GMT 19/20, lecture 12, 24/4/20

Haar measures, continued

<u>setting</u>

G is a topological group

$\forall y \in G, \quad \tau_y^{\ell/r} : x \longmapsto yx \,//\, x \mapsto xy$

then $\quad [\tau_y^{\ell/r} \mu](E) := \mu(\tau_y^{\ell/r}(E))$

μ is ℓ/r-invariant if

$$\tau_y^{\ell/r} \mu = \mu \quad \forall y \in G.$$

<u>Th. 1</u> If G is <u>compact</u> then $\exists!$ invariant probability measure μ on G.

<u>Th. 2</u> If G is <u>locally compact</u> then $\exists$ μ invariant and locally finite which is unique up to a constant.

These invariant measures are called HAAR measures on G.

Let G be a topological group that acts on some top. space X

that is, it's given

$$\tau : G \times X \longrightarrow X$$
$$(y, x) \longmapsto \tau_y x$$
$$\quad \cap \quad \cap$$
$$\quad G \quad X$$

Continuous!!

left-action
$\tau_y(\tau_{y'} x) = $
$= \tau_{yy'} x$

Given μ measure on X, $\tau_y \mu$ is the measure on X defined as before

And μ is G-invariant if $\tau_y \mu = \mu$ $\forall y \in G$.

Question: are there G-invariant meas?

Answer is NO even for G, X compact!

EX | $X := P^1 \mathbb{R} = \mathbb{R} \cup \{\infty\}$

$G :=$ projectivities on $P^1 \mathbb{R} =$

$$= \left\{ x \mapsto \frac{ax+b}{cx+d} \quad \text{with } \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \right\}$$

Theorem 3 There exists a G-inv. prob. measure in the following cases:

- G commutative + compact, X compact
- G compact, $X = G/H$ with H closed subgroup of G. and μ is unique!!

$X := \{ xH \mid x \in G \}$

$\tau_y(xH) := yxH.$

- (G compact + satisfies Weyl condition.)

Example

Let $G(n,m)$ be the Grassmannian of m-planes in $\mathbb{R}^n$

$$\boxed{G(n,m) \simeq \frac{O(n)}{O(m) \times O(n-m)}}$$

$$(A,B) \simeq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

So there exists μ prop. meas. on $G(n,m)$ s.t. $\tau_M \mu = \mu \quad \forall M \in O(n)$

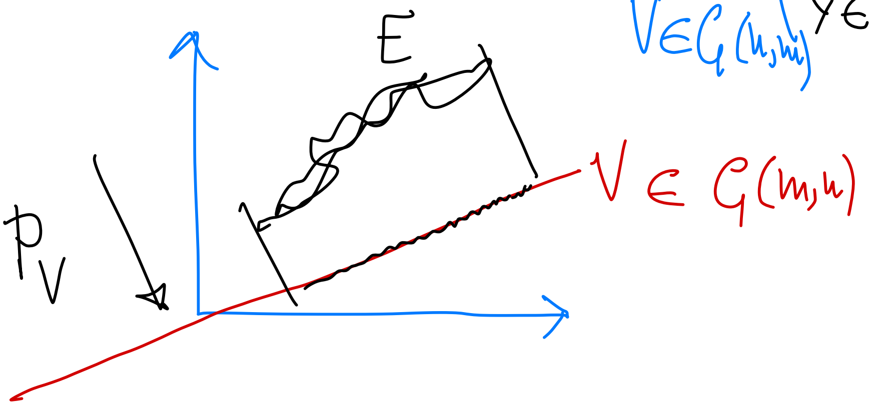
Integral geometric measures
(Favard measures)

Fix $1 \leqslant m \leqslant n$. The m-dimensional integ. geom. measure (of parameter 1) on $\mathbb{R}^n$ is

renorm. constant s.t. $\mathcal{I}_1^m = \mathcal{H}^m$ on each V

$$\mathcal{I}_1^m(E) := C_{m,n} \int_{V \in G(n,m)} \left(\int_{y \in V} \#\left(p_V^{-1}(y) \cap E\right) d\mathcal{H}^m(y) \right) \underline{dV}$$

inv. meas. on $G(n,m)$



$\mathcal{I}_1^m$ is invariant under affine isom. and agrees with $\mathcal{H}^m$ on every $V \in G(n,m)$

$\mathcal{I}_1^m$ agrees with $\mathcal{H}^m$ on every surface M of dim. m in $\mathbb{R}^d$ (of class C^1)

Some proofs....

Lemma 4 If G, X compact, G commutative then $\exists\ \mu$ G-inv. prob. meas. on X.

(in particular this proves Th. 1 for G commutative — I do not know easy proofs for G non comm.)

Proof For every $\mathcal{Y} \subset G$ let

$$\mathcal{P}_{\mathcal{Y}} := \{ \mu \in \mathcal{P}(X) \text{ s.t. } \tau_y \mu = \mu \ \ \forall y \in \mathcal{Y} \}$$

- $\mathcal{P}_{\mathcal{Y}}$ is closed (w.r.t. w^* top. of meas.)
- $\mathcal{P}_y \neq \emptyset$ for every $y \in G$.

