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Some Existence and Regularity Results for Abstract Non-Autonomous Parabolic Equations

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We study existence, uniqueness and regularity of the strict, classical and strong solutions $u \in C([0, T], E)$ of the non-autonomous evolution equation $u'(t) - A(t) u(t) = f(t)$, with the initial datum $u(0) = x$, in a Banach space E, under the classical Kato–Tanabe assumptions. The domains of the operators $A(t)$ are not needed to be dense in E. We prove necessary and sufficient conditions for existence and Hölder regularity of the solution and its derivative.

0. INTRODUCTION

Let E be a Banach space, $\{A(t)\}_{0 \leqslant t \leqslant T}$ a family of closed linear operators on E. We consider the following Cauchy problem:

$$(\mathrm{P}) \qquad \begin{cases} u'(t) - A(t) u(t) = f(t), & t \in [0, T], \\ u(0) = x \\ x \in E, f \in C([0, T], E) \text{ prescribed.} \end{cases}$$

We suppose that for each $t \in [0, T]$ $A(t)$ is the infinitesimal generator of an analytic semigroup, and moreover $A(t)$ has a domain $D(A(t))$ which varies with t and is not necessarily dense in E.

Problem (P) has been discussed by several authors under the assumption that $\overline{D(A(t))} = E$ for every $t \in [0, T]$. The case of variable domains was first studied by Kato [11], who supposed $D(A(t))$ to vary "smoothly": more precisely, a bounded operator $R(t)$ was assumed to exist, with bounded inverse $R(t)^{-1}$, such that $D(R(t) A(t) R(t)^{-1}) =$ constant; moreover $R(t)$ was

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subject to very strong differentiability properties. Similar hypotheses are also considered by Tanabe (see Section 6 of [31]), with a slight weakening of regularity assumptions about $R(t)$.

A first generalization was carried over by Sobolevski [27, 28] and Kato [12]. In [27] the evolution space E is a Hilbert space and $-A(t)$ is positive definite and self-adjoint for each $t \in [0, T]$; in [28] and [12] a Banach space situation is considered and in both papers a number $\rho > 0$ is supposed to exist, such that $D((-A(t))^{\rho}) = \text{constant}$, with the further requirement in [12] for the number ρ^{-1} to be an integer. All these papers also require a Hölder condition for $(-A(t))^{\rho}\,(-A(0))^{-\rho}$ of order $\alpha \in]1-\rho, 1[$.

Such assumptions are, in a certain sense, intermediate between the case of a constant dense domain and that of variable (dense) domains. Now it is difficult in general to examine $D((-A(t))^{\rho})$; on the other hand many examples can be made in the opposite direction, relative to domains which vary very "badly": namely, there are cases of dense domains $D(A(t))$ such that $D(A(t)) \cap D(A(s))$ is nowhere dense or even equal to $\{0\}$ for any $t \neq s$ (see, e.g., Dorroh [9], Kato [11], and Goldstein [10]).

It was therefore desirable to avoid any direct assumption about the "regularity" of the domains. A great improvement was attained by Kato and Tanabe [15], who replaced any assumption about $D(A(t))$ by a differentiability condition for $R(\lambda, A(t))$ and a Hölder condition for $(d/dt)A(t)^{-1}$. These assumptions also generalize those of Chapter 7 of Lions' book [18] (see also [19]), where the variational case in a Hilbert space E is considered. In recent years the hypotheses of [15] have been slightly modified and weakened by Tanabe [32] and Yagi [36, 37]. In all these papers Problem (P) is solved by constructing the fundamental solution with the use of integral equation techniques; the density of domains makes it possible to find solutions which are strongly differentiable in $]0, T]$, for any $x \in E$ and f Hölder continuous in $[0, T]$.

From a different point of view, Da Prato and Grisvard [6] studied Problem (P) without assuming $\overline{D(A(t))} = E$, as a special case of their theory about sums of non-commuting linear operators. They restrict themselves to the case $x = 0$ and discuss evolution both in L^p-spaces and in spaces of continuous functions with values in E and $f(0) = 0$.

In the present paper we will assume the same hypotheses of [15] and use a large part of their results, but we are mainly inspired by the techniques of [6]. We only consider evolution in spaces of continuous functions, discussing also the case $x \neq 0$ and proving existence and uniqueness of various kinds of solutions of Problem (P), namely, strong, classical and strict solutions. In particular we prove a representation formula for the solution of Problem (P), without passing through the construction of the fundamental solution.

Our formula can be heuristically derived by the following argument: we look for a solution of this kind:

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)} g(s)\,ds, \tag{0.1}$$

where $g(t)$ is a suitable (integrable) function with values in E. Of course when $A(t) = A =$ constant, this formula with $g = f$ gives the ordinary mild solution of Problem (P). Thus in the general case g may be considered as a "modification" of f. Taking the formal derivative of (0.1) we get

$$\begin{aligned} u'(t) &= A(t)\, e^{tA(t)}x + \left[\frac{\partial}{\partial t} e^{\xi A(t)}\right]_{\xi = t} x + g(t) \\ &\quad + \int_0^t A(t)\, e^{(t-s)A(t)} g(s)\,ds + \int_0^t \left[\frac{\partial}{\partial t} e^{\xi A(t)}\right]_{\xi = t-s} g(s)\,ds \\ &= A(t)\, u(t) + \left[g(t) + \int_0^t \left[\frac{\partial}{\partial t} e^{\xi A(t)}\right]_{\xi = t-s} g(s)\,ds + \left[\frac{\partial}{\partial t} e^{\xi A(t)}\right]_{\xi = t} x\right]. \end{aligned}$$

Hence, if we want (0.1) to be a solution of Problem (P), we must choose g such that

$$g(t) + \int_0^t P(t, s)\, g(s)\,ds + P(t, 0)\, x = f(t),$$

where

$$P(t, s) = \left[\frac{\partial}{\partial t} e^{\xi A(t)}\right]_{\xi = t-s}. \tag{0.2}$$

Denote by P the integral operator $\varphi \to \int_0^t P(t, s)\, \varphi(s)\,ds$; then the representation formula for the solution of Problem (P) is formally given by

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)} (1 + P)^{-1}\, [f - P(\cdot, 0)\, x](s)\,ds. \tag{0.3}$$

We also study the "maximal regularity" of the solution. We say that there is maximal regularity for the solution of Problem (P) if it has Hölder continuous first derivative for some exponent $\alpha \in]0, 1[$, whenever f is Hölder continuous with the same exponent α, provided the vectors x and $f(0)$ satisfy some suitable compatibility conditions. Here we get a necessary and sufficient condition on x and $f(0)$, which generalizes the regularity results of [6, Theorem 7.21] relative to the case $x = f(0) = 0$, and the analogous condition of Sinestrari [25, 26] and Da Prato and Sinestrari [8] in the case $A(t) = A =$ constant. When $A(t)$ is not constant, partial results in this direction are due to Poulsen [24], who proved $(\alpha - \varepsilon)$-Hölder regularity of the derivative for each $\varepsilon > 0$ provided f is α-Hölder continuous and $x = f(0) = 0$.

Let us now introduce some notations. If A is a linear operator on a Banach space E, we denote by $D(A)$ its domain and by $R(A)$ its range; also, $\rho(A)$ is the resolvent set of A, $\sigma(A)$ its spectrum, and the resolvent operator $(\lambda I - A)^{-1}$, which is defined for each $\lambda \in \rho(A)$, will be denoted by $R(\lambda, A)$.

We will consider the following Banach spaces:

$$C([0, T], E) = \{u : [0, T] \to E : u \text{ is continuous}\},$$

with norm

$$\|u\|_{C([0,T],E)} = \sup_{t \in [0,T]} \|u(t)\|_E;$$

for any $\theta \in \,]0, 1[$,

$$C^\theta([0, T], E) = \{u : [0, T] \to E : u \text{ is Hölder continuous with exponent } \theta\},$$

with norm

$$\|u\|_{C^\theta([0,T],E)} = \|u\|_{C([0,T],E)} + \sup_{t \neq s} \frac{\|u(t) - u(s)\|_E}{|t-s|^\theta};$$

$$C^1([0, T], E) = \{u : [0, T] \to E : u \text{ is strongly differentiable and } u' \in C([0, T], E)\},$$

with norm

$$\|u\|_{C^1([0,T],E)} = \|u\|_{C([0,T],E)} + \|u'\|_{C([0,T],E)};$$

for any $\theta \in \,]0, 1[$,

$$C^{1,\theta}([0, T], E) = \{u : [0, T] \to E : u \text{ is strongly differentiable and } u' \in C^\theta([0, T], E)\},$$

with norm

$$\|u\|_{C^{1,\theta}([0,T],E)} = \|u\|_{C^1([0,T],E)} + \|u'\|_{C^\theta([0,T],E)}.$$

In addition, we define the spaces

$$C(]0, T], E), \qquad C^\theta(]0, T], E), \qquad C^1(]0, T], E), \qquad C^{1,\theta}(]0, T], E)$$

as the spaces of the functions $u : \,]0, T] \to E$ belonging to

$$C([\varepsilon, T], E), \qquad C^\theta([\varepsilon, T], E), \qquad C^1([\varepsilon, T], E), \qquad C^{1,\theta}([\varepsilon, T], E)$$

for each $\varepsilon > 0$.

Similarly, for any $p \in [1, \infty]$ we define the Banach space

$$L^p(0, T; E) = \{u :]0, T[\to E : u \text{ is Bochner measurable and } \|u(\cdot)\|_E \in L^p(0, T)\},$$

with norm

$$\|u\|_{L^p(0,T;E)} = \begin{cases} \left[\int_0^T \|u(t)\|_E^p \, dt\right]^{1/p} & \text{if } p < \infty, \\ \operatorname*{ess\,sup}_{t \in]0,T[} \|u(t)\|_E & \text{if } p = \infty. \end{cases}$$

Finally if E is a Banach space, denote by $\mathscr{L}(E)$ the Banach space of all continuous linear operators with domain E and range contained in E, with norm

$$\|A\|_{\mathscr{L}(E)} = \sup_{x \in E - \{0\}} \frac{\|Ax\|_E}{\|x\|_E}.$$

Now we list our assumptions.

(I) For each $t \in [0, T]$, $A(t)$ is a closed linear operator on the Banach space E with domain $D(A(t))$, which generates an analytic semigroup $\{e^{\xi A(t)}\}_{\xi \geq 0}$; in particular:

(i) there exists $\theta_0 \in]\pi/2, \pi[$ such that

$$\rho(A(t)) \subseteq \Sigma_{\theta_0} \triangleq \{z \in \mathbf{C} : \rho e^{i\theta}, \rho \in [0, +\infty[, \theta \in]-\theta_0, \theta_0[\} \quad \forall t \in [0, T];$$

(ii) there exists $M > 0$ such that

$$\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leq M/|\lambda| \qquad \forall \lambda \in \Sigma_{\theta_0}, \quad \forall t \in [0, T].$$

(II) The operator-valued function $t \mapsto R(\lambda, A(t))$ is in $C^1([0, T], \mathscr{L}(E))$ for each $\lambda \in \Sigma_{\theta_0}$; moreover there exist $L > 0$ and $\alpha \in]0, 1]$ such that

$$\left\|\frac{\partial}{\partial t} R(\lambda, A(t))\right\|_{\mathscr{L}(E)} \leq \frac{L}{|\lambda|^\alpha} \qquad \forall \lambda \in \Sigma_{\theta_0}, \quad \forall t \in [0, T].$$

(III) There exist $B > 0$ and $\eta \in]0, 1]$ such that

$$\left\|\frac{d}{dt} A(t)^{-1} - \frac{d}{d\tau} A(\tau)^{-1}\right\|_{\mathscr{L}(E)} \leq B\,|t - \tau|^\eta \qquad \forall t, \tau \in [0, T].$$

Let us specify now what we mean as a "solution" of Problem (P). First of all we set

$$D \triangleq \{u \in C([0, T], E) : u(t) \in D(A(t)) \ \forall t \in [0, T], \text{ and}$$
$$t \mapsto A(t)\,u(t) \text{ is in } C([0, T], E)\}$$
$$D_0 \triangleq \{u \in C([0, T], E) : u(t) \in D(A(t)) \ \forall t \in [0, T], \text{ and}$$
$$t \mapsto A(t)\,u(t) \text{ is in } C([0, T], E)\}.$$

Now we define our solutions.

DEFINITION 0.1. $u : [0, T] \to E$ is a *strict solution* of Problem (P) if $u \in D$ and

$$u'(t) - A(t)\,u(t) = f(t) \qquad \forall t \in [0, T], \quad u(0) = x.$$

DEFINITION 0.2. $u : [0, T] \to E$ is a *classical solution* of Problem (P) if $u \in D_0$ and

$$u'(t) - A(t)\,u(t) = f(t) \qquad \forall t \in [0, T], \quad u(0) = x.$$

DEFINITION 0.3. $u : [0, T] \to E$ is a *strong solution* of Problem (P) if $u \in C([0, T], E)$ and there exists $\{u_n\}_{n \in \mathbf{N}} \subseteq D$ such that

(i) $u_n \to u$ in $C([0, T], E)$;

(ii) $u_n(t) - A(t)\,u_n(t) \triangleq f_n(t) \in C([0, T], E)$ and $f_n \to f$ in $C([0, T], E)$;

(iii) $u_n(0) \to x$ in E.

Remark 0.4. Hypothesis III will be used only to prove existence and regularity of classical and strict solutions, while it is not necessary for uniqueness and for what concerns strong solutions.

Remark 0.5. Yagi [37] has shown existence and uniqueness of the classical solution of Problem (P) under Hypotheses I, II and the following condition, weaker than III:

(III′) There exist $B > 0$, $k \in N$ and $\alpha_1, ..., \alpha_k$, $\beta_1, ..., \beta_k \in \mathbf{R}$ with $-1 \leqslant \alpha_i < \beta_i \leqslant 1$ for $i = 1, ..., k$, such that

$$\left\| A(t)\,R(\lambda, A(t)) \frac{d}{dt} A(t)^{-1} - A(\tau)\,R(\lambda, A(\tau)) \frac{d}{d\tau} A(\tau)^{-1} \right\|_{\mathcal{L}(E)}$$
$$\leqslant B \sum_{i=1}^{k} |\lambda|^{\alpha_i} |t - \tau|^{\beta_i} \qquad \forall \lambda \in \Sigma_{\theta_0}, \quad \forall t, \tau \in [0, T].$$

It should be observed that in the present paper condition III′ instead of III would be sufficient to get all our results about classical and strict solutions. In fact, existence is guaranteed by the fact that under this assumption the Dunford integrals, formally defining the derivative of the solution, are actually convergent; on the other hand, by a direct but very tedious calculation based upon our representation formula, it can be shown that our maximal regularity results still hold under condition III′ instead of III. However, we prefer to assume the stronger condition III, for it allows much simpler proofs, mainly in the case of maximal regularity, and in addition in concrete situations it seems more difficult to verify directly condition III′ rather than III.

Remark 0.6. It follows directly by the definitions that every strict solution of Problem (P) is also a classical solution and a strong solution. It is not true, however, that a classical solution is necessarily a strong solution (see Remark 6.5 in Section 6 below).

Remark 0.7. Kato and Tanabe [15] also consider weak solutions of Problem (P), i.e., functions $u \in C([0, T], E)$ such that

$$\int_0^T \langle u(t), \varphi'(t) - A(t)^* \varphi(t)\rangle \, dt + \int_0^T \langle f(t), \varphi(t)\rangle \, dt + \langle x, \varphi(0)\rangle = 0$$

for each $\varphi \in C^1([0, T], E^*)$ satisfying the following conditions:

(i) $\varphi(t) \in D(A(t)^*) \ \forall t \in [0, T]$, and $t \mapsto A(t)^* \varphi(t) \in C([0, T], E^*)$;

(ii) $\varphi(T) = 0$.

Here $\langle \cdot \rangle$ denotes the duality product between E and its dual space E^*, and $A(t)^*$ is the adjoint operator of $A(t)$. Then, it is easy to verify that every strict or strong solution of Problem (P) is also a weak solution in this sense. The same is true for every classical solution of Problem (P): to prove this, one has just to integrate by parts the equation in $]\varepsilon, T[$, and let $\varepsilon \to 0^+$.

Let us now describe the subjects of the following sections. In Section 1 we establish some preliminary results and give sense to formula (0.3), i.e., our candidate to be the representation formula for any solution of Problem (P). In Section 2 we derive some necessary conditions for existence of solutions of Problem (P), and prove uniqueness of such solutions. Section 3 is devoted to the basic results which are needed to get our existence and regularity theorems. In Section 4 we discuss classical solutions. Section 5 concerns strict solutions and their maximal regularity. In Section 6 we study strong solutions. Finally, in Section 7 we describe some examples and applications.

1. Preliminaries

Let A be a closed linear operator on a Banach space E, satisfying Hypothesis I; then $D(A)$, equipped with the graph norm, is itself a Banach space continuously imbedded into E. We recall the following definitions of the intermediate space $(D(A), E)_{1-\theta,\infty}$ between $D(A)$ and E (see Lions [17] and Lions and Peetre [20]):

Definition 1.1. Let E be a Banach space, and let $\theta \in]0, 1[$. If $x \in E$, x is said to be in $(D(A), E)_{1-\theta,\infty}$ if there exists $u :]0, \infty[\to D(A)$ having first derivative in the sense of distributions $u' :]0, \infty[\to E$, such that

(i) $t^{1-\theta}u(t), t^{1-\theta}Au(t), t^{1-\theta}u'(t) \in L^{\infty}(0, \infty; E)$.

(ii) $u(0) = x$.

Remark 1.2. Condition (ii) of Definition 1.1 is meaningful since it is easily seen that condition (i) implies $u \in C^{\theta}([0, \infty[, E)$. Moreover it is clear that

$$D(A) \subseteq (D(A), E)_{1-\theta,\infty} \subseteq \overline{D(A)} \qquad \forall \theta \in]0, 1[.$$

The space $(D(A), E)_{1-\theta,\infty}$ is also customarily denoted by $D_A(\theta, \infty)$. In Peetre [23] and in Butzer and Berens' book [4] (see also Da Prato and Grisvard [7]) many properties and characterizations of $D_A(\theta, \infty)$ are proved under the assumption that $D(A)$ is dense in E. In the general case we can set

$$Z = \{x \in D(A) : Ax \in \overline{D(A)}\}$$

and define the restriction of A to Z:

$$D(A') = Z$$
$$A'x = Ax \qquad \forall x \in Z.$$

Obviously, $D(A')$ is dense in $\overline{D(A)}$ which is a Banach space with the norm of E. Moreover we have:

Proposition 1.3. $(D(A'), \overline{D(A)})_{1-\theta,\infty} = (D(A), E)_{1-\theta,\infty}$ $\forall \theta \in]0, 1[$.

