Richiamo: • $Z \sim \mathcal{N}(0,1)$ $\quad f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$$\mathbb{P}(Z \in (a,b)) = \int_a^b f_Z(z)\,dz = \Phi(b) - \Phi(a).$$

• $X \sim \mathcal{N}(\mu, \sigma^2)$ $\quad X = \mu + \sigma \overset{*}{Z} \iff \underline{\frac{X-\mu}{\sigma}} = Z \sim \mathcal{N}(0,1)$

↳ standardizzazione

$$\mathbb{E}(X) = \mathbb{E}(\mu + \sigma Z) = \mu + \sigma \mathbb{E}(Z) = \mu$$

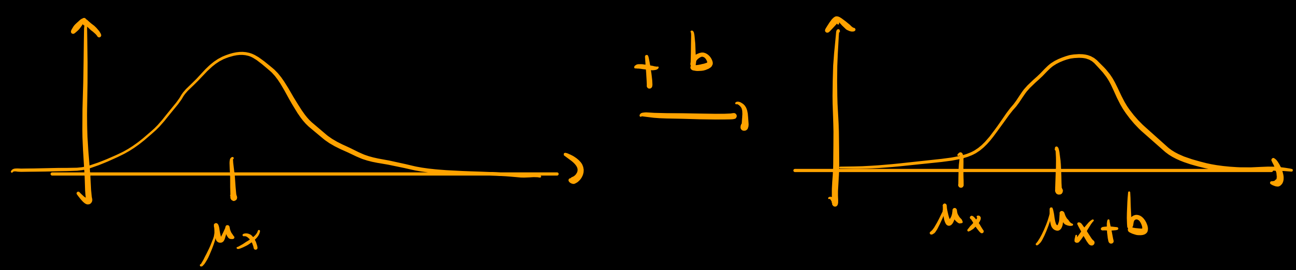
$$\mathrm{Var}(X) = \mathrm{Var}(\mu + \sigma Z) = \sigma^2 \mathrm{Var}(Z) = \sigma^2$$

Esempio: $X \sim \mathcal{N}(1, 2^2)$ $\quad Y = X + 3 \sim ?$

$X = 1 + 2Z$ * $\quad Y = X + 3 = 1 + 2Z + 3$

$$= \underset{\mu}{\underline{4}} + \underset{\sigma}{\underline{2}}Z \sim \mathcal{N}(1+3, 2^2)$$

In generale $X \sim \mathcal{N}(\mu_X, \sigma_X^2) \to X + b \sim \mathcal{N}(\mu_X + b, \sigma_X^2)$



Esempio: $X \sim \mathcal{N}(1, 2^2)$ $\quad Y = 3 \cdot X \sim ?$

$X = 1 + 2Z$ $\quad Z \sim \mathcal{N}(0,1)$

$$Y = 3 \cdot X = 3(1 + 2Z) = 3 \cdot 1 + 3 \cdot 2 Z = \underset{\mu_Y}{\underline{3}} + \underset{\sigma_Y}{\underline{6}} Z \sim \mathcal{N}(3 \cdot 1, 3^2 \cdot 2^2)$$

In generale $X \sim N(\mu_x, \sigma_x^2) \Longrightarrow a\cdot X \sim N(a\cdot\mu_x, a^2\sigma_x^2)$

Esempio: Investo $C_0 \longrightarrow C_1 = 1.1\cdot C_0 + 10\cdot Z$

Se $C_0 = 10K$

$$C_1 = 1.1\cdot 10K + 10\cdot Z \sim N(\underbrace{11K}_{\mu}, \underbrace{100}_{\sigma^2})$$

