REMARKS ON A KNOWN REGULARITY RESULT FOR FLOWS WITH NONLINEAR VISCOSITY

1 Introduction

We want to warn the reader that the results proved below are not new. However these notes may have some interest to young researchers as a very simple and clear introduction to the subject.

In the sequel we consider the Ladyzhenskaya model for the Navier-Stokes equations with shear dependent viscosity

$$\begin{cases} -\nu_0 \nabla \cdot \mathcal{D}u - \nu_1 \, \nabla \cdot (|\mathcal{D}u|^{p-2} \, \mathcal{D}u) + \nabla \, \pi = \, f(x), \\ \nabla \cdot u = \, 0 \, , \end{cases} \tag{1.1}$$

where

$$\mathcal{D}\, u = \, \nabla \, u + \, \nabla \, u^T \, .$$

Hence

$$(\mathcal{D}\, u)_{ij} = \, \frac{\partial \, u_i}{\partial \, x_j} + \, \frac{\partial \, u_i}{\partial \, x_j} \, .$$

Moreover we define

$$|\mathcal{D}\, u(x)|^2 = \, \sum_{i,j=1}^{3} \, |(\mathcal{D}\, u)_{ij}(x)|^2 \, ,$$

$$|\nabla \, u(x)|^2 \, = \, \sum_{i,j=1}^{3} \, |\frac{\partial \, u_i(x)}{\partial \, x_j}|^2$$

$$|(\nabla \, \mathcal{D}\, u)(x)|^2 \, = \, \sum_{i,j,k=1}^{3} \left| \frac{\partial}{\partial \, x_k} \, (\mathcal{D}\, u)_{ij}(x) \right|^2 \, ,$$

and similarly for $|\nabla^2 \, u(x)|^2$. Since $2 \, \partial_i \, \partial_j \, u_k = \, \partial_i \, (\mathcal{D}\, u)_{jk} + \partial_j \, (\mathcal{D}\, u)_{ki} - \partial_k \, (\mathcal{D}\, u)_{ij}$ it follows the well known result

$$c \, |\nabla^2 \, u(x)| \leq \, |(\nabla \, \mathcal{D}\, u)(x)| \leq \, C \, |\nabla^2 \, u(x)| \, . \tag{1.2}$$

In the sequel we prove the following well known regularity result.

Theorem 1.1. *Assume that* $f \in L^2(\Omega)$ *and let* (u, π) *be a local weak solution of problem* (1.1) *in* Ω *(see definition below). Fix an open set* $\Omega_0 \subset\subset \Omega$. *Then the estimate*

$$\begin{aligned} &\|D^2 \, u\|^2_{L^2(\Omega_0)} + \, \textstyle\sum_{k=1}^{3} \left\| \, |\mathcal{D}\, u|^{\frac{p-2}{2}} \, \mathcal{D} \, \frac{\partial u}{\partial x_k} \right\|^2_{L^2(\Omega_0)} + \\ &\|\nabla \, \pi\|^2_{L^{p'}(\Omega_0)} \leq \, c \, (K_1^2 + \, K_2^2 + \, \Phi) \end{aligned} \tag{1.3}$$

holds where K_1, K_2 *and* Φ *depend only on the finite quantities* $\|f\|_2\, \|\nabla u\|_2\, \|\nabla u\|_p\, \|\pi\|_{p'}$ *and* $\|D^2\theta\|^2_\infty$. *More precisely, these quantities are given by* (3.11), (3.14) *and* (2.12) *respectively.*

Moreover, if $2 < p < 6$ *the pressure satisfies the estimate*

(1.4)

$$\|\nabla\,\pi\|_{L^r(\Omega_0)} \leq c\,(1 + \|u\|_2)\,(K_1 + K_2 + \sqrt{\Phi}) + c\,(K_1 + K_2 + c\,\sqrt{\Phi})^{\frac{p}{2}} + \|f\|_2\,.$$

where r, $r > p'$, *is given by* (4.1). *This last result can be improved as shown in Section 5.*

REMARK. The cut-of function θ that appears in the definitions of K_1, K_2 and Φ is a $C^2(\Omega)$-function with compact support and such that $0 \leq \theta(x) \leq 1$ in Ω, and $\theta(x) = 1$ in Ω_0.

The above system of equations was introduced by O.A. Ladyzhenskaya. See, for instance, [3] and [4]. For $p = 3$ the above model was considered by Smagorinsky, see [7], as a turbulence model. J.-L. Lions considered similar models, in which $\mathcal{D}\,u$ is replaced by ∇u. See [5], Chap.2, n.5.