Proof. Obviously $(D(A'), \overline{D(A)})_{1-\theta,\infty} \subseteq (D(A), E)_{1-\theta,\infty}$; conversely if $x \in (D(A), E)_{1-\theta,\infty}$ let u be the vector-valued function appearing in Definition 1.1. Then if we set $w(t) = e^{tA}u(t)$ it is easy to verify that $t^{1-\theta}w(t)$, $t^{1-\theta}A'w(t)$, $t^{1-\theta}w'(t) \in L^{\infty}(0, \infty; \overline{D(A)})$ and $w(0) = x$, which means $x \in (D(A'), \overline{D(A)})_{1-\theta,\infty}$.

Remark 1.4. By Proposition 1.3 and the density of $D(A')$ in $\overline{D(A)}$ it follows that the space $D_A(\theta, \infty)$ has the same well-known characterizations which are valid (see [4]) when A has dense domain. Therefore the following equalities hold:

$$D_A(\theta, \infty) = \{x \in E : \sup_{t>0} t^{-\theta} \|e^{tA}x - x\|_E < \infty\}$$

or equivalently

$$\begin{aligned} D_A(\theta, \infty) &= \{x \in E : \sup_{\lambda>0} \lambda^{\theta} \|AR(\lambda, A)x\|_E < \infty\} \\ &= \{x \in E : \sup_{\lambda \in \rho(A)} |\lambda|^{\theta} \|AR(\lambda, A)x\|_E < \infty\}. \end{aligned}$$

Now we go back to our situation and represent the analytic semigroup $\{e^{\xi A(t)}\}_{\xi \geqslant 0}$ by a Dunford integral.

Let γ be an arbitrary continuous path contained in Σ_{θ_0} joining $+\infty e^{-i\theta}$ and $+\infty e^{i\theta}$, $\theta \in]\pi/2, \theta_0[$ being fixed. For our purposes it is convenient to choose

$$\gamma = \gamma_0 \cup \gamma_+ \cup \gamma_-,$$

where

$$\gamma_0 \triangleq \{\lambda \in \mathbf{C} : |\lambda| = 1, |\arg \lambda| \leqslant \theta\}$$
$$\gamma_1 \triangleq \{\lambda \in \mathbf{C} : \lambda = \rho e^{\pm i\theta}, \rho \geqslant 1\}.$$

For each $\tau \in]0, T]$ we define

$$\gamma_\tau \triangleq \{\lambda \in \mathbf{C} : \tau\lambda \in \gamma\}.$$

Then, for example, the following equalities hold:

$$\begin{aligned} e^{\xi A(t)} &= \frac{1}{2\pi i} \int_\gamma e^{\xi\lambda} R(\lambda, A(t))\, d\lambda, \qquad \xi > 0, \quad t \in [0, T], \\ A(t)\, e^{\xi A(t)} &= \frac{1}{2\pi i} \int_\gamma \lambda e^{\xi\lambda} R(\lambda, A(t))\, d\lambda, \qquad \xi > 0, \quad t \in [0, T], \end{aligned} \tag{1.1}$$

the integrals being absolutely convergent.

The following lemma is very useful.

LEMMA 1.5. *Under Hypotheses* I, II *we have:*

(i) *If* $x \in E$ *then* $\|e^{tA(t)}x\|_E \leqslant C\|x\|_E$ $\forall t \in [0, T]$.

(ii) *If* $x \in E$ *then* $\|A(\tau)\, e^{tA(\tau)}x\|_E \leqslant C/t\, \|x\|_E$ $\forall t \in]0, T]$, $\forall \tau \in [0, T]$.

(iii) *If* $x \in D_{A(0)}(\beta, \infty)$, $\beta \in]0, 1[$, *then* $\|A(\tau)\, e^{tA(\tau)}x\|_E \leqslant C(x)/t^{1-\beta}$ $\forall t \in]0, T]$, $\forall \tau \in [0, t[$.

(iv) *If* $x \in D(A(0))$ *then* $\|A(t)\, e^{tA(t)} x\|_E \leqslant C\, \|A(0)\, x\|_E$ $\forall t \in [0, T]$.

(v) *If* $x \in \overline{D(A(0))}$ *then* $\lim_{t \to 0^+} \|tA(t)\, e^{tA(t)} x\|_E = 0$.

Proof. (i) and (ii) are evident by (1.1).

(iii) We have

$$A(\tau)\, e^{tA(\tau)} x = A(\tau)\, e^{tA(\tau)} [x - e^{tA(0)} x + (A(0)^{-1} - A(\tau)^{-1})\, A(0)\, e^{tA(0)} x] + e^{tA(\tau)} A(0)\, e^{tA(0)} x,$$

so (iii) follows by (i), (ii) and the estimate (coming from Remark 1.4)

$$\|A(0)\, e^{tA(0)} x\|_E = \left\| \frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda t}}{|\lambda|^\beta} |\lambda|^\beta A(0)\, R(\lambda, A(0))\, x\, d\lambda \right\|_E \leqslant \frac{C(x)}{t^{1-\beta}}.$$

(iv) By (i), (ii) and

$$A(t)\, e^{tA(t)} x = A(t)\, e^{tA(t)} [A(0)^{-1} - A(t)^{-1}]\, A(0)\, x + e^{tA(t)} A(0)\, x$$

the result follows.

(v) If $x \in D(A(0))$ the result follows by (iv); the general case is a consequence of (ii) and the density of $D(A(0))$ in $\overline{D(A(0))}$.

Under hypotheses I, II it is possible to define the linear operator on E (see (0.2))

$$\begin{aligned} P(t, s) &\triangleq \left[\frac{\partial}{\partial t} e^{\xi A(t)} \right]_{\xi = t - s} \\ &= \frac{1}{2\pi i} \int_\gamma e^{\lambda(t-s)} \frac{\partial}{\partial t} R(\lambda, A(t))\, d\lambda, \qquad 0 \leqslant s < t \leqslant T. \end{aligned} \tag{1.2}$$

Observing that (see [31, Lemma 1.1])

$$\begin{aligned} \|P(t, s)\|_{\mathscr{L}(E)} &= \left\| \frac{1}{2\pi i} \int_{\gamma_{t-s}} e^{\sigma} \left[\frac{\partial}{\partial t} R(\xi, A(t)) \right]_{\xi = \sigma/(t-s)} \cdot \frac{1}{t-s}\, d\theta \right\|_{\mathscr{L}(E)} \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma} e^{\sigma} \left[\frac{\partial}{\partial t} R(\xi, A(t)) \right]_{\xi = \sigma/(t-s)} \cdot \frac{1}{t-s}\, d\sigma \right\|_{\mathscr{L}(E)} \\ &\leqslant \frac{L}{2\pi} \int_\gamma \frac{e^{\operatorname{Re} \sigma}}{|\sigma|^\alpha} |d\sigma| \cdot \frac{1}{(t-s)^{1-\alpha}}, \end{aligned}$$

we conclude that there exists $K > 0$ such that

$$\|P(t, s)\|_{\mathscr{L}(E)} \leqslant \frac{K}{(t-s)^{1-\alpha}}, \qquad 0 \leqslant s < t \leqslant T. \tag{1.3}$$

Therefore we can associate to the kernel $P(t, s)$ the integral operator P defined by

$$Pg(t) \triangleq \int_0^t P(t, s)\, g(s)\, ds, \qquad t \in [0, T], \quad g \in L^1(0, T; E). \tag{1.4}$$

The properties of the operator P will be discussed in Section 3 below; for the moment we will just remark some basic facts (whose proofs will be found in Section 3).

Remark 1.6. P is a Volterra integral operator with integrable kernel, so that $P \in \mathscr{L}(L^1(0, T; E)) \cap \mathscr{L}(L^\infty(0, T; E))$, and $1 + P$ can be inverted, with the operator $Q \triangleq (1 + P)^{-1}$ being in $\mathscr{L}(L^1(0, T; E)) \cap \mathscr{L}(L^\infty(0, T; E))$. In other words, the integral equation

$$g(t) + \int_0^t P(t, s)\, g(s)\, ds = \varphi(t), \qquad t \in [0, T], \tag{1.5}$$

can be uniquely solved in $L^1(0, T; E)$ (resp. $L^\infty(0, T; E)$) for any $\varphi \in L^1(0, T; E)$ (resp. $L^\infty(0, T; E)$).

Remark 1.7. As a trivial consequence of (1.1), the vector-valued function $t \to P(t, 0)\, x$ is in $L^1(0, T; E)$ for each $x \in E$.

Now we are able to give sense to formula (0.1); for the moment, it will define just an element of $L^\infty(0, T; E)$.

PROPOSITION 1.8. *Under Hypotheses* I, II, *let* $x \in E$ *and* $f \in C([0, T], E)$. *The formula*

$$u(t) \triangleq e^{tA(t)} x + \int_0^t e^{(t-s)A(t)} [(1 + P)^{-1} (f - P(\cdot, 0)\, x)]\, (s)\, ds \tag{F}$$

defines a vector-valued function $u \in L^\infty(0, T; E) \cap C(]0, T], E)$; *moreover, if* $x \in \overline{D(A(0))}$, *then* $u \in C([0, T], E)$.

Proof. It is a consequence of Propositions 3.4, 3.6 and 3.7 of Section 3 below.

Formula (F) will play a very important role in the following, namely, to prove existence and regularity results for classical, strict and strong solutions of Problem (P). It is possible to get another representation formula for a strict solution u of Problem (P). Introduce the operator (analogous to (1.2))

$$\begin{aligned}\tilde{P}(t, s) &\triangleq \left[\frac{\partial}{\partial s} e^{\xi A(s)} \right]_{\xi = t - s} \\ &= \frac{1}{2\pi i} \int_\gamma e^{\lambda(t-s)} \frac{\partial}{\partial s} R(\lambda, A(s))\, d\lambda, \qquad 0 \leqslant s < t \leqslant T;\end{aligned}$$

clearly an estimate similar to (1.3) holds for $\tilde{P}(t, s)$, so that all properties of Remark 1.6 still hold for the operator

$$\tilde{P}g(t) \triangleq \int_0^t \tilde{P}(t, s)\, g(s)\, ds, \qquad t \in [0, T], \quad g \in L^1(0, T; E).$$

Then we have

PROPOSITION 1.9. *Under Hypotheses* I, II, *suppose u is a strict solution of Problem* (P). *Then the following formula holds:*

$$u(t) = (1 - \tilde{P})^{-1} \left[e^{tA(0)}x + \int_0^t e^{(t-s)A(s)} f(s)\, ds \right], \qquad t \in [0, T]. \qquad (\tilde{\mathrm{F}})$$

Proof. If $t \in]0, T]$, define

$$v(s) \triangleq e^{(t-s)A(s)} u(s), \qquad s \in [0, t[;$$

then it is easy to verify that $v \in C^1([0, t[, E)$ and

$$v'(s) = \tilde{P}(t, s)\, u(s) + e^{(t-s)A(s)} f(s)$$

(see also Proposition 3.4 of Section 3 below). Consequently by integrating between 0 and $t - \varepsilon$ and letting $\varepsilon \to 0^+$, we obtain

$$u(t) - e^{tA(0)}x = \tilde{P}u(t) + \int_0^t e^{(t-s)A(s)} f(s)\, ds,$$

from which formula $(\tilde{\mathrm{F}})$ follows.

The representation formula $(\tilde{\mathrm{F}})$ will be used in Section 2 to get uniqueness results for classical, strict and strong solutions of Problem (P).

2. A Priori Estimates: Necessary Conditions

THEOREM 2.1 (A PRIORI ESTIMATE). *Under Hypotheses* I, II *let u be a strict solution of Problem* (P); *then*

$$\|u(t)\|_E \leqslant C \left\{ \|x\|_E + \int_0^t \|f(s)\|_E\, ds \right\} \qquad \forall t \in [0, T]. \qquad (2.1)$$

Proof. It is a trivial consequence of formula $(\tilde{\mathrm{F}})$ and the fact that $(1 - \tilde{P})^{-1} \in \mathscr{L}(L^\infty(0, t; E))$ for each $t \in]0, T]$.

COROLLARY 2.2. *Under Hypotheses* I, II *let u be a strong solution of Problem* (P). *Then* (2.1) *holds.*

Proof. Obvious, by Definition 0.3 and Theorem 2.1.

COROLLARY 2.3. *Under Hypothesis* I, II *let u be a classical solution of Problem* (P). *Then* (2.1) *holds.*

Proof. Fix $\varepsilon > 0$ and set

$$v_\varepsilon(t) = u(t+\varepsilon), \qquad t \in [0, T-\varepsilon].$$

Then v_ε is a strict solution of

$$v_\varepsilon'(t) = A(t+\varepsilon)\, v_\varepsilon(t) = f(t+\varepsilon), \qquad t \in [0, T-\varepsilon],$$
$$v_\varepsilon(0) = u(\varepsilon).$$

Hence by Theorem 2.1

$$\|u(t+\varepsilon)\|_E \leqslant C \left\{ \|u(\varepsilon)\|_E + \int_0^t \|f(s+\varepsilon)\|_E \, ds \right\} \qquad \forall t \in [0, T-\varepsilon];$$

as $\varepsilon \to 0^+$ the result follows.

COROLLARY 2.4 (UNIQUENESS). *Under Hypotheses* I, II, *Problem* (P) *has at most one strict* (*resp. strong, classical*) *solution.*

Proof. It follows trivially by Theorem 2.1 and Corollaries 2.2 and 2.3.

We will discuss now some necessary conditions on the vectors x and $f(0)$ for existence of strict and strong solutions; we will see in Sections 5 and 6 that such conditions are also sufficient, provided f is sufficiently smooth in the case of strict solutions.

THEOREM 2.5. *Under Hypotheses* I, II, *let* $x \in E$ *and* $f \in C([0, T], E)$, *and suppose that the strict solution of Problem* (P) *exists; then the vectors* x *and* $f(0)$ *satisfy the following condition:*

$$x \in D(A(0))$$

and

$$A(0)\, x + f(0) - \left[\frac{d}{dt} A(t)^{-1} \right]_{t=0} \cdot A(0)\, x \in \overline{D(A(0))}. \tag{2.2}$$

Proof. By Definition 0.1, we have $u \in D$ (see the Introduction); in particular, $x = u(0)$ is in $D(A(0))$, and moreover

$$\lim_{t \to 0} \frac{u(t) - x}{t} = A(0)\, x + f(0);$$

on the other hand

$$\frac{u(t)-x}{t}=\frac{A(t)^{-1}-A(0)^{-1}}{t}A(t)\,u(t)+A(0)^{-1}\frac{A(t)\,u(t)-A(0)\,x}{t},$$

hence

$$\frac{u(t)-x}{t}-\frac{A(t)^{-1}-A(0)^{-1}}{t}A(t)\,u(t)\in D(A(0)).$$

As $t\to 0^+$ we conclude that

$$A(0)\,x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x\in\overline{D(A(0))}.$$

THEOREM 2.6. *Under Hypotheses* I, II, *let* $x\in E$ *and* $f\in C([0,T],E)$, *and suppose that the strong solution of Problem* (P) *exists; then the vectors* x *and* $f(0)$ *satisfy the following condition:*

$$\begin{gathered}\exists\{x_k\}_{k\in\mathbf{N}}\subseteq D(A(0)),\qquad \exists\{y_k\}_{k\in\mathbf{N}}\subseteq E\qquad \textit{such that}\\ x_k\to x,\qquad y_k\to f(0),\\ A(0)\,x_k+y_k-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x_k\in\overline{D(A(0))}.\end{gathered}\tag{2.3}$$

In particular, $x\in\overline{D(A(0))}$.

Proof. It is an obvious consequence of Theorem 2.5 and Definition 0.3.

3. SOME BASIC RESULTS

This section contains all auxiliary results which will be needed in the following sections. Our first two lemmata have been proved in [15, Lemma 4.1]; see also Tanabe's book [33, Lemma 5.3.2].

LEMMA 3.1. *Under Hypotheses* I, II, III *we have*

$$\left\|\frac{\partial}{\partial t}R(\lambda,A(t))-\frac{\partial}{\partial\tau}R(\lambda,A(\tau))\right\|_{\mathscr{L}(E)}\leqslant C\{|t-\tau|^{\eta}+|\lambda|^{1-\alpha}\,|t-\tau|\}$$
$$\forall\lambda\in\Sigma_{\theta_0},\quad\forall t,\tau\in[0,T].$$

Proof. Since

$$\frac{\partial}{\partial t}R(\lambda,A(t))=-A(t)\,R(\lambda,A(t))\,\frac{d}{dt}A(t)^{-1}\,A(t)\,R(\lambda,A(t)),$$

we have

$$\frac{\partial}{\partial t} R(\lambda, A(t)) - \frac{\partial}{\partial \tau} R(\lambda, A(\tau))$$

$$= -[A(t) R(\lambda, A(t)) - A(\tau) R(\lambda, A(\tau))] \frac{d}{dt} A(t)^{-1} A(t) R(\lambda, A(t))$$

$$- A(\tau) (R(\lambda, A(\tau)) \left[\frac{d}{dt} A(t)^{-1} - \frac{d}{d\tau} A(\tau)^{-1} \right] A(t) R(\lambda, A(t))$$

$$- A(\tau) R(\lambda, A(\tau)) \frac{d}{d\tau} A(\tau)^{-1} [A(t) R(\lambda, A(t)) - A(\tau) R(\lambda, A(\tau))];$$

but

$$\|A(t) R(\lambda, A(t)) - A(\tau) R(\lambda, A(\tau))\|_{\mathscr{L}(E)}$$

$$= \left\| \lambda \int_{\tau}^{t} \frac{\partial}{\partial \sigma} R(\lambda, A(\sigma))\, d\sigma \right\|_{\mathscr{L}(E)} \leqslant C\,|t-\tau|\,|\lambda|^{1-\alpha},$$

and the proof is easily completed.

The next lemma concerns the operator $P(t, s)$ defined in (1.2).