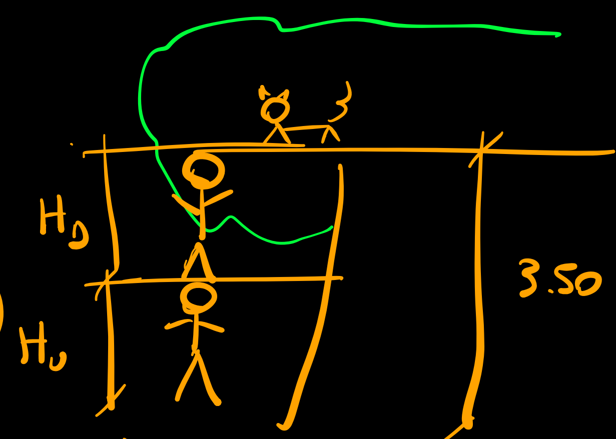
$$P(C_1 < 10K) = \Phi\left(\frac{10K - 11K}{10}\right) = \Phi(-100)$$

$X \sim N(\mu, \sigma^2)$ $\quad P(X < a) = \Phi\left(\frac{a-\mu}{\sigma}\right)$

Fatto: $X \sim N(\mu_x, \sigma_x^2) \quad Y \sim N(\mu_y, \sigma_y^2) \quad X \perp Y$

$$\Longrightarrow X + Y \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

Esempio: $\begin{cases} H_U \sim N(\underbrace{1.80}_{\mu_x}, \underbrace{0.01}_{\sigma_x^2}) \\ H_D \sim N(\underbrace{1.65}_{\mu_y}, \underbrace{0.01}_{\sigma_y^2}) \end{cases}$



$$H_{tot} = H_U + H_D \sim N(\underline{3.45}, \underline{0.02})$$

$$P(\underline{H_{tot}} \geq 3.50) = 1 - P(H_{tot} < 3.5)$$

$$= 1 - \Phi\left(\frac{3.5 - 3.45}{\sqrt{0.02}}\right) = 1 - \Phi(0.35) = \dots$$

Più in generale: $X_i \sim N(\mu_i, \sigma_i^2)$ $\{X_1, \dots X_n\}$ indipendenti

$$X_1 + X_2 + \dots + X_n \sim N(\mu_1 + \dots + \mu_n, \sigma_1^2 + \dots + \sigma_n^2)$$

Le stesse cose vale per VA di Poisson

$$\boxed{\begin{array}{l} Y_i \sim Pois(\mu_i) \quad \{Y_1, \dots Y_n\} \text{ sono indipendenti} \\ Y_1 + \dots + Y_n \sim Pois(\mu_1 + \dots + \mu_n). \end{array}}$$

Esempio: $Y_j \sim Pois(3)$ # di chiamate all'ora in un call center.

$$\# \text{ chiamate in un giorno} = Y_1 + \dots + Y_{10} \sim Pois(3 + \dots 3) \sim Pois(30).$$

Esempio (Distribuzione di $\bar{X}_n$). Siano $X_1, \dots X_n \overset{iid}{\sim} N(\mu_x, \sigma_x^2)$

$$S_n = X_1 + \dots + X_n \sim N(\underbrace{\mu_x + \dots \mu_x}_{n \text{ volte}}, \underbrace{\sigma_x^2 + \dots + \sigma_x^2}_{n \text{ volte}})$$

$$= N(n\mu_x, n\sigma_x^2)$$

$$\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n) = \frac{1}{n} \cdot S_n \sim N\left(\frac{1}{n} \cdot n\mu_x, \frac{1}{n^2} \cdot n\sigma_x^2\right)$$

$$a \cdot X \sim N(a \cdot \mu, a^2\sigma^2) \qquad a \qquad = N\left(\mu_x, \frac{1}{n}\sigma_x^2\right)$$