The above local $W^{2,2}$-regularity result is well known, even for much more general problems. The literature on this subject is wide. We just refer here to the proof of the above result given in reference [2], which is part of the proof of Lemma 3.0.5 in this same reference.

The symbol $\|\,.\,\|_p$ denotes the canonical norm in $L^p(\Omega)$. Moreover, $\|\,.\,\| = \|\,.\,\|_2$. We denote by $W^{k,p}(\Omega), k$ a positive integer and $1 < p < \infty$, the usual Sobolev space of order k, by $W^{1,p}_0(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C^\infty_0(\Omega)$ and by $W^{-1,p'}(\Omega)$ the strong dual of $W^{1,p}_0(\Omega)$, where $p' = p/(p-1)$. The canonical norms in these spaces are denoted by $\|\,.\,\|_{k,p}$.

In notation concerning duality pairings and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domain Ω, provided that the meaning remains clear.

We set

$$\mathbb{L}^p = [L^p(\Omega)]^3, \quad \mathbb{W}^{k,p} = [W^{k,p}(\Omega)]^3\,, \quad \mathbb{W}^{1,p}_0 = [W^{1,p}_0(\Omega)]^3\,.$$

We denote by c, $\overline{c}$, c_1, c_2, etc., positive constants that depend, at most, on Ω, ν_0, ν_1 and p. The same symbol c may denote different constants, even in the same equation.

Local weak solutions (u, π) to problem (1.1) in an open set Ω_1 are assumed here to belong, by definition, to the space $W^{1,p}_{loc}(\Omega_1) \times L^{p'}_{loc}(\Omega_1)$. From the local regularity point of view one studies regularity in any arbitrary, fixed, open bounded set $\Omega \subset\subset \Omega_1$. Hence

$$(u, \pi) \in W^{1,p}(\Omega) \times L^{p'}(\Omega)\,. \tag{1.5}$$

Note, in particular, that weak solutions to significant boundary value problems in a domain Ω typically satisfy the assumption (1.5). By taking into account the above remark we are lead to use here the following definition.

Definition. Assume that $f \in L^2(\Omega)$. We say that a pair (u, π) is a <u>local weak solution</u> of problem (1.1) in Ω if it satisfies (1.5) together with

$$\begin{aligned} &\tfrac{\nu_0}{2} \int_\Omega \mathcal{D}u \cdot \mathcal{D}\phi\, dx + \tfrac{\nu_1}{2} \int_\Omega |\mathcal{D}u|^{p-2}\, \mathcal{D}u \cdot \mathcal{D}\phi\, dx \\ &- \int_\Omega \pi\,(\nabla \cdot \phi)\, dx = \int_\Omega f \cdot \phi\, dx\,, \end{aligned} \tag{1.6}$$

for each $\phi \in W_0^{1,p}(\Omega)$ (or, equivalently, $\phi \in C_0^\infty(\Omega)$).

Remark. In order to extend partially the proof to boundary value problems it looks convenient to consider explicitly the pressure term in equation (1.6) and do not assume that the test-functions ϕ are divergence-free.

Let us justify the above definition, in particular the assumption (1.5). For fixing ideas we consider the non-slip boundary condition

$$u_{|\Gamma} = 0\,. \tag{1.7}$$

A formal integrations by parts show that

$$\begin{aligned} &\frac{1}{2}\int_\Omega \nu_T(u)\,\mathcal{D}\,u \cdot \mathcal{D}\,v\,dx = \\ &-\int_\Omega [\nabla\cdot(\nu_T(u)\,\mathcal{D}\,u)]\cdot v\,dx + \int_\Gamma \underline{\tau}(u)\cdot v\,d\Gamma\,, \end{aligned} \tag{1.8}$$

for each divergence free vector field v vanishing on the boundary. It readily follows that (at least formally; see below for the functional framework) u is a solution to problem (1.1), (1.7), for some π, if and only if $u \in V$ satisfies (1.11) for all $v \in V$, where V denotes the set of all divergence-free "regular" vector fields vanishing on the boundary.

The existence of π, as a distribution, follows from well known results, by using divergence free test functions $v \in C_0^\infty(\Omega)$ in equation (1.11).

The above considerations give rise to the definition of a weak solution described below. V_0 denotes the space

$$V_0 = \left\{ v \in W_0^{1,2}(\Omega) \,:\, \nabla\cdot v = 0 \quad \text{in } \Omega \right\} \tag{1.9}$$

endowed with the norm $\|\nabla u\|$. Moreover, $[\,\cdot\,]_{-1}$ denotes the strong norm in the dual space $(V_0)'$.

