Richiamo:

- $\mathbb{E}(X) = \sum_x x\, \mathbb{P}(X=x)$
- $\mathbb{E}(g(X)) = \sum_x g(x)\, \mathbb{P}(X=x)$
- $Var(X) = \mathbb{E}((X-\mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$X_1, \dots X_n$ iid (indipendenti e identicamente distr.)

- $\mathbb{E}(X_1 + \dots + X_n) = \sum_{j=1}^{n} \mathbb{E}(X_j) = n\,\mathbb{E}(X_1)$
- $Var(X_1 + \dots + X_n) = \sum Var(X_j) = n\, Var(X_1)$

- $\mathbb{E}(aX+b) = a\,\mathbb{E}(X) + b$
- $Var(aX+b) = a^2 Var(X)$

$$\Longrightarrow \quad \overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$$

$$\mathbb{E}(\overline{X}_n) = \frac{1}{n}\cdot n\,\mathbb{E}(X_1) = \mathbb{E}(X_1)$$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n^2} Var\left(\sum X_j\right)$$

$$= \frac{1}{n^2}\cdot n\, Var(X_1) = \frac{1}{n} Var(X_1)$$

Esempio (Binomiale) $\quad S_n = I_1 + \dots + I_n \sim Bin(n,p)$

$$\mathbb{E}(\overline{X}_n) = \mathbb{E}(I_1) = p = \frac{1}{2}$$

$$\mathrm{Var}(\bar{X}_n) = \frac{1}{n}\mathrm{Var}(I_1) = \frac{1}{n}\cdot\frac{1}{4} \xrightarrow{n\to\infty} 0$$

$$\mathbb{P}\left(|\bar{X}_n - \mathbb{E}(\bar{X}_n)| > \varepsilon\right) \xrightarrow{n\to\infty} 0 \quad \text{per ogni } \varepsilon > 0$$

Questo è un esempio di legge dei grandi numeri

"Se ripetiamo molte volte un esperimento con risultato X_i allora la media empirica sui risultati ottenuti è vicina a $\mathbb{E}(X_1)$"

<u>Esempio</u>: lancio un dado $n = 100$ volte

faccio la media $\bar{X}_n$ dei miei risultati

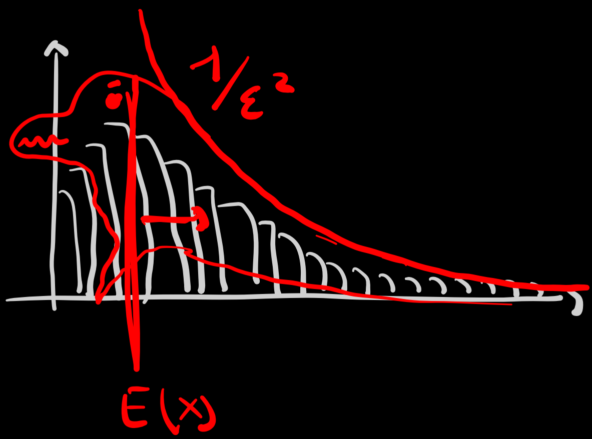
$$\bar{X}_n \approx 3.5$$

<u>Code di distribuzioni</u>

<u>Teorema</u> (Disuguaglianza di Chebyshev)

Se $\mathrm{Var}(X) = \sigma^2$ allora

$$\mathbb{P}\left(|X - \mathbb{E}(X)| > \varepsilon\right) \leq \frac{\sigma^2}{\varepsilon^2}$$



Teorema (Legge dei grandi numeri)

Sia $X_1, \dots X_n$ una lista di VA iid con la stessa distr. di X. Sia $\mu = \mathbb{E}(X)$ e $\bar{X}_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$ allora per ogni $\varepsilon > 0$ vale

$$\mathbb{P}\left(|\bar{X}_n - \mathbb{E}(X)| > \varepsilon\right) \xrightarrow{n \to \infty} 0$$

Infatti

$$\mathbb{P}\left(|\bar{X}_n - \mathbb{E}(\bar{X}_n)| > \varepsilon\right) \overset{\text{chebyshev}}{\leq} \frac{\mathrm{Var}(\bar{X}_n)}{\varepsilon^2} = \frac{1}{n} \cdot \frac{\mathrm{Var}(X)}{\varepsilon^2} \xrightarrow{n \to \infty} 0$$