Lemma 3.2. *Under Hypotheses* I, II, III *we have for* $0 \leqslant s < \tau \leqslant t \leqslant T$:

$$\|P(t, s) - P(\tau, s)\|_{\mathscr{L}(E)} \leqslant C(\delta)(t-\tau)^{\delta}\,(\tau - s)^{-1+[\eta \wedge \alpha - \delta]} \qquad \forall \delta \in\,]0, \eta \wedge \alpha[.$$

Proof. We start from the equality

$$P(t, s) - P(\tau, s) = \frac{1}{2\pi i} \int_{\gamma} e^{\lambda(t-s)} \left[\frac{\partial}{\partial t} R(\lambda, A(t)) - \frac{\partial}{\partial \tau} R(\lambda, A(\tau)) \right] d\lambda$$

$$+ \frac{1}{2\pi i} \int_{\tau-s}^{t-s} \int_{\gamma} \lambda e^{\lambda\sigma} \frac{\partial}{\partial \tau} R(\lambda, A(\tau))\, d\lambda\, d\sigma,$$

which, by Lemma 3.1, implies

$$\|P(t, s) - P(\tau, s)\|_{\mathscr{L}(E)}$$

$$\leqslant C \left[(t-\tau)(t-s)^{-2+\alpha} + (t-\tau)^{\eta}\,(t-s)^{-1} + \int_{\tau-s}^{t-s} \sigma^{-2+\alpha}\, d\sigma \right] \tag{3.1}$$

$$\leqslant C[(t-\tau)^{\eta}\,(t-s)^{-1} + (\tau-s)^{-1+\alpha} - (t-s)^{-1+\alpha}];$$

since $t-\tau < t-s$ and $\tau - s \leqslant t-s$, we get the estimate

$$\|P(t,s)-P(\tau,s)\|_{\mathscr{L}(E)}$$
$$\leqslant C[(t-\tau)^{\delta}(t-s)^{-1+\eta-\delta}[(t-\tau)(t-s)^{-1}]^{\eta-\delta}$$
$$+(\tau-s)^{-1+\alpha}[(t-\tau)(t-s)^{-1}]]$$
$$\leqslant C(t-\tau)^{\delta}(t-s)^{-\delta}(\tau-s)^{-1+\eta\wedge\alpha},$$

from which the result follows.

Our next statements concern the properties of all the operators and vector-valued functions appearing in formula (F).

(a) *The Function* $t \mapsto P(t,0)x$

PROPOSITION 3.3. *Under Hypotheses* I, II *we have:*

(i) *If* $x \in E$, *then* $P(t,0)x \in C(]0,T],E) \cap L^p(0,T;E)$ $\forall p \in [1,(1-\alpha)^{-1}[$.

(ii) *If* $x \in D_{A(0)}(\beta,\infty)$, $\beta \in [\alpha,1[$, *then* $P(t,0)x \in L^p(0,T;E)$ $\forall p \in [1,(1-\beta)^{-1}[$.

(iii) *If* $x \in D(A(0))$, *then* $P(t,0)x \in L^{\infty}(0,T;E)$ *and, as* $t \to 0^+$,

$$P(t,0)x = O(t^{\alpha}) - [e^{tA(t)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x$$
$$= O(t^{\alpha}) - [e^{tA(0)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x.$$

Under Hypotheses I, II, III *we have:*

(iv) *If* $x \in E$ *then* $P(t,0)x \in C^{\eta}(]0,T],E)$.

(v) *If* $x \in D(A(0))$ *then, as* $t-\tau \to 0^+$,

$$P(t,0)x - P(\tau,0)x$$
$$= O((t-\tau)^{\alpha\wedge\eta}) - [e^{tA(\tau)}-e^{\tau A(\tau)}]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x$$
$$= O((t-\tau)^{\alpha\wedge\eta}) - [e^{tA(0)}-e^{\tau A(0)}]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x.$$

Proof. (i) Suppose $t > \tau \geqslant \varepsilon$. The equality

$$P(t,0)x - P(\tau,0)x = \frac{1}{2\pi i}\int_{\gamma} e^{\lambda t}\left[\frac{\partial}{\partial t}R(\lambda,A(t)) - \frac{\partial}{\partial \tau}R(\lambda,A(\tau))\right]x\,d\lambda$$
$$+\frac{1}{2\pi i}\int_{\gamma}[e^{\lambda t}-e^{\lambda t}]\frac{\partial}{\partial \tau}R(\lambda,A(\tau))x\,d\lambda \qquad (3.2)$$

implies the continuity of $t \to P(t, 0)x$ on $]0, T]$. The L^p-integrability when $p \in [1, (1-\alpha)^{-1}[$ is a consequence of (1.3).

(ii) We have

$$P(t,0)x = P(t,0)[x - e^{tA(0)}x] + P(t,0)[A(0)^{-1} - A(t)^{-1}]A(0)e^{tA(0)}x + \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda t}}{\lambda}\frac{\partial}{\partial t}R(\lambda, A(t))A(0)e^{tA(0)}x\,d\lambda + [1 - e^{tA(t)}]\frac{d}{dt}A(t)^{-1}A(0)e^{tA(0)}x,$$

which implies, by Lemma 1.5(iii),

$$\|P(t,0)x\|_E \leqslant Ct^{-1+\alpha+\beta} + Ct^{-1+\beta} \leqslant Ct^{-1+\beta};$$

this proves (ii).

(iii) As in (ii) we have

$$P(t,0)x = P(t,0)[A(0)^{-1} - A(t)^{-1}]A(0)x + \frac{1}{2\pi i}\int_\lambda \frac{e^{\lambda t}}{\lambda}\frac{\partial}{\partial t}R(\lambda, A(t))A(0)x\,d\lambda + [1 - e^{tA(t)}]\frac{d}{dt}A(t)^{-1}A(0)x;$$

therefore as $t \to 0^+$

$$P(t,0) = O(t^\alpha) - [e^{tA(t)} - 1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x = O(t^\alpha) - [e^{tA(0)} - 1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x,$$

and recalling (i), (iii) follows.

(iv) Suppose $t > \tau \geqslant \varepsilon$. By (3.2) and Lemma 3.1,

$$\|P(t,0)x - P(\tau,0)x\|_E \leqslant C[(t-\tau)^\eta \varepsilon^{-1} + (t-\tau)\varepsilon^{-2+\alpha}] + \left\|\frac{1}{2\pi i}\int_\gamma\int_\tau^t \lambda e^{\lambda\sigma}\frac{\partial}{\partial\tau}R(\lambda, A(\tau))\,d\sigma\,d\lambda\right\|_E \leqslant C(\varepsilon)(t-\tau)^\eta.$$

(v) Suppose $t > \tau > 0$. By (3.2), as in (iii) we get

$$
\begin{aligned}
&P(t, 0)\,x - P(\tau, 0)\,x \\
&= \frac{1}{2\pi i}\int_\gamma \left[e^{\lambda t} \left\{ \left[\frac{\partial}{\partial t} R(\lambda, A(t)) - \frac{\partial}{\partial \tau} R(\lambda, A(\tau)) \right] [A(0)^{-1} - A(t)^{-1}] \right.\right. \\
&\quad + \frac{\partial}{\partial \tau} R(\lambda, A(\tau))[A(\tau)^{-1} - A(t)^{-1}] + \frac{1}{\lambda}\left[\frac{\partial}{\partial t} R(\lambda, A(t)) - \frac{\partial}{\partial \tau} R(\lambda, A(\tau)) \right] \\
&\quad + \frac{1}{\lambda}\left[\frac{d}{dt} A(t)^{-1} - \frac{d}{d\tau} A(\tau)^{-1} \right] - [R(\lambda, A(t)) - R(\lambda, A(\tau))]\, \frac{d}{d\tau} A(\tau)^{-1} \\
&\quad \left. - R(\lambda, A(t)) \left[\frac{d}{dt} A(t)^{-1} - \frac{d}{d\tau} A(\tau)^{-1} \right] \right\} \\
&\quad + \int_\tau^t \lambda e^{\lambda\sigma} \left\{ \frac{\partial}{\partial \tau} R(\lambda, A(\tau))[A(0)^{-1} - A(\tau)^{-1}] + \frac{1}{\lambda}\frac{\partial}{\partial \tau} R(\lambda, A(\tau)) \right. \\
&\quad + \frac{1}{\lambda}\frac{d}{d\tau} A(\tau)^{-1} - R(\lambda, A(\tau)) \left[\frac{d}{d\tau} A(\tau)^{-1} \right. \\
&\quad \left.\left.\left. - \left[\frac{d}{dt} A(t)^{-1} \right]_{t=0} \right] \right\} A(0)\, x\, d\lambda - [e^{tA(\tau)} - e^{\tau A(\tau)}] \left[\frac{d}{dt} A(t)^{-1} \right]_{t=0} A(0)\, x;
\end{aligned}
$$

consequently it is easy to deduce that, as $t - \tau \to 0^+$,

$$
\begin{aligned}
&P(t, 0)\,x - P(\tau, 0)\,x \\
&= O((t-\tau)^{\alpha}) + O((t-\tau)^{\eta}) - [e^{tA(\tau)} - e^{\tau A(\tau)}] \left[\frac{d}{dt} A(t)^{-1} \right]_{t=0} A(0)\, x,
\end{aligned}
$$

and the proof is complete.

(b) *The Function* $t \mapsto e^{tA(t)}x$

Proposition 3.4. *Let* $x \in E$. *Under Hypotheses* I, II *we have:*

(i) $e^{tA(t)}x \in D(A(t))\ \forall t \in \,]0, T]$; $e^{tA(t)}x \in L^\infty(0, T; E) \cap C^1(]0, T], E)$ *and*

$$\frac{d}{dt} e^{tA(t)}x = A(t)\, e^{tA(t)}x + P(t, 0)\, x \qquad \forall t \in \,]0, T].$$

(ii) $e^{tA(t)}x \in C([0, T], E)$ *if and only if* $x \in \overline{D(A(0))}$; *in this case* $[e^{tA(t)}x]_{t=0} = x$.

(iii) *If* $\beta \in \,]0, \alpha]$, *then* $e^{tA(t)}x \in C^\beta([0, T], E)$ *if and only if* $x \in D_{A(0)}(\beta, \infty)$.

(iv) *If* $\beta \in \,]\alpha, 1[$ *and* $x \in D_{A(0)}(\beta, \infty)$, *then* $e^{tA(t)}x \in C^{\beta}([0, T], E)$.

(v) *If* $x \in D(A(0))$, *then as* $t \to 0^+$

$$\frac{e^{tA(t)}-1}{t}x = o(1) + \frac{e^{tA(0)}-1}{t}x - [e^{tA(0)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x$$

and

$$\begin{aligned} A(t)\,e^{tA(t)}x &= o(1) + e^{tA(t)}A(0)\,x - tA(t)\,e^{tA(t)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x \\ &= o(1) + e^{tA(0)}A(0)\,x - tA(0)\,e^{tA(0)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x. \end{aligned}$$

Under Hypotheses I, II, III *we have:*

(vi) $e^{tA(t)}x \in C^{1,\eta}(]0, T], E)$.

Proof. (i) Obviously, $e^{tA(t)}x \in D(A(t))$ $\forall t \in \,]0, T]$. Next, $e^{tA(t)}x \in L^{\infty}(0, T; E)$ by Lemma 1.5(i); finally, suppose $t, \tau \geqslant \varepsilon$ and let $\tau \to t$. By Lebesgue's Theorem

$$\begin{aligned} \frac{e^{tA(t)}-e^{\tau A(\tau)}}{t-\tau}x &= \frac{e^{tA(t)}-e^{tA(\tau)}}{t-\tau}x + \frac{e^{tA(\tau)}-e^{\tau A(\tau)}}{t-\tau}x \\ &\to A(t)\,e^{tA(t)}x + P(t,0)\,x. \end{aligned}$$

We have to prove that $A(t)\,e^{tA(t)}x + P(t,0)\,x \in C(]0, T], E)$. The continuity of $P(t,0)\,x$ follows by Proposition 3.3(i), while the equality

$$\begin{aligned} A(t)\,e^{tA(t)}x - A(\tau)\,e^{A(\tau)}x &= \frac{1}{2\pi i}\int_{\gamma} \lambda e^{t\lambda}[R(\lambda, A(t)) - R(\lambda, A(\tau))]\,x\,d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{\gamma} [e^{t\lambda} - e^{\tau\lambda}]\,R(\lambda, A(\tau))\,x\,d\lambda \end{aligned}$$

implies the continuity of $A(t)\,e^{tA(t)}x$, via Lebesgue's Theorem.

(ii) It is enough to show that

$$\lim_{t \to 0^+} \|e^{tA(t)}x - x\|_E = 0 \qquad \text{if and only if} \quad x \in \overline{D(A(0))}.$$

We have as $t \to 0^+$

$$e^{tA(t)}x - x = O(t^{\alpha}) + [e^{tA(0)} - 1]\,x;$$

now it is easy to verify that

$$\lim_{t\to 0^+} \|e^{tA(0)}x - x\|_E = 0 \qquad \text{if and only if} \quad x \in \overline{D(A(0))},$$

and (ii) follows.

(iii)–(iv) Let $x \in D_{A(0)}(\beta, \infty)$ and suppose $t > \tau \geqslant 0$. We have, as $t - \tau \to 0^+$,

$$\begin{aligned}
&[e^{tA(t)} - e^{\tau A(\tau)}]\,x \\
&\quad = [e^{tA(t)} - e^{tA(\tau)}]\,x + [(e^{tA(\tau)} - e^{\tau A(\tau)}) - (e^{tA(0)} - e^{\tau A(0)})]\,x \\
&\qquad + [e^{tA(0)} - e^{\tau A(0)}]\,x \\
&\quad = \frac{1}{2\pi i}\int_\gamma \Bigg[e^{t\lambda} \bigg\{ \int_\tau^t \frac{\partial}{\partial\sigma} R(\lambda, A(\sigma))[x - e^{\sigma A(0)}x \\
&\qquad + [A(0)^{-1} - A(\sigma)^{-1}]\,A(0)\,e^{\sigma A(0)}x\Big]\,d\sigma \\
&\qquad + \int_\tau^t \bigg[\frac{1}{\lambda}\frac{\partial}{\partial\sigma} R(\lambda, A(\sigma)) + \frac{1}{\lambda}\frac{d}{d\sigma}A(\sigma)^{-1} \\
&\qquad - R(\lambda, A(\sigma))\frac{d}{d\sigma}A(\sigma)^{-1}\bigg]\,A(0)\,e^{\sigma A(0)}x\,d\sigma\bigg\} \\
&\qquad + \int_\tau^t \lambda e^{\lambda\sigma}\int_0^\tau \bigg[\frac{\partial}{\partial r} R(\lambda, A(r))[x - e^{rA(0)} + [A(0)^{-1} - A(r)^{-1}]\,A(0)\,e^{rA(0)}x] \\
&\qquad + \bigg[\frac{1}{\lambda}\frac{\partial}{\partial r} R(\lambda, A(r)) + \frac{1}{\lambda}\frac{d}{dr}A(r)^{-1} \\
&\qquad - R(\lambda, A(r))\frac{d}{dr}A(r)^{-1}\bigg]\,A(0)\,e^{rA(0)}x\bigg]\,dr\,d\sigma\Bigg]\,d\lambda \\
&\qquad + e^{\tau A(0)}[e^{(t-\tau)A(0)} - 1]\,x \\
&\quad = O((t-\tau)^{\alpha+\beta}) + O((t-\tau)^\beta) = O((t-\tau)^\beta),
\end{aligned}$$

so $e^{tA(t)}x \in C^\beta([0, T], E)$. Suppose now that $e^{tA(t)}x \in C^\beta([0, T], E)$; then, in particular, by (ii) we get, as $t \to 0^+$.

$$\begin{aligned}
[e^{tA(0)} - 1]\,x &= [e^{tA(0)} - e^{tA(t)}]\,x + [e^{tA(t)} - 1]\,x \\
&= O(t^\alpha) + O(t^\beta) = O(t^{\beta\wedge\alpha}),
\end{aligned}$$

hence we deduce that $x \in D_{A(0)}(\beta \wedge \alpha, \infty)$, and (iii), (iv) are proved.

(v) Suppose $t>0$. Then, as $t \to 0^+$,

$$\begin{aligned}
&\frac{e^{tA(t)}-1}{t}x \\
&\quad = \frac{e^{tA(t)}-1}{t}\left[A(0)^{-1}-A(t)^{-1}+t\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}\right]A(0)x \\
&\qquad + \left[\frac{e^{tA(t)}-1}{t}A(t)^{-1}-\frac{e^{tA(0)}-1}{t}A(0)^{-1}\right]A(0)x \\
&\qquad + \frac{e^{tA(0)}-1}{t}x-[e^{tA(t)}-e^{tA(0)}]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x \\
&\qquad - [e^{tA(0)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x \\
&\quad = o(1)+O(t^\alpha)+\frac{e^{tA(0)}-1}{t}x-[e^{tA(0)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x.
\end{aligned}$$

Similarly we have, as $t \to 0^+$,

$$\begin{aligned}
&A(t)\,e^{tA(t)}x \\
&\quad = tA(t)\,e^{tA(t)}\left[\frac{A(0)^{-1}-A(t)^{-1}}{t}+\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}\right]A(0)x \\
&\qquad + [e^{tA(t)}-e^{tA(0)}]A(0)x+e^{tA(0)}A(0)x \\
&\qquad - [tA(t)\,e^{tA(t)}-tA(0)\,e^{tA(0)}]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x \\
&\qquad - tA(0)\,e^{tA(0)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x \\
&\quad = o(1)+O(t^\alpha)+e^{tA(0)}A(0)x-tA(0)\,e^{tA(0)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)x,
\end{aligned}$$

which proves (v).

(vi) By Proposition 3.3(iv), $P(t,0)x \in C^n(]0,T],E)$, so by (i) it is enough to prove that $A(t)\,e^{tA(t)}x \in C^n(]0,T],E)$. Indeed, if $t>\tau \geqslant \varepsilon$ we have

$$\begin{aligned}
&\|A(t)\,e^{tA(t)}x-A(\tau)\,e^{\tau A(\tau)}x\|_E \\
&\quad \leqslant \|A(t)\,e^{tA(t)}x-A(\tau)\,e^{tA(\tau)}x\|_E+\|A(\tau)\,e^{tA(\tau)}x-A(\tau)\,e^{\tau A(\tau)}x\|_E \\
&\quad \leqslant C[(t-\tau)\,\varepsilon^{-2+\alpha}+(t-\tau)\,\varepsilon^{-2}].
\end{aligned}$$

The proof is complete.

(c) *The Operator* $P\varphi(t) = \int_0^t P(t, s)\, \varphi(s)\, ds$

PROPOSITION 3.5. *Under Hypotheses* I, II *we have:*

(i) $P \in \mathscr{L}(L^p(0, T; E))$ $\forall p \in [1, +\infty[$; $P \in \mathscr{L}(C([0, T], E))$.

(ii) *If* $\varphi \in C(]0, T], E) \cap L^1(0, T; E)$, *then* $P\varphi \in C(]0, T], E)$.