We set

$$V = \{ v \in V_0 \,:\, \|\mathcal{D}v\|_p < \infty \}\,,$$

endowed with the norm

$$\|v\|_V = \|\nabla v\|_2 + \|\mathcal{D}v\|_p\,.$$

It should be remarked that, by appealing to inequalities of Korn's type, we can verify that $V = \{ v \in V_2 \,:\, \|\nabla\, v\|_p < \infty \}$ and also that $\|\nabla v\|_2 + \|\mathcal{D}v\|_p$ and $\|\nabla v\|_2 + \|\nabla v\|_p$ are equivalent norms in V.

Weak solutions exist under the assumption

$$f \in (V_0)'\,. \tag{1.10}$$

Definition 1.1. *We say that u is a* weak solution *to problem* (1.1), (1.7) *if $u \in V$ satisfies*

$$\frac{1}{2}\int_\Omega \nu_T(u)\,\mathcal{D}u \cdot \mathcal{D}v\,dx = \int_\Omega f\cdot v\,dx\,, \tag{1.11}$$

for all $v \in V$.

By defining $< A\,u,\, v >$, for each pair $u,\, v \in V$, as the left hand side of (1.11), the operator $A:\, V \to V'$ satisfies the assumptions in the Theorems 2.1 and 2.2, Chap.2, Sect.2, [5]. This shows existence and uniqueness of the weak solution.

By replacing v by u in equation (1.11) one gets

$$\nu_0\, \|\nabla u\|^2 +\, \nu_1\, \|\mathcal{D}u\|_p^p = < f, u >_\Omega\,, \tag{1.12}$$

where the symbols $< \cdot, \cdot >$ denote "duality pairings". Note that the left hand side of equation (1.12) is just $< A\,u,\, u >$. This shows that the assumption (2.3) in the above Theorem 2.1, reference [5], holds.

From (1.12) there readily follows the basic estimate

$$\frac{\nu_0^2}{2}\, \|\nabla u\|^2 +\, \nu_0\, \nu_1\, \|\mathcal{D}u\|_p^p \leq\, c_n\, [f]_{-1}^2\,, \tag{1.13}$$

where the constant c_n depends only on n. Recall that

$$\|\nabla u\|_p \leq\, c_{n,p}\, \|\mathcal{D}u\|_p\,.$$

By restriction of (1.11) to divergence-free test-functions v with compact support in Ω, and by (1.8), there follows the existence of a distribution π (determined up to a constant) such that

$$\nabla\, \pi = -\, \nabla \cdot \big[\nu_0 \nabla\, u + \nu_1\, |\mathcal{D}u|^{p-2}\, \mathcal{D}u\,\big] +\, f\,. \tag{1.14}$$

Equation (1.14) shows that the first equation (1.1) holds in the distributional sense.

On the other hand, it is well known (see [6]) that if

$$\nabla\, \pi =\, \nabla\, \cdot\, U$$

for some $U \in L^\alpha(\Omega)$, $\alpha >\, 1$, then

$$\|\pi\|_{L^\alpha_\#(\Omega)} \leq\, c\, \|U\|_{L^\alpha(\Omega)}\,, \tag{1.15}$$

where $L^\alpha_\# =\, L^\alpha/\mathbb{R}$. Hence, from equations (1.14) and (1.13), it readily follows that

$$\pi \in L^{p'}(\Omega).$$

2 An estimate for the velocity in terms of the pressure

In this section we prove the estimate (2.11).

We start by recalling the following result. Let U, V be two arbitrary vectors in $\mathbb{R}^N, N \geq 1$, and $p \geq 2$. Then

$$\begin{aligned} &\big(|U|^{p-2}U -\, |V|^{p-2}V\big) \cdot\, (U -\, V) \geq \frac{1}{2}\, \big(|U|^{p-2} +\, |V|^{p-2}\big)\, |U -\, V|^2\,, \\ &\big|\, |U|^{p-2}U -\, |V|^{p-2}V\big| \leq\, \frac{p-1}{2}\, \big(|U|^{p-2} +\, |V|^{p-2}\big)\, |U -\, V|\,. \end{aligned} \tag{2.1}$$

We assume in the sequel that test functions ϕ have compact support in Ω and that the translation's amplitudes $|h|$ are such that the translations of the test functions used in the sequel have compact support in Ω. Without loss of generality we assume that translations h are given in the x_1 direction. We write in general

$$g_h(x) = g(x_1 + h, x_2, x_3)$$

for an arbitrary function g.

