SIAM J. CONTROL AND OPTIMIZATION
Vol. 34, No. 1, pp. 1–30, January 1996

29

INFINITE-HORIZON LINEAR-QUADRATIC REGULATOR PROBLEMS FOR NONAUTONOMOUS PARABOLIC SYSTEMS WITH BOUNDARY CONTROL*

PAOLO ACQUISTAPACE† AND BRUNELLO TERRENI‡

Abstract. This paper concerns the classical linear-quadratic regulator problem for general nonautonomous parabolic systems with boundary control over infinite time horizon from the point of view of semigroup theory. Under appropriate assumptions we prove existence and uniqueness of the optimal pair, as well as existence, uniqueness, and further properties of the solution of the associated Riccati equation. Several examples are discussed in detail.

Key words. optimal control, parabolic systems, boundary control, infinite horizon, Riccati equation

AMS subject classifications. 49J20, 49N10, 49L20, 34G20

Introduction. This paper concerns the classical linear-quadratic regulator (LQR) problem for general nonautonomous parabolic systems with boundary control over infinite time horizon. Under appropriate assumptions we prove here existence and uniqueness of the optimal pair as well as existence, uniqueness, and further properties of the solution of the associated Riccati equation. Such results generalize the similar ones known in the autonomous case [F3], [LT2], [BDDM] and those of [DI3] relative to nonautonomous problems with distributed control; they also constitute a development of the theory of [AFT] concerning the case of finite time horizon.

Our assumptions are, generally speaking, not uniform with respect to t, with few exceptions concerning the spectra of the elliptic operators $A(t)$ appearing in the state equation and the regularity of the Green maps $G(t)$ associated with them: see Hypotheses 1.1 and 1.3 below. In particular, we do not assume any global exponential estimate for the evolution operator $U(t,s)$ or any boundedness for $G(t)$ and the operators appearing in the cost functional.

On the other hand, some uniform requirements arise in the study of certain features of the Riccati equation. Thus, in order to construct a minimal solution $P_\infty(t)$ of such an equation and to solve the synthesis, we need the "finite cost condition" (Hypothesis 2.2), which is necessary and sufficient; moreover a uniform version of this condition (Hypothesis 3.1) is necessary and sufficient for the existence of a bounded solution of the Riccati equation. Further uniform assumptions (Hypotheses 3.4, 3.5, 3.6 and 3.9) guarantee other properties, such as stability of the optimal state and uniqueness of $P_\infty(t)$. The periodic case is also analyzed. All these results seem to be new even in the case of distributed control of [DI3].

We now list some notations. If X is a Hilbert space, we denote the inner product and the norm of X by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$. If Y is another Hilbert space, $\mathcal{L}(X,Y)$ is the Banach space of bounded linear operators from X into Y, and $|\cdot|_{\mathcal{L}(X,Y)}$ denotes its usual norm; we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,X)$.

If $A : D(A) \subseteq X \to Y$ is a closed linear operator with dense domain, the adjoint operator $A^* : D(A^*) \subseteq Y^* \to X^*$ is defined in the usual way. In particular we denote by $\Sigma(X)$ the set of operators $A \in \mathcal{L}(X)$ such that $A = A^*$, and we set

$$\Sigma^+(X) := \{A \in \Sigma(X) : (Ax,x)_X \geq 0 \,\forall x \in X\},$$
$$\Sigma^{++}(X) := \{A \in \Sigma^+(X) : \exists \nu > 0 : (Ax,x)_X \geq \nu\|x\|_X \,\forall x \in X\}.$$

If $I \subseteq \mathbb{R}$ is an interval, we will use the spaces $L^p(I,X) := \{f : I \to X : f$ is strongly

---

* Received by the editors February 10, 1993; accepted for publication (in revised form) August 9, 1994.

† Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, 56127 Pisa, Italy.

‡ Dipartimento di Matematica Pura ed Applicata, Università dell'Aquila, Via Vetoio, 67010 Coppito (L'Aquila), Italy.

measurable and $\int_I \|f(t)\|_X^p \, dt < \infty\}$ $(1 \le p < \infty)$, and $L^\infty(I,X)$, $C(I,X)$, whose definitions are similar. Finally we will also use the spaces

$$L^p_{\rm loc}(I,X) := \bigcap_{J \subset\subset I} L^p(J,X) \quad (1 \le p \le \infty).$$

1. Problem formulation and hypotheses.

1.1. Abstract formulation of a parabolic differential system. Let $\Omega \subset R^d$ be a bounded open domain with boundary $\partial\Omega$ of class C^2, and consider the following parabolic system:

$$\begin{cases} y_t(t,x) = \mathcal{A}(t,x,D)y(t,x) & \text{in } [0,\infty[\times \Omega, \\ \mathcal{B}(t,x,D)y(t,x) = u(t,x) & \text{in } [0,\infty[\times \partial\Omega, \\ y(0,x) = y_0(x) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here the strongly elliptic differential operators $\{\mathcal{A}(t,\cdot,D)\}_{t\ge0}$ and the boundary operators $\{\mathcal{B}(t,\cdot,D)\}_{t\ge0}$ are assumed to be such that the abstract hypotheses listed in the next subsection are satisfied (see, for instance, the conditions of [AFT, §2.2]).

For each $t \ge 0$ we define $A(t)$ as the realization in $L^2(\Omega)$ of the operator $\mathcal{A}(t,\cdot,D)$ with homogeneous boundary conditions determined by $\mathcal{B}(t,\cdot,D)$, i.e.,

$$\begin{aligned} D_{A(t)} &:= \{y \in L^2(\Omega) \mid \mathcal{A}(t,\cdot,D)y \in L^2(\Omega) \text{ and } \mathcal{B}(t,\cdot,D)y = 0 \text{ on } \partial\Omega\}, \\ A(t)y &:= \mathcal{A}(t,\cdot,D)y \quad \forall y \in D_{A(t)}. \end{aligned}$$

If we choose $\lambda_0 \in \mathbb{R}$ large enough, we can define simultaneously (for $t \ge 0$) the fractional powers $(\lambda_0 - A(t))^\alpha$ with $0 < \alpha < 1$. We also require that for each $u \in L^2(\partial\Omega)$ we can solve (in the sense of [AFT, §2.4]) simultaneously for $t \ge 0$ the following elliptic problems:

$$\begin{cases} \lambda_0\Phi - \mathcal{A}(t,\cdot,D)\Phi = 0 & \text{in } \Omega, \\ \mathcal{B}(t,\cdot,D)\Phi = u & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

in other words, we can define the map $G : [0,\infty) \times L^2(\partial\Omega) \to L^2(\Omega)$ as $G(t)u := \Phi$, where Φ is the unique solution of problem (1.2). In the next section we shall need certain regularity properties for $G(t)$: for instance, we shall assume that

$$t \to (\lambda_0 - A(t))^\alpha G(t) \in L^\infty_{\rm loc}([0,\infty[; \mathcal{L}(L^2(\partial\Omega), L^2(\Omega)))$$

for some $\alpha \in]0,1[$. It is shown in [AFT] that systems of type (1.1) fulfill this condition.

We remark also that the map G depends on the initial choice of λ_0.

Let

$$z(t) := e^{-\lambda_0 t} y(t),$$

where y solves problem (1.1); then z solves the following problem:

$$\begin{cases} z_t = (\mathcal{A}(t,\cdot,D) - \lambda_0)z & \text{in } [0,\infty[\times \Omega, \\ \mathcal{B}(t,\cdot,D)z = e^{-\lambda_0 t}u(t) & \text{on } [0,\infty[\times \partial\Omega, \\ z(0) = y_0 & \text{in } \Omega, \end{cases}$$

so using the representation formula proved in [AFT, §2.5] we have

$$z(t) = U_{\lambda_0}(t,0)y_0 + \int_0^t U_{\lambda_0}(t,s)(\lambda_0 - A(s))G(s)u(s)e^{-\lambda_0 s} \, ds. \tag{1.3}$$

Here $U_{\lambda_0}(t,s) := e^{-\lambda_0(t-s)}U(t,s)$ where $U(t,s)$ is the evolution operator associated with $\{A(t)\}_{t\geq 0}$. Thus

$$y(t) = U(t,0)y_0 + \int_0^t U(t,s)(\lambda_0 - A(s))G(s)u(s)\,ds. \tag{1.4}$$

We must recall that formula (1.4) is very useful for our calculations, but we understand that its exact form is

$$y(t) = U(t,0)y_0 + \int_0^t [(\lambda_0 - A(s)^*)^{1-\alpha}U(t,s)^*]^*((\lambda_0 - A(s))^\alpha G(s)u(s)\,ds; \tag{1.5}$$

for more details we refer to [AFT, (2.80)–(2.74)].

Throughout this paper, equation (1.4) or (1.5) will be considered the state equation for our abstract control problem.

1.2. Standing assumptions. In the following discussion we will consider three Hilbert spaces: H (state space), U (control space), and V (space of observations), and we will study the optimal control of equation (1.4) over an unbounded time interval I (which could be $[T_0,\infty[$ for some $T_0 \in \mathbb{R}$, or even $\mathbb{R}$, as for instance in the periodic case). We will consider equation (1.4) as an abstract evolution equation in the Hilbert space H, subject to the abstract assumptions listed below. Thus, equation (1.4) can also cover concrete problems different from those explicitly described above and in the examples in §4. We assume the following hypotheses.

Hypothesis 1.1. $\{A(t)\}_{t\in I}$ is a family of infinitesimal generators of analytic semigroups in H; the spectrum of $A(t)$ is such that the fractional powers $(\lambda_0 - A(t))^\alpha$ are well defined for any $\alpha > 0$, simultaneously with respect to $t \in I$, for some fixed $\lambda_0 \in \mathbb{R}$.

Hypothesis 1.2. The assumptions of [AFT] hold locally, i.e., over every bounded interval $J \subset\subset I$, possibly with constants depending on the interval. More precisely, given $J = [a,b] \subset I$, we assume

(i) the evolution operators $U(t,s)$ and $U(t,s)^*$ are strongly continuous in $\bar{\Delta}$, where $\Delta := \{(t,s) \in [a,b]^2 : t > s\}$, and there exists $M_0 > 0$ such that

$$|U(t,s)|_{\mathcal{L}(H)} + |U(t,s)^*|_{\mathcal{L}(H)} \leq M_0 \quad \forall\,(t,s) \in \bar{\Delta};$$

(ii) for every $\beta,\ \mu \in [-1,1]$ and $(t,s) \in \Delta$ the operators

$$(\lambda_0 - A(t))^\beta U(t,s)(\lambda_0 - A(s))^{-\mu}, \qquad (\lambda_0 - A(s)^*)^\beta U(t,s)^*(\lambda_0 - A(t)^*)^{-\mu}$$

have continuous extensions to H, the maps

$$(t,s) \to (\lambda_0 - A(t))^\beta U(t,s)(\lambda_0 - A(s))^{-\mu}, \quad (t,s) \to (\lambda_0 - A(s)^*)^\beta U(t,s)^*(\lambda_0 - A(t)^*)^{-\mu}$$

are strongly continuous, and there exists $M_{\beta,\mu} > 0$ such that

$$\begin{aligned} &|(\lambda_0 - A(t))^\beta U(t,s)(\lambda_0 - A(s))^{-\mu}|_{\mathcal{L}(H)} + |(\lambda_0 - A(s)^*)^\beta U(t,s)^*(\lambda_0 - A(t)^*)^{-\mu}|_{\mathcal{L}(H)} \\ &\quad \leq M_{\beta,\mu}[(t-s)^{\mu-\beta} + 1] \quad \forall\,(t,s) \in \Delta. \end{aligned} \tag{1.6}$$

Hypothesis 1.3. There exists $\alpha \in\,]0,1]$ such that, for each $t \in I$, $G(t)$ maps U into the domain of $(\lambda_0 - A(t))^\alpha$, and $(\lambda_0 - A(\cdot))^\alpha G(\cdot)$ is strongly measurable and bounded over each $[a,b] \subset I$.

Remark 1.4. (i) The only novelty with respect to [AFT] is the uniformity with respect to t in Hypotheses 1.1 and 1.3 for the choices of λ_0 and α, respectively. In particular, we do not assume any global exponential estimate for $U(t,s)$ and $(\lambda_0 - A(t))^\beta U(t,s)(\lambda_0 - A(s))^{-\mu}$ or uniform boundedness for $(\lambda_0 - A(t)^*)^\alpha G(t)$, $C(t)$, $N(t)$, and $N(t)^{-1}$ (defined below). Under these assumptions, the representation formulas (1.4), (1.5) can be studied as in [AFT] over any bounded time interval $[a,b]$.

(ii) In §2 the sentence "c depends on $[a,b]$" will mean that c in fact depends, besides $[a,b]$ itself, on all constants involved in estimating functions and operators defined in $[a,b]$. In §3 some questions of stability are treated, and there we shall assume the necessary uniformities and point out the independence of the constants.

(iii) We formulated Hypothesis 1.2 in terms of the evolution operators $U(t,s)$ and $U(t,s)^*$, rather than of the family $\{A(t)\}_{t\geq 0}$; this choice is motivated by the existence in the literature of many independent sets of assumptions on the family $\{A(t)\}_{t\geq 0}$, each of which implies Hypothesis 1.2. In [AT1] and [A1] one can find a review of these assumptions.

1.3. Formulation of the infinite-horizon LQR problem. Given $t_0 \in I$ and $y_0 \in H$, we will consider the problem of minimizing the cost functional

$$J_{t_0,\infty}(u) := \int_{t_0}^{\infty} [\|C(t)y(t)\|_V^2 + (N(t)u(t),u(t))_U]\,dt \tag{1.7}$$

over all $u \in L^2_{\mathrm{loc}}(t_0,\infty;U)$, subject to the state equation (1.5).