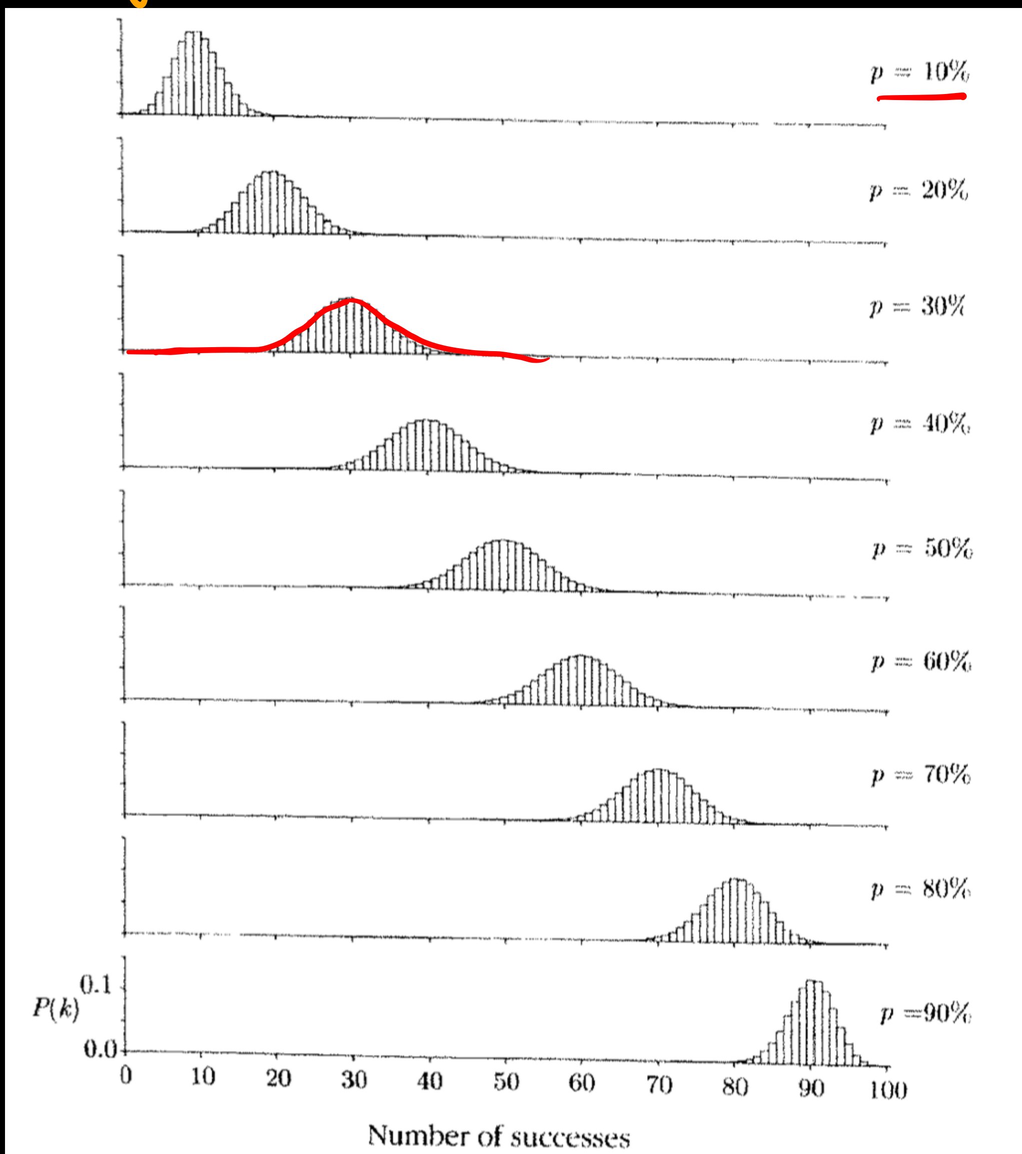
Approssimazione normale della distr. Binomiale

$X \sim Bin(n, p)$ conta # successi n prove indip. con probabilità di successo p.