Next consider the equation (1.6) with ϕ replaced by the admissible test functions $\phi_h(x) = \phi(x+h)$. This leads to the equation (2.2) below with $x_1 - h$ replaced by x_1. Then we obtain (2.2) simply by doing the change of variables $x_1 \rightarrow x_1 - h$.

$$\begin{aligned} &\tfrac{\nu_0}{2} \int \mathcal{D}u_{-h} \cdot \mathcal{D}\phi \, dx + \tfrac{\nu_1}{2} \int |\mathcal{D}u_{-h}|^{p-2} \, \mathcal{D}u_{-h} \cdot \mathcal{D}\phi \, dx \\ &- \int \pi_{-h} \, (\nabla \cdot \phi) \, dx = \int f \cdot \phi_h \, dx \,. \end{aligned} \tag{2.2}$$

Next, by taking the difference, side by side, between equation (1.6) and (2.2) we get

$$\begin{aligned} &\tfrac{\nu_0}{2} \int (\mathcal{D}(u - u_{-h})) \cdot \mathcal{D}\phi \, dx \\ &+ \tfrac{\nu_1}{2} \int \big(|\mathcal{D}u|^{p-2} \mathcal{D}u - |\mathcal{D}u_{-h}|^{p-2} \mathcal{D}u_{-h} \big) \cdot \mathcal{D}\phi \, dx \\ &- \int (\pi - \pi_{-h}) \, (\nabla \cdot \phi) \, dx = - \int f \cdot (\phi_h - \phi) \, dx \,. \end{aligned} \tag{2.3}$$

A classical result shows that

$$| \int f \cdot (\phi_h - \phi) \, dx | \leq h \, \|f\|_2 \, \|\nabla \phi\|_2 \,. \tag{2.4}$$

Now we replace in (2.3) the test functions ϕ by $(u - u_{-h}) \, \theta^2$, where $\theta \in C^2_0(\Omega)$ is no-negative and has compact support in Ω. The translation's amplitudes $|h|$ are such that the h-translations of the θ functions have compact support inside Ω. Due to the identity

$$\mathcal{D}\, v \cdot \mathcal{D}(\theta^2 \, v) = |\mathcal{D}(\theta \, v)|^2 - |v \otimes (\nabla \, \theta) + (\nabla \, \theta) \otimes v|^2$$

one has

$$\int \mathcal{D}v \cdot \mathcal{D}(\theta^2 \, v) \, dx = \int |\mathcal{D} \, (\theta \, v)|^2 \, dx - \int |v \otimes (\nabla \, \theta) + (\nabla \, \theta) \otimes v|^2 \, dx \,.$$

Moreover, if θ has compact support in Ω, a direct calculation shows that

$$\begin{aligned} &\int |\mathcal{D} \, (\theta \, v)|^2 \, dx = 2 \int |\nabla \, v|^2 \, \theta^2 \, dx + 2 \int |\nabla \cdot v|^2 \, \theta^2 \, dx \\ &+ 2 \int |v|^2 \, |\nabla \, \theta|^2 \, dx - \int |v|^2 \, \Delta \, \theta^2 \, dx + 2 \int |v \cdot \nabla \, \theta|^2 \, dx \,. \end{aligned} \tag{2.5}$$

Note that equation (2.5) is a Korn's type "inequality" (see, for instance, [2] Lemma 3.0.1).

By using equation (2.5) with $v = u - u_{-h}$ and by appealing to the inequalities (2.1) in order to treat the ν_1-term one easily gets from (2.3) that

(2.6)
$$
\begin{aligned}
&\nu_0 \int |\nabla(u - u_{-h})|^2\, \theta^2\, dx \\
&+ \tfrac{\nu_1}{4} \int \left(|\mathcal{D}u|^{p-2} + |\mathcal{D}u_{-h}|^{p-2}\right) (|\mathcal{D}u - \mathcal{D}u_{-h}|^2\, \theta^2\, dx \leq \\
&c\,\nu_1\,(p-1) \int \left(|\mathcal{D}u|^{p-2} + |\mathcal{D}u_{-h}|^{p-2}\right) (|\mathcal{D}u - \mathcal{D}u_{-h}|\, |u - u_{-h}|\, |\nabla\theta^2|\, dx \\
&- \int (\pi - \pi_{-h})\, (u - u_{-h}) \cdot \nabla\theta^2\, dx + \\
&c\, h\, \|f\|_2 \left(\int |\nabla(u - u_{-h})|^2\, \theta^2\, dx\right)^{\frac{1}{2}} + c\, h^2\, \|f\|_2\, \|\nabla u\|_2\, \|\nabla\theta^2\|_\infty + \\
&\tfrac{\nu_0}{2} \int (u - u_{-h})^2\, |\Delta\theta|^2\, dx\,.
\end{aligned}
$$

We have estimated the f-term as follows

$$
\begin{aligned}
&|\int f \cdot (\phi_h - \phi)\, dx| \leq h\, \|f\|_2 \left(\int |\nabla u - \nabla u_{-h}|^2\, \theta^2\, dx\right)^{\frac{1}{2}} + \\
&h^2\, \|f\|_2\, \|\nabla u\|_2\, \|\nabla\theta^2\|_\infty\,,
\end{aligned}
\tag{2.7}
$$

by appealing to (2.4). For convenience we assume that $\theta(x) \leq 1$ everywhere.