Concerning $C(\cdot)$ and $N(\cdot)$ we assume the following hypothesis.

Hypothesis 1.5. For every interval $[a,b] \subset I$,

$$C(\cdot) \in L^\infty([a,b];\mathcal{L}(H,V)) \quad \text{and} \quad N(\cdot) \in L^\infty([a,b];\Sigma^{++}(U)).$$

(This means that there exists $\nu > 0$, possibly depending on $[a,b]$, such that $N(t) \geq \nu\ \forall t \in [a,b]$, i.e., $(N(t)u,u)_U \geq \nu\|u\|_U^2\ \forall u \in U$, $\forall t \in [a,b]$.)

2. Solutions of the Riccati equation and related questions. The aim of this section is to solve the integral Riccati equation

$$\begin{aligned} Q(s) &= U(t,s)^*Q(t)U(t,s) \\ &\quad + \int_s^t U(r,s)^*[C(r)^*C(r) \\ &\qquad - Q(r)(\lambda_0 - A(r))G(r)N(r)^{-1}G(r)^*(\lambda_0 - A(r)^*)Q(r)]U(r,s)\,dr, \end{aligned} \tag{2.1}$$

with $s,\ t \in I,\ s \leq t$, and to prove some further results related to this equation. We follow here along the lines of [F3], *mutatis mutandis*; in particular we shall need a new local existence result (see Theorem 2.5 below).

We recall that a solution $Q(\cdot)$ of equation (2.1) is an operator-valued function $Q \in C_s(I;\Sigma^+(H))$ such that (i) $t \to (\lambda_0 - A(t)^*)^{1-\alpha}Q(t)$ is well defined and strongly continuous from I into $\mathcal{L}(H)$, and (ii) $Q(\cdot)$ satisfies the following meaningful version of (2.1):

$$\begin{aligned} Q(s) &= U(t,s)^*Q(t)U(t,s) \\ &\quad + \int_s^t U(r,t)^*[C(r)^*C(r) \\ &\qquad - [(\lambda_0 - A(r)^*)^{1-\alpha}Q(r)]^*K(r)[(\lambda_0 - A(r)^*)^{1-\alpha}Q(r)]]U(r,s)\,dr \end{aligned}$$

with $s,\ t \in I,\ s \le t$, and where

$$K(r) := [(\lambda_0 - A(r))^\alpha G(r)] N(r)^{-1} [(\lambda_0 - A(r))^\alpha G(r)]^*;$$

but in order to simplify our calculations and for sake of clearness we will always use equation (2.1).

2.1. Definition of the operator $P_\infty(t)$. Let us denote by $P_T(t)$ the solution, defined for every $t \le T$ (and $t \in I$; we will not repeat this detail in the following discussion), of Riccati equation (2.1) with value 0 at $t = T$:

$$P_T(t) = \int_t^T U(r,t)^* [C(r)^* C(r) - [(\lambda_0 - A(r)^*)^{1-\alpha} P_T(r)]^* K(r) [(\lambda_0 - A(r)^*)^{1-\alpha} P_T(r)]] U(r,s)\, dr. \tag{2.2}$$

This equation was solved in [AFT, Thm. 3.13]. We recall that

$$(P_T(t_0) y_0, y_0)_H = \min_u J_{t_0,T}(u) \tag{2.3}$$

under the condition $y(t_0) = y_0$, where for each $u \in L^2_{\text{loc}}(t_0, \infty; U)$ the functional $J_{t_0,T}(u)$ is defined as

$$J_{t_0,T}(u) = \int_{t_0}^T [\|C(t)y(t)\|_V^2 + (N(t)u(t), u(t))_U]\, dt$$

and y satisfies the state equation (1.4).

Let us introduce the following important condition:

$$\left\{ \begin{array}{l} \text{there exists } c = c(t_0) > 0 \text{ such that to each } y_0 \in H \text{ there corresponds} \\ \text{a control } u = u(y_0) \in L^2_{\text{loc}}(t_0, \infty; U) \text{ for which } J_{t_0,\infty}(u) \le c\|y_0\|_H^2; \end{array} \right. \tag{2.4_{t_0}}$$

in other words condition (2.4_{t_0}) requires the existence of an admissible control with respect to a given $t_0 \in I$ for each initial state $y_0 \in H$.

The following lemma holds.

LEMMA 2.1. *Under Hypotheses* 1.1 – 1.3 *and* 1.5, *we have*

(i) $P_{T_1}(t) \le P_{T_2}(t)$ *for each* $t \le T_1 \le T_2$;

(ii) *condition* (2.4_{t_1}) *implies condition* (2.4_t) *for each* $t \le t_1$;

(iii) *if condition* (2.4_{t_0}) *holds, then* $\sup_{T \ge t_0} |P_T(t_0)|_{\mathcal{L}(H)} < \infty$ *and there exists* $P_\infty(t_0) \in \Sigma^+(H)$ *such that* $P_T(t_0) \uparrow P_\infty(t_0)$ *strongly as* $T \uparrow \infty$.

(iv) *if condition* (2.4_{t_0}) *holds, then for each fixed* $\tau_0 < t_0$ *we have*

$$\sup_{\tau_0 \le t \le t_0, T > t_0} |P_T(t)|_{\mathcal{L}(H)} < \infty; \tag{2.5}$$

moreover $P_\infty(t)$ *is well defined for each* $t \le t_0$, *and*

$$P_\infty(s) \le U(t,s)^* P_\infty(t) U(t,s) + \int_s^t U(r,s)^* C(r)^* C(r) U(r,s)\, dr; \tag{2.6}$$

in particular, for each fixed $\tau_0 < t_0$ *we have*

$$\sup_{\tau_0 \le t \le t_0} |P_\infty(t)|_{\mathcal{L}(H)} < \infty. \tag{2.7}$$

Proof. It is standard and follows from [F1], [AFT]. □

Our main assumption in order to obtain a solution of equation (2.1) is the following hypothesis.

Hypothesis 2.2. Condition (2.4_t) is satisfied for every $t \in I$.

By the preceding lemma, if Hypothesis 2.2 holds, we can define $P_\infty(t)$ for each $t \in I$; this operator-valued function is the candidate solution of the Riccati equation.

Remark 2.3. As we will see, Hypothesis 2.2 is necessary and sufficient to construct $P_\infty(t)$ and to solve the synthesis; hence it is important to know when it is satisfied in concrete examples. Two general remarks in this direction are the following:

(i) if system (1.4) is exactly controllable at 0 in finite time, starting from any time t_0 and initial position y_0, then Hypothesis 2.2 holds;

(ii) if (1.4) is exponentially stabilizable, starting from any t_0, and if $C(t)$ and $N(t)$ are uniformly bounded, then Hypothesis 2.2 holds.

The analysis of these properties in concrete cases is under investigation; however, for certain classes of systems property (ii), hence Hypothesis 2.2, has been proved to hold true (see, for instance, Example 4.2 below). Note also that in the case of periodic systems it is sufficient to show that (2.4_t) is satisfied for some $t \in \mathbb{R}$, because this implies that (2.4_s) holds for each $s < t$ and thus for each s by periodicity.

2.2. A priori bound on $(\lambda_0 - A(t)^*)^\mu P_T(t)$. The following result plays a basic role in solving the Riccati equation (2.1). The same result was proved in [F1] in the case of autonomous parabolic systems, but the proof given in [F1] cannot be extended (at least in an obvious way) to the present case; thus, the proof given here is new. See also [F2], where a similar proof provides the a priori bound needed to get a global solution over a finite time horizon.

LEMMA 2.4. *Assume Hypotheses* 1.1 – 1.3, 1.5, *and* 2.2. *Then for each* $\beta \in]0, 1/2[$ *and each interval* $[a, b] \subset I$, *we have*

$$\sup_{a \le t \le b+1, T > b+2} |(\lambda_0 - A(t)^*)^\beta P_T(t)(\lambda_0 - A(t))^\beta|_{\mathcal{L}(H)} =: c_1(\beta, [a, b]) < +\infty, \tag{2.8}$$

$$\sup_{a \le t \le b+1} |(\lambda_0 - A(t)^*)^\beta P_\infty(t)(\lambda_0 - A(t))^\beta|_{\mathcal{L}(H)} < +\infty. \tag{2.9}$$

Proof. Let us fix an interval $[a, b] \subset I$. Fix $t \in [a, b+1]$, $x \in D((\lambda_0 - A(t))^\beta)$ and set $y_1 := U(b+2, t)(\lambda_0 - A(t))^\beta x$. By Hypothesis 2.2 there exists a control $\bar{u}$ belonging to $L^2_{\text{loc}}(b+2, \infty; U)$ such that $J_{b+2,\infty}(\bar{u}) \le c\|y_1\|_H^2$; hence by Hypothesis 1.2(ii) we also have $J_{b+2,\infty}(\bar{u}) \le c\|x\|_H^2$. Consider the control $\tilde{u} \in L^2_{\text{loc}}(t, \infty; U)$ defined by

$$\tilde{u}(s) = \begin{cases} 0 & \text{for } t \le s < b+2, \\ \bar{u}(s) & \text{for } b+2 \le s < \infty. \end{cases}$$

Using (2.3) we have

$$\begin{aligned} &(P_T(t)(\lambda_0 - A(t))^\beta x, (\lambda_0 - A(t))^\beta x)_H \\ &\quad \le J_{t,T}(\tilde{u}) \\ &\quad = \int_t^{b+2} \|C(s)U(s,t)(\lambda_0 - A(t))^\beta x\|_V^2 \, ds + J_{b+2,\infty}(\bar{u}) \\ &\quad \le \left[c\int_t^{b+2} |U(s,t)(\lambda_0 - A(t))^\beta|^2_{\mathcal{L}(H)} \, ds + c \right] \|x\|_H^2 \le c\|x\|_H^2. \end{aligned} \tag{2.10}$$

Recalling that $P_T(t) \geq 0$, by (2.10) we easily obtain (2.8); moreover, as $P_\infty(t) \geq 0$ too, (2.9) follows by letting $T \to \infty$ in (2.10). □

We state now the following local existence theorem, proved in the Appendix. Its proof is the nonautonomous version of that of [F2, Lem. 2.1].

THEOREM 2.5. *Assume Hypotheses* 1.1 – 1.3, 1.5, *and* 2.2; *fix* $\beta \in]1/2 - \alpha, 1/2[$, $t \in I$, $r_0 > 0$, *and let* $Q_t \in \Sigma^+(H)$ *be such that* $|(\lambda_0 - A(t)^*)^\beta Q_t (\lambda_0 - A(t))^\beta|_{\mathcal{L}(H)} \leq r_0$. *Then there exists* $\tau_0 = \tau_0(r_0, \beta) > 0$ *such that the Riccati equation*

$$Q(s) = U(t,s)^* Q_t U(t,s) + \int_s^t U(r,s)^* [C(r)^* C(r) - Q(r)(\lambda_0 - A(r))G(r)N(r)^{-1}G(r)^*(\lambda_0 - A(r)^*)Q(r)]U(r,s)\,dr \tag{2.11}$$

has a unique solution $Q(\cdot)$ *in* $[t - \tau_0, t[$ *such that* $Q(s) \in \Sigma^+(H)$ *for each* $s \in [t - \tau_0, t[$ *and*

$$|(\lambda_0 - A(s)^*)^{1-\alpha} Q(s)|_{\mathcal{L}(H)} \leq c(\beta, \alpha, r_0)(t-s)^{\beta+\alpha-1} \quad \forall s \in [t - \tau_0, t[. \tag{2.12}$$

LEMMA 2.6. *Assume Hypotheses* 1.1 – 1.3, 1.5 *and* 2.2. *Then for each* $\mu \in]0,1[$ *and each interval* $[a,b] \subset I$, *we have*

$$\sup_{a \leq s \leq b, T > b+2} |(\lambda_0 - A(s)^*)^\mu P_T(s)|_{\mathcal{L}(H)} := c_2(\mu, [a,b]) < \infty. \tag{2.13}$$

Proof. Using Theorem 2.5 and estimate (2.8) (having fixed any $\beta \in]0, 1/2[$), we find two constants $\tau \in]0,1]$ and $c' > 0$, depending only on $[a,b]$ and on the constant c_1 of (2.8), such that for each $s, t \in [a, b+1]$ with $s < t$ and $t - s \leq \tau$ we have

$$(t-s)^{1-\alpha-\beta} |(\lambda_0 - A(s)^*)^{1-\alpha} P_T(s)|_{\mathcal{L}(H)} \leq c'. \tag{2.14}$$

Fix $s \in [a,b]$ and choose $t_1 := s + \tau/2 < t_2 := s + \tau \leq b+1$. As $P_T(s)$, in particular, solves (2.2) for $s \leq t_1$ with final datum $P_T(t_1)$, we deduce

$$P_T(s) = U(t_1, s)^* P_T(t_1) U(t_1, s) + \int_s^{t_1} U(r,s)^* C(r)^* C(r) U(r,s)\,ds - \int_s^{t_1} U(r,s)^* [(\lambda_0 - A(r)^*)^{1-\alpha} P_T(r)]^* K(r) [(\lambda_0 - A(r)^*)^{1-\alpha} P_T(r)] U(r,s)\,dr.$$

Applying the operator $(\lambda_0 - A(s)^*)^\mu$ to both sides and using Hypothesis 1.2(ii) (and the estimate (2.12) with $t = t_2$), we get

$$|(\lambda_0 - A(s)^*)^\mu P_T(s)|_{\mathcal{L}(H)} \leq \frac{c}{(t_1 - s)^\mu} |P_T(t_1)|_{\mathcal{L}(H)} + \int_s^{t_1} \frac{c}{(r-s)^\mu}\,ds \sup_{r \in [a,b]} |C(r)|^2_{\mathcal{L}(H,V)} + c \int_s^{t_1} \frac{c_2^2}{(r-s)^\mu (t_2 - r)^{2(1-\alpha-\beta)}}\,dr \leq c(\tau, c', \beta, [a,b], \mu),$$

and the proof is complete. □

2.3. Existence of solutions of the Riccati equation. We prove here the following result.

THEOREM 2.7. *Assume Hypotheses* 1.1 – 1.3, 1.5, *and* 2.2. *Then*

(i) *for each* $\mu \in]0,1[$ *and* $t \in I$ *the operator* $P_\infty(t)$, *defined in Lemma* 2.1, *maps* H *into the domain of* $(\lambda_0 - A(t)^*)^\mu$ *and*

$$(\lambda_0 - A(t)^*)^\mu P_T(t) \to (\lambda_0 - A(t)^*)^\mu P_\infty(t) \quad \textit{strongly as } T \to \infty; \tag{2.15}$$

(ii) *the operator* $P_\infty(\cdot)$ *is a solution of equation* (2.1); *i.e., for each* $t,\ s \in I$ *with* $s \le t$ *we have*

$$\begin{aligned} P_\infty(s) &= U(t,s)^* P_\infty(t) U(t,s) \\ &\quad + \int_s^t U(r,s)^*[C(r)^*C(r) \\ &\qquad\qquad - P_\infty(r)(\lambda_0 - A(r))G(r)N(r)^{-1}G(r)^*(\lambda_0 - A(r)^*)P_\infty(r)] \\ &\quad \times U(r,s)\,dr. \end{aligned}$$

Remark 2.8. (i) It is possible to show that the convergence in (2.15) is uniform in t over bounded intervals. However, we omit the proof because we do not need this result in what follows.

