GMT 19/20
Lecture 2 13/3/20

Approaches to Plateau Pb.

1] "Set theoretic" (easy for $d=1$
general case
Reifenberg (mid 1960s
+ others)

2] "Parametric approach"
(works fine for $d=1$, $d=2$, not for
$d>2$. Douglas & Radó mid 1930s)

3] "Measure theoretic // distributional"
- Define classes of "generalized" surfaces with good compactness properties

- ....

definition based on measure theory and reminds the def. of Sobolev functions

- Finite Perimeter Sets (DeGiorgi, late 1950s)

ORIENTED surfaces of codim. 1, i.e., $d = n-1$

- Integral currents (Federer-Fleming Early 1960s)

ORIENTED surfaces of arbitrary dim. and cod. ($1 \leq d \leq n-1$)

I will describe the construction of these classes of gen. surf.

Prove existence of sol. of Pl. Pb.

Regularity? | Is quite delicate and not yet fully established

I will only prove minimal reg.

(Min. surfaces will be closed rectifiable sets)

<u>Rem</u> Usually "min. surf." means surface with $H=0$ (that is, sol. of the E-L eq. associated to the area functional)

In this course "min. surf." means minimizer of the area funct. (in a given class).

Examples of non regular minimal surfaces // examples where Pl. Pb. has no solution in the class of surfaces of class C^1

<u>Ex 1</u> In $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ $\quad d=2 \quad n=4$

$$\Gamma := (S^1 \times \{0\}) \cup (\{0\} \times S^1)$$

"then the sol. of P.P. is

"singular" at $(0,0)$

$$\Sigma := (D^2 \times \{0\}) \cup (\{0\} \times D^2) \text{"}$$

More precisely: if $\{\Sigma_n\}$ is a seq. of surfaces of class C^1 with $\partial \Sigma_n = \Gamma$

<u>oriented</u> and is a <u>minimizing seq.</u>

$$\left(\text{area}(\Sigma_n) \longrightarrow \inf \{ \text{area}(\Sigma) \mid \partial\Sigma = \Gamma \} \right)$$

Then

$$\Sigma_n \longrightarrow \Sigma$$

(w.r.t. Hausdorff distance)

$\boxed{Ex\,2}$ In $\mathbb{R}^4 \simeq \mathbb{C}\times\mathbb{C}$ $d=2$ $n=4$

$$\Gamma = \{ (w^3, w^2) \mid w \in S^1 \subset \mathbb{C} \}$$

smooth curve with no self-intersect.

" The sol. of P.P. is

$$\Sigma := \{ (w^3, w^2) \mid w \in D^2 \subset \mathbb{C} \} \text{"}$$

singular at $(0,0)$

As before, any minimizing seq. of ~~sur~~ oriented surfaces Σ_n of class C^1 converge to Σ (in d_H)

P.P. has no solution of class C^1

Ex 3 | $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ $\qquad n = 2m$, $d = 2k$

D regular domain in $\mathbb{C}^k$

U open neighb. of D

$f : U \longrightarrow \mathbb{C}^m$ holomorphic

$$\Gamma := \{ f(s) \mid s \in \partial D \}$$

$2k-1$ dimensional surface regular if f is injective on ∂D and $\operatorname{rank}_{\mathbb{C}}(\nabla f(s)) = k$ $\forall s \in \partial D$

"Then the sol. of P.P. is

$$\Sigma := \{ f(s) \mid s \in D \}, "$$

Σ is singular at all points $f(s)$ s.t. $\operatorname{rank}_{\mathbb{C}}(\nabla f(s)) < k$

Proof Using Kähler form + Wirtinger inequality

Ex 4 $n \geq 8$, $d = n-1$

$$\mathbb{R}^{2m} = \mathbb{R}^m \times \mathbb{R}^m \qquad (n = 2m)$$

$$\Gamma := \{(x,y) \mid |x| = |y| = 1\} = S^{m-1} \times S^{m-1}$$

if $m \geq 4$ analytic surface of dim $n-2$

Then the sol. of P.P. is the cone

$$\Sigma := \{(x,y) \mid |x| = |y| \leq 1\}$$

singular at $(0,0)$

Σ is known as Simon's cone

Proof due to Bombieri–DeGiorgi–Giu

Simple proof due to DePhilippis–E. Paoli

For $d = n-1$ (codim. one case)

the singular set of sol. of P.P. with any (reg) Γ
has codim. 7 at most in the
surface.