Quando n è grande vediamo per distr. Binomiale

n = 100

diversi p



⇒ tutti gli istogrammi hanno una forma comune sono approssimabili da una funzione a "campana".

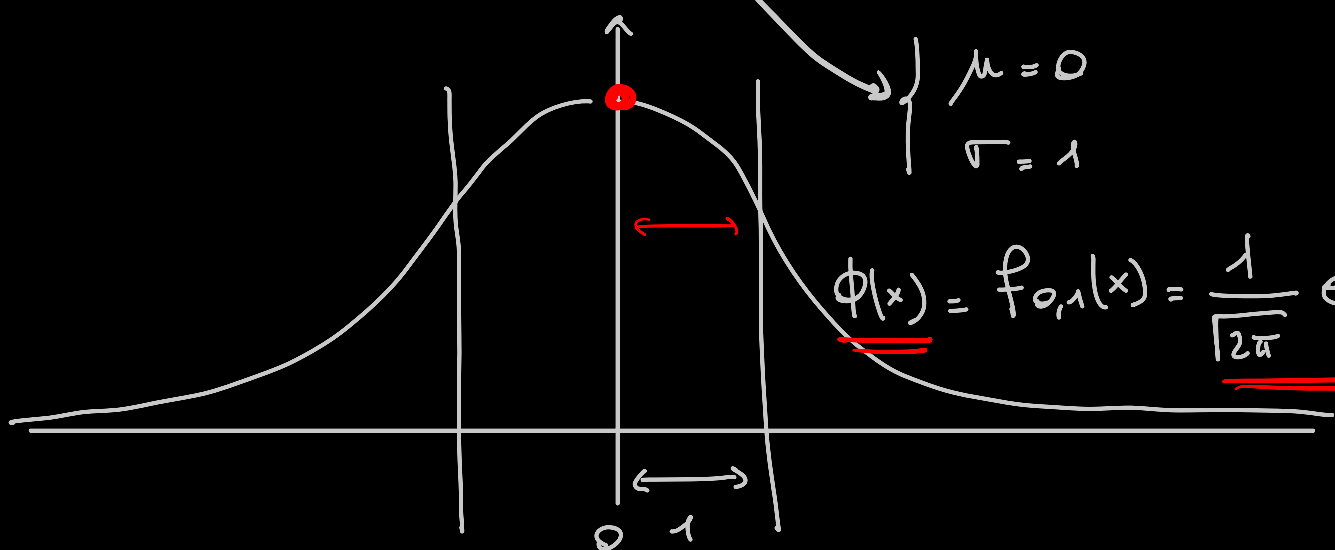
Una famiglia di funzioni "a campana" può essere scritta:

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\pi = 3.1415\ldots$
$e = 2.7183\ldots$
(Distribuzione normale)

2 parametri: $\begin{cases} \mu \in \mathbb{R} & \text{media} \\ \sigma \in \mathbb{R} & \text{deviazione standard.} \end{cases}$