(iii) *If* $\varphi \in C(]0, T], E) \cap L^p(0, T; E)$, $p \in]\alpha^{-1}, +\infty]$, *then* $P\varphi \in C([0, T], E)$ *and* $P\varphi(0) = 0$.

Under Hypotheses I, II, III *we have:*

(iv) *If* $\varphi \in C(]0, T], E) \cap L^1(0, T; E)$, *then* $P\varphi \in C^\delta(]0, T], E)$ $\forall \delta \in]0, \eta[\cap]0, \alpha]$.

(v) *If* $\varphi \in L^\infty(0, T; E)$ *then* $P\varphi \in C^\delta([0, T], E)$ $\forall \delta \in]0, \eta[\cap]0, \alpha[$.

Proof. (i) A standard calculation shows that

$$\|P\varphi\|_{L^p(0,T;E)} \leqslant \frac{KT^\alpha}{\alpha} \|\varphi\|_{L^p(0,T;E)} \qquad \forall p \in [1, +\infty[,$$

where K is the constant appearing in (1.3). Hence it is sufficient to prove that $P\varphi \in C([0, T], E)$ whenever $\varphi \in C([0, T], E)$. This is a consequence of Lebesgue's Theorem and of the following equality, which is true for any $t > \tau \geqslant 0$ as $t - \tau \to 0^+$:

$$\begin{aligned} P\varphi(t) - P\varphi(\tau) &= \int_\tau^t P(t, s)\, \varphi(s)\, ds + \int_0^\tau [P(t, s) - P(\tau, s)]\, \varphi(s)\, ds \\ &= O((t-\tau)^\alpha) + \int_0^\tau \frac{1}{2\pi i} \int_\gamma \Big[e^{\lambda(t-s)} \Big(\frac{\partial}{\partial t} R(\lambda, A(t)) - \frac{\partial}{\partial \tau} R(\lambda, A(\tau)) \Big) \\ &\quad + (e^{\lambda(t-s)} - e^{\lambda(\tau-s)}) \frac{\partial}{\partial \tau} R(\lambda, A(\tau)) \Big]\, \varphi(s)\, d\lambda\, ds. \end{aligned} \tag{3.3}$$

(ii) Let $t > 0$, and let us show that

$$\lim_{\tau \to t} \|P\varphi(t) - P\varphi(\tau)\|_E = 0.$$

We can suppose $t, \tau \geqslant \varepsilon$ and, for example, $t > \tau$; then we have

$$\begin{aligned} &\|P\varphi(t) - P\varphi(\tau)\|_E \\ &\quad \leqslant C(t-\tau)^\alpha \|\varphi\|_{C([\varepsilon, T], E)} + \left\| \int_0^\tau [P(t, s) - P(\tau, s)]\, \varphi(s)\, ds \right\|_E, \end{aligned}$$

so we have to show that the last term on the right-hand side goes to 0 as $\tau \to t^-$. First of all we prove that

$$\lim_{\tau \to t^-} \|[P(t, s) - P(\tau, s)]\, \varphi(s)\|_E = 0 \qquad \forall s \in\,]0, t[. \tag{3.4}$$

We start from the equality

$$\begin{aligned}
&[P(t, s) - P(\tau, s)]\, \varphi(s) \\
&\quad = \frac{1}{2\pi i} \int_\gamma e^{\lambda(t-s)} \left[\frac{\partial}{\partial t} R(\lambda, A(t)) - \frac{\partial}{\partial \tau} R(\lambda, A(\tau))\right] \varphi(s)\, d\lambda \\
&\qquad + \frac{1}{2\pi i} \int_\gamma [e^{\lambda(t-s)} - e^{\lambda(\tau-s)}] \frac{\partial}{\partial \tau} R(\lambda, A(\tau))\, \varphi(s)\, d\lambda;
\end{aligned}$$

for each $\lambda \in \gamma$ the first integrand on the right-hand side goes to 0 as $\tau \to t^-$ and is dominated by const $\cdot\, |\lambda|^{-\alpha} \cdot \exp(\operatorname{Re} \lambda(t-s))$; hence its integral over γ goes to 0 as $\tau \to t^-$. For each $\lambda \in \gamma$ the second integrand also goes to 0 as $\tau \to t^-$, and we have

$$\begin{aligned}
&\left\| \frac{1}{2\pi i} [e^{\lambda(t-s)} - e^{\lambda(\tau-s)}] \frac{\partial}{\partial \tau} R(\lambda, A(\tau))\, \varphi(s) \right\|_E \\
&\qquad \leqslant C e^{\operatorname{Re} \lambda(\tau-s)} |\lambda|^{-\alpha} \qquad \forall \lambda \in \gamma, \quad \forall \tau \in\,]s, t[.
\end{aligned}$$

Now the (integrable) functions $\{F_\tau(\lambda)\}_{\tau \in]s,t]}$, defined by

$$F_\tau(\lambda) = C e^{\operatorname{Re} \lambda(\tau-s)} |\lambda|^{-\alpha}, \qquad \lambda \in \gamma,$$

converge to $F_t(\lambda)$ as $\tau \to t^-$, and moreover $\int_\gamma F_\tau(\lambda)\, d\lambda \to \int_\gamma F_t(\lambda)\, d\lambda$ as $\tau \to t^-$. This implies that (3.4) holds.

Next, we can write

$$\int_0^\tau [P(t, s) - P(\tau, s)]\, \varphi(s)\, ds = \int_0^t \chi_{[0,\tau[}(s) [P(t, s) - P(\tau, s)]\, \varphi(s)\, ds,$$

and we have

$$\|\chi_{[0,\tau[}(s)[P(t, s) - P(\tau, s)]\, \varphi(s)\|_E \leqslant G_\tau(s) \qquad \forall s \in\,]0, t[, \quad \forall \tau \in\,]0, t[,$$

where

$$G_\tau(s) = \begin{cases} C \cdot (\varepsilon/2)^{-1+\alpha} \|\varphi(s)\|_E & \text{if} \quad s \in\,]0, \varepsilon/2[, \\ C \cdot [(t-s)^{-1+\alpha} + (\tau-s)^{-1+\alpha}]\, \|\varphi\|_{C([\varepsilon/2, T], E)} & \text{if} \quad s \in [\varepsilon/2, \tau[, \\ 0 & \text{if} \quad s \in [\tau, t[. \end{cases}$$

The functions $\{G_\tau(s)\}_{\tau\in[0,t[}$ have the following properties:

$$\lim_{\tau\to t^-} G_\tau(s) = G_t(s) \qquad \forall s \in]0, t[,$$

$$\lim_{\tau\to t^-} \int_0^t G_\tau(s)\, ds = \int_0^t G_t(s)\, ds.$$

This shows that

$$\lim_{\tau\to t^-} \left\| \int_0^\tau [P(t, s) - P(\tau, s)]\, \varphi(s)\, ds \right\|_E = 0,$$

and (ii) is proved.

(iii) It is sufficient to prove continuity at $t = 0$. The estimate

$$\|P\varphi(t)\|_E \leqslant K \int_0^t \|\varphi(s)\|_E\, (t-s)^{-1+\alpha}\, ds \leqslant K \cdot t^{(\alpha p-1)/p}\, \|\varphi\|_{L^p(0,T;E)}$$

implies the result.

(iv) Suppose $t > \tau \geqslant \varepsilon$. Then (3.3) and (3.1) imply that

$$\begin{aligned}
\|P\varphi(t) - P\varphi(\tau)\|_E &\leqslant C(t-\tau)^\alpha\, \|\varphi\|_{C([\varepsilon,T],E)} \\
&\quad + C \int_0^\tau \left[(t-\tau)^\eta\, (t-s)^{-1} + \int_{\tau-s}^{t-s} \sigma^{-2+\alpha}\, d\sigma \right] \cdot \|\varphi(s)\|_E\, ds \\
&\leqslant C(t-\tau)^\alpha\, \|\varphi\|_{C([\varepsilon,T],E)} \\
&\quad + C[(t-\tau)^\eta \cdot \varepsilon^{-1} + (t-\tau) \cdot \varepsilon^{-2+\alpha}]\, \|\varphi\|_{L^1(0,\varepsilon/2;E)} \\
&\quad + C \left[(t-\tau)^\eta \log \frac{t-\varepsilon/2}{t-\tau} + (\tau-\varepsilon/2)^\alpha + (t-\tau)^\alpha \right. \\
&\quad \left. - (t-\varepsilon/2)^\alpha \right] \|\varphi\|_{C([\varepsilon/2,T],E)} \\
&\leqslant C(\varepsilon, \delta)(t-\tau)^\delta \qquad \forall \delta \in]0, \eta[\cap]0, \alpha].
\end{aligned}$$

(v) Suppose $t > \tau \geqslant 0$. As in (iv) we get

$$\begin{aligned}
\|P\varphi(t) - P\varphi(\tau)\|_E &\leqslant C\, \|\varphi\|_{L^\infty(0,T;E)} \left[(t-\tau)^\alpha + \int_0^\tau (t-\tau)^\eta\, (t-s)^{-1}\, ds \right. \\
&\quad \left. + \int_0^\tau \int_{\tau-s}^{t-s} \sigma^{-2+\alpha}\, d\sigma\, ds \right] \\
&\leqslant C(\delta)(t-\tau)^\delta \qquad \forall \delta \in]0, \eta[\cap]0, \alpha].
\end{aligned}$$

(d) *The Operator* $Q \triangleq (1+P)^{-1}$

Let us consider now the operator $1 + P$; the next proposition proves that it is invertible and describes the properties of $Q = (1+P)^{-1}$ (see Remark 1.6).

PROPOSITION 3.6. *Under Hypotheses* I, II *we have:*

(i) *Q exists and* $Q \in \mathscr{L}(L^p(0, T; E))$ $\forall p \in [1, +\infty]$; $Q \in \mathscr{L}(C([0, T], E))$.

(ii) *If* $\varphi \in C(]0, T], E) \cap L^1(0, T; E)$, *then* $Q\varphi \in C(]0, T], E)$.

Under Hypotheses I, II, III *we have:*

(iii) *If* $\varphi \in L^1(0, T; E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \eta[\cap]0, \alpha]$, *then* $Q\varphi \in C^\delta(]0, T], E)$.

(iv) *If* $\varphi \in C^\delta([0, T], E)$, $\delta \in]0, \eta[\cap]0, \alpha]$, *then* $Q\varphi \in C^\delta([0, T], E)$.

Proof. (i) We confine ourselves to the case of $C([0, T], E)$, since in the case of $L^p(0, T; E)$ the proof is identical.

For each $\omega > 0$, define a new norm over $C([0, T], E)$ by

$$\|f\|_{\omega,\infty} \triangleq \sup_{t \in [0,T]} \|e^{-\omega t} f(t)\|_E.$$

Obviously

$$\|f\|_{\omega,\infty} \leqslant \|f\|_{C([0,T],E)} \leqslant e^{\omega T} \|f\|_{\omega,\infty} \qquad \forall f \in C([0, T], E), \quad \forall \omega > 0; \tag{3.5}$$

moreover it is easily seen that

$$\|P\varphi\|_{\omega,\infty} \leqslant K \int_0^T e^{-\omega\tau} \tau^{-1+\alpha}\, d\tau \cdot \|\varphi\|_{\omega,\infty} \qquad \forall \varphi \in C([0, T], E).$$

Set $M(\omega) = \int_0^T e^{-\omega\tau} \tau^{-1+\alpha}\, d\tau$; it is clear that

$$\lim_{\omega \to \infty} M(\omega) = 0,$$

hence there exists ω_0 such that $K \cdot M(\omega) < 1$ $\forall \omega \geqslant \omega_0$. Choosing $\omega \geqslant \omega_0$ we conclude that $(1 + P)$ is an isomorphism over $C([0, T], E)$ with the norm $\|\cdot\|_{\omega,\infty}$. By (3.5) we get the result.

(ii) The following argument is in [15, proof of Lemma 3.2]. We have $(1+P)^{-1} f = f + \sum_{n=1}^{\infty} (-P)^n f$; define

$$P_1(t, s) \triangleq P(t, s), \qquad P_n(t, s) = \int_s^t P_1(t, \rho)\, P_{n-1}(\rho, s)\, d\rho \qquad \forall n > 1;$$

then it is easy to verify by induction that

$$[(-P)^n f](t) = \int_0^t P_n(t, s) f(s)\, ds,$$

and again by induction we get

$$\|P_n(t, s)\|_{\mathscr{L}(E)} \leqslant \frac{K^n \Gamma(\alpha)^n}{\Gamma(n\alpha)} (t-s)^{n\alpha - 1}.$$

Hence

$$\sum_{n=1}^{\infty} \|P_n(t, s)\|_{\mathscr{L}(E)} \leqslant (t-s)^{-1+\alpha}\, T^{-\alpha} \sum_{n=1}^{\infty} \frac{[KT^\alpha \Gamma(\alpha)]^n}{\Gamma(n\alpha)} \triangleq \frac{A}{(t-s)^{1-\alpha}}; \tag{3.6}$$

and if we set

$$R(t, s) \triangleq \sum_{n=1}^{\infty} P_n(t, s)$$

we check

$$Qf(t) = f(t) + \int_0^t R(t, s) f(s)\, ds,$$

$$\|R(t, s)\|_{\mathscr{L}(E)} \leqslant \frac{A}{(t-s)^{1-\alpha}}.$$

Hence it suffices to show that $t \mapsto \int_0^t R(t, s) f(s)\, ds$ is in $C(]0, T], E)$. By (3.6), via Lebesgue's Theorem, we get

$$\int_0^t R(t, s) f(s)\, ds = \sum_{n=1}^{\infty} \int_0^t P_n(t, s) f(s)\, ds, \qquad \forall t \in]0, T].$$

so that it is enough to prove that the series on the right-hand side converges uniformly in $[\varepsilon, T]$ for any $\varepsilon > 0$. But

$$\lim_{k\to\infty} \sup_{t\in[\varepsilon,T]} \sum_{n=k}^{\infty} \int_0^t \|P_n(t, s)\|_{\mathscr{L}(E)} \|f(s)\|_E\, ds$$

$$\leqslant \lim_{k\to\infty} T^{-\alpha} \frac{[KT^\alpha \Gamma(\alpha)]^n}{\Gamma(n\alpha)} \sup_{t\in[\varepsilon,T]} \int_0^t \|f(s)\|_E (t-s)^{-1+\alpha}\, ds = 0,$$

and (ii) follows.

(iii) It is a consequence of (ii), Proposition 3.5(iv) and the integral equation (1.5).

(iv) It is a consequence of (i), Proposition 3.5(v) and the integral equation (1.5).

(e) *The Operator* $T\varphi(t) \triangleq \int_0^t e^{(t-s)A(t)}\varphi(s)\,ds$

PROPOSITION 3.7. *Under Hypotheses* I, II *we have:*

(i) *If* $\varphi \in L^1(0, T; E)$, *then* $T\varphi \in C([0, T], E)$ *and* $T\varphi(0) = 0$.

(ii) *If* $\varphi \in L^p(0, T; E)$, $p \in]1, \infty]$, *then* $T\varphi \in C^\delta([0, T], E)$ $\forall \delta \in]0, 1 - 1/p[$.

(iii) *If* $\varphi \in C(]0, T], E) \cap L^1(0, T; E)$, *then* $T\varphi \in C^\delta(]0, T], E)$ $\forall \delta \in]0, 1[$.

(iv) *If* $\varphi \in L^1(0, T; E) \cap C^\delta(]0, T], E)$, $\delta \in]0, 1[$, *then* $T\varphi(t) \in D(A(t))$ $\forall t \in]0, T]$, *and*

$$A(t)\,T\varphi(t) = \int_0^t A(t)\,e^{(t-s)A(t)}[\varphi(s) - \varphi(t)]\,ds + [e^{tA(t)} - 1]\,\varphi(t) \qquad \forall t \in]0, T]. \tag{3.7}$$

(v) *If* $\varphi \in L^1(0, T; E) \cap C^\delta(]0, T], E)$, $\delta \in]0, 1[$, *then* $T\varphi \in C^1(]0, T], E)$ *and*

$$(T\varphi)'\,(t) = \int_0^t A(t)\,e^{(t-s)A(t)}[\varphi(s) - \varphi(t)]\,ds + e^{tA(t)}\varphi(t) + \int_0^t P(t, s)\,\varphi(s)\,ds \qquad \forall t \in]0, T]. \tag{3.8}$$

Under Hypotheses I, II, III *we have:*

(vi) *If* $\varphi \in L^1(0, T; E) \cap C^\delta(]0, T], E)$, $\delta \in]0, \eta[\cap]0, \alpha]$, *then* $T\varphi \in C^{1,\delta}(]0, T], E)$.

(vii) *If* $\varphi \in C^\delta([0, T], E)$, $\delta \in]0, \eta[\cap]0, \alpha]$, *and* $\varphi(0) = 0$, *then* $T\varphi \in C^{1,\delta}([0, T], E)$ *and* $(T\varphi)'\,(0) = 0$.

Proof. (i) It is an easy consequence of Lebesgue's Theorem.

(ii) Suppose $t > \tau \geqslant 0$. Then

$$T\varphi(t) - T\varphi(\tau) = \int_\tau^t e^{(t-s)A(t)}\varphi(s)\,ds + \int_0^\tau [e^{(t-s)A(t)} - e^{(t-s)A(\tau)}]\,\varphi(s)\,ds + \int_0^\tau \int_{\tau-s}^{t-s} A(\tau)\,e^{\sigma A(\tau)}\,d\sigma\varphi(s)\,ds; \tag{3.9}$$

hence

$$\|T\varphi(t)-T\varphi(\tau)\|_E \leqslant C\int_\tau^t \|\varphi(s)\|_E\,ds + C(t-\tau)\int_0^\tau \|\varphi(s)\|_E\,(t-s)^{-1+\alpha}\,ds$$

$$+\,C\int_0^\tau \|\varphi(s)\|_E \log\left[1+\frac{t-\tau}{\tau-s}\right] ds. \tag{3.10}$$

By (3.10) we have

$$\|T\varphi(t)-T\varphi(\tau)\|_E$$

$$\leqslant C(\delta)\,\|\varphi\|_{L^p(0,T;E)}\left\{(t-\tau)^{1/q}+(t-\tau)^{\alpha+\delta}\left[\int_0^\tau (\tau-s)^{-\delta q}\,ds\right]^{1/q}\right.$$

$$\left.+\,(t-\tau)^{\delta}\left[\int_0^\tau (\tau-s)^{-\delta q}\,ds\right]^{1/q}\right\},$$

where $q=p/(p-1)$. This proves (ii).