(ii) In the special case where the resolvent of $A(t)$ is compact, the proof of Theorem 2.7 is very simple (see Theorem 2 in [F1]). However, we prefer to deal here with the general case, where the proof is considerably more difficult, since it is easy to construct examples with lack of compactness (see the remark at the end of Example 4.1).

The proof of Theorem 2.7 is based on the following lemma, which has also other applications (see, for instance, [F3]). Consider the following Riccati equation for $s \in [t_0, t[$, with fixed $t_0, t \in I$:

$$\begin{aligned} Q(s) &= U(t,s)^* \bar{Q}_t U(t,s) \\ &\quad + \int_s^t U(r,s)^*[C(r)^*C(r) \\ &\qquad - Q(r)(\lambda_0 - A(r)^*)G(r)N(r)^{-1}G(r)^*(\lambda_0 - A(r)^*)Q(r)]U(r,s)dr \end{aligned} \tag{2.16}$$

under the assumption that $\bar{Q}_t \in \Sigma^+(H)$ and that the operator $(\lambda_0 - A(t)^*)^\beta \bar{Q}_t(t)(\lambda_0 - A(t))^\beta$ belongs to $\mathcal{L}(H)$; denote by $Q(s;\bar{Q}_t)$ its solution in a suitable interval $[t - r_0, t[$, given by Theorem 2.5.

LEMMA 2.9. *Assume Hypotheses* 1.1 – 1.3, 1.5, *and* 2.2, *and fix* $\beta \in]0,1/2[$. *Let* $\{Q_{t,n}\}_{n>n_0}$ *be a family of operators in* $\Sigma^+(H)$ *such that*

(i) *there exists a constant* $c_3 > 0$ *such that*

$$|(\lambda_0 - A(t)^*)^\beta Q_{t,n}(\lambda_0 - A(t))^\beta|_{\mathcal{L}(H)} \le c_3 \quad \forall n > n_0;$$

(ii) $Q_{t,n}$ *converges strongly as* $n \to \infty$ *to an operator* $Q_t \in \Sigma^+(H)$ *for which the operator* $(\lambda_0 - A(t)^*)^\beta Q_t (\lambda_0 - A(t))^\beta$ *also belongs to* $\mathcal{L}(H)$.

Then for each $s \in [t - r_0, t[$

$$(\lambda_0 - A(s)^*)^{1-\alpha} Q(s; Q_{t,n}) \to (\lambda_0 - A(s)^*)^{1-\alpha} Q(s; Q_t) \quad \textit{strongly as } n \to \infty.$$

Proof of Lemma 2.9. This proof is adapted from [F3]. We denote by $[\Gamma(Q)](s)$ and $[\Gamma_n(Q)](s)$ the right-hand side of (2.16) when we consider the final data Q_t and $Q_{t,n}$, respectively. Thus equation (2.16) can be rewritten as

$$Q(s) = [\Gamma(Q)](s), \qquad s \in [t_0, t], \tag{2.17}$$

if the final datum is Q_t, and

$$Q(s) = [\Gamma_n(Q)](s), \qquad s \in [t_0, t], \tag{2.17$_n$}$$

if the final datum is $Q_{t,n}$.

Next set $\gamma := \min\{1-\alpha-\beta, \beta\}$ and consider the following space:

$$\begin{aligned} X(t-r_0,t) := \{ Q : [t-r_0,t] \to \Sigma^+(H) :\ & Q(s) \text{ maps } H \text{ into the domain of} \\ & (\lambda_0 - A(s)^*)^{1-\alpha} \text{for each } s \in [t-r_0, t[, \text{ and both } (\lambda_0 - A(\cdot)^*)^{1-\alpha}Q(\cdot) \\ & \text{and its adjoint are strongly continuous in } [t-r_0,t[; \text{ moreover} \\ & |(\lambda_0 - A(s)^*)^{1-\alpha}Q(s)|_{\mathcal{L}(H)} \\ & \quad \le c(Q)[1+(t-s)^{-(1-\alpha-\beta)}] \quad \forall\, s \in [t-r_0, t[, \\ & |(\lambda_0 - A(s)^*)^{1-\alpha}Q(s)U(s,r)(\lambda_0 - A(s))^{\beta}|_{\mathcal{L}(H)} \\ & \quad \le c(Q)(t-r)^{\gamma}[1+(t-s)^{-(1-\alpha-\beta)}](s-r)^{-\beta} \\ & \qquad \forall\, s \in [t-r_0, t[, \forall\, r \in [0,s[\}, \end{aligned}$$

endowed with the norm

$$|Q|_X := \max\{A, B\},$$

where

$$A := \sup_{s \in [t-r_0, t]} (t-s)^{1-\alpha-\beta} |(\lambda_0 - A(s)^*)^{1-\alpha} Q(s)|_{\mathcal{L}(H)},$$

$$\begin{aligned} B := \sup_{t-r_0 \le r < s < t} & \frac{(s-r)^{\beta}[1+(t-s)^{1-\alpha-\beta}]}{(t-r)^{\gamma}} \\ & \times |(\lambda_0 - A(s)^*)^{1-\alpha} Q(s) U(s,r)(\lambda_0 - A(r))^{\beta}|_{\mathcal{L}(H)}. \end{aligned}$$

It can be easily shown, arguing as in the Appendix below, that the maps Γ_n and Γ are equicontractions on any sufficiently large ball of $X(t_0,t)$, provided that r_0 is suitably small. Hence, possibly replacing the constant c_3 in (i) by a larger one, we may say that Γ and Γ_n are equicontractions on the ball

$$B(t-r_0, t; c_3) := \{Q \in X(t-r_0,t) : |Q|_X \le c_3\}. \tag{2.18}$$

Now we apply the contraction principle to equations (2.17$_n$) in the ball $B(t-r_0,t;c_3)$ uniformly with respect to n. Namely, let $Q_0(\cdot)$ be the initial iteration point in $B(t-r_0,t;c_3)$ for Γ and Γ_n; then, remarking that

$$\lim_{k\to\infty} |Q(\cdot\,, Q_{t,n}) - [(\Gamma_n)^k(Q)]|_X = 0 \quad \text{uniformly with respect to } n, \tag{2.19}$$

$$\lim_{k\to\infty} |Q(\cdot\,, Q_t) - [(\Gamma)^k(Q)]|_X = 0, \tag{2.20}$$

we deduce for each $k \in \mathbb{N}^+$, $s \in [t-r_0, t]$, and $x \in H$

$$\begin{aligned} & \|(\lambda_0 - A(s)^*)^{1-\alpha} Q(s; Q_{t,n})x - (\lambda_0 - A(s)^*)^{1-\alpha} Q(s; Q_t)x\|_H \\ & \le |Q(\cdot; Q_{t,n}) - [(\Gamma_n)^k(Q_0)]|_X \, \|x\|_H \\ & + \|(\lambda_0 - A(s)^*)^{1-\alpha}[(\Gamma_n)^k(Q_0)](s)x - (\lambda_0 - A(s)^*)^{1-\alpha}[(\Gamma)^k(Q_0)](s)x\|_H \\ & + |[(\Gamma)^n(Q_0)] - Q(\cdot\,; Q_t)|_X \, \|x\|_H. \end{aligned} \tag{2.21}$$

Thus we just need to show that for each $k \in \mathbb{N}^+$ and $s \in [t - r_0, t]$

$$(\lambda_0 - A(s)^*)^{1-\alpha}[(\Gamma_n)^k(Q_0)](s) \to (\lambda_0 - A(s)^*)^{1-\alpha}[(\Gamma)^k(Q_0)](s) \quad \text{strongly as } n \to \infty.$$

This result is obviously true when $k = 1$, since for each $s \in [t - r_0, t]$ and $x \in H$

$$\begin{aligned} &(\lambda_0 - A(s)^*)^{1-\alpha}[[\Gamma_n(Q_0)](s) - [\Gamma(Q_0)](s)]x \\ &\quad = (\lambda_0 - A(s)^*)^{1-\alpha} U(t,s)^*(Q_{t,n} - Q_t)U(t,s)x; \end{aligned} \tag{2.22}$$

on the other hand, if the result is true for the integer $k - 1$, then we have

$$\begin{aligned} &(\lambda_0 - A(s)^*)^{1-\alpha}[[(\Gamma_n)^k(Q_0)](s) - [(\Gamma)^k(Q_0)](s)]x \\ &\quad = (\lambda_0 - A(s)^*)^{1-\alpha} U(t,s)^*(Q_{t,n} - Q_t)U(t,s)x \\ &\qquad - \int_s^t (\lambda_0 - A(s)^*)^{1-\alpha} U(r,s)^*[(\lambda_0 - A(r)^*)^{1-\alpha}(\Gamma_n)^{k-1}(Q_0)(r)]^* \\ &\qquad \times K(r)(\lambda_0 - A(r)^*)^{1-\alpha}[(\Gamma_n)^{k-1}(Q_0)(r) - (\Gamma)^{k-1}(Q_0)(r)]U(r,s)x\,dr \\ &\qquad - \int_s^t (\lambda_0 - A(s)^*)^{1-\alpha} U(r,s)^*[(\lambda_0 - A(r)^*)^{1-\alpha}[(\Gamma_n)^{k-1}(Q_0)(r) - (\Gamma)^{k-1}(Q_0)(r)]]^* \\ &\qquad \times K(r)(\lambda_0 - A(r)^*)^{1-\alpha}(\Gamma)^{k-1}(Q_0)(r)U(r,s)x\,dr, \end{aligned} \tag{2.23}$$

and remarking that $(\Gamma_n)^{k-1}(Q_0)$ and $(\Gamma)^{k-1}(Q_0)$ belong to $B(t - r_0, t; c_3)$ by the induction hypothesis we get the result for the integer k. This proves Lemma 2.9. □

Proof of Theorem 2.7. Fix $t, t_0 \in I$ with $t > t_0$. We have to show that

$$P_\infty(s) = Q(s, P_\infty(t)) \quad \forall\, s \in [t_0, t[. \tag{2.24}$$

We apply Lemma 2.9 with $Q_t = P_\infty(t)$, $Q_{t,n} = P_n(t)$ (i.e., the solution of equation (2.2) with final time $T = n$); this is allowed by Lemmas 2.4 and 2.1. As a consequence we get

$$(\lambda_0 - A(s)^*)^{1-\alpha}[Q(s, P_n(t)) - Q(s, P_\infty(t))] \to 0 \quad \text{strongly in } [t - r_0, t[\text{ as } n \to \infty.$$

On the other hand we have

$$Q(s, P_n(t)) \equiv P_n(s) \to P_\infty(s) \quad \text{strongly in } [t - r_0, t] \text{ as } n \to \infty,$$

and (2.24) follows for each $s \in [t - r_0, t[$. The same result for all $s \in [t_0, t[$ follows now by standard uniqueness arguments. □

2.4. Minimality property of P_∞. Let $\hat{P} \in C_s(I, \Sigma^+(H))$ be any solution of equation (2.1), and consider the evolution operator $\hat{\Phi}(t, r)$ corresponding to $\hat{P}(\cdot)$, i.e., the operator-valued function defined for r, $t \in I$, $r \le t$, by the following equation:

$$\begin{aligned} \hat{\Phi}(t,r) &= U(t,r) \\ &\quad - \int_r^t U(t,s)(\lambda_0 - A(s))G(s)N(s)^{-1}G(s)^*(\lambda_0 - A(s)^*)\hat{P}(s)\hat{\Phi}(s,r)\,ds. \end{aligned} \tag{2.25}$$

We know (see, e.g., [G], [LT1]) that for each $s,\ t \in I$ with $s \leq t$ the following identities hold:

$$\hat{P}(s) = \hat{\Phi}(t,s)^* \hat{P}(t) \hat{\Phi}(t,s) + \int_s^t \hat{\Phi}(v,s)^*$$
$$\times [C(v)^* C(v) + \hat{P}(v)(\lambda_0 - A(v)) G(v) N(v)^{-1} G(v)(\lambda_0 - A(v)^*) \hat{P}(v)] \hat{\Phi}(v,s)\, dv, \tag{2.26}$$

$$\hat{P}(s) = U(t,s)^* \hat{P}(t) \hat{\Phi}(t,s) + \int_s^t U(v,s)^* C(v)^* C(v) \hat{\Phi}(v,s)\, dv. \tag{2.27}$$

We have the following proposition.