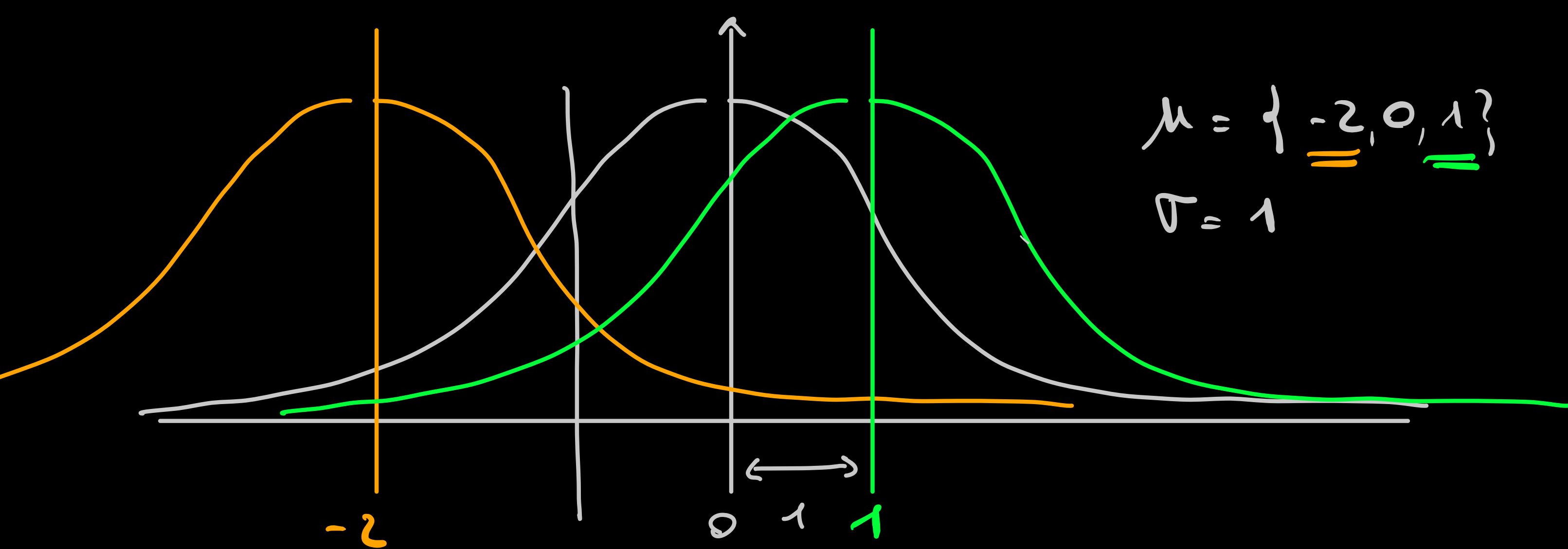
Distribuzione normale standard



$\begin{cases} \mu = 0 \\ \sigma = 1 \end{cases}$

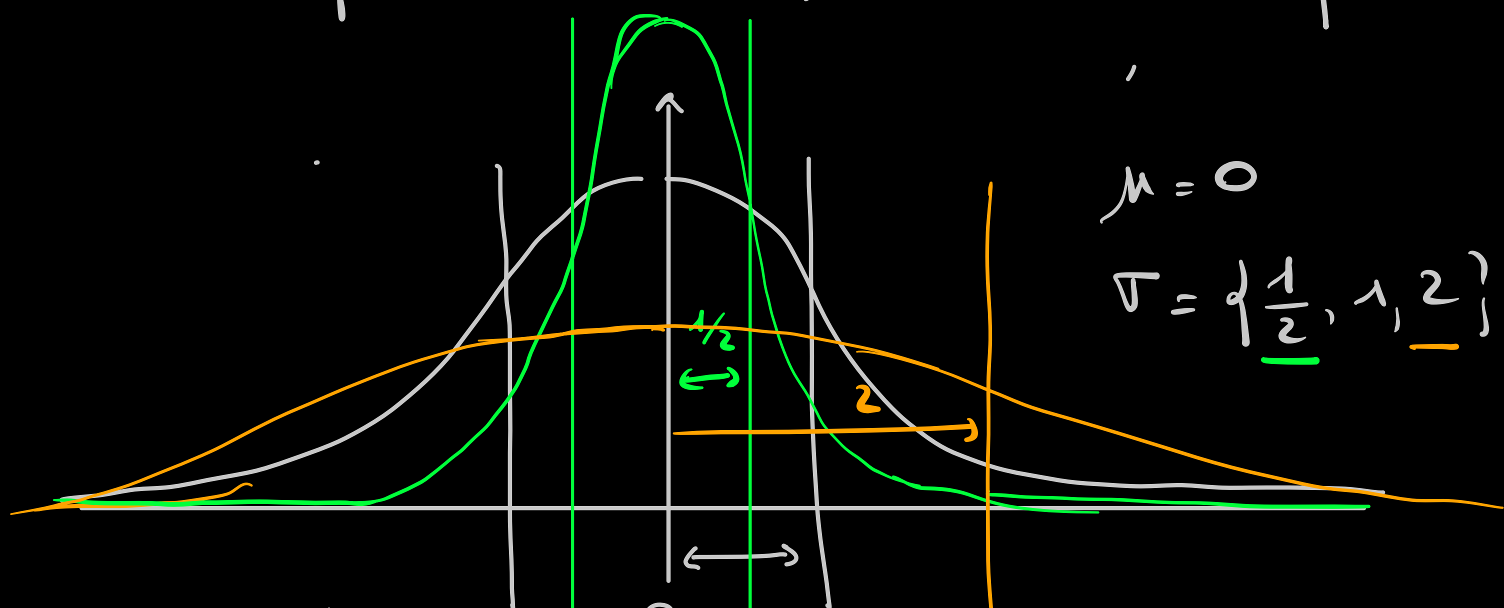
$$\phi(x) = f_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