(iii) Suppose $t>\tau\geqslant\varepsilon$. Then (3.9) yields, $\forall\delta\in\,]0,1[$,

$$\|T\varphi(t)-T\varphi(\tau)\|_E$$

$$\leqslant C(t-\tau)\,\|\varphi\|_{C([\varepsilon,T],E)} + C(t-\tau)\int_0^\tau \|\varphi(s)\|_E\,(t-s)^{-1+\alpha}\,ds$$

$$+\,C\int_0^\tau\int_{\tau-s}^{t-s}\sigma^{-1}\,d\sigma\,\|\varphi(s)\|_E\,ds$$

$$\leqslant C(t-\tau)\left[\|\varphi\|_{C([\varepsilon,T],E)}+\left[\frac{\varepsilon}{2}\right]^{-1+\alpha}\|\varphi\|_{L^1(0,\varepsilon/2;E)}\right.$$

$$\left.+\,\tau^{\alpha}\,\|\varphi\|_{C([\varepsilon/2,T],E)}+\left[\frac{\varepsilon}{2}\right]^{-1}\|\varphi\|_{L^1(0,\varepsilon/2;E)}\right]$$

$$+\,C\,\|\varphi\|_{C([\varepsilon/2,T],E)}\int_{\varepsilon/2}^{\tau}\log\left[1+\frac{t-\tau}{\tau-s}\right]ds\leqslant C(\varepsilon,\delta)(t-\tau)^{\delta}.$$

(iv) Fix $t>0$ and observe that

$$T\varphi(t)=\int_0^t e^{(t-s)A(t)}[\varphi(s)-\varphi(t)]\,ds+[e^{tA(t)}-1]\,A(t)^{-1}\,\varphi(t).$$

The fact that $T\varphi(t)\in D(A(t))$ is easily proved by observing that the integral in (3.7) is absolutely convergent since, choosing $\varepsilon<t$, we can write

$$\|A(t)\,e^{(t-s)A(t)}[\varphi(s)-\varphi(t)]\|_E \leqslant \begin{cases} \dfrac{C(\varepsilon)}{t} & \text{if} \quad s\in\,]0,\varepsilon], \\[2ex] \dfrac{C(\varepsilon)}{(t-s)^{1-\delta}} & \text{if} \quad s\in\,]\varepsilon,t[. \end{cases}$$

(v) Suppose $t>\tau\geqslant\varepsilon$. Then

$$\begin{aligned}
&\frac{T\varphi(t)-T\varphi(\tau)}{t-\tau}\\
&\quad=\frac{1}{t-\tau}\int_\tau^t e^{(t-s)A(t)}[\varphi(s)-\varphi(t)]\,ds+\frac{e^{(t-\tau)A(t)}-1}{t-\tau}A(t)^{-1}\,\varphi(t)\\
&\qquad+\int_0^\tau \frac{e^{(t-s)A(t)}-e^{(\tau-s)A(t)}}{t-\tau}\,[\varphi(s)-\varphi(\tau)]\,ds\\
&\qquad+\frac{e^{(t-\tau)A(t)}-1}{t-\tau}\,[e^{\tau A(t)}-1]\,A(t)^{-1}\,[\varphi(\tau)-\varphi(t)]\\
&\qquad+\frac{e^{(t-\tau)A(t)}-1}{t-\tau}\,[e^{\tau A(t)}-e^{tA(t)}]\,A(t)^{-1}\,\varphi(t)\\
&\qquad+\frac{e^{(t-\tau)A(t)}-1}{t-\tau}\,[e^{tA(t)}-1]\,A(t)^{-1}\,\varphi(t)\\
&\qquad+\int_0^\tau \frac{e^{(\tau-s)A(t)}-e^{(\tau-s)A(\tau)}}{t-\tau}\,\varphi(s)\,ds \triangleq \sum_{i=1}^{7} B_i.
\end{aligned}$$

Terms B_1, B_4 and B_5 on the right-hand side go to 0 as $t-\tau\to 0^+$; moreover, by summing terms B_2 and B_6 we check

$$\frac{e^{(t-\tau)A(t)}-1}{t-\tau}A(t)^{-1}\,e^{tA(t)}\varphi(t),$$

which converges to $e^{tA(t)}\varphi(t)$, as it is easily seen.

Term B_7, by Lebesgue's Theorem, goes to $\int_0^t P(t,s)\,\varphi(s)\,ds$.

Finally, term B_3 is equal to

$$\int_0^\tau \frac{1}{2\pi i}\int_\gamma \frac{e^{\lambda(t-s)}-e^{\lambda(\tau-s)}}{t-\tau}R(\lambda,A(t))[\varphi(s)-\varphi(\tau)]\,d\lambda\,ds;$$

now we have, as $t-\tau\to 0^+$,

$$\begin{aligned}
&\frac{e^{\lambda(t-s)}-e^{\lambda(\tau-s)}}{t-\tau}R(\lambda,A(t))[\varphi(s)-\varphi(\tau)]\\
&\qquad\to \lambda e^{\lambda(t-s)}R(\lambda,A(t))[\varphi(s)-\varphi(t)] \qquad \forall\lambda\in\gamma,\quad \forall s\in\,]0,t[,
\end{aligned}$$

and

$$\left\| \frac{e^{\lambda(t-s)} - e^{\lambda(\tau-s)}}{t-\tau} R(\lambda, A(t))[\varphi(s) - \varphi(\tau)] \right\|_E$$
$$\leqslant C e^{\operatorname{Re}\lambda(t-s)} \|\varphi(s) - \varphi(\tau)\|_E \triangleq \psi_\tau(\lambda),$$

where the (integrable) functions $\{\Psi_\tau(\lambda)\}_{\tau \in [\varepsilon, t]}$ satisfy, as $\tau \to t^-$,

$$\Psi_\tau(\lambda) \to \Psi_t(\lambda) \qquad \forall \lambda \in \gamma,$$
$$\int_\gamma \Psi_\tau(\lambda)\, d\lambda \to \int_\gamma \Psi_t(\lambda)\, d\lambda.$$

This implies that, as $\tau \to t^-$,

$$\frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda(t-s)} - e^{\lambda(\tau-s)}}{t-\tau} R(\lambda, A(t))[\varphi(s) - \varphi(\tau)]\, d\lambda$$
$$\to \frac{1}{2\pi i} \int_\gamma \lambda e^{\lambda(t-s)} R(\lambda, A(t))[\varphi(s) - \varphi(t)]\, d\lambda \qquad \forall s \in]0, t[.$$

A similar argument proves that, as $\tau \to t^-$,

$$\int_0^\tau \frac{1}{2\pi i} \int_\gamma \frac{e^{\lambda(t-s)} - e^{\lambda(\tau-s)}}{t-\tau} R(\lambda, A(t))[\varphi(s) - \varphi(\tau)]\, d\lambda\, ds$$
$$\to \int_0^t \frac{1}{2\pi i} \int_\gamma \lambda e^{\lambda(t-s)} R(\lambda, A(t))[\varphi(s) - \varphi(t)]\, d\lambda\, ds$$

and this proves that, as $\tau \to t^-$,

$$\frac{T\varphi(t) - T\varphi(\tau)}{t-\tau} \to \int_0^t A(t)\, e^{(t-s)A(t)}[\varphi(s) - \varphi(t)]\, ds + e^{tA(t)}\varphi(t)$$
$$+ \int_0^t P(t, s)\, \varphi(s)\, ds.$$

It remains to prove that the derivative of $T\varphi(t)$ is continuous at every $t > 0$. By Proposition 3.5(ii), $t \mapsto \int_0^t P(t, s)\, \varphi(s)\, ds$ is continuous; now we will show that the remaining terms are Hölder-continuous in $]0, T]$. Suppose $t > \tau \geqslant \varepsilon$. Then

$$\int_0^t A(t)\, e^{(t-s)A(t)}[\varphi(s) - \varphi(t)]\, ds - \int_0^\tau A(\tau)\, e^{(\tau-s)A(\tau)}[\varphi(s) - \varphi(\tau)]\, ds$$
$$= \int_\tau^t A(t)\, e^{(t-s)A(t)}[\varphi(s) - \varphi(t)]\, ds$$

$$+\int_0^\tau [A(t)\,e^{(t-s)A(t)} - A(\tau)\,e^{(t-s)A(\tau)}][\varphi(s)-\varphi(t)]\,ds$$
$$+\,[e^{tA(\tau)} - e^{\tau A(\tau)}][\varphi(\tau)-\varphi(t)]$$
$$+\int_0^\tau [A(\tau)\,e^{(t-s)A(\tau)} - A(\tau)\,e^{(\tau-s)A(\tau)}][\varphi(s)-\varphi(\tau)]\,ds,$$

which implies

$$\begin{aligned}
&\left\| \int_0^t A(t)\,e^{(t-s)A(t)}[\varphi(s)-\varphi(t)]\,ds - \int_0^\tau A(\tau)\,e^{(\tau-s)A(\tau)}[\varphi(s)-\varphi(\tau)]\,ds \right\|_E \\
&\quad\leqslant C(\varepsilon)(t-\tau)^\delta + C(t-\tau)\int_0^{\varepsilon/2} \frac{\|\varphi(s)-\varphi(t)\|_E}{(t-s)^{2-\alpha}}\,ds \\
&\qquad + C(\varepsilon)(t-\tau)\int_{\varepsilon/2}^{\tau} \frac{ds}{(t-s)^{2-\alpha-\delta}} \\
&\qquad + C\log\left(1+\frac{t-\tau}{\varepsilon}\right)\|\varphi(t)-\varphi(\tau)\|_E \\
&\qquad + C(t-\tau)\int_0^{\varepsilon/2} \frac{\|\varphi(s)-\varphi(\tau)\|_E}{(\tau-s)^2}\,ds + C(\varepsilon)\int_{\varepsilon/2}^{\tau}\int_{\tau-s}^{t-s} \frac{d\sigma}{\sigma^2}\,(\tau-s)^\delta\,ds \\
&\quad\leqslant C(\varepsilon)(t-\tau)^\delta. \qquad (3.11)
\end{aligned}$$

On the other hand

$$\begin{aligned}
&e^{tA(t)}\varphi(t) - e^{\tau A(\tau)}\varphi(\tau) \\
&\quad= e^{tA(t)}[\varphi(t)-\varphi(\tau)] + [e^{tA(t)} - e^{tA(\tau)}]\,\varphi(\tau) + [e^{tA(\tau)} - e^{\tau A(\tau)}]\,\varphi(\tau);
\end{aligned}$$

hence

$$\begin{aligned}
&\|e^{tA(t)}\varphi(t) - e^{\tau A(\tau)}\varphi(\tau)\|_E \\
&\quad\leqslant C(\varepsilon)(t-\tau)^\delta + C(t-\tau)[\varepsilon^{-1+\alpha} + \varepsilon^{-1}]\,\|\varphi\|_{C([\varepsilon,T],E)} \\
&\quad\leqslant C(\varepsilon)(t-\tau)^\delta, \qquad (3.12)
\end{aligned}$$

and (v) is proved.

(vi) It is a consequence of (3.11), (3.12) and Proposition 3.5(iv).

(vii) Suppose $t > \tau \geqslant 0$. As in (3.11) we have

$$\left\| \int_0^t A(t)\, e^{(t-s)A(t)} [\varphi(s) - \varphi(t)]\, ds - \int_0^\tau A(\tau)\, e^{(\tau-s)A(\tau)} [\varphi(s) - \varphi(\tau)]\, ds \right\|_E$$

$$\leqslant C(t-\tau)^\delta + C \int_0^\tau \frac{t-\tau}{(t-s)^{2-\alpha-\delta}}\, ds + C\, \|\varphi(t) - \varphi(\tau)\|_E$$

$$+ C \int_0^\tau \int_{\tau-s}^{t-s} \frac{d\sigma}{\sigma^2} (\tau - s)^\delta\, ds \leqslant C(t-\tau)^\delta,$$

and as in (3.12) we get

$$\|e^{tA(t)}\varphi(t) - e^{\tau A(\tau)}\varphi(\tau)\|_E$$

$$\leqslant C\, \|\varphi(t) - \varphi(\tau)\|_E + C \left[\frac{(t-\tau)\,\tau^\delta}{t^{1-\alpha}} + \int_\tau^t \frac{d\sigma}{\sigma} \cdot \tau^\delta \right] \leqslant C(t-\tau)^\delta.$$

By Proposition 3.5(v) and (iii) the proof is complete.

4. Classical Solutions

In Section 2 we have shown under Hypotheses I and II the uniqueness of the classical solution of Problem (P). We will prove now that under Hypotheses I, II, III a classical solution of (P) does exist, and can be represented by formula (F), provided $x \in \overline{D(A(0))}$ and f is Hölder-continuous in $]0, T]$.

Theorem 4.1. *Under Hypotheses* I, II, III *suppose* $x \in \overline{D(A(0))}$ *and* $f \in C([0,T], E) \cap C^\sigma(]0,T], E)$, $\sigma \in]0,1]$; *then the vector-valued function* $u(t)$ *defined by*

$$u(t) = e^{tA(t)}x + \int_0^t e^{(t-s)A(t)} [(1+P)^{-1} (f - P(\cdot, 0)\, x)]\, (s)\, ds, \qquad t \in [0, T], \tag{4.1}$$

is the unique classical solution of (P). *Moreover,* $u \in C^{1,\sigma\wedge\delta}(]0,T], E)$ $\forall \delta \in]0, \eta[\cap]0, \alpha]$.

Proof. First, we observe that $u \in C([0,T], E)$ by Propositions 3.4(ii), 3.3(i), 3.6(i) and (ii), 3.7(i); in particular $u(0) = x$. Next, by Propositions 3.4(i), 3.3(iv), 3.6(iii) and 3.7(iv), we have $u(t) \in D(A(t))$ $\forall t \in]0, T]$. It remains to show that

$$u \in C^{1,\sigma\wedge\delta}(]0,T], E) \qquad \forall \delta \in]0, \eta[\cap]0, \alpha]$$

and

$$u'(t) - A(t)\, u(t) = f(t) \qquad \forall t \in \,]0, T].$$

Propositions 3.4(i) and (vi) imply that $e^{tA(t)}x \in C^{1,\eta}(]0, T], E)$ and

$$\frac{d}{dt} e^{tA(t)}x = A(t)\, e^{tA(t)}x + P(t, 0)\, x. \tag{4.2}$$

Define

$$g = (1 + P)^{-1}\, (f - P(\cdot, 0)\, x),$$

and observe that $f - P(\cdot, 0)\, x \in L^1(0, T; E) \cap C^{\sigma \wedge \eta}(]0, T], E)$ by Proposition 3.3(iv), so that Proposition 3.6(iii) yields $g \in L^1(0, T; E) \cap C^{\sigma \wedge \delta}(]0, T], E)$ $\forall \delta \in 0, \eta[\, \cap \,]0, \alpha]$. Thus, by Proposition 3.7(vi) $Tg \in C^{1, \sigma \wedge \delta}(]0, T], E)$ $\forall \delta \in \,]0, \eta[\, \cap \,]0, \alpha]$ and

$$\frac{d}{dt} Tg(t) = \int_0^t A(t)\, e^{(t-s)A(t)}[\, g(s) - g(t)]\, ds + e^{tA(t)} g(t)$$
$$+ \int_0^t P(t, s)\, g(s)\, ds. \tag{4.3}$$

By (4.2), (4.3) and (3.7) we deduce

$$u'(t) - A(t)\, u(t) = P(t, 0)\, x + \int_0^t P(t, s)\, g(s)\, ds + g(t) \qquad \forall t \in \,]0, T],$$

and the proof is complete since $(1 + P)\, g = f - P(\cdot, 0)\, x$.

Remark 4.2. If in Theorem 4.1 we suppose $x \in \overline{D(A(0))}$ and $f \in L^1(0, T; E) \cap C^{\sigma}(]0, T], E)$ only, then the function $u(t)$ defined by (4.1) is still the unique classical solution of Problem (P), and all properties stated in Theorem 4.1 still hold (with the same proof).

Remark 4.3. In Theorem 4.1 it suffices to suppose that $f \in C([0, T], E)$ and there exists $t_0 \in \,]0, T]$ such that the oscillation $\omega(\cdot)$ of f satisfies

$$\int_0^{t_0} \frac{\omega(\tau)}{\tau}\, d\tau < +\infty. \tag{4.4}$$

This assumption, together with $x \in \overline{D(A(0))}$, still guarantees that (4.1) is the unique classical solution of Problem (P) (but not, of course, its Hölder regularity). We omit the proof, which is similar to the previous one; in particular condition (4.4) assures the absolute convergence of all Dunford integrals involved. This generalizes a result of Crandall and Pazy [5] relative to the case $A(t) = A =$ constant.

We note that the only assumption $f \in C([0, T], E)$ is not sufficient to guarantee the existence of the classical solution, even in the case $A(t) = A$ and $x = 0$. If, for instance, E is reflexive and $A(t) = A$ is unbounded, then a continuous f does exist, such that Problem (P) has no classical solutions (see Baillon [3] and Travis [35]; see also Da Prato and Grisvard [7]).

5. Strict Solutions and Maximal Regularity

In Section 2 we have shown under Hypotheses I and II the uniqueness of the strict solution of Problem (P); moreover, we know that condition (2.2) is necessary for the existence of such a solution.

In this section we will show that under Hypotheses I, II, III condition (2.2) is also sufficient for the existence of a strict solution of Problem (P), provided f is Hölder-continuous in $[0, T]$.

Theorem 5.1. *Under Hypotheses* I, II, III *let* $x \in D(A(0))$ *and* $f \in C^{\sigma}([0, T], E)$, $\sigma \in]0, 1]$, *and suppose that* x *and* $f(0)$ *verify*

$$A(0)\,x + f(0) - \left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} A(0)\,x \in \overline{D(A(0))}. \tag{5.1}$$

Then the vector-valued function $u(t)$ *defined by* (4.1) *is the unique strict solution of Problem* (P). *Moreover,* $u \in C^{1, \delta \wedge \sigma}(]0, T], E)$ $\forall \delta \in]0, \eta[\cap]0, \alpha]$.

