator curl to both sides of equation (3.4) one gets, with obvious notation,

$$\partial_t\zeta - \nu\Delta\zeta + (u\cdot\nabla)\zeta + \sum c(Du)(D\omega) = 0. \tag{2.8}$$

Next, multiply both sides of the above equation by $|\zeta|^{p-2}\zeta$, integrate in Ω, and take into account the Theorem 2.1. An obvious extension of the above argument gives

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p + \nu\frac{2(2p-3)}{p^2}\int \left|\nabla|\zeta|^{\frac{p}{2}}\right|^2 dx \leq \int |\nabla u|\,|\nabla\omega|\,|\zeta|^{p-1}dx \\ + \nu\int_\Gamma |\zeta|^{p-2}(\partial_i\zeta_j)n_i\zeta_j d\Gamma. \end{aligned} \tag{2.9}$$

Finally, by applying the operator curl to the equation (2.8), and by following devices similar to that used in obtaining (2.9), we get the estimate

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + 4\nu\frac{2(2p-3)}{p^2}\int \left|\nabla|\chi|^{\frac{p}{2}}\right|^2 dx$$
$$\leq \int \big(|Du|\,|D^2\omega| + |D^2u|\,|D\omega|\big)|\chi|^{p-1}dx + \nu\int_\Gamma |\chi|^{p-2}(\partial_i\chi_j)n_i\chi_j d\Gamma. \qquad (2.10)$$

The estimates obtained in this section are not sufficient to extend Theorem 1.1 to general boundaries. Hence, in order to prove our results, we have to confine ourselves to flat-boundaries. This is done in the next section. In Section 4 we comment on the obstacles that prevent us from considering non-flat boundaries.

3. The 3-D flat-boundary case. Proof of Theorem 1.1

In this section we consider a cubic domain $Q = (]0,1[)^3$, and impose our boundary conditions only on two opposite faces. On the other faces we assume periodicity, as a device to avoid unessential technical difficulties. By working in this simple context, we concentrate on the basic ideas of proofs. We set

$$\Gamma = \{x : 0 \leq x_1, x_2 \leq 1, \quad \text{and} \quad x_3 = 0 \quad \text{or} \quad x_3 = 1\}.$$

The boundary condition (1.5) will be imposed on Γ. The problem is assumed to be periodic, with period equal to 1, both in the x_1 and the x_2 directions.

In the flat boundary case, the boundary conditions (1.5) (as well as (1.4)) are simply

$$u_3 = \omega_1 = \omega_2 = 0 \quad \text{on} \quad \Gamma. \qquad (3.1)$$

Further, from $\omega_1 = \omega_2 = 0$ on Γ and $\operatorname{div}\omega = 0$ it follows that

$$\partial_3\omega_3 = 0 \quad \text{on} \quad \Gamma. \qquad (3.2)$$

Proposition 3.1. *Assume that $\omega_1 = \omega_2 = 0$ on Γ and $\operatorname{div}\omega = 0$ in Ω. Then*

$$(\partial_i\omega_j)n_i\omega_j = 0 \quad \textit{on} \quad \Gamma. \qquad (3.3)$$

The result follows by appealing to (3.2) and to $n_1 = n_2 = 0$.

Lemma 3.1. *Let ω be a vector field in Q such that $\omega_1 = \omega_2 = 0$ on Γ, and set $\zeta = \operatorname{curl}\omega$. Then $\zeta_3 = 0$ on Γ.*

The following lemma was inspired by [41].

Lemma 3.2. *Let u be a vector field in Q, and $\omega = \operatorname{curl} u$. Assume that $u_3 = \omega_1 = \omega_2 = 0$ on Γ. Then the vector fields $(u\cdot\nabla)\omega$ and $(\omega\cdot\nabla)u$ are normal to Γ.*

The proof is left to the reader. Note that $\partial_3 u_1 = \omega_2 + \partial_1 u_3 = 0$ on Γ, and similarly for $\partial_3 u_2$.

Lemma 3.3. *Assume, in addition to the hypothesis of Lemma 3.2, that ω satisfies the equation*

$$\partial_t \omega - \nu \Delta \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = 0, \tag{3.4}$$

where $\nu > 0$. Then

$$(\operatorname{curl} \zeta) \times \underline{n} = 0$$

on Γ, where $\zeta = \operatorname{curl} \omega$.