Cambio il parametro μ $\longrightarrow$ spostiamo il "centro"



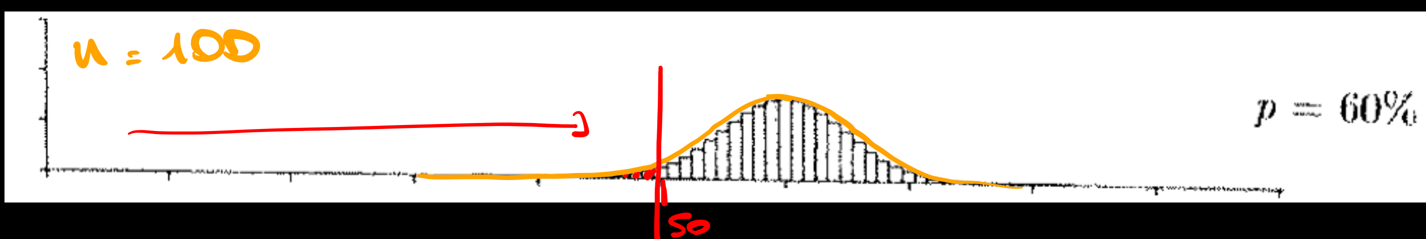
$\mu = \{-2, 0, 1\}$
$\sigma = 1$

Cambio il parametro $\sigma \to$ variamo la dispersione



Perché ci interessa?

Esempio: Sondaggio in popolazione $n = 100$ $p_A = 0.6$



Esperimento a prove ripetute con $n = 100$ con $p = 0.6$

$$\mathbb{P}(A \text{ perde nel sondaggio}) = \sum_{k=0}^{49} \binom{100}{k} p_A^k (1-p_A)^{100-k} =$$

Se trovassimo μ, σ tali che la funzione $f_{\mu\sigma}$ approssima l'istogramma della binomiale.

potremmo approssimare.

$$\mathbb{P}(A \text{ perde nel sondaggio}) \approx \int_0^{49} f_{\mu,\sigma^2}(x)\,dx$$

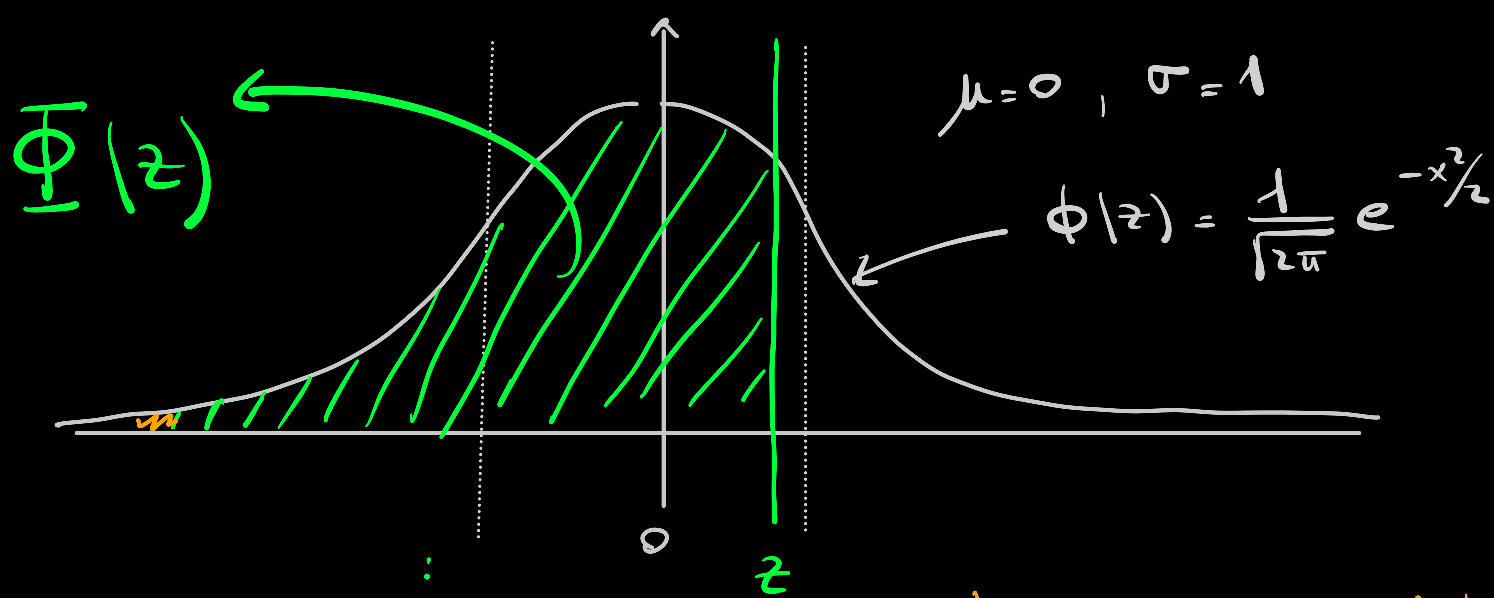
Nota: siccome questo approssima la distr. binomiale