PROPOSITION 2.10. *Assume Hypotheses* 1.1 – 1.3, *and* 1.5. *Then*

(i) *equation* (2.1) *has a solution if and only if Hypothesis* 2.2 *holds*;

(ii) *if this is the case, the function* $P_\infty(\cdot)$ *defined in Lemma* 2.1 *is the minimal solution of equation* (2.1); *i.e., for any solution* $\hat{P}(\cdot)$ *of equation* (2.1) *we have*

$$P_\infty(t) \leq \hat{P}(t) \quad \forall t \in I.$$

Proof. (i) Theorem 2.7 shows the if part of the proposition. Conversely, if $\hat{P}(\cdot)$ is a solution of (2.1) and $y_0 \in H,\ t_1 \in I$ are given, we consider the control

$$\hat{u}(t) = -N(t)^{-1} G(t)^* (\lambda_0 - A(t)^*) \hat{P}(t) \hat{\Phi}(t,t_1) y_0, \qquad t \geq t_1.$$

A simple calculation shows that the corresponding state is $\hat{y}(t) = \hat{\Phi}(t,t_1)y_0$. Using equation (2.26) we easily obtain

$$(\hat{P}(t_1)y_0, y_0)_H = (\hat{P}(t)\hat{y}(t), \hat{y}(t))_H + \int_{t_1}^t [\|C(v)\hat{y}(v)\|_V^2 + (N(v)\hat{u}(v), \hat{u}(v))_U]\, dv.$$

Hence for each $t \geq t_1$ we have

$$\int_{t_1}^t [\|C(v)\hat{y}(v)\|_V^2 + (N(v)\hat{u}(v), \hat{u}(v))_U]\, dv \leq (\hat{P}(t_1)y_0, y_0)_H;$$

consequently

$$J_{t_1,\infty}(\hat{u}) \leq (\hat{P}(t_1)y_0, y_0)_H \leq |\hat{P}(t_1)|_{\mathcal{L}(H)} \|y_0\|_H^2. \tag{2.28}$$

By the local coercivity of $N(\cdot)$ (Hypothesis 1.5) we then get $\hat{u} \in L^2_{\text{loc}}(t_1, \infty; U)$, so condition (2.4_{t_1}) holds.

(ii) Using (2.3) we obtain

$$(P_T(t_1)y_0, y_0)_H \leq J_{t_1,T}(u) \quad \forall t_1 \in I,\ \forall T > t_1,\ \forall y_0 \in H,\ \forall u \in L^2_{\text{loc}}(t_1, \infty; U),$$

and consequently we have

$$(P_T(t_1)y_0, y_0)_H \leq J_{t_1,\infty}(u) \quad \forall t_1 \in I,\ \forall T > t_1,\ \forall y_0 \in H,\ \forall u \in L^2_{\text{loc}}(t_1, \infty; U);$$

letting $T \to \infty$ we obtain

$$(P_\infty(t_1)y_0, y_0)_H \leq J_{t_1,\infty}(u) \quad \forall t_1 \in I,\ \forall y_0 \in H,\ \forall u \in L^2_{\text{loc}}(t_1, \infty; U) \tag{2.29}$$

so that, in particular, by (2.28)

$$(P_\infty(t_1)y_0, y_0)_H \leq J_{t_1,\infty}(\hat{u}) \leq (\hat{P}(t_1)y_0, y_0)_H \quad \forall t_1 \in I,\ \forall y_0 \in H, \tag{2.30}$$

and the result follows. □

2.5. Synthesis of the infinite-horizon LQR problem. We use the properties of the operator $P_\infty(\cdot)$ to solve the problem of the synthesis. We have the following theorem.

THEOREM 2.11. *Assume Hypotheses* 1.1 – 1.3, 1.5, *and* 2.2. *Let* $t_0 \in I$ *and* $y_0 \in H$ *be given. Then*

(i) *there exists a unique optimal control* $u^* \in L^2_{\rm loc}(t_0,\infty;U)$ *for problem* (1.7);

(ii) *if* (u^*,y^*) *is the optimal pair and* $P_\infty(\cdot)$ *is defined by Lemma* 2.1, *then*

$$u^*(t) = -N(t)^{-1}G(t)^*(\lambda_0 - A(t)^*)P_\infty(t)y^*(t) \quad \forall t \ge t_0;$$

(iii) *the optimal cost is*

$$J_{t_0,\infty}(u^*) = (P_\infty(t_0)y_0, y_0)_H;$$

(iv) *the optimal state is given by*

$$y^*(t) = \Phi_\infty(t,t_0)y_0,$$

where $\Phi_\infty(t,s)$ *is the evolution operator defined by equation* (2.25) *with* $P_\infty(\cdot)$ *in place of* $\hat P(t)$.

Proof. Given $t_0 \in I$ and $y_0 \in H$, set

$$u^*(t) := -N(t)^{-1}G(t)^*(\lambda_0 - A(t)^*)P_\infty(t)\Phi_\infty(t,t_0)y_0, \qquad t \ge t_0;$$

by the same arguments in the proof of Proposition 2.10 we easily see that $u^* \in L^2_{\rm loc}(t_0,\infty;U)$ is an admissible control with respect to t_0, whereas $y^*(t) := \Phi_\infty(t,t_0)y_0$ is the state corresponding to u^*. By (2.29) and (2.30), with $\hat P$ replaced by P_∞, $\hat u$ by u^*, and t_1 by t_0, we obtain

$$(P_\infty(t_0)y_0, y_0)_H = \min_u J_{t_0,\infty}(u) = J_{t_0,\infty}(u^*);$$

i.e., u^* is an optimal control.

Finally it is clear that Hypothesis 1.5 on $N(\cdot)$ implies the strict convexity of $J_{t_0,\infty}$, so the optimal control is unique. □

3. Further properties of solutions of the Riccati equation.

3.1. Bounded solutions. In many cases it is important to know whether some bounded solution of Riccati equation (2.1) exists. In order to obtain boundedness we have to assume some uniformity in Hypothesis 2.2. Thus, following [DI3], we introduce a stronger version of that assumption.

Hypothesis 3.1. There exists a constant $\bar c > 0$ such that for each $t \in I$ and $y_0 \in H$ there exists a control $u \in L^2_{\rm loc}(t,\infty;U)$ such that

$$J_{t,\infty}(u) < \bar c\,\|y_0\|_H^2.$$

We have the following proposition.

PROPOSITION 3.2. *Assume Hypotheses* 1.1 – 1.3, *and* 1.5. *Then there exists a bounded solution of Riccati equation* (2.1) *if and only if Hypothesis* 3.1 *holds.*

Proof. If Hypothesis 3.1 holds, then in particular Hypothesis 2.2 holds too, so that by the results of §2 the function $P_\infty(\cdot)$, defined in Lemma 2.1, is a solution of equation (2.1). In addition we have for each $t \in I$, $T > t$, and $y \in H$

$$(P_T(t)y, y)_H \le \min_u J_{t,T}(u) \le \bar c\,\|y\|_H^2;$$

thus letting $T \to \infty$ we get

$$(P_\infty(t)y, y)_H \le \bar{c}\,\|y\|_H^2 \quad \forall\, t \in I, \forall\, y \in H,$$

and recalling that $P_\infty(t) \ge 0$ we obtain

$$\sup_{t\in I} |P_\infty(t)|_{\mathcal{L}(H)} \le \bar{c}.$$

Conversely, if there exists a bounded solution $\hat{P}(\cdot)$ of (2.1), then we can repeat the argument of Proposition 2.10(i), and (2.28) shows that Hypothesis 3.1 holds with

$$\bar{c} = \sup_{t\in I} |P_\infty(t)|_{\mathcal{L}(H)}.$$

Remark 3.3. Hypothesis 3.1 is fulfilled in several cases.

(i) In the periodic case of [L1], [F], [DI2] (see §3.4), if condition (2.4_t) is satisfied for some $t \in \mathbb{R}$, then Hypothesis 3.1 holds.

(ii) If system (1.4) is stabilizable, i.e., there exists $K \in L^\infty(I, \mathcal{L}(H, U))$ such that the evolution operator associated with the family $\{[A-(\lambda_0-A)GK](t)\}$ is stable, and in addition the operators $C(\cdot)$, $N(\cdot)$ are bounded, then Hypothesis 3.1 holds (compare with the comments after Hypothesis (H3) in [DI3]).

(iii) In Example 4.2 below, Hypothesis 3.1 holds naturally.

3.2. Stability of the perturbed evolution operator. Theorem 2.11 shows, under suitable assumptions, the existence of a unique optimal pair (u^*, y^*) for problem (1.7) with $t_0 = 0$; we also know that

$$y^* = \Phi_\infty(\cdot, 0)y_0, \qquad u^* = -[N^{-1}G^*(\lambda_0 - A^*)P_\infty y^*].$$

(From now on we will drop the indication of the variable t if unnecessary.) Here $P_\infty(t)$ is the minimal solution of Riccati equation (2.1) and $\Phi_\infty(t,s)$ is the evolution operator associated to the closed-loop operator family

$$\{A - G(\lambda_0 - A)N^{-1}G^*(\lambda_0 - A^*)P_\infty\}$$

by the integral equation (2.25); in other words, $\Phi_\infty(t,s)$ is the solution, for $t, s \in I$, $t \ge s$, of

$$\begin{aligned}\Phi_\infty(t,s) &= U(t,s)\\ &\quad - \int_s^t U(t,r)(\lambda_0 - A(r))G(r)N(r)^{-1}G(r)^*(\lambda_0 - A(r)^*)P_\infty(r)\Phi_\infty(r,s)\,dr.\end{aligned} \tag{3.1}$$

In this subsection, following the ideas of [DI1], [DI2] and [BDDM, Chap. IV.2, §3.2], we will prove a stability result for $y^*(t) = \Phi_\infty(t,0)$ as $t \to \infty$. In order to do this we have to assume that Hypotheses 1.2, 1.3, and 1.5 hold uniformly over the time interval I. More precisely we formulate the following hypothesis.

Hypothesis 3.4. (i) The evolution operators $U(t,s)$ and $U(t,s)^*$ are strongly continuous in $\bar{\Delta}_I$, where $\Delta_I := \{(t,s) \in I^2 \,;\, t > s\}$ and there exist $M_0 > 0$ and $\omega \in \mathbb{R}$ such that

$$|U(t,s)|_{\mathcal{L}(H)} + |U(t,s)^*|_{\mathcal{L}(H)} \le M_0 \exp(\omega_0(t-s)) \quad \forall\,(t,s) \in \Delta_I;$$

(ii) for each β, $\mu \in [-1,1]$ and $(t,s) \in \Delta_I$, the operators

$$(\lambda_0 - A(t))^\beta U(t,s)(\lambda_0 - A(s))^{-\mu}, \qquad (\lambda_0 - A(s)^*)^\beta U(t,s)^*(\lambda_0 - A(t)^*)^{-\mu}$$

have continuous extensions to H, the maps

$$(t,s) \to (\lambda_0 - A(t))^{\beta} U(t,s)(\lambda_0 - A(s))^{-\mu}, \quad (t,s) \to (\lambda_0 - A(s)^*)^{\beta} U(t,s)^* (\lambda_0 - A(t)^*)^{-\mu}$$

are strongly continuous, and there exists $M_{\beta,\mu} > 0$ such that

$$\begin{aligned} &|(\lambda_0 - A(t))^{\beta} U(t,s)(\lambda_0 - A(s))^{-\mu}|_{\mathcal{L}(H)} + |(\lambda_0 - A(s)^*)^{\beta} U(t,s)^* (\lambda_0 - A(t)^*)^{-\mu}|_{\mathcal{L}(H)} \\ &\quad \le M_{\beta,\mu} [(t-s)^{\mu-\beta} + 1] \exp(\omega_0 (t-s)) \quad \forall (t,s) \in \Delta_I. \end{aligned}$$

Hypothesis 3.5. There exists $\alpha \in]0,1]$ such that, for each $t \in I$, $G(t)$ maps U into the domain of $(\lambda_0 - A(t))^{\alpha}$, and $(\lambda_0 - A(\cdot))^{\alpha} G(\cdot)$ is strongly measurable and bounded over I.

Hypothesis 3.6. We have

$$C(\cdot) \in L^{\infty}(I; \mathcal{L}(H,V)), \qquad N(\cdot) \in L^{\infty}(I; \Sigma^{++}(U)).$$

(This means that there exists $\nu > 0$ such that $N(t) \ge \nu$, $\forall t \in I$.)

Under the assumption listed above we can revisit the proof of Lemma 2.6, and we get the following lemma.