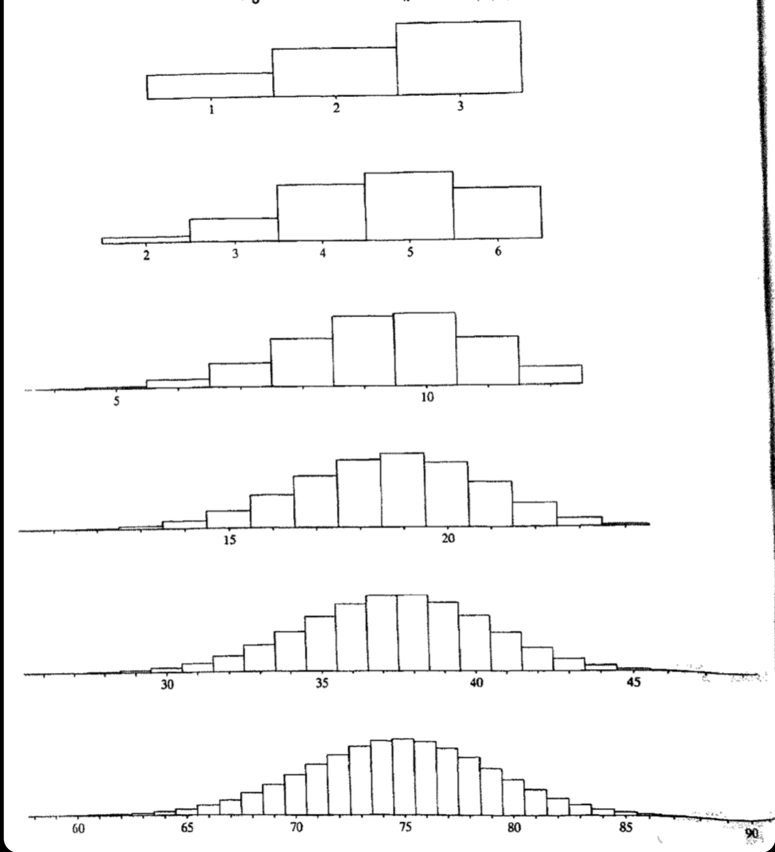


Figure 4. Distribution of S_n for $n = 1, 2, 4, 8, 16, 32$.

Messaggio: distribuzione di somme di VA indipendenti è **sempre** normale ($n \gg 1$)

Richiamo: $S_n = \mathbb{1}_1 + \dots + \mathbb{1}_n \sim Bin(n, p)$

per $n \gg 1$

$$S_n \approx \mathcal{N}(\mu_\#, \sigma_\#^2)$$

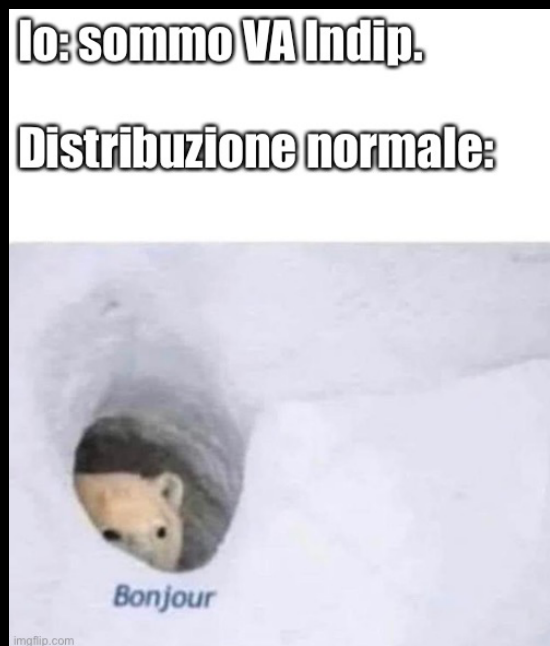
$\mu_\# = \mathbb{E}(S_n) = n \cdot p$ $\quad \sigma_\#^2 = Var(S_n) = n \cdot p(1-p)$

Teorema (CLT per somme)

Sia $X_1, \dots, X_n$ VA iid

$$S_n = X_1 + \dots + X_n$$

$$\Rightarrow \begin{cases} \mathbb{E}(S_n) = n\,\mathbb{E}(X_1) = n\mu \\ Var(S_n) = n\,Var(X_1) = n\sigma^2 \end{cases}$$



Allora $S_n \approx \mathcal{N}(\mu_\#, \sigma_\#^2)$ con $\begin{cases} \mu_\# = n \cdot \mu \\ \sigma_\#^2 = n\sigma^2 \end{cases}$

Cioè: $\mathbb{P}(A < S_n < B) \approx \Phi\left(\frac{B - \mu_\#}{\sigma_\#}\right) - \Phi\left(\frac{A - \mu_\#}{\sigma_\#}\right)$

Esempio: Lancio un dado $n = 900$ volte.