We define the nonnegative quantities A_0, A_1, R_1 and B as follows.

$$
A_0^2 = \int |\nabla(u - u_{-h})|^2\, \theta^2\, dx\,,
$$

$$
A_1^2 = \int \left(|\mathcal{D}u|^{p-2} + |\mathcal{D}u_{-h}|^{p-2}\right) |\mathcal{D}u - \mathcal{D}u_{-h}|^2\, \theta^2\, dx\,,
$$

$$
R_1 = \int \left(|\mathcal{D}u|^{p-2} + |\mathcal{D}u_{-h}|^{p-2}\right) |\mathcal{D}u - \mathcal{D}u_{-h}|\, |u - u_{-h}|\, |\nabla\theta|\, \theta\, dx\,,
$$

$$
B = \int (\pi - \pi_{-h})\, (u - u_{-h}) \cdot \theta\, \nabla\theta\, dx\,.
$$

From (2.6) it follows that

$$
\begin{aligned}
&\nu_0\, A_0^2 + \nu_1\, A_1^2 \leq c\,\nu_1\, R_1 + c\,|B| + \\
&\tfrac{c}{\nu_0}\, h^2\, \|f\|_2^2 + c\, h^2\, \|f\|_2\, \|\nabla u\|_2\, \|\nabla\theta^2\|_\infty + c\,\nu_0\, h^2\, \|\nabla u\|_2^2\, \|\nabla\theta\|_\infty^2\,.
\end{aligned}
\tag{2.8}
$$

On the other hand

$$
\begin{aligned}
&|\mathcal{D}u - \mathcal{D}u_{-h}|\, |u - u_{-h}|\, |\nabla\,\theta|\, |\theta| \leq \\
&\tfrac{\epsilon}{2}\, |\mathcal{D}u - \mathcal{D}u_{-h}|^2\, \theta^2 + \tfrac{1}{2\,\epsilon}\, |u - u_{-h}|^2\, |\nabla\,\theta|^2\,.
\end{aligned}
\tag{2.9}
$$

Hence

$$
R_1 \leq \frac{\epsilon}{2}\, A_1^2 + \frac{1}{2\,\epsilon}\, h^2\, \|\nabla u\|_p^p\, \|\nabla\theta\|_\infty^2\,,
\tag{2.10}
$$

where $\epsilon > 0$ is arbitrary. It follows that(sss)

$$
\nu_0\, A_0^2 + \nu_1\, A_1^2 \leq c\,|B| + \Phi\, h^2,
\tag{2.11}
$$

where

$$\Phi = \tfrac{c}{\nu_1}\,\|\nabla u\|_p^p\,\|\nabla\theta\|_\infty^2 + \tfrac{c}{\nu_0}\,\|f\|_2^2 \tag{2.12}$$
$$+\,c\,\|f\|_2\,\|\nabla u\|_2\,\|\nabla\theta^2\|_\infty + c\,\nu_0\,\|\nabla u\|_2^2\,\|\nabla\theta\|_\infty^2\,.$$

Note that Φ is finite.

3 An estimate for the pressure in terms of the velocity

Here we prove the estimate (3.15) which, together with (2.11), leads to the desired results.

From equation (2.3) with ϕ replaced by $\phi\,\theta$, where θ is as above and (for instance) $\phi \in C^\infty(\Omega)$, one easily obtains

$$-\int \nabla\,[(\pi - \pi_{-h})\,\theta]\,\cdot\phi\,dx = -\int (\pi - \pi_{-h})\,\phi\cdot(\nabla\theta)\,dx+$$
$$\tfrac{\nu_0}{2}\int (\mathcal{D}(u - u_{-h}))\,\theta\cdot\mathcal{D}\phi\,dx + \nu_0\int (\mathcal{D}(u - u_{-h}))\cdot(\phi\otimes\nabla\theta)\,dx+$$
$$\tfrac{\nu_1}{2}\int \left(|\mathcal{D}u|^{p-2}\mathcal{D}u - |\mathcal{D}u_{-h}|^{p-2}\mathcal{D}u_{-h}\right)\,\theta\cdot\mathcal{D}\phi\,dx+ \tag{3.1}$$
$$\nu_1\int \left(|\mathcal{D}u|^{p-2}\mathcal{D}u - |\mathcal{D}u_{-h}|^{p-2}\mathcal{D}u_{-h}\right)\cdot(\phi\otimes\nabla\theta)\,dx+$$
$$\int f\cdot((\theta\,\phi)_h - (\theta\phi))\,dx =$$
$$I_1 + \dots + I_6\,.$$