Proof. By Theorem 4.1 $u(t)$ is the unique classical solution of Problem (P), so it is enough to prove that $u'(t)$ exists at $t = 0$ and $u' \in C([0, T], E)$, for this will also imply that $A(t)\,u(t) \in C([0, T], E)$ and

$$u'(t) - A(t)\,u(t) = f(t) \qquad \forall t \in [0, T].$$

Put

$$g = (1 + P)^{-1}\,(f - P(\cdot, 0)\,x),$$

then, by Proposition 3.4(v), as $t \to 0^+$,

$$\begin{aligned}\frac{u(t) - x}{t} &= \frac{e^{tA(t)} - 1}{t} x + \frac{1}{t} \int_0^t e^{(t-s)A(t)} g(s)\,ds \\ &= o(1) + \frac{e^{tA(0)} - 1}{t} x - [e^{tA(0)} - 1]\left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} A(0)\,x + \frac{1}{t} Tg(t).\end{aligned} \tag{5.2}$$

Since $f \in C^{\sigma}([0,T],E)$, by Propositions 3.5(iv) and 3.3(iii) we check, as $t \to 0^+$,

$$
\begin{aligned}
\frac{1}{t}\,Tg(t) &= \frac{1}{t}\,T(-Pg+f-f(0)-P(\cdot,0)\,x)(t) + \frac{e^{tA(t)}-1}{t}A(t)^{-1}f(0) \\
&= O(t^{\delta}) + O(t^{\sigma}) + \frac{1}{t}\int_0^t [e^{(t-s)A(t)} - e^{(t-s)A(0)}][-P(s,0)\,x]\,ds + O(t^{\alpha}) \\
&\quad + \frac{1}{t}\int_0^t [e^{tA(0)} - e^{(t-s)A(0)}]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x\,ds + O(t^{\alpha}) \\
&\quad + \frac{e^{tA(0)}-1}{t}A(0)^{-1}f(0) \\
&= O(t^{\delta\wedge\sigma}) + e^{tA(0)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x \\
&\quad + \frac{e^{tA(0)}-1}{t}A(0)^{-1}\left[f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x\right] \\
&\qquad\qquad \forall \delta \in\,]0,\eta[\, \cap\,]0,\alpha].
\end{aligned}
$$

By (5.1) and (5.2) we deduce that

$$\exists u'(0) = A(0)\,x + f(0). \tag{5.3}$$

On the other hand we recall that

$$
\begin{aligned}
u'(t) &= A(t)\,e^{tA(t)}x + P(t,0)\,x + \int_0^t A(t)\,e^{(t-s)A(t)}[g(s)-g(t)]\,ds \\
&\quad + e^{tA(t)}g(t) + \int_0^t P(t,s)\,g(s)\,ds \qquad \forall t \in\,]0,T];
\end{aligned}
$$

now Proposition 3.4(v) implies

$$
\begin{aligned}
A(t)\,e^{tA(t)}x &= o(1) + e^{tA(0)}A(0)\,x \\
&\quad - tA(0)\,e^{tA(0)}\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x \\
&\quad \text{as} \quad t \to 0^+,
\end{aligned} \tag{5.4}
$$

while Proposition 3.3(iii) yields

$$\begin{aligned}P(t,0)\,x &= o(1) - [e^{tA(0)} - 1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x \\ &= o(1) - [e^{tA(t)} - 1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x \qquad \text{as} \quad t \to 0^+.\end{aligned} \tag{5.5}$$

Moreover, recalling Propositions 3.3(iv) and 3.5(v) we have, $\forall \delta \in\,]0, \eta[\, \cap\,]0, \alpha]$, as $t \to 0^+$,

$$\begin{aligned}&\int_0^t A(t)\,e^{(t-s)A(t)}[g(s) - g(t)]\,ds \\ &\quad = \int_0^t A(t)\,e^{(t-s)A(t)}[-Pg(s) + Pg(t) + f(s) - f(t) \\ &\qquad - P(s,0)\,x + P(t,0)\,x]\,ds \\ &\quad = O(t^\delta) + O(t^\sigma) + \int_0^t [A(t)\,e^{(t-s)A(t)} - A(s)\,e^{(t-s)A(s)}] \\ &\qquad \times [-P(s,0)\,x + P(t,0)\,x]\,ds + \int_0^t A(s)\,e^{(t-s)A(s)}\Bigg[O((t-s)^{\eta \wedge \alpha}) \\ &\qquad + [e^{sA(s)} - e^{tA(s)}]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x\Bigg]\,ds \\ &\quad = O(t^{\delta \wedge \sigma}) + \int_0^t [A(s)\,e^{tA(s)}(1 - e^{(t-s)A(s)}) - A(0)\,e^{tA(0)}(1 - e^{(t-s)A(0)}) \\ &\qquad + A(0)\,e^{tA(0)}(1 - e^{(t-s)A(0)})]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x\,ds,\end{aligned}$$

so that

$$\begin{aligned}&\int_0^t A(t)\,e^{(t-s)A(t)}[g(s) - g(t)]\,ds \\ &\quad = o(1) + [tA(0)\,e^{tA(0)} - e^{tA(0)}(e^{tA(0)} - 1)]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x \\ &\qquad \text{as} \quad t \to 0^+.\end{aligned} \tag{5.6}$$

Next, by Proposition 3.3(iii) we have, $\forall\delta\in\,]0,\eta[\,\cap\,]0,\alpha]$,

$$e^{tA(t)}g(t)=e^{tA(t)}[-Pg(t)+f(t)-f(0)-P(t,0)\,x]+e^{tA(t)}f(0)$$
$$=o(1)+e^{tA(0)}f(0)+e^{tA(0)}[e^{tA(0)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x$$
$$\text{as}\quad t\to 0^+;$$

hence

$$e^{tA(t)}g(t)=o(1)+e^{tA(0)}f(0)+e^{tA(0)}[e^{tA(0)}-1]\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x$$
$$\text{as}\quad t\to 0^+. \tag{5.7}$$

Finally, since $g\in L^\infty(0,T;E)$ by Proposition 3.3(iii), estimate (1.3) yields

$$\int_0^t P(t,s)\,g(s)\,ds=O(t^\alpha)\qquad\text{as}\quad t\to 0^+. \tag{5.8}$$

Now (5,4), (5.5), (5.6), (5.7) and (5.8) imply

$$u'(t)=o(1)+e^{tA(0)}\left[A(0)\,x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x\right]$$
$$+\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x\qquad\text{as}\quad t\to 0^+,$$

and the result follows by (5.3).

Remark 5.2. As in the case of classical solutions, Theorem 5.1 still holds (except for the Hölder regularity of $u'(t)$) assuming $f\in C([0,T],E)$ and condition (4.4) instead of Hölder continuity for f.

About maximal regularity of the strict solutions we have the following result:

THEOREM 5.3. *Under Hypotheses* I, II, III *let* $x\in D(A(0))$ *and* $f\in C^\delta([0,T],E)$, $\delta\in\,]0,\eta[\,\cap\,]0,\alpha]$, *and suppose u is a strict solution of Problem* (P). *Then* $u\in C^{1,\delta}([0,T],E)$ *if and only if the vectors x and $f(0)$ verify the following condition:*

$$A(0)\,x+f(0)-\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}A(0)\,x\in D_{A(0)}(\delta,\infty). \tag{5.9}$$

Proof. Consider the following problem

$$z'(t) - A(t)\,z(t) = A(0)\,x + f(0) - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x, \qquad t \in [0, T],$$

$$z(0) = 0$$

which has a unique strict solution z(t) by Theorem 5.1; $z(t)$ can be represented by

$$z(t) = \int_0^t e^{(t-s)A(t)} h(s)\,ds, \qquad t \in [0, T], \tag{5.10}$$

where $h(t)$ is the solution of the integral equation

$$h(t) + \int_0^t P(t, s)\,h(s)\,ds = f(0) + A(0)\,x - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)\,x. \tag{5.11}$$

Now define

$$w(t) \triangleq u(t) - A(t)^{-1} A(0)\,x - z(t); \tag{5.12}$$

we claim that $w \in C^{1,\sigma}([0, T], E)$ for any $\sigma \in\,]0, \eta[\, \cap\,]0, \alpha]$. Indeed, $w(t)$ is the strict solution of

$$w'(t) - A(t)\,w(t) = f(t) - f(0) - \left[\frac{d}{dt}A(t)^{-1} - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}\right] A(0)\,x,$$
$$t \in [0, T],$$

$$w(0) = 0$$

and can be represented by

$$w(t) = \int_0^t e^{(t-s)A(t)} k(s)\,ds, \qquad t \in [0, T],$$

where

$$k(t) + \int_0^t P(t, s)\,k(s)\,ds$$
$$= f(t) - f(0) - \left[\frac{d}{dt}A(t)^{-1} - \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}\right] A(0)\,x;$$

thus by Proposition 3.6(iv) we have $k \in C^{\sigma}([0, T], E)$ for any $\sigma \in\,]0, \eta[\, \cap$

$]0, \alpha]$ and, in particular, $k(0)=0$. Hence our claim is proved by Proposition 3.7(vii).

Then, by (5.12) we deduce that $u \in C^{1,\delta}([0, T], E)$ if and only if $z \in C^{1,\delta}([0, T], E)$. Newt, by (5.10) and Proposition 3.7(v), we have

$$\begin{aligned} z'(t) = &\int_0^t A(t)\, e^{(t-s)A(t)}[(h(s)-h(0))-(h(t)-h(0))]\, ds \\ &+ e^{tA(t)}[h(t)-h(0)] + e^{tA(t)}h(0) \\ &+ \int_0^t P(t,s)[h(s)-h(0)]\, ds + \int_0^t P(t,s)\, h(0)\, ds \\ = &\frac{d}{dt}[T(h(t)-h(0))] + e^{tA(t)}h(0) + P(h(0)) \qquad \forall t \in [0, T]. \end{aligned}$$

Since, by (5.11) and Proposition 3.6(iv), $h \in C^{\sigma}([0, T], E)$ $\forall \sigma \in]0, \eta[\cap]0, \alpha]$, we deduce, by Propositions 3.7(vii) and 3.5(v), that $z \in C^{1,\delta}([0, T], E)$ if and only if $e^{tA(t)}h(0) \in C^{\delta}([0, T], E)$. Therefore, by Proposition 3.4(iii) we conclude that $u \in C^{1,\delta}([0, T], E)$ if and only if $h(0) \in D_{A(0)}(\delta, \infty)$.

The proof is complete, since

$$h(0) = A(0)\, x + f(0) - \left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} A(0)\, x.$$

6. Strong Solutions

We know from Section 2 that under Hypotheses I and II there is at most one strong solution of Problem (P), and a necessary condition for the existence of such a solution is the following:

$$\begin{gathered} \exists \{x_n\}_{n\in\mathbf{N}} \subseteq D(A(0)) \text{ and } \{y_n\}_{n\in\mathbf{N}} \subseteq E \text{ such that:} \\ x_n \to x \quad \text{in } E, \qquad y_n \to f(0) \quad \text{in } E, \\ A(0)\, x_n + y_n - \left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} A(0)\, x_n \in \overline{D(A(0))}. \end{gathered} \tag{6.1}$$

In this section we will prove that condition (6.1) is also sufficient for the existence of a strong solution.

Condition (6.1) seems somewhat involved: thus we will see some simpler conditions which imply (6.1), being therefore sufficient (but not necessary) for the existence of a strong solution. First of all we need the following lemma.

LEMMA 6.1. *Under Hypotheses* I, II *let* $x \in D(A(0))$ *and* $g \in C^{\delta}([0, T], E)$, $\delta \in]0, 1[$; *define*

$$z \triangleq \left[\frac{d}{dt} A(t)^{-1}\right]_{t=0} A(0)\, x$$

$$v(t) \triangleq e^{tA(t)} x + \int_0^t e^{(t-s)A(t)} [g(s) + e^{sA(s)} z]\, ds, \qquad t \in [0, T].$$

Then $v \in D$ *(see the Introduction) if and only if* $A(0)\, x + g(0) \in \overline{D(A(0))}$; *if this is the case, then we have*

$$\begin{aligned} &v'(t) - A(t)\, v(t) \\ &\quad = P(t, 0)\, x + e^{tA(t)} z + g(t) + \int_0^t P(t, s)[g(s) + e^{sA(s)} z]\, ds \\ &\quad \in C([0, T], E), \qquad t \in [0, T], \\ &v(0) = x \in D(A(0)). \end{aligned} \tag{6.2}$$

Proof. Obviously $v \in C([0, T], E)$, by Propositions 3.4(ii) and 3.7(i), with $v(0) = x$. Moreover, by Propositions 3.4(i) and 3.7(v) we have $v \in C^1(]0, T], E)$ and

$$\begin{aligned} v'(t) = {} & A(t)\, e^{tA(t)} x + P(t, 0)\, x + \int_0^t A(t)\, e^{(t-s)A(t)} [g(s) - g(t)]\, ds \\ & + e^{tA(t)} g(t) + \int_0^t P(t, s)\, g(s)\, ds \\ & + \int_0^t A(t)\, e^{(t-s)A(t)} [e^{sA(s)} - e^{sA(t)} + e^{sA(t)} - e^{tA(t)}]\, z\, ds \\ & + e^{2tA(t)} z + \int_0^t P(t, s)\, e^{sA(s)} z\, ds, \qquad t \in]0, T]. \end{aligned}$$

Hence Propositions 3.4(v) and 3.3(iii) yield, as $t \to 0^+$,

$$\begin{aligned} v'(t) &= o(1) + e^{tA(t)} A(0)\, x - tA(t)\, e^{tA(t)} z - [e^{tA(t)} - 1]\, z + e^{tA(t)} g(0) \\ &\quad + tA(t)\, e^{tA(t)} z - [[e^{tA(t)} - e^{2tA(t)}]\, z + e^{2tA(t)} z] \\ &= z + o(1) + e^{tA(0)} [A(0)\, x + g(0)]. \end{aligned}$$

Similarly, Proposition 3.4(v) implies, as $t \to 0^+$,

$$\frac{v(t)-x}{t} = o(1) + \frac{e^{tA(t)}-1}{t} x + \frac{e^{tA(t)}-1}{t} A(t)^{-1} g(0) + e^{tA(t)} z$$

$$= z + o(1) + \frac{e^{tA(0)}-1}{t} A(0)^{-1} [A(0)x + g(0)].$$

This shows that $v \in C^1([0,T],E)$ if and only if $A(0)x + g(0) \in \overline{D(A(0))}$. Next, we have for each $t \in \,]0,T]$,

$$A(t)v(t) = A(t)e^{tA(t)}x + \int_0^t A(t)e^{(t-s)A(t)}[g(s)-g(t)]\,ds$$

$$+ [e^{tA(t)}-1]g(t) + \int_0^t A(t)e^{(t-s)A(t)}[e^{sA(s)} - e^{sA(t)}$$

$$+ e^{sA(t)} - e^{tA(t)}]z\,ds + [e^{2tA(t)} - e^{tA(t)}]z,$$

which implies, as $t \to 0^+$,

$$A(t)v(t) = -g(0) + o(1) + e^{tA(0)}[A(0)x + g(0)].$$

Hence $A(t)v(t) \in C([0,T],E)$ if and only if $A(0)x + g(0) + \overline{D(A(0))}$. Finally, it is clear that if $A(0)x + g(0) \in \overline{D(A(0))}$, then $v'(t)$ and $A(t)v(t)$ are in $C([0,T],E)$ and

$$v'(t) - A(t)v(t) = P(t,0)x + e^{tA(t)}z + g(t)$$

$$+ \int_0^t P(t,s)[g(s) + e^{sA(s)}z]\,ds, \qquad t \in [0,T].$$

Observe that the continuity of the right-hand side also follows directly by Propositions 3.3(iii) and 3.5(iii). The proof is complete.

THEOREM 6.2. *Under Hypotheses* I, II, *let* $x \in \overline{D(A(0))}$ *and* $f \in C([0,T],E)$, *and suppose that* x *and* $f(0)$ *verify* (6.1). *Then the vector-valued function* $u(t)$ *defined by* (4.1) *is the unique strong solution of Problem* (P).

Proof. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq D(A(0))$ and $\{y_n\}_{n\in\mathbb{N}} \subseteq E$ the sequences appearing in (6.1). Define

$$z_n \triangleq \left[\frac{d}{dt}A(t)^{-1}\right]_{t=0} A(0)x_n.$$

Consider the functions

$$\varphi_n(t) = [1+P]^{-1}\Bigg[-P(t,0)\,x_n - e^{tA(t)}z_n - \int_0^t P(t,s)[e^{sA(s)}z_n]\,ds + y_n - f(0) + f(t)\Bigg], \qquad n \in \mathbf{N}.$$

By Propositions 3.6(i), 3.3(iii) and 3.5(iii), $\varphi_n \in C([0,T],E)$. Since $C^\delta([0,T],E)$ is a dense subspace of $C([0,T],E)$, for any $n \in \mathbf{N}$ there exists $g_n \in C^\delta([0,T],E)$ such that

$$\|g_n - \varphi_n\|_{C([0,T],E)} \leqslant 1/n, \qquad g_n(0) = \varphi_n(0) = y_n - z_n.$$

Put

$$v_n(t) \triangleq e^{tA(t)}x_n + \int_0^t e^{(t-s)A(t)}[g_n(s) + e^{sA(s)}z_n]\,ds;$$

by Lemma 6.1 v_n is the strict solution of the problem

$$\begin{aligned} v_n'(t) - A(t)\,v_n(t) &= [(1+P)\,g_n](t) + P(t,0)\,x_n + e^{tA(t)}z_n \\ &\quad + \int_0^t P(t,s)\,e^{sA(s)}z_n\,ds, \qquad t \in [0,T], \\ v_n(0) &= x_n. \end{aligned} \tag{6.3}$$

Now as $n \to +\infty$ we have

$$\begin{aligned} &[(1+P)\,g_n](t) + P(t,0)\,x_n + e^{tA(t)}z_n + \int_0^t P(t,s)\,e^{sA(s)}z_n\,ds \\ &\quad = [(1+P)(g_n - \varphi_n)](t) + y_n - f(0) + f(t) \to f(t) \qquad \text{in} \quad C([0,T],E); \\ &\qquad\qquad x_n \to x \qquad \text{in} \quad E. \end{aligned} \tag{6.4}$$

On the other hand it is easily seen that

$$\begin{aligned} &g_n(s) + e^{sA(s)}z_n \\ &\quad = g_n(s) + [1+P]^{-1}\,[1+P]\,e^{sA(s)}z_n \\ &\quad = (g_n - \varphi_n)(s) + [1+P]^{-1}\,(y_n - f(0) + f(s) - P(s,0)\,x_n), \qquad s \in [0,T]; \end{aligned}$$

therefore, as $n \to +\infty$,

$$g_n(s) + e^{sA(s)}z_n \to (1+P)^{-1}\,[f(s) - P(s,0)\,x] \qquad \text{in} \quad L^1(0,T;E),$$

which implies, as $n \to +\infty$,

$$v_n(t) \to e^{tA(t)}x + \int_0^t e^{(t-s)A(t)}[(1+P)^{-1}(f - P(\cdot, 0)x)](s)\, ds$$
$$\text{in} \quad C([0, T], E). \tag{6.5}$$

By (6.5), (6.3) and (6.4) we conclude that u is the strong solution of Problem (P).