Take any $\mu_0 \in \mathcal{P}(X)$. For $n = 1, 2, \dots$

set

$$\mu_n := \frac{1}{n+1} \sum_{m=0}^{n} \underbrace{(\tau_y)^m \mu_0}_{= \tau_{my}\mu_0}$$

then $$\tau_y \mu_n - \mu_n = \frac{1}{n+1}\left((\tau_y)^{n+1} \mu_0 - \mu_0 \right)$$

then $\|\tau_y \mu_m - \mu_n\| \leq \frac{2}{n+1}$

Let μ be an acc. point of μ_n

Then $\|\tau_y \mu - \mu\| \leq \liminf \|\tau_y \mu_n - \mu_n\| = 0$

$\Rightarrow \mu \in \mathcal{P}_y$.

- If $\mathcal{P}_{\mathcal{Y}} \neq \emptyset$ then $\mathcal{P}_{\mathcal{Y} \cup \{y\}} \neq \emptyset$ $\forall y \in G$.

  Proof as before: start with $\mu_0 \in \mathcal{P}_{\mathcal{Y}}$

$$\boxed{\begin{array}{l} G \text{ commutative} \Rightarrow \text{ if } \mu_0 \in \mathcal{P}_y \\ \text{then } \mu_n \in \mathcal{P}_y \quad \forall n \quad \dots \end{array}}$$

- $\mathcal{P}_{\mathcal{Y}} \neq \emptyset$ $\forall \mathcal{Y}$ finite
- $\mathcal{P}_G := \bigcap_{y \in G} \mathcal{P}_y \neq \emptyset$

KNOWN lemma on compact sets

Lemma 5

If G is compact and commutative, there exists at most ONE invariant prob. meas. μ on G.

Proof Assume μ_1, μ_2 are invariant prop. meas. Then

$$\mu_1 = \mu_1 * \mu_2 = \mu_2$$

Where for any μ_1, μ_2 meas. on G

$$\mu_1 * \mu_2 (E) := \mu_1 \times \mu_2 \left(\{ (x_1, x_2) \mid x_1 + x_2 \in G \} \right)$$

$$= \int \mu_1 (E - x_2) \, d\mu_2 (x_2)$$

$$= \int \tau_{-x_2} \mu_1 (E) \, d\mu_2 (x_2)$$

Lemma 6 If μ is an inv. measure on G compact and H is a closed subgroup of G then

$$\tilde{\mu}\Big(\overset{\overset{G/H}{\cup}}{E}\Big) := \mu\Big(\overset{\overset{G}{\cup}}{p^{-1}(E)}\Big)$$

is an G-invariant measure on G/H.

Lipschitz maps

X, Y metric spaces

$f : X \to Y$

$$\mathrm{lip}(f) := \cancel{\inf}\ \min \{ L \in [0,+\infty] \mid d_Y(f(x_1), f(x_2)) \le L\, d_X(x_1, x_2)\ \forall x_1, x_2 \in X \}$$

Relevance of Lipschitz maps

- Compactness (by Arzelà-Ascoli Th.)

  If $f_n : X \to Y$ are unif. Lipschitz ($\mathrm{lip}(f_n) \le L < +\infty$) and Y compact ~~(+ X separable)~~ ? then, up to subseq., $f_n \to f$ uniformly & $\mathrm{lip}(f) \le L$.

  X compact?

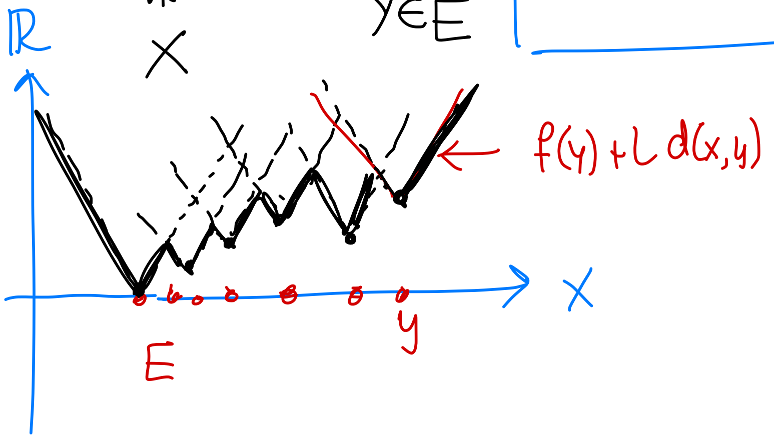
- Extension properties

  McShane lemma: If $f : E \to \mathbb{R}$, $E \subset X$, is Lipschitz, then $\exists\, F : X \to \mathbb{R}$ extension of f with $\mathrm{lip}(F) = \mathrm{lip}\, f$.

This provide an extension F ~~for~~ for $f : E \to \mathbb{R}^m$ with $\mathrm{Lip}(F) < +\infty$ but $\mathrm{Lip}(F) \ne \mathrm{lip}(f)$

Proof

Take $F(x) := \inf_{y \in E} \boxed{f(y) + \overset{\text{Lip}(f)}{\overset{\shortparallel}{L}} \cdot d(x,y)}$, $x \in X$



(Also $\tilde{F}(x) := \sup_{y \in Y} \big(f(y) - L\, d(x,y)\big)$ works)

Kirszbraum Theorem

If X, Y are Hilbert spaces and $f : E \subset X \longrightarrow Y$ is Lipschitz, then $\exists$ $F : X \longrightarrow Y$ extension of f with $\text{Lip}(F) = \text{Lip}(f)$

Ex $X = Y = \mathbb{R}^2$, $E \subset X$, $\# E = 3$.

then F may be NOT the affine extension of f.