Next we estimate the I_j terms. Since

$$I_1 = -\int \pi\,(\phi\cdot\nabla\,\theta - \phi_h\cdot\nabla\,\theta_h)\,dx$$

it readily follows that

$$|I_1| \le c\,h\,\|\pi\|_{p'}\,\|D^2\theta\|_\infty\,\|\nabla\,\phi\|_p\,. \tag{3.2}$$

On the other hand

$$|I_2| \le c\,\nu_0\left(\int |\mathcal{D}u - \mathcal{D}u_{-h}|^2\,\theta^2\,dx\right)^{\frac{1}{2}}\|\mathcal{D}\phi\|_2\,. \tag{3.3}$$

Next, by the second inequality (2.1),

$$|I_4| \le \nu_1\,\tfrac{p-1}{4}\int \left(|\mathcal{D}\,u|^{\frac{p-2}{2}} + |\mathcal{D}\,u_{-h}|^{\frac{p-2}{2}}\right)\times \tag{3.4}$$
$$\left[\left(|\mathcal{D}\,u|^{\frac{p-2}{2}} + |\mathcal{D}\,u_{-h}|^{\frac{p-2}{2}}\right)|\mathcal{D}u - \mathcal{D}u_{-h}|\,\theta\right]\,|\mathcal{D}\,\phi|\,dx\,.$$

by taking into account that

$$\frac{p-2}{2p} + \frac{1}{2} + \frac{1}{p} = 1\,,$$

and by appealing to Hőlder's inequality one gets

$$|I_4| \le \nu_1 \frac{p-1}{2} \|\mathcal{D}u\|_p^{\frac{p-2}{2}} A_1 \|\mathcal{D}\phi\|_p \,. \tag{3.5}$$

Next we estimate I_5. By a suitable translation one has

$$I_5 = \int |\mathcal{D}u|^{p-2}\mathcal{D}u \cdot (\phi \otimes \nabla\theta - \phi_h \otimes \nabla\theta_h)\, dx \,.$$

It readily follows that

$$|I_5| \le c\,\nu_1\, |h|\, \|\mathcal{D}u\|_{p'}^{p-1} \|D^2\theta\|_\infty \|\nabla\phi\|_p \,. \tag{3.6}$$

This estimate is sufficient to our purposes, however we rather prefer to appeal to the estimate

$$|I_5| \le \nu_1 \frac{p-1}{2} \|\mathcal{D}u\|_p^{\frac{p-2}{2}} A_1 \|\nabla\phi\|_2 \,. \tag{3.7}$$

For the proof see the appendix. Similarly,

$$|I_3| \le c\,\nu_0\, |h|\, \|\mathcal{D}u\|_2 \|D^2\theta\|_\infty \|\nabla\phi\|_2 \,. \tag{3.8}$$

Finally, by appealing to (2.4) one shows that

$$|I_6| \le |h|\, \|f\|_2 \|\nabla\theta\|_\infty \|\nabla\,\phi\|_2 \,. \tag{3.9}$$

From (3.1) together with the above estimates for the I_j terms one gets

$$\|\nabla[(\pi - \pi_{-h})\,\theta]\|_{-1,p'} \le c\,|h|\, K_1 + c\,\nu_0\, A_0 + c\,\nu_1\, A_1 \,, \tag{3.10}$$

where

$$K_1 = \|D^2\,\theta\|_\infty \left(1 + \|f\|_2 + \|\pi\|_{p'} + \nu_0\, \|\nabla\,u\|_2 + \nu_1\, \|\nabla\,u\|_p^{\frac{p-2}{2}}\right) \,. \tag{3.11}$$

Well known results, see [6], show the existence of a constant c such that for all $g \in L^q(\Omega)$ one has

$$\|g\|_q \le c\,(\|g\|_{-1,\,q} + \|\nabla\,g\|_{-1,\,q}) \,.$$

From this result together with (3.10) one shows that

$$\|(\pi - \pi_{-h})\,\theta\|_{p'} \le c\,|h|\, K_1 + c\,\nu_0\, A_0 + c\,\nu_1\, A_1 \,. \tag{3.12}$$

Note that

$$\|(\pi - \pi_{-h})\,\theta\|_{-1,p'} \le c\,|h|\,(1 + \|\nabla\,\theta\|_\infty)\, \|\pi\|_{p'} \,.$$