Since ω is normal to the boundary, so is $\partial_t \omega$. By appealing to Lemma 3.2, it follows that $-\Delta \omega = \operatorname{curl} \zeta$ is normal to the boundary.

Proposition 3.2. *Under the assumptions of Lemma 3.3 one has*

$$(\partial_i \zeta_j) n_i \zeta_j = 0 \quad on \quad \Gamma. \tag{3.5}$$

The thesis follows from Lemmas 3.1 and 3.3. Note that $\partial_3 \zeta_1 = \chi_2 + \partial_1 \zeta_3$, and similarly for $\partial_3 \zeta_2$.

Proposition 3.3. *Set $\chi = \operatorname{curl} \zeta$. Under the assumptions of Lemma 3.3 one has*

$$(\partial_i \chi_j) n_i \chi_j = 0 \quad on \quad \Gamma. \tag{3.6}$$

By Lemma 3.3, one has $\chi \times \underline{n} = 0$. Hence $\chi_1 = \chi_2 = 0$ on Γ. Further, by appealing to $\operatorname{div} \chi = 0$, it follows that $(\partial_3 \chi_3)\chi_3 = 0$.

Lemma 3.4. *Let u, ω, ζ, χ be as above. Then, for each non-negative integer k one has the following norm-equivalence results.*

$$\|\omega\|_{k,p} \simeq \|u\|_{k+1,p}\,; \quad \|\zeta\|_{k,p} \simeq \|u\|_{k+2,p}\,; \quad \|\chi\|_{k,p} \simeq \|u\|_{k+3,p}\,.$$

The first claim follows from $\operatorname{curl} u = \omega$ and $\operatorname{div} u = 0$ in Q, together with the boundary condition $u \cdot \underline{n} = 0$. The second claim follows from $-\Delta u = \zeta$ in Q, together with the boundary conditions $\partial_3 u_1 = \partial_3 u_2 = u_3 = 0$. Finally, the third claim follows from the second claim, by taking into account that $\operatorname{curl} \zeta = \chi$ and $\operatorname{div} \zeta = 0$ in Q, and that $\zeta \cdot \underline{n} = 0$ on Γ (by Lemma 3.1).

By appealing to equations (3.3), (3.5) and (3.6) one obtains the following theorem.

Theorem 3.5. *If the boundary is flat, the boundary integrals in equations* (2.7), (2.9) *and* (2.10) *vanish.*

From the continuous immersion of $W^{1,2}$ in L^6 it follows that

$$\|g\|_{3p}^p \le c\big(\|\nabla |g|^{\frac{p}{2}}\|_2^2 + \|g\|_p^p\big). \tag{3.7}$$

We may use this estimate in equations (2.7), (2.9) and (2.10). From (2.9) and (3.7), one gets

$$\frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p + c\nu\|\zeta\|_{3p}^p \le \int |\nabla u|\,|\nabla\omega|\,|\zeta|^{p-1}dx + c\nu\|\zeta\|_p^p, \tag{3.8}$$

where we assume that $p > \frac{3}{2}$. Further, the integral on the right-hand side of (3.8) is bounded by $\|\nabla u\|_\infty \|\nabla\omega\|_p \|\zeta\|_p^{p-1}$. Since $W^{1,p} \subset L^\infty$, for $p > 3$, the following result holds.

Theorem 3.6. *Assume that $p > 3$. Then*

$$\frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p + c\nu\|\zeta\|_{3p}^p \le c\|\zeta\|_p^{p+1} + c\nu\|\zeta\|_p^p. \tag{3.9}$$

Similarly, from (2.10) we obtain, if $p > \frac{3}{2}$,

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + c\nu\|\chi\|_{3p}^p \le \int \big(|Du||D^2\omega| + |D^2u|^2\big)\,|\chi|^{p-1}dx + c\nu\|\chi\|_p^p. \tag{3.10}$$

Note that, for $p > \frac{3}{2}$, one has $\|Du\|_\infty \le c\|Du\|_{2,p}$ and

$$\big\|\,|D^2u|^2\,\big\|_p = \|D^2u\|_{2p}^2 \le c\|D^3u\|_p^2,$$

since $W^{1,p} \subset L^{2p}$. So, the following result holds.