$$\sum_{k=0}^{n} \mathbb{P}(\#\text{successi} = k) = 1 \implies \int_{-\infty}^{\infty} f_{\mu,\sigma^2}(x)\, dx = 1$$

Per calcolare gli integrali cominciamo con $\mu=0, \sigma=1$

scriviamo

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)\, dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$



questo integrale non è esplicito, ma è tabellato

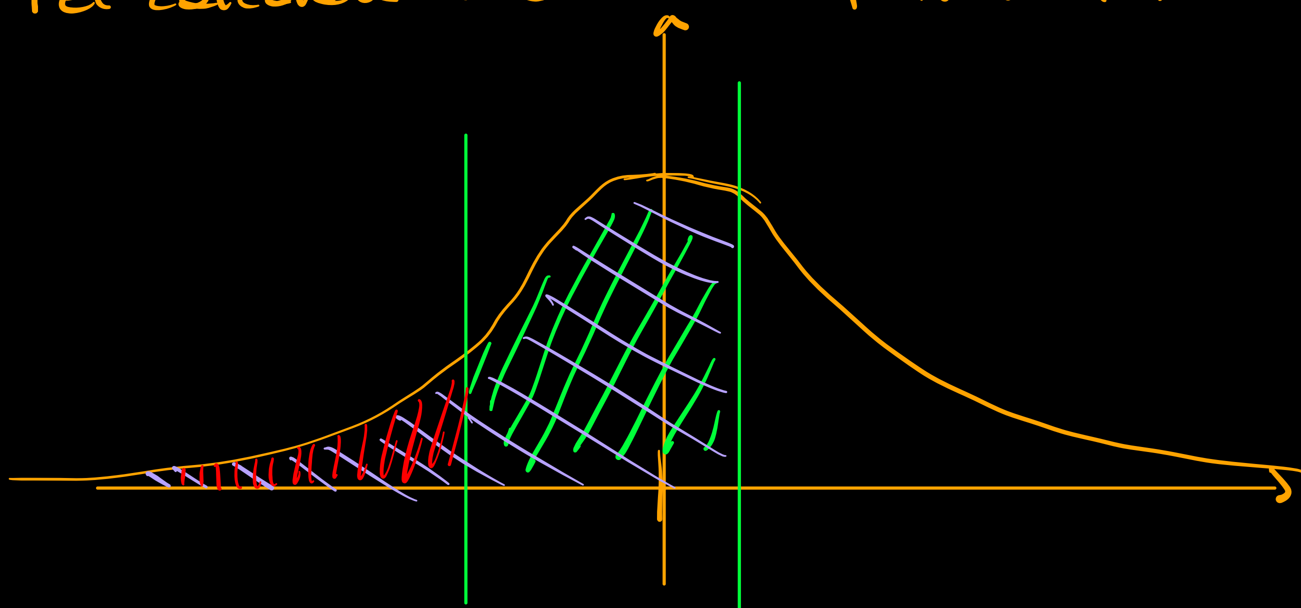
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
[illegible]	.5398	.5438	.5478	.5517	.5557	.5596	.5636	[illegible]	.5714	.5753
0.2	[illegible]	.5832	.5871	.5910	.5948	.5987	.6026	[illegible]	.6103	.6141
0.3	[illegible]	.6217	.6255	.6293	.6331	.6368	[illegible]	.6443	.6480	.6517
[illegible]	.6554	.6591	.6628	.6664	.6700	.6736	.6772	[illegible]	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Esempio: $\Phi(0.37) = \int_{-\infty}^{0.37} \phi(x)\,dx =$