LEMMA 3.7. *Assume Hypotheses* 1.1, 3.4–3.6, *and* 3.1. *Then for each* $\mu \in]0,1[$ *we have*

$$\sup_{s \in I} |(\lambda_0 - A(s)^*)^{\mu} P_{\infty}(s)|_{\mathcal{L}(H)} < \infty.$$

Proof. Fix $t \in I$, $0 \le \beta < 1/2$, $x \in D[(\lambda_0 - A(t))^{\beta}]$, and set $y_1 := U(t+1,t)(\lambda_0 - A(t))^{\beta} x$. By Hypothesis 3.1 there exists a control $\bar{u} \in L^2_{\mathrm{loc}}(t+1,\infty;U)$ such that

$$J_{t+1,\infty}(\bar{u}) \le \bar{c}\, \|y_1\|_H^2,$$

and by Hypothesis 3.4(ii) we also have

$$J_{t+1,\infty}(\bar{u}) \le c\, \|x\|_H^2. \tag{3.2}$$

Consider the control $\tilde{u} \in L^2_{\mathrm{loc}}(t,\infty;U)$ defined by

$$\tilde{u}(s) = \begin{cases} 0 & \text{if } t \le s < t+1, \\ \bar{u}(s) & \text{if } t+1 \le s < \infty. \end{cases}$$

Using Theorem 2.11(iii) and (3.2) we have

$$\begin{aligned} &(P_{\infty}(t)(\lambda_0 - A(t))^{\beta} x, (\lambda_0 - A(t))^{\beta} x)_H \le J_{t,\infty}(\tilde{u}) \\ &\quad = \int_t^{t+1} \|C(s) U(s,t)(\lambda_0 - A(t))^{\beta} x\|_V^2 \, ds + J_{t+1,\infty}(\bar{u}) \\ &\quad \le \left(\|C\|^2_{L^{\infty}(I;\mathcal{L}(V,H))} \int_t^{t+1} |U(s,t)(\lambda_0 - A(t))^{\beta}|^2_{\mathcal{L}(H)} \, ds + c \right) \|x\|_H^2 \\ &\quad \le \left(|C|^2_{L^{\infty}(I;\mathcal{L}(V,H))} M^2_{\beta,0} \exp(\omega_0) + c \right) \|x\|_H^2 \le c \|x\|_H^2. \end{aligned}$$

Recalling that $P_{\infty}(t) \ge 0$, the above estimate shows that

$$\sup_{t \in I} |(\lambda_0 - A(t)^*)^{\beta} P_{\infty}(t) (\lambda_0 - A(t))^{\beta}|_{\mathcal{L}(H)} =: L < +\infty.$$

We now repeat the argument of the proof of Lemma 2.6; invoking Theorem 2.5 and noting that our assumptions are uniform in t now, we find two constants $\tau \in]0,1]$ and $c' > 0$, depending on L and β but independent of $t \in I$, for which the analogue of (2.14) holds, i.e.,

$$(t-s)^{1-\alpha-\beta}|(\lambda_0 - A(s)^*)^{1-\alpha}P_\infty(s)|_{\mathcal{L}(H)} \le c' \quad \forall s \in I \cap [t-\tau, t[.$$

Arguing as in the proof of Lemma 2.6, one arrives easily at the estimate

$$\sup_{s\in I} |(\lambda_0 - A(s)^*)^\mu P_\infty(s)|_{\mathcal{L}(H)} \le c(\tau, c', \beta, L, \mu),$$

which concludes the proof. □

Using the result of Lemma 3.7 it is easy to show that the evolution operator $\Phi_\infty(t,s)$ has an exponential growth. Namely, we have the following lemma.

LEMMA 3.8. *Under Hypotheses* 1.1, 3.4–3.6, *and* 3.1, *there exist* $M_1 > 0$ *and* $\omega_1 > \omega_0$ *such that*

$$|\Phi_\infty(t,s)|_{\mathcal{L}(H)} \le M_1 \exp(\omega_1(t-s)) \quad \forall (t,s) \in \bar{\Delta}_I.$$

Proof. The result follows easily by equation (3.1), using our assumptions and the result of Lemma 3.7. □

It is important, in some applications, to give conditions under which the evolution operator Φ_∞ is exponentially stable, i.e., there exist $M > 0$ and $\gamma > 0$ such that

$$\|\Phi_\infty(t,0)\|_{\mathcal{L}(H)} \le M \exp(-\gamma t).$$

A simple situation where this occurs is when the operator $C(s)$ is invertible for each $s \in I$, and C^{-1} belongs to $L^\infty(I; \mathcal{L}(V,H))$ (see [BDDM]); indeed, if this is the case, we fix $x \in H$ and argue as in the proof of Proposition 2.10, replacing t_1 with t_0, $\hat{\Phi}$ with Φ_∞, $\hat{P}$ with P_∞, $\hat{y}$ with $y^* := \Phi(\cdot,0)x$, and $\hat{u}$ with u^*. Then we obtain, for each $t \in I$,

$$\int_{t_0}^{t} [\|Cy^*\|_V^2 + \|N^{-1/2}G^*(\lambda_0 - A^*)P_\infty y^*\|_H^2]\, ds \le (P_\infty(0)x, x)_H \le \bar{c}\, \|x\|_H^2.$$

So we have

$$C(\cdot)\Phi_\infty(\cdot,0)x = Cy^* \in L^2(t_0,\infty;V), \tag{3.3}$$

$$N^{-1/2}G^*(\lambda_0 - A^*)P_\infty y^* \in L^2(t_0,\infty;H), \tag{3.4}$$

and by (3.3) we deduce that $\Phi_\infty(\cdot,0)x \in L^2(I;H)$; thus by the classical results of Datko [D], we obtain the exponential stability of Φ_∞.

A sufficient condition yielding the same property, even if C is not invertible, is given by the following detectability condition [F1], [DI1], [DI3].

Hypothesis 3.9. The family $\{(A,C)\}$ is detectable; this means that there exists a mapping $K : I \to \mathcal{L}(V,H)$, strongly measurable and bounded, such that the evolution operator $U_{A-KC}(t,s)$ associated with $\{A - KC\}$ is stable; i.e., there exist two constants $M_2 > 0$, $\omega_2 > 0$ such that

$$|U_{A-KC}(t,s)|_{\mathcal{L}(H)} \le M_2 \exp(-\omega_2(t-s)) \quad \forall (t,s) \in \Delta_I. \tag{3.5}$$

LEMMA 3.10. *Assume Hypotheses* 3.4, 3.6, *and* 3.9. *Then there exists a constant* $c > 0$ *such that for each* $(t,s) \in \Delta_I$ *the operator* $U_{A-KC}(t,s)(\lambda_0 - A(s))^{1-\alpha}$ *has a continuous extension to* H *and*

$$|U_{A-KC}(t,s)(\lambda_0 - A(s))^{1-\alpha}|_{\mathcal{L}(H)} \le c(t-s)^{\alpha-1}\exp(-\omega_2(t-s)) \quad \forall (t,s) \in \Delta_I.$$

Proof. By Hypotheses 3.6 and 3.9 it follows that $KC \in L^\infty(I;\mathcal{L}(H))$ and the construction of $U_{A-KC}(t,s)$ is standard. Next, for each $s \in I$ set

$$V(t,s)x := U_{A-KC}(t,s)(\lambda_0 - A(s))^{1-\alpha}x, \qquad t \geq s,\ x \in D((\lambda_0 - A(s))^{1-\alpha});$$

then it is immediately seen that

$$V(t,s)x = U(t,s)(\lambda_0 - A(s))^{1-\alpha}x + \int_s^t U(t,r)K(r)C(r)V(r,s)x\,dr, \qquad (t,s) \in \Delta_I.$$

Using Hypothesis 3.4 we easily get

$$\|V(t,s)x\|_H \leq c(t-s)^{\alpha-1}\|x\|_H \quad \forall\,(t,s) \in \Delta_I \text{ with } t-s \leq 1;$$

hence, taking (3.3) into account we easily get the result. □

As a simple consequence of the above lemma we have the following theorem.

THEOREM 3.11. *Assume Hypotheses* 1.1, 3.1, 3.4 – 3.6, *and* 3.9. *Then* $\Phi_\infty(\cdot\,,0)$ *is exponentially stable.*

Proof. We have

$$U_{A-KC}(t,s) = U(t,s) + \int_s^t U(t,r)K(r)C(r)U_{A-KC}(r,s)\,dr, \quad (t,s) \in I. \tag{3.6}$$

Comparing with (3.1) we easily obtain for each $(t,s) \in I$

$$\Phi_\infty(t,s) = U_{A-KC}(t,s) - \int_s^t U_{A-KC}(t,r)[KC - (\lambda_0 - A)GN^{-1}G^*(\lambda_0 - A^*)P_\infty](r)\Phi_\infty(r,s)\,dr.$$

Now using (3.5), Lemma 3.10, Hypothesis 3.6, and the boundedness of K, by Young's inequality we deduce that $\Phi_\infty(\cdot\,,0) \in L^2(I;H)$, and finally the exponential stability is a consequence of the results of Datko [D]. □

3.3. Uniqueness of the solution of the Riccati equation. By Proposition 2.10 it is clear that if a bounded solution $\hat{P}$ of equation (2.1) exists, then P_∞ also is bounded. Under suitable assumptions on the LQR system we are able to show uniqueness of bounded solutions.

We have the following result, which generalizes [F1, Thm. 4].

THEOREM 3.12. *Assume Hypotheses* 1.1 – 1.3, 1.5, *and* 3.1; *in addition, assume that the optimal trajectory* $y^*(\cdot)$ *is stable. Then the only bounded solution of equation* (2.1) *is* P_∞.

Proof. By Proposition 3.2 we know that P_∞ is a bounded solution of (2.1). Now let $\hat{P}$ be another bounded solution of (2.1); by Proposition 2.10 we know that

$$P_\infty(t) \leq \hat{P}(t) \quad \forall t \in I,$$

so it is sufficient to prove the converse inequality.

Fix $y_0 \in H$ and $t_1 \in I$; by [AFT, Thm. 3.14] we deduce that

$$(\hat{P}(t_1)y_0, y_0)_H \leq J_{t_1,t}(u) + (\hat{P}(t)y(t), y(t))_H \quad \forall u \in L^2(t_1,t;U),\ \forall t > t_1, \tag{3.7}$$

where $y(\cdot)$ satisfies the state equation (1.4).

We apply (3.7), using the optimal control $u^* = -N^{-1}G^*(\lambda_0 - A^*)P_\infty y^*$; we recall that the optimal trajectory is given by $y^*(t) = \Phi_\infty(t, t_1)y_0$. We obtain

$$\begin{aligned}(\hat{P}(t_1)y_0, y_0)_H &\leq \int_{t_1}^{t} [\|Cy^*\|_V^2 + (Nu^*, u^*)_U]\, dv + (\hat{P}(t)y^*(t), y^*(t))_H \\ &\leq J_{t_1,\infty}(u^*) + (\hat{P}(t)y^*(t), y^*(t))_H \\ &= (P_\infty(t_1)y_0, y_0)_H + (\hat{P}(t)y^*(t), y^*(t))_H.\end{aligned}$$

By assumption we have $y^*(t) \to 0$ as $t \to \infty$, whereas $\hat{P}$ is bounded; hence, as $t \to \infty$ we obtain

$$(\hat{P}(t_1)y_0, y_0)_H \leq (P_\infty(t_1)y_0, y_0)_H.$$

This shows that $\hat{P} \leq P_\infty$. □

3.4. Periodic case and autonomous case. We consider now two special cases of equation (2.1): the periodic case and the time-invariant case. We assume the following hypothesis.

Hypothesis 3.13. There exists $\vartheta > 0$ such that $A(t+\vartheta) = A(t)$, $G(t+\vartheta) = G(t)$, $C(t+\vartheta) = C(t)$, and $N(t+\vartheta) = N(t)$ for all $t \in \mathbb{R}$. If this is the case we say that the system is ϑ-periodic.

Remark 3.14. If the system is ϑ-periodic, then

(i) evidently all assumptions concerning the uniform behaviour of the operators follow from the local assumptions listed in §1;

(ii) if $\hat{P}(t)$ is a bounded solution of (2.1), then $\hat{P}_\vartheta(t) := \hat{P}(t+\vartheta)$, $t \in \mathbb{R}$, is also a bounded solution;

(iii) some stabilizability results for equation (1.4) can be found in [L2].

As in [DI3, Prop. 3.4] we have the following proposition.

PROPOSITION 3.15. *Assume Hypotheses* 1.1 – 1.3, 1.5, 2.2, *and* 3.13. *Then the minimal solution* P_∞ *of* (2.1) *is* ϑ*-periodic. If Hypothesis* 3.9 *holds too, then* P_∞ *is the unique nonnegative* ϑ*-periodic solution of* (2.1) *and the corresponding optimal trajectory for problem* (1.7) *is exponentially stable.*

Proof. The periodicity of P_∞ follows from the same argument as in [DI3, Prop. 3.4]; the stability of the optimal trajectory is a direct consequence of Theorem 3.11. □

Finally assume that A, G, C, and N are independent of t. Then our assumptions correspond to those assumed by Flandoli [F1], and the corresponding result is the following proposition.

PROPOSITION 3.16. *Suppose that A is the infinitesimal generator of an analytic semigroup* e^{tA}, *and let* $G \in \mathcal{L}(U, D((\lambda_0 - A)^\alpha))$, $C \in \mathcal{L}(U, V)$, $N, N^{-1} \in \Sigma^+(U)$. *In addition assume that condition* (2.4_0) *holds (i.e., there exists an admissible control). Then* $P_\infty(t) \equiv P_\infty$ *is independent of t, and it is the minimal solution of the algebraic Riccati equation*

$$A^*Q + QA + C^*C - Q(\lambda_0 - A)^{1-\alpha}[(\lambda_0 - A)^\alpha G]N^{-1}[(\lambda_0 - A)^\alpha G]^*(\lambda_0 - A^*)^{1-\alpha}Q = 0. \tag{3.8}$$

Furthermore, if (A, C) *is detectable, then* P_∞ *is the unique nonnegative solution of* (3.8).
□

4. Examples.

4.1. A finite-dimensional example. Consider the family of 2×2 matrices $\{A(t)\} = \{(1+t)A_1\}_{t\geq 0}$, where

$$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

In $\mathbb{R}^2$ consider the state equation

$$y'(t) = A(t)y(t) + B(t)u(t), \qquad t \geq 0, \quad y(0) = (x_1, x_2) \in \mathbb{R}^2, \tag{4.1}$$

where $u \in L^2_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^2)$ is the control and $B(t) = b(t)I$ (I is the identity matrix), with $b(\cdot)$ a nonzero continuous function with polynomial growth as $t \to \infty$. We want to minimize the quadratic cost functional given by

$$J_{0,\infty}(u) = \int_0^\infty [\|C(t)y(t)\|^2_{\mathbb{R}^2} + (N(t)u(t), u(t))_{\mathbb{R}^2}]\, dt,$$

with y, u subject to equation (4.1); here $C(t) = (2\sqrt{2(1+t)^2 - 1})I$, $N(t) = b(t)^2 I$.