Theorem 3.7. *Assume that $p > \frac{3}{2}$. Then*

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + c\nu\|\chi\|_{3p}^p \le c\|\chi\|_p^{p+1} + c\nu\|\chi\|_p^p. \tag{3.11}$$

To avoid a useless dependence on ν, fix a value ν_0 and assume that $\nu \le \nu_0$. Constants c may depend on ν_0.

From comparison theorems for ordinary differential equations applied to (3.11), it follows that $\|\chi(t)\|_p \le y(t)$, where $y(t)$ satisfies

$$y' = cy^2 + cy, \quad y(0) = y_0 =: \|\chi(0)\|_p. \tag{3.12}$$

The solution to the above equation is given by

$$\frac{y}{1+y} = \frac{y_0}{1+y_0}e^{ct}.$$

Note that $y(t)$ is no-negative, increasing, and goes to ∞ as t goes to T^*, where T^* is defined by

$$e^{cT^*} = \frac{1+y_0}{y_0}.$$

Further, we fix a value $T_0 \in (0, T^*)$. Then $y(t) \leq y(T_0)$ for each $t \leq T_0$. For instance, define T_0 by

$$e^{cT_0} = \frac{1}{2}\left(1 + \frac{1+y_0}{y_0}\right).$$

It follows that $y(T_0) = 1 + 2y_0$. Hence

$$\|\chi\|_{L^\infty(0,T_0;L^p)} \leq 1 + 2\|\chi_0\|_p. \tag{3.13}$$

Next, we turn back to the equation (3.11). By integrating it over $(0,t)$, for $t \in (0, T_0)$, and by using (3.13), with a straightforward manipulation we show that

$$\|\chi\|_{L^\infty(0,T_0;L^p)} + \nu^{\frac{1}{p}}\|\chi\|_{L^p(0,T_0;L^{3p})} \leq M_0. \tag{3.14}$$

The reader may verify that, with the above choice of T_0,

$$M_0^p \leq c\|\chi(0)\|_p^p + c\big(1 + \|\chi(0)\|_p^{p+1}\big).$$

Denote by N_0 positive constants that depend on M_0. One has

$$\|\omega\|_{L^\infty(0,T_0;W^{2,p})} + \nu^{\frac{1}{p}}\|\omega\|_{L^p(0,T_0;W^{2,3p})} \leq N_0. \tag{3.15}$$

Consequently,

$$\|u\|_{L^\infty(0,T_0;W^{3,p})} + \nu^{\frac{1}{p}}\|u\|_{L^p(0,T_0;W^{3,3p})} \leq N_0. \tag{3.16}$$

From (3.15), it follows that

$$\|\nu\Delta\omega\|_{L^\infty(0,T_0;L^p)} \leq \nu N_0. \tag{3.17}$$

Further,

$$\|(u\cdot\nabla)\omega\|_{L^\infty(0,T_0;W^{1,p})} + \|(\omega\cdot\nabla)u\|_{L^\infty(0,T_0;W^{2,p})} \leq N_0. \tag{3.18}$$

From (3.4), together with the above estimates it follows, in particular, that

$$\|\partial_t\omega\|_{L^\infty(0,T_0;L^p)} + \|\partial_t u\|_{L^\infty(0,T_0;W^{1,p})} \leq N_0. \tag{3.19}$$

Since $\|\cdot\|_{s,p} \leq c\|\cdot\|_{2,p}^{\frac{s}{2}}\|\cdot\|_{0,p}^{1-\frac{s}{2}}$, for $0 < s < 2$, it follows from (3.15) and (3.19), together with

$$\omega(t) - \omega(\tau) = \int_\tau^t \partial_s\omega(s)ds,$$

that

$$\|\omega\|_{C^{1-\frac{s}{2}}([0,T_0];W^{s,p})} \leq c\|\omega\|_{L^\infty(0,T_0;W^{2,p})}^{\frac{s}{2}}\|\partial_t\omega\|_{L^\infty(0,T_0;L^p)}^{1-\frac{s}{2}} \leq N_0. \tag{3.20}$$

From the above estimates, together with the uniqueness of the strong solution to the Euler equations (1.2), (1.3), one obtains the first equation (1.6), and also the following property

$$\partial_t u^\nu \rightharpoonup \partial_t u \quad \text{in} \quad L^\infty(0,T_0;W^{1,p}) \quad \text{weak-*}.$$

Note that we may pass to the limit directly in equation (3.4), as $\nu \to 0$. The second equation (1.6) follows by appealing to Ascoli–Arzela's compact embedding theorem.