On the other hand (ole)

$$|B| \le |h|\, K_2\, \|(\pi - \pi_{-h})\,\theta\|_{p'} \,, \tag{3.13}$$

where

$$K_2 = \|\nabla\theta\|_\infty \|\nabla u\|_p \,. \tag{3.14}$$

From (3.12) and (3.13) one gets

$$|B| \leq c\,|h|\,K_2\,(|h|\,K_1 + \nu_0\,A_0 + \nu_1\,A_1)\,. \tag{3.15}$$

Finally from (2.11) and (3.15) it follows that

$$\frac{\nu_0}{2}\left(\frac{A_0}{h}\right)^2 + \frac{\nu_1}{2}\,(p-1)\left(\frac{A_1}{h}\right)^2 \leq c\,K_2\,(K_1 + \nu_0\,\frac{A_0}{|h|} + \nu_1\,\frac{A_1}{|h|}) + \Phi\,. \tag{3.16}$$

This shows that the left hand side of (3.16) is bounded. In particular

$$\left(\frac{A_0}{h}\right)^2 + \left(\frac{A_1}{h}\right)^2 \leq c\,[K_2^2 + K_1\,K_2 + \Phi] = \widetilde{C}\,. \tag{3.17}$$

For convenience assume that the constants c may depend on ν_0 and ν_1. Fix an open set $\Omega_0 \subset\subset \Omega$ and set $\overline{h} = \mathit{dist}\,(\Omega_0,\,\partial\,\Omega)$. Fix the function θ equal to 1 on Ω_0 and with compact support in Ω. From (3.17) it follows that

$$\begin{aligned} &\int_{\Omega_0} \left|\nabla(\tfrac{u-\,u_{-h}}{h})\right|^2\,dx + \\ &\int_{\Omega_0} \left(|\mathcal{D}u|^{p-2} + |\mathcal{D}u_{-h}|^{p-2}\right)\left|\mathcal{D}(\tfrac{u-\,u_{-h}}{h})\right|^2\,dx \leq \widetilde{C}\,. \end{aligned} \tag{3.18}$$

Note that the h-translations can be done in all the directions $x_k,\ k = 1,2,3$. Next we pass to the limit in (3.18), as $h \to 0$. Clearly, $\nabla\,u^{-h} \to \nabla\,u$ almost everywhere in Ω. A classical result on differential quotients proves the estimate (3.19) below for the first term on the left hand side. In particular,

$$\nabla\frac{u - u^{-h}}{h} \to \nabla\frac{\partial u}{\partial x_k}\,,$$

almost everywhere in Ω. The above considerations, together with the nonnegativity of the integrands that appear in (3.18), allow us to pass to the limit by using Fatou's lemma. This yields

$$\|D^2\,u\|^2_{L^2(\Omega_0)} + \sum_{k=1}^{3}\left\|\,|\mathcal{D}\,u|^{\frac{p-2}{2}}\,\mathcal{D}\,\frac{\partial u}{\partial x_k}\right\|^2_{L^2(\Omega_0)} \leq \widetilde{C}\,. \tag{3.19}$$

Finally, from (3.12) and (3.16) one gets

$$\left\|\frac{(\pi - \pi_{-h})}{h}\,\theta\right\|^2_{p'} \leq c\,(K_1^2 + K_2^2 + \Phi)\,. \tag{3.20}$$

Hence the complete estimate (1.3) holds.

4 FURTHER regularity FOR THE PRESSURE

Set

$$r = \frac{12}{p+4}\,. \tag{4.1}$$

Note that $r \leq 2$ since $p \geq 2$. By assuming that

$$2 < p < 6\,,$$

one has the strict inequality

$$r > p'\,.$$

By a Sobolev embedding theorem one has (for instance, in Ω_0)

$$\|\nabla u\|_6 \leq c\,(\|D^2\,u\| + \|u\|)\,.$$

Note that for solutions to the non-slip boundary value problem we may drop everywhere the term $\|u\|$.