In this situation the eigenvalues of the matrix $A(t)$ are $(1+t, \pm(1+t)i)$ and

$$U(t,s) = \exp([(t^2 - s^2)/2 + (t-s)]A_1).$$

For a given $t_0 \geq 0$ an admissible feedback control relative to t_0 is easily found by choosing $\hat{u}(t) = K(t)\hat{y}(t)$, with

$$K(t) = b(t)^{-1}(t+1) \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix},$$

so all our assumptions hold locally over the time interval $[0, +\infty[$.

The Riccati equation (2.1) becomes, in this simple situation,

$$P'(t) + (1+t)[A_1^* P(t) + P(t)A_1] + 4[2(1+t)^2 - 1]I - P(t)^2 = 0, \qquad t \geq 0. \tag{4.2}$$

The nonnegative symmetric solution $P_T(\cdot)$ of equation (4.2) over the interval $[0,T]$, with final datum $P(T) = 0$, is given by

$$P_T(t) = 4(1+t)I - v_T(t)^{-1}I,$$

where

$$v_T(t) = \frac{\exp[3(T^2 - t^2) + 6(T-t)]}{4(1+T)} - \int_t^T \exp[3(s^2 - t^2) + 6(s-t)]\, ds.$$

It is easily seen that $v_T(t) > 0$ for each $t \in [0,T]$ and that $\lim_{T\to\infty} v_T(t) = +\infty$; hence for each $t \geq 0$

$$P_T(t) \uparrow P_\infty(t) := 4(1+t)I \quad \text{as } T \uparrow \infty.$$

The optimal control u^* is given by $u^*(t) = -4b(t)^{-1}(1+t)Iy^*(t)$, and the optimal trajectory y^* is the solution of the closed-loop system

$$y'(t) = (1+t)A_2 y, \quad t \geq 0, \quad y(0) = (x_1, x_2),$$

where

$$A_2 = \begin{pmatrix} -3 & 1 \\ -1 & -3 \end{pmatrix}.$$

We remark that the optimal trajectory is stable. Furthermore it can be seen that $P_\infty(t) = 4(1+t)I$ $(t \geq 0)$ is the only positive solution of the Riccati equation (4.2).

An infinite-dimensional example can be easily obtained by the above example, by just adding to it, as a direct sum, a control problem with unbounded time-invariant operators. In a similar way one can easily arrange things in such a way that the resolvent operator of $A(t)$ is not compact for any $t \geq 0$, using, for instance, multiplicative operators in infinite-dimensional spaces (compare with Remark 2.8(ii)).

4.2. Parabolic equations in noncylindrical domains. Let Ω_0 be a bounded open set of $\mathbb{R}^n$ with smooth boundary Γ_0. Following [DZ], [A2] we consider the family of mappings $\{T_t(\cdot) : \mathbb{R}^n \to \mathbb{R}^n,\ t \geq 0\}$ associated with a family of regular vector fields $\{V(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n,\ t \geq 0\}$ by the dynamic system

$$\frac{\partial}{\partial t} T_t(x) = V(t, T_t(x)), \quad T_0(x) = x, \quad t \geq 0,\ x \in \mathbb{R}^n.$$

Consider the sets $\Omega_t := T_t(\Omega_0)$ with boundary $\Gamma_t := T_t(\Gamma)$, and the evolution domain $Q = \cup_{t>0}\{t\} \times \Omega_t$ with boundary $\Sigma = \cup_{t>0}\{t\} \times \Gamma_t$.

We want to apply the results of the preceding sections to the following problem: minimize among all $u \in L^2_{\mathrm{loc}}(\Sigma)$ the functional

$$J(u) = \int_0^\infty \int_{\Omega_t} |y(t,\xi)|^2 d\xi\, dt + \int_\Sigma |u(t,\xi)|^2\, d\Sigma(t,\xi), \tag{4.3}$$

where y is the solution of the parabolic boundary problem

$$\begin{cases} y_t(t,\xi) = \Delta y(t,\xi), & t \geq 0,\ \xi \in \Omega_t, \\ y(t,\xi) = u(t,\xi) \quad \text{or} \ \dfrac{\partial y}{\partial \nu_t}(t,\xi) = u(t,\xi), & t \geq 0,\ \xi \in \Gamma_t, \\ y(0,\xi) = y_0(\xi), & \xi \in \Omega_0. \end{cases} \tag{4.4}$$

(ν_t is the outward normal to Γ_t.)

Denote by DT_t the Jacobian matrix of T_t and by J_t its determinant; then the change of variable

$$z(t,x) = y(t, T_t(x)), \quad t \geq 0, \quad x \in \Omega_0, \quad v(t,x) = u(t, T_t(x)), \quad t \geq 0, \quad x \in \Gamma_0,$$

transforms problem (4.3), (4.4) into the following one: minimize among all $v \in L^2_{\mathrm{loc}}(\partial\Omega \times \mathbb{R}^+)$ the functional

$$\begin{aligned} J_0(v) = {} & \int_0^\infty \int_{\Omega_0} |z(t,x)|^2 J_t(x)\, dx\, dt \\ & + \int_0^\infty \int_{\partial\Omega_0} |v(t,\xi)|^2 J_t(x) B(t,x)\, dH_{n-1}(x)\, dt, \end{aligned} \tag{4.5}$$

where z is the solution of the parabolic boundary problem

$$\begin{cases} z_t(t,x) = \mathcal{A}(t,x,D)y, & t \geq 0,\ x \in \Omega_0, \\ z(t,x) = v(t,x) \quad \text{or} \ \dfrac{\partial z(t,x)}{\partial \nu_{A(t)}} = v(t,x)\beta(t,x), & t \geq 0,\ x \in \Gamma_0, \\ z(0,x) = y_0(x), & x \in \Omega_0, \end{cases} \tag{4.6}$$

with

$$\begin{aligned} B(t,x) &:= \sqrt{1+[(V(t,x),n_t(x))_{\mathbb{R}^n}]^2}, \\ \mathcal{A}(t,x,D)w &:= -J_t(x)^{-1}\operatorname{div}((DT_t(x)^{-1})(DT_t(x)^{-1})^*\cdot Dw(x)) \\ &\quad + ((DT_t(x)^{-1})\cdot Dw(x), V(t,x))_{\mathbb{R}^n}, \\ \beta(t,x) &:= J_t(x)|(DT_t(x)^{-1})^*\cdot n_0(x)|, \\ \nu_{A(t)} &:= (DT_t(x)^{-1})(DT_t(x)^{-1})^*\cdot \nu_0(x). \end{aligned}$$

Problem (4.5) – (4.6) can be studied with the methods of this paper. It is shown in [DZ] (in the case of Dirichlet boundary control) and in [A2] (in the case of Neumann boundary control) that an admissible control exists; more precisely, in both cases Hypothesis 3.1 holds true.

4.3. Strongly damped wave equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the Dirichlet or Neumann boundary control problem for the damped wave equation in $]0,\infty[\times\Omega$:

$$\text{(4.7)}\qquad \begin{cases} y_{tt}(t,x) = \Delta y(t,x) + \rho(t)\Delta y_t(t,x), & t>0,\ x\in\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = w_0(x), & x\in\Omega, \\ By(t,x) = u(t,x), \quad t>0,\ x\in\partial\Omega & \left[B = I \text{ or } B = \dfrac{\partial}{\partial\nu}\right]; \end{cases}$$

here ρ is a scalar function belonging to $C^{\varepsilon+1/2}([0,\infty[)$; the data y_0, w_0 belong to $H^1(\Omega)$ and $L^2(\Omega)$, respectively; and Δ is the Laplace operator. The cost functional

$$\text{(4.8)}\qquad \begin{aligned} J(u) = \int_0^\infty \{ &\|C_1(t)y(t,\cdot)\|^2_{L^2(\Omega)} + \|C_2(t)y_t(t,\cdot)\|^2_{L^2(\Omega)} + \|C_3(t)Dy(t,\cdot)\|^2_{L^2(\Omega)} \\ &+ (N_1(t)u(t,\cdot),u(t,\cdot))_{L^2(\partial\Omega)} + (N_2(t)u_t(t,\cdot),u_t(t,\cdot))_{L^2(\partial\Omega)}\}\,dt \end{aligned}$$

has to be minimized among all $u \in W^{1,2}_{\mathrm{loc}}(0,\infty;L^2(\partial\Omega))$, with y subject to (4.7); the operators C_1, C_2, C_3 and N_1, N_2 belong to $L^\infty_{\mathrm{loc}}(0,\infty;\mathcal{L}(L^2(\Omega)))$ and $L^\infty_{\mathrm{loc}}(0,\infty;\Sigma^{++}(L^2(\partial\Omega)))$, respectively.

In order to apply the results of this paper we rewrite problem (4.7) – (4.8) in abstract form. Define

$$\text{(4.9)}\qquad \begin{cases} D_A = \{y\in H^2(\Omega) : By = 0 \text{ on } \partial\Omega\}, \\ Ay = \Delta y, \end{cases}$$

$$\text{(4.10)}\qquad G: L^2(\partial\Omega)\to L^2(\Omega), \qquad Gu = z \Leftrightarrow \begin{cases} \Delta z = z & \text{in } \Omega, \\ Bz = u & \text{in } \partial\Omega. \end{cases}$$

Then, following [B2], it is easy to see that if $u\in W^{2,2}_{\mathrm{loc}}(0,\infty;L^2(\partial\Omega))$ then the function

$$Z := \begin{pmatrix} y \\ y_t \end{pmatrix} - G\begin{pmatrix} u \\ u_t \end{pmatrix}$$

solves

$$\text{(4.11)}\qquad \begin{cases} Z_t = \begin{pmatrix} 0 & 1 \\ \Delta & \rho(t)\Delta \end{pmatrix} Z + \begin{pmatrix} 0 \\ Gu + \rho(t)Gu_t - Gu_{tt} \end{pmatrix}, & t>0,\ x\in\Omega, \\ Z(0) = \begin{pmatrix} y_0 - Gu(0) \\ w_0 - Gu_t(0) \end{pmatrix}, \quad x\in\Omega; \quad BZ = 0, & t>0,\ x\in\partial\Omega. \end{cases}$$

Now set $H := H^1(\Omega) \times L^2(\Omega)$, $U := L^2(\partial\Omega)$, $V := H$, and

$$\begin{cases} D_{\mathcal{A}(t)} = \left\{ \begin{pmatrix} y \\ w \end{pmatrix} \in H : y + \rho(t) w \in D_A \right\}, \\ \mathcal{A}(t) \begin{pmatrix} y \\ w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta & \rho(t)\Delta \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix}; \end{cases}$$

then it is shown in [L3] that (i) the operators $\mathcal{A}(t)$ generate analytic semigroups in H, and (ii) they satisfy the assumptions of [AT1], [AT2] and [AT3,§6]; this in turn yields that the evolution operator $U_{\mathcal{A}}(t,s)$ associated with $\{\mathcal{A}(t)\}$ exists and fulfills the hypotheses of this paper. (We remark that by choosing $H = L^2(\Omega) \times L^2(\Omega)$, as done in [B1], [B2] in the autonomous case, we still would have (i), but (ii) would no longer be true.)