Next, we write

$$\partial_t \omega - \partial_t \omega^\nu + (u - u^\nu)\cdot\nabla\omega + u^\nu \cdot \nabla(\omega - \omega^\nu)$$
$$+\omega^\nu \cdot \nabla(u^\nu - u) + (\omega^\nu - \omega)\cdot \nabla u = -\nu\Delta\omega^\nu.$$

From the previous results it follows that the non-linear terms on the left-hand side of the above equation go to zero in $C([0,T_0];W^{s,p})$, as ν goes to zero, for any $s<1$. In view of (3.17), equation (1.8) follows. In particular, the above non-linear terms go to zero in $C([0,T_0];L^{3p})$, if $p \geq 2$. On the other hand, from (3.15), it follows that

$$\|\nu\Delta\omega\|_{L^p(0,T_0;L^{3p})} \leq N_0 \nu^{\frac{1}{p'}}. \tag{3.21}$$

Hence (1.8) holds.

If in the above argument we appeal to (3.9) instead of (3.11), we obtain similar results, where $W^{3,p}$ is replaced by $W^{2,p}$, and $s < 2$. In this case it must be $p > 3$ everywhere.

4. On the 3-D non-flat boundary case

The obstacle that prevents us from extending to non-flat boundaries the proof of Theorem 1.1 is that, in such a more general case, Lemma 3.2 does not hold. This lemma was used to show that the boundary integrals vanish. In this section we show an attempt to overcome this obstacle, by trying to control the boundary integrals. However, this device seems not sufficient. Nevertheless, it looks interesting by itself, and we would like to present it to the reader. Basically, it allows to lower the highest order of the derivatives appearing in the boundary integrals.

Lemma 4.1. *Let the boundary Γ be a surface of class $C^k, k \geq 2$. Then, at any point $x_0 \in \Gamma$, the component of $(u\cdot\nabla)\omega - (\omega\cdot\nabla)u$ in any tangential direction τ has the form*

$$\big((u\cdot\nabla)\omega - (\omega\cdot\nabla)u\big)\cdot\tau = \sum a_{ij}(x)u_i\omega_j, \tag{4.1}$$

where the coefficients a_{ij} are of class C^{k-2} on Γ. Consequently,

$$\nu \operatorname{curl}\zeta\cdot\tau = -\nu(\Delta\omega)\cdot\tau = \sum a_{ij}(x)u_i\omega_j. \tag{4.2}$$

The straightforward proof is left to the reader. This lemma allows us to estimate the boundary integrals for each positive ν. However, we are interested in letting $\nu \to 0$. To fix ideas, let us consider (2.9). By appealing to (2.4), and by using arguments similar to that leading to (3.8), we get

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|\zeta\|_p^p + \nu\frac{2(2p-3)}{p^2}\int \left|\nabla|\zeta|^{\frac{p}{2}}\right|^2 dx \\ \leq \int |\nabla u|\,|\nabla\omega|\,|\zeta|^{p-1}dx + \nu\int_\Gamma |\zeta|^{p-2}(\operatorname{curl}\zeta)\times\underline{n}\cdot\zeta d\Gamma \\ -\nu\int_\Gamma |\zeta|^{p-2}(\partial_j n_i)\zeta_i\zeta_j d\Gamma,\end{aligned} \tag{4.3}$$

By (4.2), we find, with obvious notation,

$$\nu\int_\Gamma |\zeta|^{p-2}(\operatorname{curl}\zeta)\times n\cdot\zeta d\Gamma \leq c\int_\Gamma |\zeta|^{p-1}\sum|a(x)||u||\omega|d\Gamma. \tag{4.4}$$

In this way we drop the higher order derivatives in the boundary integral. However this is obtained at the cost of losing multiplication by ν. Actually, for each fixed positive ν we can estimate the boundary integral by the two $\nu-$ terms that appear in the left-hand side of (4.3). However, if ν goes to zero, the coefficient ν in the boundary integrals looks crucial. A similar argument can be applied to (2.10) instead of (2.9).