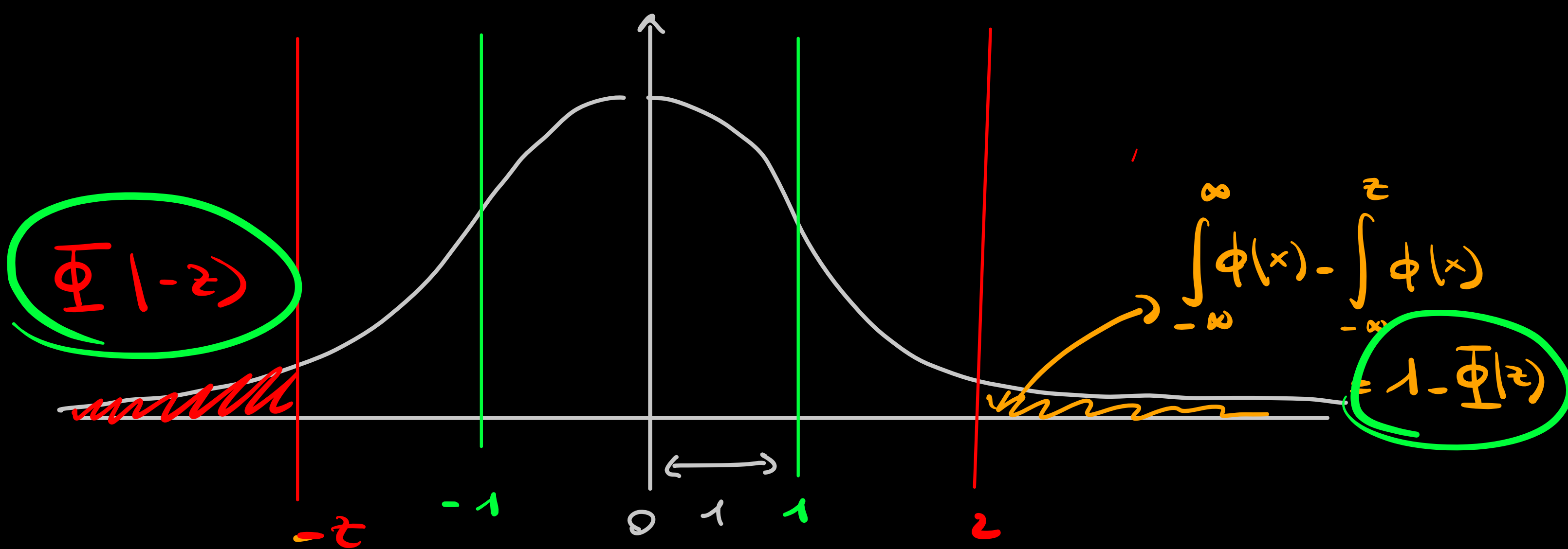
$= \text{Tabelle}(0.3 \times 7) = 06443$

Per calcolare l'area di ϕ in un intervallo (z, z')



$$\int_{z}^{z'} \phi(x)\,dx = \int_{-\infty}^{z'} \phi(x)\,dx - \int_{-\infty}^{z} \phi(x)\,dx = \Phi(z') - \Phi(z)$$

Un caso speciale importante: $z' = -z$



$$\int_{-z}^{z} \phi(x)\,dx = \int_{-\infty}^{z} \phi(x)\,dx - \int_{-\infty}^{-z} \phi(x)\,dx = \Phi(z) - \Phi(-z)$$

$$= \Phi(z) - (1 - \Phi(z)) = 2\,\Phi(z) - 1$$