Consequently we can write the mild form of (4.11), and after integrating by parts we get

$$Z(t) = U_{\mathcal{A}}(t,0) \begin{pmatrix} y_0 \\ w_0 \end{pmatrix} - \begin{pmatrix} 0 \\ Gu'(t) \end{pmatrix} + \int_0^t U_{\mathcal{A}}(t,s) \begin{pmatrix} 0 \\ Gu(s) + \rho(s)u'(s) \end{pmatrix} ds$$
$$- \int_0^t U_{\mathcal{A}}(t,s)\mathcal{A}(s) G \begin{pmatrix} 0 \\ u'(s) \end{pmatrix} ds, \qquad t \geq 0$$

(where the last term has to be interpreted as in (1.4) – (1.5)); by density we see that this formula holds for all $u \in W^{1,2}_{\mathrm{loc}}(0,\infty; L^2(\partial\Omega))$. Hence, setting $L := \binom{G}{0}$, $M := \binom{0}{G}$ for the sake of simplicity, we obtain, for $Y := \binom{y}{y_t} \in L^2_{\mathrm{loc}}(0,\infty; H)$,

$$Y(t) = U_{\mathcal{A}}(t,0) \begin{pmatrix} y_0 \\ w_0 \end{pmatrix} + Lu(t) + \int_0^t U_{\mathcal{A}}(t,s) M u(s)\, ds$$
$$+ \int_0^t U_{\mathcal{A}}(t,s)[\rho(s) - \mathcal{A}(s)] M u'(s)\, ds.$$

Now, as in [B1], [B2], we regard the control u as an auxiliary component of the state and define u' as a new control; namely, we set $v := u'$, $X := (Y, u)$, $\bar{H} := H \times U$, $\bar{U} := U$, and $\bar{V} := \bar{H}$ and look for the state equation satisfied by X. As shown in [B1,§2], $X(t)$ is the mild solution of

$$\begin{cases} X'(t) = \mathcal{B}(t) X(t) + Q(t) v(t), & t > 0, \\ X(0) = X_0, \end{cases}$$

where

$$\begin{cases} D_{\mathcal{B}(t)} = \left\{ \begin{pmatrix} Y \\ u \end{pmatrix} \in \bar{H} : Y - Lu \in D_{\mathcal{A}(t)} \right\} \\ \mathcal{B}(t) = \begin{pmatrix} \mathcal{A}(t) & M - \mathcal{A}(t) L \\ 0 & 0 \end{pmatrix}, \end{cases} \tag{4.12}$$

$$Q(t) = \begin{pmatrix} L + [\rho(t) - \mathcal{A}(t)] M \\ 1 \end{pmatrix}, \qquad X_0 = \begin{pmatrix} Y(0) \\ u(0) \end{pmatrix}.$$

It is easy to see that the operators $\mathcal{B}(t)$ possess in $\bar{H}$ the same properties enjoyed by the operators $\mathcal{A}(t)$ in H; in particular $\rho(\mathcal{B}(t)) = \rho(\mathcal{A}(t))$ and

$$[\lambda - \mathcal{B}(t)]^{-1}$$
$$= \begin{pmatrix} [\lambda - \mathcal{A}(t)]^{-1} & [\lambda - \mathcal{A}(t)]^{-1} M - \mathcal{A}(t)[\lambda - \mathcal{A}(t)]^{-1} L \\ 0 & 1 \end{pmatrix} \quad \forall \lambda \in \rho(\mathcal{B}(t)).$$

In addition the evolution operator $U_{\mathcal{B}}(t,s)$ associated with $\{\mathcal{B}(t)\}$ exists; it fulfills the hypotheses of this paper and has the following explicit representation:

$$U_{\mathcal{B}}(t,s) = \begin{pmatrix} U_{\mathcal{A}}(t,s) & \int_s^t U_{\mathcal{A}}(t,\sigma)M\,d\sigma + [1 - U_{\mathcal{A}}(t,s)]L \\ 0 & 1 \end{pmatrix}.$$

The operator $Q(t)$ may be also written (improperly but usually) as $Q(t) = [1 - \mathcal{B}(t)]\mathcal{G}(t)$, where $\mathcal{G}(t) = [1 - \mathcal{B}(t)]^{-1}Q(t)$ is given, after some manipulations, by

$$\mathcal{G}(t) = \begin{pmatrix} L + M + \rho(t)[1 - \mathcal{A}(t)]^{-1}M \\ 1 \end{pmatrix}.$$

Hence the state $X(t)$ solves the equation

$$X(t) = U_{\mathcal{B}}(t,0)X_0 + \int_0^t U_{\mathcal{B}}(t,s)[1 - \mathcal{B}(s)]\mathcal{G}(s)v(s)\,ds, \qquad t \geq 0. \tag{4.13}$$

We remark that, conversely, if $X(t)$ is given by this formula, then, setting $X(t) = (Y(t), u(t))$, the second component of X' gives $u' = v$, and from the first component it is easy to go back to (4.11) and hence to the solution y of the original problem.

Concerning the cost functional $J(u)$, we can rewrite it as $\bar{J}(v)$, where

$$\bar{J}(v) = \int_0^\infty \left\{ \|C(t)X(t)\|_{\bar{H}}^2 + (N(t)v(t), v(t))_{\bar{U}} \right\} dt, \tag{4.14}$$

with $C(t)$ and $N(t)$ given by

$$C(t)\begin{pmatrix} Y \\ u \end{pmatrix} \equiv C(t)\begin{pmatrix} Y_1 \\ Y_2 \\ u \end{pmatrix} = C_1(t)Y_1 + C_2(t)Y_2 + C_3(t)DY_1 + [N_1(t)]^{1/2}u,$$
$$N(t)v = N_2(t)v.$$

Thus the original control problem (4.7)–(4.8) is equivalent to minimizing $\bar{J}(v)$ among all $v \in L^2_{\text{loc}}(0,\infty;\bar{U})$, with X subject to equation (4.13).

In order to apply the theory of this paper we still need to verify Hypothesis 1.3 for $\mathcal{G}(t)$ (and this follows by the results of [B1], [B2]) and the finite cost condition (Hypothesis 2.2). Concerning the latter, in the case of Dirichlet boundary control it is satisfied by choosing $u = 0$, as the following proposition shows.

PROPOSITION 4.1. *Let $J(u)$ be given by* (4.8), *where y satisfies* (4.7) *with $B = I$, and assume that $\rho_1 \geq \rho(t) \geq \rho_0 > 0$ for each $t > 0$. Then we have $J(0) < \infty$.*

Proof. Multiply the partial differential equation (PDE) in (4.7) by y_t and integrate over Ω; then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |y_t(t,x)|^2\,dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega |Dy(t,x)|^2\,dx - \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx.$$

Integrating over $]0,T[$ we get

$$\int_\Omega |y_t(T,x)|^2\,dx + \int_\Omega |Dy(T,x)|^2\,dx$$
$$= \int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx - 2\int_0^T \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx\,dt;$$

this implies that

$$
\begin{aligned}
(4.15)\quad & \sup_{T>0}\left[\int_\Omega |y_t(T,x)|^2\,dx + \int_\Omega |Dy(T,x)|^2\,dx\right] + \int_0^\infty \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx\,dt \\
& \leq c\left[\int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx\right].
\end{aligned}
$$

As $\rho(t) \geq \rho_0$, by the Poincaré inequality we also get

$$
(4.16)\qquad \int_0^\infty \int_\Omega |y_t(t,x)|^2\,dx\,dt \leq c\left[\int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx\right].
$$

On the other hand, multiplying the PDE in (4.7) by y and integrating over $]0,T[\times\Omega$, we obtain after some integrations by parts

$$
\begin{aligned}
&\int_\Omega y_t(T,x)y(T,x)\,dx - \int_\Omega w_0 y_0\,dx \\
&\quad = \int_0^T \int_\Omega |y_t(t,x)|^2\,dx\,dt - \int_0^T \int_\Omega |Dy(t,x)|^2\,dx\,dt \\
&\qquad - \int_0^T \rho(t)\int_\Omega Dy_t(t,x)\cdot Dy(t,x)\,dx\,dt,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_0^T \int_\Omega |Dy(t,x)|^2\,dx\,dt \\
&\quad \leq \int_\Omega w_0 y_0\,dx + \frac{1}{2}\int_\Omega |y(T,x)|^2\,dx + \frac{1}{2}\int_\Omega |y_t(T,x)|^2\,dx \\
&\qquad + \frac{1}{\eta}\rho_1 \int_0^T \int_\Omega |Dy_t|^2\,dx\,dt + \eta\rho_1 \int_0^T \int_\Omega |Dy|^2\,dx\,dt;
\end{aligned}
$$

hence if η is sufficiently small, again using the Poincaré inequality we get

$$
\begin{aligned}
&\int_0^\infty \int_\Omega |y(t,x)|^2\,dx\,dt + \int_0^\infty \int_\Omega |Dy(t,x)|^2\,dx\,dt \\
&\quad \leq c\left[\int_\Omega w_0 y_0\,dx + \int_\Omega |y(T,x)|^2\,dx + \int_\Omega |y_t(T,x)|^2\,dx + \int_0^T \int_\Omega |Dy_t|^2\,dx\,dt\right],
\end{aligned}
$$

and by (4.15) we finally obtain

$$
\int_0^\infty \int_\Omega |y|^2\,dx\,dt + \int_0^\infty \int_\Omega |Dy|^2\,dx\,dt \leq c\left[\int_\Omega |w_0|^2\,dx + \int_\Omega |y_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx\right].
$$
(4.17)

The result now follows by (4.8), (4.17), and (4.16). □

Remark 4.2. The above proposition and the results of [D] imply that the evolution operator $U_{\mathcal{A}}(t,s)$ associated with $\{\mathcal{A}(t)\}$ is exponentially stable; i.e., it satisfies

$$
\|U_{\mathcal{A}}(t,s)\|_{\mathcal{L}(H)} \leq c\,e^{-\beta(t-s)} \quad \forall\, 0 \leq s \leq t
$$

for some $\beta > 0$.

In the case of Neumann boundary control, the finite cost condition is fulfilled too; indeed, we have the following proposition.

PROPOSITION 4.3. *Let $J(u)$ be given by* (4.8), *where y satisfies* (4.7) *with $B = \partial/\partial\nu$, and assume that $\rho_1 \geq \rho(t) \geq \rho_0 > 0$ for each $t > 0$. Then there exists $u \in L^2(0, \infty; L^2(\partial\Omega))$ such that $J(u) < \infty$.*

Proof. Multiply the PDE in (4.7) by y_t and integrate over Ω; then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |y_t(t,x)|^2\,dx = \int_{\partial\Omega} u(t,x)y_t(t,x)\,d\sigma_x + \rho(t)\int_{\partial\Omega} u_t(t,x)y_t(t,x)\,d\sigma_x$$
$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |Dy(t,x)|^2\,dx - \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx.$$

Choose the feedback control $u = -y|_{\partial\Omega}$; we then have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |y_t(t,x)|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |Dy(t,x)|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_{\partial\Omega} |y(t,x)|^2\,d\sigma_x$$
$$= -\rho(t)\int_{\partial\Omega} |y_t(t,x)|^2\,d\sigma_x - \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx$$

so that integrating over $]0,T[$ we get

$$\int_\Omega |y_t(T,x)|^2\,dx + \int_\Omega |Dy(T,x)|^2\,dx + \int_{\partial\Omega} |y(T,x)|^2\,d\sigma_x$$
$$= \int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx + \int_{\partial\Omega} |y_0|^2\,d\sigma_x$$
$$-2\int_0^T \rho(t)\int_{\partial\Omega} |y_t(t,x)|^2\,d\sigma_x\,dt - 2\int_0^T \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx\,dt.$$

This implies that

$$\begin{aligned}\sup_{T>0}&\left[\int_\Omega |y_t(T,x)|^2\,dx + \int_\Omega |Dy(T,x)|^2\,dx + \int_{\partial\Omega} |y(T,x)|^2\,d\sigma_x\right]\\ &+ \int_0^\infty \rho(t)\int_\Omega |Dy_t(t,x)|^2\,dx\,dt + \int_0^\infty \rho(t)\int_{\partial\Omega} |y_t(t,x)|^2\,d\sigma_x\,dt\\ &\leq c\left[\int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx + \int_{\partial\Omega} |y_0|^2\,d\sigma_x\right].\end{aligned} \tag{4.18}$$

On the other hand, multiplying the PDE in (4.7) by y and integrating over $]0,T[\times\Omega$, we obtain after some integrations by parts

$$\int_\Omega y_t(T,x)y(T,x)\,dx - \int_\Omega w_0y_0\,dx + \int_0^T\int_{\partial\Omega} |y(t,x)|^2\,d\sigma_x\,dt$$
$$= \int_0^T\int_\Omega |y_t(t,x)|^2\,dx\,dt - \int_0^T\int_\Omega |Dy(t,x)|^2\,dx\,dt$$
$$-\int_0^T \rho(t)\int_{\partial\Omega} y_t(t,x)y(t,x)\,d\sigma_x\,dt - \int_0^T \rho(t)\int_\Omega Dy_t(t,x)\cdot Dy(t,x)\,dx\,dt;$$

hence by (4.18) one easily finds that

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \int_\Omega |Dy(t,x)|^2\,dx\,dt + \frac{1}{2}\int_0^\infty \int_{\partial\Omega} |y(t,x)|^2\,d\sigma_x\,dt \\
&\le \int_\Omega |y(T,x)|^2\,dx + c\left[\int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx + \int_\Omega |y_0|^2\,dx\right. \\
&\qquad\left. + \int_{\partial\Omega} |y_0|^2\,d\sigma_x + \int_0^\infty \int_\Omega |y_t(t,x)|^2\,dx\,dt\right].
\end{aligned} \tag{4.19}$$

Now we have the following lemma.

LEMMA 4.4. *There exists $c > 0$ such that*

$$\|f\|_{L^2(\Omega)} \le c\left[\|Df\|_{L^2(\Omega)} + \|f\|_{L^2(\partial\Omega)}\right] \quad \forall\, f \in H^1(\Omega).$$

Proof. The proof is by contradiction; otherwise there should exist a sequence $\{f_k\}$ in $H^1(\Omega)$ such that

$$\|f_k\|_{L^2(\Omega)} > k\left[\|Df_k\|_{L^2(\Omega)} + \|f_k\|_{L^2(\partial\Omega)}\right] \quad \forall\, k \in \mathbb{N}^+.$$

In particular, the right member is not zero (since in that case $f_k \equiv 0$). Then if we set

$$g_k(x) = \frac{f_k(x)}{\|Df_k\|_{L^2(\Omega)} + \|f_k\|_{L^2(\partial\Omega)}},$$

we have

$$\|g_k\|_{L^2(\Omega)} > k, \quad \|Dg_k\|_{L^2(\Omega)} + \|g_k\|_{L^2(\partial\Omega)} = 1 \quad \forall\, k \in \mathbb{N}^+.$$

Hence for a suitable subsequence we get $Dg_k \to z$ weakly in $L^2(\Omega)$ and $g_k \to w$ weakly in $L^2(\partial\Omega)$.

Now let $\varphi \in L^2(\Omega)$ and take the solution $\Phi \in H^2(\Omega) \cap H_0^1(\Omega)$ of $\Delta\Phi = \varphi$ in Ω. Then as $k \to \infty$ we have

$$\begin{aligned}
\int_\Omega g_k\varphi\,dx &= \int_\Omega g_k\,\Delta\Phi\,dx \\
&= \int_{\partial\Omega} g_k\frac{\partial\Phi}{\partial\nu}d\sigma_x - \int_\Omega Dg_k\cdot D\Phi\,dx \to \int_{\partial\Omega} w\frac{\partial\Phi}{\partial\nu}d\sigma_x - \int_\Omega z\cdot D\Phi\,dx,
\end{aligned}$$

which implies that $\{g_k\}$ is weakly convergent in $L^2(\Omega)$, but this is impossible since $\{g_k\}$ is not bounded in $L^2(\Omega)$.