In part. if $n < 8$ then
sol of P.P. are <u>analytic</u>

(several authors in 1960s–70s)

For $2 \leq d \leq n-2$ then
the singular set has codim.
at most 2 in the surface

(several authors 1970s – 2010s)

Recap. of basic measure theory
(there are other courses for details)

Classes of measures

0| Outer measures on a set X

$\mu : \mathcal{P}(X) \longrightarrow [0,+\infty]$ s.t.

- $\mu(\emptyset)=0$

monotonicity

- $E \subset E' \Rightarrow \mu(E) \leq \mu(E')$
- if $\{E_i\}$ is countable then

$$\mu\Big(\bigcup_i E_i\Big) \leq \sum_i \mu(E_i)$$

σ - <u>sub</u> additivity

Easy to construct and provide examples for next class.

1| Borel measures on a ~~Topological~~ Space X
metric, locally compact, separable
that is, σ-additive measures on the Borel σ-algebra $\mathcal{B}(X)$.

Important remark

I always confine myself to Borel sets and Borel funct/maps

NO NEED TO GO TO LARGER CLASSES OF SETS // MAPS

2| Signed/vector-valued Borel measures on X as above

Borel Measures

μ σ-add. measure on $\mathcal{B}(X)$
with X metric space
+ locally compact + separable
(ex. open/closed subset of $\mathbb{R}^n$)

Notation

$\lambda \ll \mu$ means λ is absolutely continuous w.r.t. μ

$\lambda \perp \mu$ λ and μ are mutually singular

$\mathbb{M}(\mu) = \|\mu\| := \mu(X)$ mass of μ

given $f : X \to [0,+\infty]$ Borel meas.

$f \cdot \mu\,(E) := \int_E f \, d\mu$ $\|f\mu\| = \|f\|_{L^1(\mu}$

$\mu \llcorner F := 1_F \cdot \mu$ i.e. $\mu(E) := \mu(E \cap F)$

Facts

$\boxed{\text{Theorem}}$ If μ is (locally) finite then μ is regular that is $\forall E \subset \beta(X)$

$$\mu(E) = \sup \left\{ \mu(K) \;\middle|\; \begin{matrix} K \text{ compact} \\ K \subset E \end{matrix} \right\}$$

$$= \inf \left\{ \mu(A) \;\middle|\; \begin{matrix} A \text{ open} \\ A \supset E \end{matrix} \right\}$$

$\boxed{\text{Theorem}}$ (Hahn-Radon-Nikodym-Leb.)

λ, μ finite measures.

Then

- $\lambda = \lambda_a + \lambda_s$ with $\lambda_a \ll \mu$, $\lambda_s \perp \mu$
- this decomp. is unique,
- $\lambda_a = f \cdot \mu$ with $f \in L^1(\mu)$, $f \geq 0$

If in addition $X \subset \mathbb{R}^n$ (or the measure μ is asymptotically doubling) then for μ-a.e. x

$$f(x) := \lim_{r \to 0} \frac{\lambda(\overline{B(x,r)})}{\mu(\overline{B(x,r)})}$$

$$= \lim_{r \to 0} \frac{\lambda_a(\overline{B(x,r)})}{\mu(\overline{B(x,r)})}$$

and
$$\lim_{r \to 0} \frac{\lambda_s(\overline{B(x,r)})}{\mu(\overline{B(x,r)})} = 0$$

μ asympt. doubling means

$$\limsup_{r \to 0} \frac{\mu(\overline{B(x,2r)})}{\mu(\overline{B(x,r)})} < +\infty \text{ for } \mu\text{-a.e. } x$$

Perhaps I will prove this last statm.