$X_i \sim \text{Unif}(\{1, \dots, 6\})$. $\quad S_n = \sum_{j=1}^{900} X_j \approx \mathcal{N}(\mu_\#, \sigma_\#^2)$ CLT

$$\mathbb{P}(3050 \le S_n \le 3250) \simeq \Phi\left(\frac{3250 - \mu_\#}{\sigma_\#}\right) - \Phi\left(\frac{3050 - \mu_\#}{\sigma_\#}\right)$$

$\mu_\# = n\,\mathbb{E}(X_1) = 900 \cdot 3.5$

$\sigma_\#^2 = n\,\text{Var}(X_1) = 900 \cdot 1.7^2$

$$= \Phi\left(\frac{3250 - 900 \cdot 3.5}{\sqrt{900 \cdot 1.7^2}}\right) - \Phi\left(\frac{3050 - 900 \cdot 3.5}{\sqrt{900 \cdot 1.7^2}}\right)$$

$$= \Phi(2) - \Phi(-2) = 95\%$$

Esempio: Compagnia di assicurazione $n = 25\,000$ polizze

$X_i =$ richieste di rimborso in un anno della polizza i

$\mathbb{E}(X_i) = 320 \quad SD(X_i) = 540 \quad S_n = \sum_{j=1}^{n} X_j \sim \mathcal{N}()$

$$\mathbb{P}(S_n > 8.3\ 10^6) = 1 - \mathbb{P}(S_n \le 8.3\ 10^6)$$

$$\approx 1 - \Phi\left(\frac{8.3 \cdot 10^6 - \mu_\#}{\sigma_\#}\right)$$

$\mu_\# = n\,\mathbb{E}(X_i) =$

$= 25000 \cdot 320$

$\sigma_\#^2 = 25000 \cdot 540$

$$\approx 1 - \Phi(3.51) = 0.00023.$$

Stessa cosa può essere fatta per $\bar{X}_n = \frac{1}{n} S_n$.

Teorema (CLT per la media)

Sia $X_1, \dots X_n$ VA iid, $S_n = X_1 + \dots + X_n$, $\bar{X}_n = \frac{1}{n} S_n$

è approssimativamente normale $\bar{X}_n \approx N(\mu_p, \sigma_p^2)$

$$\Longrightarrow \begin{cases} \underline{\mu_p} = \mathbb{E}(\bar{X}_n) = \mathbb{E}\left(\frac{S_n}{n}\right) = \frac{1}{n}\mathbb{E}(S_n) = \frac{1}{n} \cdot n \cdot \mu = \underline{\mu} \\ \underline{\sigma_p^2} = \mathrm{Var}(\bar{X}_n) = \mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\mathrm{Var}(S_n) = \frac{1}{n^2} n \cdot \sigma^2 = \underline{\frac{1}{n}\sigma^2} \end{cases}$$

Cioè: $\mathbb{P}(\alpha \le \bar{X}_n \le \beta) = \Phi\left(\frac{\beta - \mu_p}{\sigma_p}\right) - \Phi\left(\frac{\alpha - \mu_p}{\sigma_p}\right)$

Esempio: 900 lanci di un dado.

$\mu = \mathbb{E}(X_i) = 3.5$

$\sigma^2 = \mathrm{Var}(X_i) = \frac{2}{3}$

$$\mathbb{P}(\underbrace{3.39}_{\alpha} \le \bar{X}_n \le \underbrace{3.61}_{\beta}) \overset{CLT}{\approx} \Phi\left(\frac{3.61 - \mu_p}{\sigma_p}\right) - \Phi\left(\frac{3.39 - \mu_p}{\sigma_p}\right)$$

$$= \Phi\left(\frac{3.61 - 3.5}{\sqrt{\frac{2}{3} \cdot \frac{1}{900}}}\right) - \Phi\left(\frac{3.39 - 3.5}{\sqrt{\frac{2}{3} \cdot \frac{1}{900}}}\right)$$

$$= \Phi(2) - \Phi(-2) = 0.95.$$