Straightforward calculations show that

$$\frac{\partial}{\partial\,x_k}\,\left(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\right) =$$

(4.2)

$$|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,\frac{\partial u}{\partial x_k} + (p-2)\,|\mathcal{D}\,u|^{p-4}\,\left(\mathcal{D}\,u\cdot\mathcal{D}\,\frac{\partial u}{\partial x_k}\right)\,\mathcal{D}\,u\,.$$

Hence

$$(4.3)\qquad |\,\frac{\partial}{\partial x_k}\,\left(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\right)\,| \leq c\,|\mathcal{D}\,u|^{p-2}\,\left|\mathcal{D}\,\frac{\partial\,u}{\partial x_k}\right|\,,$$

almost everywhere in Ω. So, by Hőlder's inequality,

$$(4.4)\qquad |\,\nabla\,\left(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\right)\|_r \leq \|\mathcal{D}\,u\|_6^{\frac{p-2}{2}}\,\sum_{k=1}^{3}\left\|\,|\mathcal{D}\,u|^{\frac{p-2}{2}}\,\mathcal{D}\,\frac{\partial\,u}{\partial x_k}\right\|\,.$$

Consequently,

(4.5)

$$|\,\nabla\,\left(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\right)\|_r \leq c\,\left(K_1 + K_2 + \sqrt{\Phi} + \|u\|\right)^{\frac{p-2}{2}}\,(K_1 + K_2 + \sqrt{\Phi})\,.$$

From the expression of $\nabla\,\pi$ obtained from (1.1) and from estimates already proved the estimate (1.4) follows.

Let now $p \geq 2$ be arbitrarily large. The above result can be improved by appealing to the following know result (see [1], Lemma 2.5): Local solutions to problem (1.1) satisfy

$$(4.6)\qquad \nabla\,u \in W^{1,3p}_{loc}(\Omega)\,.$$

Define

$$(4.7)\qquad s = \frac{3\,p}{2p-1}\,.$$

By replacing in the above argument the norm $\|\nabla u\|_6$ by the norm $\|\nabla u\|_{3p}$ one proves (4.4) with r replaced by s and 6 replaced by $3p$. This easily leads to

$$\nabla\,(\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u),\ \nabla\,\pi \in L^s(\Omega_0)\,.$$

Finally, if $2 < p < 3$, then $\nabla u \in L^{\frac{3p}{3-p}}$; see [1], Lemma 2.3. Hence, by arguing as above, we show that (4.4) holds with s replaced by

$$\overline{s} = \frac{6\,p}{(3-p)(p-2)+3p}\,. \tag{4.8}$$

Remark. In [?] we prove that $\nabla \pi \in L^m(\Omega)$ up to the boundary, where $m = 6(4-p)/(8-p)$. Note that $2 \geq \overline{s} > s > r > m > p'$ for $p > 2$ and that all the above coefficients are equal to 2 when $p = 2$.

5 Appendix

Here we prove (3.7). We claim that there is a real no-negative function $g(x)$ such that

$$|\nabla\,\theta(x)| \leq g(x)\,|\theta(x)| \tag{5.1}$$

in Ω. The function g depends on the distance

$$d = dist\{\Omega_0,\,\partial\,\Omega\}$$

and belongs to $L^\alpha(\Omega)$ for each $\alpha > 1$.

Clearly, $|I_5|$ is bounded by the right hand side of (3.5) if in this last expression we replace $\theta\,|D\,\phi|$ by $|\nabla\,\theta|\,|\phi|$. Hence, by (5.1), this last quantity can be replaced by $g(x)\,|\theta(x)|\,|\phi|$. Arguing as in the previous section in order to prove (3.5), one gets

$$|I_5| \leq c\,\nu_1\,\|g\|_{L^{\frac{6}{5}}}\,\|\mathcal{D}u\|_p^{\frac{p-2}{2}}\,A_1\,\|\phi\|_6\,.$$

The estimate (3.7) follows by a Sobolev's embedding theorem.

Next we sketch the proof of (5.1). It is immediate that the proof can be essentially reduced to a one-dimensional problem and to the case $d = 1$. In the sequel the point $t = 0$ represents a point in $\partial\,\Omega_0$ and $t = 1$ a point in $\partial\,\Omega$. Consider the function $\theta(t) = (1-x)^m$ for $\frac{1}{2} \leq x \leq 1$, $\theta(t) = 1$ for $x \leq 0$ and $\theta(t) = 0$ for $x \geq 1$. Moreover θ is defined in $[0,\,\frac{1}{2}]$ as a non-increasing function, of class C^∞ for $x < 1$. Clearly θ is of class C^{m-1}, hence C^2 if $m = 3$. Equation (5.1) follows since

$$|\theta'(t)| \leq \frac{m}{1-t}\,\theta(t)\,,$$

and the function $g_0(t) = \frac{m}{1-t}$ belongs to L^α if $\alpha > 1$.

We establish here the following result, which could be useful in forthcoming work. There is a constant c, which depends on d, such that (grath2)

$$|\nabla\,\theta(x)|^2 \leq c\,|\theta(x)| \tag{5.2}$$

in Ω. This follows from the inequality $|\theta'(t)|^2 \leq c\,\theta(t)$, if $m \geq 2$.

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