About regularity of strong solutions, we have the following result:

THEOREM 6.3. *Under Hypotheses* I, II, *let* $x \in \overline{D(A(0))}$ *and* $f \in C([0, T], E)$, *and suppose* u *is a strong solution of Problem* (P). *Then we have:*

(i) $u \in C^\delta(]0, T], E)$ *for any* $\delta \in]0, 1[$;

(ii) *if* $\beta \in]0, \alpha[$, *then* $u \in C^\beta([0, T], E)$ *if and only if* $x \in D_{A(0)}(\beta, \infty)$;

(iii) *if* $\beta \in [\alpha, 1[$, *and* $x \in D_{A(0)}(\beta, \infty)$, *then* $u \in C^\delta([0, T], E)$ *for any* $\delta \in]0, \beta[$.

Proof. (i) It is a consequence of Propositions 3.4(i), 3.3(i), 3.6(i) and 3.7(iii).

(ii) If $x \in D_{A(0)}(\beta, \infty)$, then by Propositions 3.4(iii), 3.3(i), 3.6(i) and 3.7(ii) $u \in C^\beta([0, T], E)$. Suppose conversely $u \in C^\beta([0, T], E)$; then Propositions 3.3(i), 3.6(i) and 3.7(ii) imply that

$$\int_0^t e^{(t-s)A(t)}[(1+P)^{-1}(f - P(\cdot, 0)x)](s)\, ds \in C^\delta([0, T], E) \qquad \forall \delta \in]0, \alpha[;$$

hence, by the representation formula (4.1) one deduces that

$$e^{tA(t)}x \in C^\beta([0, T], E),$$

and Proposition 3.4(iv) implies $x \in D_{A(0)}(\beta, \infty)$.

(iii) It follows by Propositions 3.4(iii), 3.3(ii), 3.6(i) and 3.7(ii).

Remark 6.4. Here are some conditions, simpler than (6.1), which are sufficient, but not necessary, for the existence of a strong solutions of Problem (P).

Obviously (6.1) is true when $D(A(0))$ is dense in E; thus in this case the strong solution of Problem (P) always exists whenever $x \in E$ and $f \in C([0, T], E)$.

More generally, (6.1) holds for any $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$, provided

$$R\left(\left[\frac{d}{dt}A(t)^{-1}\right]_{t=0}\right) \subseteq \overline{D(A(0))}; \tag{6.6}$$

indeed, if (6.6) holds, then

$$x + A(0)^{-1}f(0) \in \overline{D(A(0))} = \overline{D(A(0)^2)},$$

hence there exists $\{w_n\}_{n\in\mathbf{N}} \subseteq D(A(0)^2)$ such that $w_n \to x + A(0)^{-1}f(0)$ in E as $n \to \infty$. Defining

$$x_n = w_n - A(0)^{-1}f(0), \qquad y_n = f(0) \qquad \forall n \in \mathbf{N},$$

it is easy to verify that (6.1) is true.

Finally, we observe that (6.6) is obviously true if there exists $\{t_n\}_{n\in\mathbf{N}} \subseteq [0, T]$ such that

$$t_n \to 0^+, \qquad D(A(t_n)) \subseteq \overline{D(A(0))} \qquad \forall n \in \mathbf{N}; \tag{6.7}$$

for instance, this is the case when $D(A(t))$ does not depend on t. Hence (6.7) is also a sufficient condition for (6.1) to hold whenever $x \in \overline{D(A(0))}$ and $f \in C([0, T], E)$.

Remark 6.5. A classical solution of Problem (P) is not necessarily a strong solution. Indeed, suppose that the hypotheses of Theorem 4.1 hold, but condition (6.1) is not true; then the classical solution of Problem (P) does exist, but it is not a strong solution, for if it were, then by Theorem 2.6 condition (6.1) would also hold: a contradiction.

7. Examples

(a) *First Example*

Set $E = C[0, 1]$, $\|u\|_E = \sup_{x\in[0,1]} |u(x)|$, and define for each $t \in [0, T]$:

$$\begin{aligned} &D(A(t)) = \{u \in C^2[0, 1] : u(0) = 0, \alpha(t)\, u(1) + \beta(t)\, u'(1) = 0\} \\ &A(t)\, u = u'' \end{aligned} \tag{7.1}$$

$\alpha(\cdot)$ and $\beta(\cdot)$ being two real functions in $C^1[0, T]$ such that

$$\alpha(t) \geqslant 0, \qquad \beta(t) \geqslant 0, \qquad \inf_{t\in[0,T]} (\alpha + \beta) > 0. \tag{7.2}$$

PROPOSITION 7.1. *We have*

$$\text{(i)}\quad \overline{D(A(t))} = \begin{cases} \{u \in C[0,1] : u(0) = u(1) = 0\} & \text{if } \beta(t) = 0, \\ \{u \in C[0,1] : u(0) = 0\} & \text{if } \beta(t) \neq 0. \end{cases}$$

In particular, $D(A(t))$ is never dense in E.

(ii) $\sigma(A(t)) \subseteq]-\infty, 0]$; *moreover, if* $0 < \theta < \pi$ *and* $\lambda \in \Sigma_\theta \triangleq \{\lambda \in \mathbf{C} - \{0\} : |\arg \lambda| \leqslant \theta\}$, *then* $\lambda \in \rho(A(t))$ *and*

$$\|R(\lambda, A(t))\|_{\mathscr{L}(E)} \leqslant M/|\lambda|.$$

Proof. (i) Obvious.

(ii) If $\lambda \in \Sigma_\theta$ with $|\theta| < \pi$, then the problem

$$\lambda u - u'' = f \in E, \qquad x \in [0, 1],$$
$$u(0) = 0$$
$$\alpha(t)\, u(1) + \beta(t)\, u'(1) = 0$$

has the unique solution

$$u(x, t) = \int_0^1 K_t(x, \tau) f(\tau)\, d\tau,$$

where (assuming $\operatorname{Re} \sqrt{\lambda} \geqslant 0$)

$$K_t(x, \tau) = \begin{cases} \dfrac{sh \sqrt{\lambda}\,\tau}{\sqrt{\lambda}} \dfrac{\alpha(t)\, sh \sqrt{\lambda}(1-x) + \sqrt{\lambda} \cdot \beta(t)\, ch \sqrt{\lambda}(1-x)}{\alpha(t)\, sh \sqrt{\lambda} + \sqrt{\lambda}\, \beta(t)\, ch \sqrt{\lambda}} & \text{if } \tau \leqslant x, \\[2ex] \dfrac{sh \sqrt{\lambda}\, x}{\sqrt{\lambda}} \dfrac{\alpha(t)\, sh \sqrt{\lambda}(1-\tau) + \sqrt{\lambda}\, \beta(t)\, ch \sqrt{\lambda}(1-\tau)}{\alpha(t)\, sh \sqrt{\lambda} + \sqrt{\lambda}\, \beta(t)\, ch \sqrt{\lambda}} & \text{if } \tau \geqslant x. \end{cases}$$

Then

$$\|R(\lambda, A(t)) f\|_E \leqslant \|f\|_E \cdot \sup_{x \in [0,1]} \int_0^1 |K_t(x, \tau)|\, d\tau. \tag{7.3}$$

Setting $\rho = \operatorname{Re}\sqrt{\lambda}$, $\sigma = \operatorname{Im}\sqrt{\lambda}$, we have $\rho > 0$ and

$$|K_t(x,\tau)| \leqslant \begin{cases} \dfrac{ch\rho\tau}{|\sqrt{\lambda}|} \cdot ch\rho(1-x) \dfrac{\alpha(t) + |\sqrt{\lambda}| \cdot \beta(t)}{|\alpha(t)\, sh\sqrt{\lambda} + \sqrt{\lambda} \cdot \beta(t)\, ch\sqrt{\lambda}|} & \text{if} \quad \tau \leqslant x, \\ \dfrac{ch\rho x}{|\sqrt{\lambda}|} ch\rho(1-\tau) \dfrac{\alpha(t) + |\sqrt{\lambda}|\beta(t)}{|\alpha(t)\, sh\sqrt{\lambda} + \sqrt{\lambda}\,\beta(t)\, ch\sqrt{\lambda}|} & \text{if} \quad \tau \geqslant x; \end{cases}$$

hence

$$\begin{aligned} \int_0^1 |K_t(x,\tau)|\, d\tau &\leqslant [sh\rho x ch\rho(1-x) + ch\rho x sh\rho(1-x)] \\ &\quad \times \frac{\alpha(t) + |\sqrt{\lambda}| \cdot \beta(t)}{\rho|\sqrt{\lambda}| \cdot |\alpha(t)\, sh\sqrt{\lambda} + \sqrt{\lambda}\,\beta(t)\, ch\sqrt{\lambda}|} \\ &= \frac{sh\rho}{\rho|\sqrt{\lambda}|} \frac{\alpha(t) + |\sqrt{\lambda}|\beta(t)}{|\alpha(t)\, sh\sqrt{\lambda} + \sqrt{\lambda}\,\beta(t)\, ch\sqrt{\lambda}|}. \end{aligned}$$

On the other hand a direct calculation shows that

$$\begin{aligned} &|\alpha(t)\, sh\sqrt{\lambda} + \sqrt{\lambda}\,\beta(t)\, ch\sqrt{\lambda}| \\ &\quad = \left| \frac{e^\rho}{2} \{[(\alpha+\rho\beta)\cos\sigma - \sigma\beta\sin\sigma] + i[(\alpha+\rho\beta)\sin\sigma + \sigma\beta\cos\sigma]\} \right. \\ &\qquad \left. + \frac{e^{-\rho}}{2} \{[(\rho\beta-\alpha)\cos\sigma + \sigma\beta\sin\sigma] + i[(\alpha-\rho\beta)\sin\sigma + \sigma\beta\cos\sigma]\} \right| \\ &\quad \geqslant \left| \frac{e^\rho}{2} \{[(\alpha+\rho\beta)\cos\sigma - \sigma\beta\sin\sigma] + i[(\alpha+\rho\beta)\sin\sigma + \sigma\beta\cos\sigma]\} \right| \\ &\qquad - \left| \frac{e^{-\rho}}{2} \{[(\rho\beta-\alpha)\cos\sigma + \sigma\beta\sin\sigma] + i[(\alpha-\rho\beta)\sin\sigma + \sigma\beta\cos\sigma]\} \right| \\ &\quad = \frac{e^\rho}{2} [(\alpha+\rho\beta)^2 + \sigma^2\beta^2]^{1/2} - \frac{e^{-\rho}}{2} [(\alpha-\rho\beta)^2 + \sigma^2\beta^2]^{1/2} \\ &\quad \geqslant sh\rho[\alpha^2 + \rho^2\beta^2 + 2\rho\alpha\beta]^{1/2}, \end{aligned}$$

which implies

$$|\alpha(t)\, sh\sqrt{\lambda} + \sqrt{\lambda}\,\beta(t)\, ch\sqrt{\lambda}| \geqslant sh\rho[\alpha(t) + \rho\beta(t)], \tag{7.4}$$

and consequently

$$\sup_{x\in[0,1]} \int_0^1 |K_t(x,\tau)|\, d\tau \leqslant \frac{sh\rho}{\rho\,|\sqrt{\lambda}\,|} \frac{\alpha(t)+|\sqrt{\lambda}\,|\,\beta(t)}{sh\rho[\alpha(t)+\rho\beta(t)]}. \tag{7.5}$$

Since $|\arg\sqrt{\lambda}\,| \leqslant \theta/2 < \pi/2$, we have

$$0<\rho\leqslant|\sqrt{\lambda}\,|\leqslant\left[1+tg^2\frac{\theta}{2}\right]^{1/2}\cdot\rho \qquad \forall\lambda\in\Sigma_\theta; \tag{7.6}$$

hence the result follows by (7.3) and (7.5).

PROPOSITION 7.2. *If* $|\lambda|\geqslant\varepsilon>0$ *and* $\lambda-\varepsilon\in\Sigma_\theta$ *then* $t\mapsto R(\lambda,A(t))$ *is in* $C^1([0,T],\mathscr{L}(E))$ *and satisfies*

$$\left\|\frac{\partial}{\partial t}R(\lambda,A(t))\right\|_{\mathscr{L}(E)}\leqslant\frac{L}{|\lambda|^{1/2}}.$$

Proof. We can rewrite $u=R(\lambda,A(t))f$ as

$$[R(\lambda,A(t))f](x)$$

$$=\frac{sh\sqrt{\lambda}x}{\sqrt{\lambda}}\frac{\alpha(t)\int_0^1 sh\sqrt{\lambda}(1-\tau)f(\tau)d\tau+\sqrt{\lambda}\beta(t)\int_0^1 ch\sqrt{\lambda}(1-\tau)f(\tau)\,d\tau}{\alpha(t)\,sh\sqrt{\lambda}+\sqrt{\lambda}\beta(t)\,ch\sqrt{\lambda}}$$

$$-\frac{1}{\sqrt{\lambda}}\int_0^x sh\,\sqrt{\lambda}(x-\tau)f(\tau)\,d\tau,$$

and an easy calculation yields

$$\left[\frac{\partial}{\partial t}R(\lambda,A(t))f\right](x)$$

$$=sh\,\sqrt{\lambda}x\,\frac{\beta'(t)\,\alpha(t)-\alpha'(t)\,\beta(t)}{[\alpha(t)\,sh\,\sqrt{\lambda}+\sqrt{\lambda}\,\beta(t)\,ch\,\sqrt{\lambda}\,]^2}\cdot\int_0^1 f(\tau)\,sh\,\sqrt{\lambda}\tau\,d\tau. \tag{7.7}$$

Thus, remembering (7.4) and (7.6), we get

$$\left\|\frac{\partial}{\partial t}R(\lambda,A(t))\right\|_{\mathscr{L}(E)}$$

$$\leqslant \sup_{x\in[0,1]}|sh\,\sqrt{\lambda}x|\cdot\frac{sh\rho}{\rho}\cdot\frac{C}{(sh\rho)^2\,[\alpha(t)+\rho\beta(t)]^2}$$

$$=\sup_{x\in[0,1]}[sh^2\rho x+\sin^2\sigma x]^{1/2}\cdot\frac{C}{\rho sh\rho[\alpha(t)+\rho\beta(t)]^2}$$

$$\leqslant C\frac{sh\rho+\rho}{\rho sh\rho[\alpha(t)+\rho\beta(t)]^2}\leqslant\frac{C}{\rho[\alpha(t)+\rho\beta(t)]^2}\cdot\frac{1}{\varepsilon\wedge 1}$$

and the result follows easily.

PROPOSITION 7.3. *If* $\alpha, \beta \in C^{1,\eta}[0, T]$, $\eta \in]0, 1[$, *then* $t \mapsto R(1, A(t))$ *is in* $C^{1,\eta}([0, T], \mathscr{L}(E))$.

Proof. It is an easy consequence of (7.7).

PROPOSITION 7.4. *Let* $\varphi \in D(A(0))$ *and* $f \in C([0, T], E) = C([0, T] \times [0, 1])$. *Condition* (5.1), *with* $A(0)$ *replaced by* $A(0) - 1$, *holds if and only if*

$$\beta(0) \neq 0 \qquad \textit{and} \qquad f(0, 0) + \varphi''(0) = 0$$

or

$$\beta(0) = 0 \qquad \textit{and} \qquad f(0, 0) + \varphi''(0) = f(0, 1) + \varphi''(1) + \frac{\beta'(0)}{\alpha(0)} \varphi'(1) = 0.$$

Proof. Condition (5.1) in the present situation can be rewritten as

$$[A(0) - 1]\, \varphi + f(0, \cdot) + \left[\frac{d}{dt} R(1, A(t))\right]_{t=0} [A(0) - 1]\, \varphi \in \overline{D(A(0))};$$

if it holds, then by (7.7) and Proposition 7.1(i), it becomes

$$\begin{cases} \varphi''(0) + f(0, 0) = 0 & \text{if} \quad \beta(0) \neq 0, \\ \varphi''(0) + f(0, 0) = 0 & \\ \varphi''(1) + f(0, 1) + \dfrac{\beta'(0)}{\alpha(0)} \varphi'(1) = 0 & \text{if} \quad \beta(0) = 0. \end{cases}$$

The converse is also easy.

PROPOSITION 7.5. *Let* $\varphi \in \overline{D(A(0))}$ *and* $f \in C([0, T], E) = C([0, T] \times [0, 1])$. *Condition* (6.1) *with* $A(0)$ *replaced by* $A(0) - 1$, *is always true.*

Proof. Condition (6.1) in the present situation becomes

$$\begin{cases} \exists \{\varphi_n\}_{n \in \mathbf{N}}, \quad \{g_n\}_{n \in \mathbf{N}} \subseteq C[0, 1] \quad \text{such that} \\ \varphi_n \to \varphi \quad \text{and} \quad g_n \to f(0, \cdot) \quad \text{uniformly in} \quad [0, 1]; \\ \varphi_n \in D(A(0)) \quad \forall n \in \mathbf{N}; \\ [A(0) - 1]\, \varphi_n + g_n + \left[\dfrac{d}{dt} R(1, A(t))\right]_{t=0} [A(0) - 1]\, \varphi_n \in \overline{D(A(0))} \quad \forall n \in \mathbf{N}. \end{cases}$$

By Propositions 7.4 and 7.1(i), it can be rewritten as:

$$\begin{cases} \exists\{\varphi_n\}_{n\in\mathbf{N}}, \quad \{g_n\}_{n\in\mathbf{N}}\subseteq C[0,1] \quad \text{such that} \\ \varphi_n\to\varphi \quad \text{and} \quad g_n\to f(0,\cdot) \quad \text{uniformly in } [0,1]; \\ \begin{cases} \left.\begin{aligned} \varphi_n(0)=\varphi_n(1)=0 \\ \varphi_n''(0)+g_n(0)=\varphi_n''(1)+g_n(1)+\frac{\beta'(0)}{\alpha(0)}\varphi_n'(1)=0 \end{aligned}\right\} \forall n\in\mathbf{N}, \\ \qquad\qquad \text{if } \beta(0)=0. \\ \left.\begin{aligned} \varphi_n(0)=\alpha(0)\,\varphi_n(1)+\beta(0)\,\varphi_n'(1)=0 \\ \varphi_n''(0)+g_n(0)=0 \end{aligned}\right\} \forall n\in\mathbf{N}, \\ \qquad\qquad \text{if } \beta(0)\neq 0. \end{cases} \end{cases}$$

Now if $f\in C([0,T]\times[0,1])$ and $\varphi\in\overline{D(A(0))}$ it is clear that such a condition can always be satisfied.