We end this section by showing that, in the non-flat boundary case, the tangential component of $(u\cdot\nabla)\omega - (\omega\cdot\nabla)u$ is not necessarily equal to zero, which is the thesis of Lemma 3.2. We show a quite simple counter-example due to C. R. Grisanti. For convenience we use here the notation (x, y, z). Let be $\Omega = \{(x, y, z) : z < -x^2\}$. The vector field $(2x, 0, 1)$ is normal to the boundary and the vector fields $\tau^1 = (-1, 0, 2x)$ and $\tau^2 = (0, 1, 0)$ are independent, and tangential to the boundary. Note that the above vector fields are not normalized. Define

$$u = e^{-2z}(1, -4xy, -2x).$$

One has $u\cdot\underline{n} = \omega\times\underline{n} = 0$ on Γ; moreover $\operatorname{div} u = 0$ and

$$\operatorname{curl} u = \omega = -4e^{-2z}(2xy, 0, y)$$

on Ω. Furthermore,

$$[(u\cdot\nabla)\omega]\cdot\tau^1 = -[(\omega\cdot\nabla)u]\cdot\tau^1 = 8ye^{4x^2}.$$

Hence, $[(u\cdot\nabla)\omega - (\omega\cdot\nabla)u]\cdot\tau^1$ does not vanish on Γ, for $y \neq 0$.

Remark 4.1. Note that the normal derivative of the tangential component of the above vector field u does not vanish on the boundary. For instance, $\partial_z(u\cdot\tau^1) = -2$ at the origin. This fact is at odds with the statement of Proposition 4.1 in [41].

5. Non-flat boundary in the 2-D case

In the 2-D case the equation (3.4) is replaced by

$$\partial_t\omega - \nu\Delta\omega + u\cdot\nabla\omega = 0, \tag{5.1}$$

where the vector field $\omega = \operatorname{curl} u$, orthogonal to the plane motion, is identified with the scalar $\omega = \partial_1 u_2 - \partial_2 u_1$. Similarly, ζ is a vector lying in the plane, and χ a scalar. In the 2-D case, the boundary conditions corresponding to (1.5) are simply given by

$$u \cdot \underline{n} = 0, \quad \omega = 0. \tag{5.2}$$

The next result corresponds to Proposition 3.2.

Proposition 5.1. *Let u be a vector field in Ω, and let $\omega = \operatorname{curl} u$. Assume that* (5.1) *holds on Γ. Then the vector field $u \cdot \nabla\omega$ vanishes on Γ.*

The result follows from the fact that $u \cdot \nabla\omega$ is a tangential derivative of ω.

Proposition 5.2. *Assume, in addition to the set up in Proposition 5.1, that ω satisfies the equation* (5.1)*, where $\nu > 0$. Then*

$$\chi = \operatorname{curl} \zeta = 0$$

on Γ, where $\zeta = \operatorname{curl} \omega$.

It thus follows that the boundary integral in equation (2.10) vanishes, since in 2-D it is given by

$$\int_\Gamma |\chi|^{p-2} (\partial_i \chi) n_i \chi d\Gamma.$$

So, in the 2-D case, (2.10) reads

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + \nu \frac{2(2p-3)}{p^2} \int \left|\nabla |\chi|^{\frac{p}{2}}\right|^2 dx \le \int \big(|Du|\,|D^2\omega| + |D^2 u|\,|D\omega|\big) |\chi|^{p-1} dx. \tag{5.3}$$

Since $W_0^{1,2} \subset L^q$, for each finite q, it readily follows that

$$\int \left|\nabla |\chi|^{\frac{p}{2}}\right|^2 dx \ge c_q \|\chi\|_q^p.$$

We have to take into account that $\frac{pq}{2}$ is arbitrarily large. Hence the following result holds.

Theorem 5.1. *Let Ω be a regular, bounded, simply-connected, open set in $\mathbb{R}^2$. Then, for each $p > \frac{3}{2}$ and each finite q, one has*

$$\frac{1}{p}\frac{d}{dt}\|\chi\|_p^p + c\nu \int \left|\nabla |\chi|^{\frac{p}{2}}\right|^2 dx + c_q \nu \|\chi\|_q^p \le \int \big(|Du|\,|D^2\omega| + |D^2 u|\,|D\omega|\big) |\chi|^{p-1} dx. \tag{5.4}$$

Actually, we may replace $|D^2\omega|$ by $|D\zeta|$ and $|D^2 u|$ by $|\zeta|$. In analogy to the 3-D flat boundary case, the estimate (5.4) leads to corresponding convergence results, local in time. Actually, the results can be proved globally in time by following ideas already known in considering 2-D problems. This is left to the interested reader.

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