Remark 7.6. $R([(d/dt)\,R(1,A(t))]_{t=0})\subseteq\overline{D(A(0))}$ if and only if $\beta(0)\neq 0$ or $\beta(0)=\beta'(0)=0$. Indeed, suppose $\beta(0)\neq 0$; then, if $x\in E$, the function $\Psi\triangleq[(d/dt)\,R(1,A(t))]_{t=0}f$ is in $\overline{D(A(0))}$ since $\Psi(0)=0$; on the other hand if $\beta(0)=\beta'(0)=0$, by (7.7) we derive that $[(d/dt)\,R(1,A(t))]_{t=0}\equiv 0$. Suppose conversely that $R([(d/dt)\,R(1,A(t))]_{t=0})\subseteq\overline{D(A(0))}$; then if $\beta(0)=0$, by (7.7) and Proposition 7.1(i) we must have

$$\left[\left[\frac{d}{dt}R(1,A(t))\right]_{t=0}f\right](1)=\frac{\beta'(0)}{\alpha(0)\,sh1}\int_0^t f(\tau)\,sh\tau\,d\tau=0 \qquad \forall f\in C[0,1],$$

which implies $\beta'(0)=0$. This shows in particular that condition (6.6) is actually stronger than (6.1).

By Propositions 7.1, 7.2, 7.3, 7.4 and 7.5 it follows that the operators $\{A(t)-1\}_{t\in[0,T]}$, with $A(t)$ defined in (7.1), verify Hypotheses I, II and possibly III of the Introduction. Hence all results of the previous sections are applicable to the problem

$$\begin{cases} u_t(t,x)-u_{xx}(t,x)+\lambda u(t,x)=f(t,x), & (t,x)\in[0,T]\times[0,1]; \\ u(t,0)=0, & t\in[0,T]; \\ \alpha(t)\,u(t,1)+\beta(t)\,u_x(t,1)=0, & t\in[0,T]; \\ u(0,x)=\varphi(x), & x\in[0,1], \end{cases}$$

where $\lambda\in\mathbf{C}$, $f\in C([0,T]\times[0,1])$, $\varphi\in C[0,1]$.

(b) *Second Example*

Let Ω be a bounded open set of $\mathbf{R}^n$, $n \geqslant 2$, with boundary of class C^2. Consider the differential operator with complex-valued coefficients:

$$A(t, x, D) = \sum_{i,j=1}^{n} a_{ij}(t, x)\, D_{x_i} D_{x_j} + \sum_{i=1}^{n} b_i(t, x)\, D_{x_i} + c(t, x)\, I, \qquad (t, x) \in [0, T] \times \bar{\Omega},$$

under the following assumptions:

(A.1) (Strong uniform ellipticity). There exists $E > 0$ such that

$$\mathrm{Re} \sum_{i,j=1}^{n} a_{ij}(t, x)\, \xi_i \xi_j \geqslant E\, |\xi|^2 \qquad \forall \xi \in \mathbf{R}^n, \quad \forall (t, x) \in [0, T] \times \bar{\Omega}.$$

(A.2) For each $t \in [0, T]$ the functions $a_{ij}(t, \cdot)$, $b_i(t, \cdot)$, $c(t, \cdot)$ are in $C(\bar{\Omega})$, with bounds independent on t.

Consider also the boundary differential operator with complex-valued coefficients:

$$B(t, x, D) = \sum_{i=1}^{n} \beta_i(t, x)\, D_{x_i} + \alpha(t, x)\, I, \qquad (t, x) \in [0, T] \times \partial\Omega,$$

under the following assumptions:

(B.1) (Normality condition, see [1]). For each $x \in \partial\Omega$ let $\nu = \nu(x)$ the outward normal unit vector of $\partial\Omega$ at x. Denoting with $\bar{B}(t, x, D)$ the principal part of $B(t, x, D)$, the following condition holds:

$$\bar{B}(t, x, \nu) \neq 0 \qquad \forall (t, x) \in [0, T] \times \partial\Omega.$$

(B.2) For each $t \in [0, T]$, the functions $\beta_i(t, \cdot)$, $\alpha(t, \cdot)$ are in $C^1(\partial\Omega)$, with norms bounded independently on t.

Suppose, moreover, that $A(t, x, D)$ and $B(t, x, D)$ satisfy the following conditions:

(AB.1) (Complementing condition, see [1, 2]) For each $x \in \partial\Omega$, let ξ be an arbitrary (real) vector tangent to $\partial\Omega$ at x, and let ν be, as above, the outward normal unit vector of $\partial\Omega$ at x. Then, condition (A.1) implies that, for each $t \in [0, T]$, the equation in the complex variable τ

$$\sum_{i,j=1}^{n} a_{ij}(t, x)(\xi_i + \tau\nu_i)(\xi_j + \tau\nu_j) = 0$$

has exactly one root $\tau^+ = \tau^+(t, x, \xi)$ with positive imaginary part ("root condition"; see Morrey [21, p. 255]). The complementing condition says that the polynomial $\tau \mapsto \bar{B}(t, x, \xi + \tau\nu)$, where $\bar{B}$ is the principal part of B, must not be divisible by $(\tau - \tau^+)$; in other words, it is required that

$$\bar{B}(t, x, \xi + \tau^+\nu) \neq 0 \qquad \forall (t, x) \in [0, T] \times \partial\Omega.$$

(AB.2) The functions $a_{ij}(\cdot, x)$, $b_i(\cdot, x)$, $c(\cdot, x)$, $\beta_i(\cdot, x)$, $\alpha(\cdot, x)$ are in $C^1[0, T]$ uniformly in x, i.e., their derivatives with respect to t have moduli of continuity which do not depend on x.

(AB.3) The functions $a_{ij}(\cdot, x)$, $b_i(\cdot, x)$, $c(\cdot, x)$, $\beta_i(\cdot, x)$, $\alpha(\cdot, x)$ are in $C^{1,\eta}[0, T]$, $\eta \in]0, 1]$, uniformly in x, i.e., the Hölder norms of their derivatives with respect to t are bounded independently on x.

Define now

$$E = C(\bar{\Omega}), \qquad \|f\|_E = \sup_{x \in \bar{\Omega}} |f(x)|,$$

and set for each $t \in [0, T]$

$$\begin{aligned} D(A(t)) = \{&u \in C(\bar{\Omega}) \cap H^{2,q}(\Omega) \text{ for some } q > n : A(t, \cdot, D)\, u \in C(\bar{\Omega}) \\ &\text{and } B(t, \cdot, D)\, u = 0 \text{ on } \partial\Omega\} \\ A(t)\, u = A&(t, \cdot, D)\, u. \end{aligned} \tag{7.8}$$

We observe that, by Sobolev's imbedding Theorem, the condition $B(t, \cdot, D)\, u = 0$ on $\partial\Omega$ is meaningful; moreover, well-known regularity results in L^p-spaces (see Agmon [1]) imply that

$$D(A(t)) \subseteq \bigcap_{p \in [1, \infty[} H^{2,p}(\Omega).$$

Note that $D(A(t))$ may be not dense in E, since the boundary condition may reduce to a Dirichlet one if $\beta_i(t, x) \equiv 0$ in $[0, T] \times \partial\Omega$, $i = 1, \dots, n$. Following Stewart [30], we will verify now that under the previous assumptions there exists $\lambda_0 > 0$ such that the operators $\{A(t) - \lambda_0\}_{t \in [0, T]}$ satisfy Hypotheses I, II and possibly III of the Introduction. We will sketch most of the proofs; details can be found in [30]. First of all, consider two differential operators $A(x, D)$ and $B(x, D)$, independent on t, and satisfying Hypotheses (A.1), (A.2), (B.1), (B.2), (AB.1), and from now on let $q > n$ be fixed. If $\lambda \in \mathbf{C}$, consider the stationary problem

$$\begin{aligned} \lambda u - A(\cdot, D)\, u &= f \qquad \text{in } \Omega, \\ B(\cdot, D)\, u &= g \qquad \text{on } \partial\Omega, \\ f \in L^q(\Omega), \qquad & g \in H^{1-1/q,q}(\partial\Omega). \end{aligned} \tag{7.9}$$

It is well known (see Agmon [1]; see also Theorems 3.8.1–3.8.2 and Lemma 5.3.3 of [33]) that this problem has a unique solution $u \in H^{2,q}(\Omega)$, provided λ belongs to a suitable sector (depending on q) $\Sigma_{\theta_0,\lambda_0} \triangleq \{\lambda \in \mathbf{C} : |\lambda| \geqslant \lambda_0, |\arg \lambda| \leqslant \theta_0\}$, with $\lambda_0 > 0$ and $\theta_0 \in]\pi/2, \pi]$. Moreover, the following estimate holds:

$$|\lambda| \cdot \|u\|_{L^q(\Omega)} + |\lambda|^{1/2} \|Du\|_{L^q(\Omega)} + \|D^2 u\|_{L^q(\Omega)} \leqslant C\{\|f\|_{L^q(\Omega)} + \|g\|_{H^{1-1/q,q}(\partial\Omega)}\}.$$

Consider now a function $\phi \in C_0^\infty(\mathbf{R}^n)$ with support contained in $B(0, 1)$ and such that $\phi = 1$ on $B(0, \frac{1}{2})$. For each $x_0 \in \bar{\Omega}$ and $r > 0$ define

$$\phi_r(x) = \phi_{r,x_0}(x) \triangleq \phi\left(\frac{x - x_0}{r}\right), \qquad G_r = G_{r,x_0} \triangleq B(x_0, r) \cap \Omega.$$

In [30, p. 306], the following key inequality is proved:

LEMMA 7.7. *Under the above assumptions, there exist* $\lambda_0 > 0$, $\theta_0 \in]\pi/2, \pi]$, $r_0 > 0$ *and* $C_0 > 0$ *such that, for each* K *sufficiently large, the solution* u *of* (7.9) *satisfies the following inequality for each* $\lambda \in \Sigma_{\theta_0,\lambda_0}$ *and* $r \leqslant r_\lambda \triangleq (Kr_0/2) |\lambda|^{-1/2}$:

$$\begin{aligned} &|\lambda| \cdot \|u\|_{C(\bar{\Omega})} + |\lambda|^{1/2} \|Du\|_{C(\bar{\Omega})} + |\lambda|^{n/2q} \sup_{x_0 \in \bar{\Omega}} \|D^2 u\|_{L^q(G_{r,x_0})} \\ &\quad \leqslant C_0 |\lambda|^{n/2q} [\sup_{x_0 \in \bar{\Omega}} \|f\|_{L^q(G_{2r,x_0})} + \sup_{x_0 \in \bar{\Omega}} \inf\{\|w \cdot \phi_{2r,x_0}\|_{H^{1,q}(G_{2r,x_0})} : \\ &\quad w \in H^{1,p}(\Omega), w = g \text{ on } \partial\Omega\}]. \end{aligned}$$

Proof. See the proof of Theorem 1 in [30].

A first consequence of Lemma 7.7 is the following

PROPOSITION 7.8. *For each* $t \in [0, T]$, *let* $A(t)$ *be the operator defined by* (7.8). *Under Hypotheses* (A.1), (A.2), (B.1), (B.2), (AB.1) *and* (AB.2), *there exist* $\theta_0 \in]\pi/2, \pi]$, $\lambda_0 > 0$ *such that* $\Sigma_{\theta_0,\lambda_0} \subseteq \rho(A(t))$ *and*

$$\|R(\lambda, A(t))f\|_{C(\bar{\Omega})} \leqslant \frac{C}{|\lambda|} \|f\|_{C(\bar{\Omega})} \qquad \forall \lambda \in \Sigma_{\theta_0,\lambda_0}, \quad \forall f \in C(\bar{\Omega}).$$

Proof. Suppose $u \in D(A(t))$; then by Lemma 7.7 it follows easily that

$$|\lambda| \, \|u\|_{C(\bar{\Omega})} \leqslant C \, \|[\lambda - A(t)] \, u\|_{C(\bar{\Omega})};$$

thus it remains to prove that $\lambda - A(t)$ is surjective. Take $f \in C(\bar{\Omega})$; then in particular $f \in L^q(\Omega)$, and therefore there exists $u \in H^{2,q}(\Omega)$ satisfying the problem, analogous to (7.9),

$$[\lambda - A(t, \cdot, D)]\, u = f \quad \text{in} \quad \Omega,$$
$$B(t, \cdot, D)\, u = 0 \quad \text{on} \quad \partial\Omega.$$

It follows that $u \in C^1(\bar{\Omega})$, and this implies $A(t, \cdot, D)\, u = \lambda u - f \in C(\bar{\Omega})$. This shows that $u \in D(A(t))$ and the proof is complete.

Remark 7.9. Under the hypotheses of Proposition 7.8, if $u(t) \triangleq R(\lambda, A(t)) f$, Lemma 7.7 yields in particular:

$$|\lambda|\, \|u(t)\|_{C(\bar{\Omega})} + |\lambda|^{1/2}\, \|Du(t)\|_{C(\bar{\Omega})} + \sup_{x_0 \in \bar{\Omega}} \|D^2 u(t)\|_{L^q(G_{r_\lambda, x_0})} \leqslant C\, \|f\|_{C(\bar{\Omega})}. \tag{7.10}$$

PROPOSITION 7.10. *Under Hypotheses* (A.1), (A.2), (B.1), (B.2), (AB.1) *and* (AB.2), *suppose* $\lambda \in \Sigma_{\theta_0, \lambda_0}$. *Then the function* $u(t) = R(\lambda, A(t)) f$ *is differentiable in* $C(\bar{\Omega})$ *for each* $f \in C(\bar{\Omega})$, *and*

$$\left\| \frac{\partial}{\partial t} R(\lambda, A(t)) f \right\|_{C(\bar{\Omega})} \leqslant \frac{C}{|\lambda|} \|f\|_{C(\bar{\Omega})} \qquad \forall f \in C(\bar{\Omega}).$$

Proof. Fix $t, s \in [0, T]$; then $u(t) - u(s)$ is a solution of

$$[\lambda - A(t, \cdot, D)](u(t) - u(s)) = [A(t, \cdot, D) - A(s, \cdot, D)]\, u(s) \quad \text{in} \quad \Omega,$$
$$B(t, \cdot, D)(u(t) - u(s)) = -\,[B(t, \cdot, D) - B(s, \cdot, D)]\, u(s) \quad \text{on} \quad \partial\Omega, \tag{7.11}$$

and by Lemma 7.7, (7.10) and (AB.2), it can be deduced that

$$|\lambda|\, \|u(t) - u(s)\|_{C(\bar{\Omega})} = O(|t - s|)\, \|f\|_{C(\bar{\Omega})} \quad \text{as} \quad s \to t. \tag{7.12}$$

Let now $\dot{A}(t, x, D)$ and $\dot{B}(t, x, D)$ be the differential operators whose coefficients are the derivatives with respect to t of the corresponding ones of $A(t, x, D)$ and $B(t, x, D)$; let $w(t)$ be the solution of the following problem, similar to (7.9):

$$\begin{aligned} [\lambda - A(t, \cdot, D)]\, w(t) &= \dot{A}(t, \cdot, D)\, u(t) \quad \text{in} \quad \Omega, \\ B(t, \cdot, D)\, w(t) &= -\dot{B}(t, \cdot, D)\, u(t) \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{7.13}$$

Then if we apply Lemma 7.7 to $v(t, s) \triangleq (u(t) - u(s))/(t - s) - w(t)$, by using (7.12) we easily get

$$|\lambda| \cdot \|v(t, s)\|_{C(\bar{\Omega})} = o(1)\, \|f\|_{C(\bar{\Omega})} \quad \text{as} \quad s \to t.$$

This shows that $(d/dt)\,u(t)$ exists in $C(\bar{\Omega})$, and

$$w(t)=\frac{d}{dt}u(t)=\frac{\partial}{\partial t}R(\lambda,A(t))f.$$

Applying again Lemma 7.7 and (7.10), one sees that

$$|\lambda|\left\|\frac{\partial}{\partial t}R(\lambda,A(t))f\right\|_{C(\bar{\Omega})}\leqslant C\,\|f\|_{C(\bar{\Omega})}.$$

PROPOSITION 7.11. *Under the hypotheses of Proposition* (7.10), *suppose moreover, that* (AB.3) *holds. Then for each* $t,s\in[0,T]$ *we have*

$$\left\|\frac{d}{dt}R(\lambda_0,A(t))f-\frac{d}{ds}R(\lambda_0,A(s))f\right\|_{C(\bar{\Omega})}\leqslant C\,|t-s|^{\eta}\,\|f\|_{C(\bar{\Omega})}\qquad\forall f\in C(\bar{\Omega}).$$

Proof. Set $w(t)\triangleq(d/dt)\,R(\lambda_0,A(t))f$. Then $w(t)-w(s)$ is the solution of the following problem, similar to (7.11):

$$\begin{aligned}[]
[\lambda_0-A(t,\cdot,D)](w(t)-w(s))&=[\dot{A}(t,\cdot,D)-\dot{A}(s,\cdot,D)]\,w(s) &&\text{in}\quad \Omega,\\
B(t,\cdot,D)(w(t)-u(s))&=-[\dot{B}(t,\cdot,D)-\dot{B}(s,\cdot,D)]\,w(s) &&\text{on}\quad \partial\Omega.
\end{aligned}$$

By Lemma 7.7 and Proposition 7.10 one deduces that

$$\|w(t)-w(s)\|_{C(\bar{\Omega})}\leqslant C\,|t-s|^{\eta}\,\|f\|_{C(\bar{\Omega})}.$$

By Propositions 7.8, 7.10 and 7.11 we conclude that all results of the previous sections are applicable to the problem

$$\begin{aligned}
u_t(t,x)-A(t,x,D)\,u(t,x)+\lambda u(t,x)&=f(t,x), &&(t,x)\in[0,T]\times\Omega,\\
B(t,x,D)\,u(t,x)&=0, &&(t,x)\in[0,T]\times\partial\Omega,\\
u(0,x)&=\varphi(x), &&x\in\bar{\Omega},
\end{aligned}$$

where $\lambda\in\mathbf{C}$, $f\in C([0,T]\times\bar{\Omega})$, $\varphi\in C(\bar{\Omega})$.

Remark 7.12. The same example can be discussed in a more general situation, i.e., by considering an unbounded open set Ω and a differential operator $A(t,x,D)$ of order $2m\geqslant 2$. The assumptions (A.1), (A.2), (B.1), (B.2), (AB.1), (AB.2) and (AB.3) have to be suitably modified, and in this case the Banach space E will consist of the functions $u\in C(\bar{\Omega})$ tending to 0 as $|x|\to+\infty$. For the details see [30].

Remark 7.13. The first example is not a special case of the second one. Indeed, assumptions (B.1) and (AB.1) do not hold, since the principal part of $B(t,x,D)$ vanishes at $x=0$.

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