Let us return to (4.19); by Lemma 4.4 and (4.18) we obtain

$$\begin{aligned}
&\int_0^\infty \int_\Omega |Dy(t,x)|^2\,dx\,dt + \int_0^\infty \int_{\partial\Omega} |y(t,x)|^2\,d\sigma_x\,dt \\
&\quad + \int_0^\infty \int_\Omega |Dy_t(t,x)|^2\,dx\,dt + \int_0^\infty \int_{\partial\Omega} |y_t(t,x)|^2\,d\sigma_x\,dt \\
&\le c\left[\int_\Omega |w_0|^2\,dx + \int_\Omega |Dy_0|^2\,dx + \int_\Omega |y_0|^2\,dx + \int_{\partial\Omega} |y_0|^2\,d\sigma x\right],
\end{aligned}$$

and consequently the choice $u = y|_{\partial\Omega}$ implies $J(u) < \infty$. □

Remark 4.4. We have in fact verified that Hypothesis 3.1 holds too.

Remark 4.5. A more general approach to problem (4.7) – (4.8) in the autonomous case, which allows one to take controls $u \in L^2(0,\infty; L^2(\partial\Omega))$, can be found in [LLP], [T].

4.4. Structurally damped plate equation. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial\Omega$. Consider the following Dirichlet or Neumann boundary control problem for the structurally damped plate equation in $]0, \infty[\times \Omega$:

$$(4.20)\quad \begin{cases} y_{tt}(t,x) = -\Delta^2 y(t,x) + \rho(t)\Delta y_t(t,x), & t > 0,\ x \in \Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = w_0(x), & x \in \Omega, \\ By(t,x) = 0, \quad By_t(t,x) = u(t,x), & t > 0,\ x \in \partial\Omega \ \left[B = I \text{ or } B = \dfrac{\partial}{\partial\nu}\right]. \end{cases}$$

Here ρ is a scalar function belonging to $C^\varepsilon([0,\infty[)$; the data y_0, w_0 belong to $H^2(\Omega)$ and $L^2(\Omega)$, respectively. The cost functional

$$(4.21)\quad \begin{aligned} J(u) = \int_0^\infty \{ &\|y(t,\cdot)\|^2_{L^2(\Omega)} + \|y_t(t,\cdot)\|^2_{L^2(\Omega)} + \|Dy(t,\cdot)\|^2_{L^2(\Omega)} \\ &+ \|D^2 y(t,\cdot)\|^2_{L^2(\Omega)} \\ &+ \|u(t,\cdot)\|^2_{L^2(\partial\Omega)} + \|u_t(t,\cdot)\|^2_{L^2(\partial\Omega)}\}\, dt \end{aligned}$$

has to be minimized among all $u \in W^{1,2}_{\text{loc}}(0,\infty; L^2(\partial\Omega))$, with y subject to (4.20). Following the same method of the preceding example, we define A, G as in (4.9), (4.10), set $H = D_A \times L^2(\Omega)$, $U = L^2(\partial\Omega)$, and finally rewrite the problem in abstract form. It turns out that if $u \in W^{2,2}_{\text{loc}}(0,\infty; L^2(\partial\Omega))$, the function

$$Z := \begin{pmatrix} y \\ y_t \end{pmatrix} - G\begin{pmatrix} u \\ u_t \end{pmatrix}$$

solves

$$\begin{cases} Z'(t) = \mathcal{A}(t)Z(t) + F(t), & t > 0, \\ Z(0) = Z_0, \end{cases}$$

where

$$\mathcal{A}(t) := \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho(t)\Delta \end{pmatrix}, \qquad Z_0 := \begin{pmatrix} y_0 + (1-A)^{-1}Gu(0) \\ w_0 + (1-A)^{-1}Gu'(0) \end{pmatrix},$$

$$F(t) := \begin{pmatrix} 0 \\ -2Gu(t) + \rho(t)Gu'(t) + (1-A)^{-1}[Gu''(t) - \rho(t)Gu'(t) + Gu(t)] \end{pmatrix}.$$

As $\mathcal{A}(t)$ fulfills the assumptions of [AT1], [AT2], and [AT3, §6], there exists its evolution operator $U_{\mathcal{A}}(t,s)$; hence setting $Y := \binom{y}{y_t}$ and integrating by parts we get

$$\begin{aligned} Y(t) &= Z(t) - \begin{pmatrix} (1-A)^{-1}Gu(t) \\ (1-A)^{-1}Gu'(t) \end{pmatrix} \\ &= U_{\mathcal{A}}(t,0)\begin{pmatrix} y_0 + (1-A)^{-1}Gu(0) \\ w_0 \end{pmatrix} - \begin{pmatrix} (1-A)^{-1}Gu(t) \\ 0 \end{pmatrix} \\ &\quad + \int_0^t U_{\mathcal{A}}(t,s)\begin{pmatrix} (1-A)^{-1}Gu'(s) \\ (1-A)^{-1}Gu(s) - 2Gu(s) \end{pmatrix} ds, \end{aligned}$$

i.e., defining

$$L := \begin{pmatrix} (1-A)^{-1}G \\ 0 \end{pmatrix}, \quad M := \begin{pmatrix} 0 \\ 2G-(1-A)^{-1}G \end{pmatrix}, \quad Y_0 := \begin{pmatrix} y_0 + (1-A)^{-1}Gu(0) \\ w_0 \end{pmatrix},$$

$$Y(t) = U_{\mathcal{A}}(t,0)(Y_0 - Lu(0)) - Lu(t) + \int_0^t U_{\mathcal{A}}(t,s)(Lu'(s) + Mu(s))\, ds.$$

This formula holds for $u \in W^{1,2}_{\text{loc}}(0,\infty; L^2(\partial\Omega))$ as well.

Now, as in the preceding example, we set $\bar{H} := H \times U$, $\bar{U} := U$, and $v(t) := u'(t)$, $X(t) := (Y(t), u(t))$. The state X satisfies

$$\begin{cases} X'(t) = \mathcal{B}(t)X(t) + Qv(t), & t > 0, \\ X(0) = X_0, \end{cases}$$

where $\mathcal{B}(t)$ is defined as in (4.12) and

$$Q := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad X_0 := \begin{pmatrix} Y_0 \\ u(0) \end{pmatrix}.$$

Arguing as in the preceding example we arrive again to the state equation (4.13) for $X(t)$, where now

$$\mathcal{G}(t) := \begin{pmatrix} (1-\mathcal{A}(t))^{-1}M - \mathcal{A}(t)(1-\mathcal{A}(t))^{-1}L \\ 1 \end{pmatrix}.$$

Note that $\mathcal{G}(t)$ is uniformly bounded in $]0,\infty[$ as an element of $\mathcal{L}(\bar{U}, D_{\mathcal{A}(t)})$. The cost functional (4.21) transforms into

$$\bar{J}(v) = \int_0^\infty \{\|X(t)\|^2_{\bar{H}} + \|v(t)\|^2_{\bar{U}}\}\, dt, \tag{4.22}$$

and our abstract theory applies to the control problem (4.13), (4.22), provided that we verify the finite cost condition (Hypothesis 2.2). Now it turns out that in the case of Dirichlet boundary conditions ($B = I$) one can choose the control $u = 0$, whereas in the case of Neumann boundary conditions $\left(B = \frac{\partial}{\partial\nu}\right)$ one can choose the feedback control $u = y + y_t$. The proof that the cost is finite can be done by adapting the arguments of Propositions 4.1 and 4.3.

Remark 4.6. A more general approach to problem (4.20) – (4.21) in the autonomous case, which allows one to take controls $u \in L^2(0,\infty; L^2(\partial\Omega))$, can be found in [LLP], [T].

Appendix: Proof of Theorem 2.5. We are going to use the contraction principle on a suitable Banach space. For fixed $T_0, T \in \mathbb{R}$ with $T_0 < T$ we set, as in the proof of Lemma 2.9,

$$\begin{aligned} X(T_0,T) := \{\, P : [T_0,T] \to \Sigma(H) \text{ such that} \\ &\text{(i) } (\lambda_0 - A^*(\cdot))^{1-\alpha}P(\cdot) \in C_s([T_0,T], \mathcal{L}(H)); \\ &\text{(ii) } |[\lambda_0 - A(t)^*]^{1-\alpha}P(t)|_{\mathcal{L}(H)} \le c[1+(T-t)^{\beta+\alpha-1}]\ \forall t \in [T_0,T[; \\ &\text{(iii) } |[\lambda_0 - A(t)^*]^{1-\alpha}P(t)U(t,s)[\lambda_0 - A(s)]^\beta|_{\mathcal{L}(H)} \\ &\quad \le c(T-s)^\gamma[1+(T-t)^{\beta+\alpha-1}](t-s)^{-\beta}\ \forall T_0 \le s < t < T\} \end{aligned} \tag{A.1}$$

with $\gamma := \min\{1-\alpha-\beta, \beta\}$. We endow $X(T_0,T)$ by its natural norm, i.e.,

$$\|P\|_{X(T_0,T)} := \max\{A, B\},$$

where

$$A := \sup_{t\in[T_0,T[} [1+(T-t)^{1-\alpha-\beta}]|[\lambda_0 - A(t)^*]^{1-\alpha}P(t)|_{\mathcal{L}(H)},$$

$$B := \sup_{T_0\le s<t<T} \frac{[1+(T-t)^{1-\alpha-\beta}](t-s)^{\beta}}{(T-s)^{\gamma}}|[\lambda_0 - A(t)^*]^{1-\alpha}P(t)U(t,s)[\lambda_0 - A(s)]^{\beta}|_{\mathcal{L}(H)}.$$

We also set

$$B(\rho) := \{P \in X(T_0,T) : \|P\|_{X(T_0,T)} \le \rho\}.$$

Theorem 2.5 will be a consequence of the following lemma.

LEMMA A.1. *For each $\rho_0 > 0$ there exist $T_0 < T$ and $\rho > 0$ such that for any P_T satisfying*

$$|[\lambda_0 - A(T)^*]^{\beta}P_T[\lambda_0 - A(T)]^{\beta}|_{\mathcal{L}(H)} \le \rho_0$$

the Riccati equation

$$\begin{aligned} P(t) &= U(T,t)^*P_TU(T,t) + \int_t^T U(r,t)^* \\ &\quad \times [C(r)^*C(r) - P(r)(\lambda_0 - A(r))G(r)N(r)^{-1}G(r)^*(\lambda_0 - A(r)^*)P(r)]U(r,t)\,dr, \\ &\qquad t \in [T_0, T[, \end{aligned} \tag{A.2}$$

has a unique solution $P(\cdot)$ in $B(\rho)$.

Proof. First set

$$Q_T = (\lambda_0 - A(T)^*)^{\beta}P_T(\lambda_0 - A(T))^{\beta}. \tag{A.3}$$

Now fix $\rho_0 > 0$ and let P_T be such that $|Q_T|_{\mathcal{L}(H)} \le \rho_0$. Consider the map Γ defined on $B(\rho)$ in the following way:

$$\begin{aligned} &[\Gamma(P)](t) \\ &\quad = U(T,t)^*P_TU(T,t) + \int_t^T U(r,t)^* \\ &\qquad \times [C(r)^*C(r) - [(\lambda_0 - A(r)^*)^{1-\alpha}P(r)]^*K(r)[(\lambda_0 - A(r)^*)^{1-\alpha}P(r)]]U(r,t)\,dr, \end{aligned} \tag{A.4}$$

where $t \in [T_0, T[$ and

$$K(r) := [(\lambda_0 - A(r))^{\alpha}G(r)]^*N(r)^{-1}[[(\lambda_0 - A(r))^{\alpha}G(r)]^*. \tag{A.5}$$

We remark that $K(\cdot) \in L^{\infty}(]T_0,T[,\mathcal{L}(H))$ by Hypothesis 1.3.

We will show that for suitable T_0 and ρ (independent of the choice of P_T) the map Γ is a contraction in $B(\rho)$.

We start with the following estimate, which is true for $t < r < T$ and follows by (A.1(iii)) and (1.6):

$$\begin{aligned} &|(\lambda_0 - A(r)^*)^{1-\alpha}P(r)U(r,t)(\lambda_0 - A(t))^{1-\alpha}|_{\mathcal{L}(H)} \\ &\le \left|(\lambda_0 - A(r)^*)^{1-\alpha}P(r)U\left(r,\frac{r+t}{2}\right)\left(\lambda_0 - A\left(\frac{r+t}{2}\right)\right)^{\beta}\right|_{\mathcal{L}(H)} \\ &\quad \times \left|\left(\lambda_0 - A\left(\frac{r+t}{2}\right)\right)^{-\beta}U\left(\frac{r+t}{2},t\right)(\lambda_0 - A(t))^{1-\alpha}\right|_{\mathcal{L}(H)} \\ &\le c\rho(T-t)^{\gamma}[1+(T-r)^{\alpha+\beta-1}](r-t)^{\alpha-1}. \end{aligned} \tag{A.6}$$

By (1.6), (A.5), and (A.4) we deduce (with Q_T and $K(r)$ given by (A.3) and (A.5))

$$\begin{aligned}
&|(\lambda_0 - A(t)^*)^{1-\alpha}\Gamma(P)(t)|_{\mathcal{L}(H)} \\
&\quad \le |(\lambda_0 - A(t)^*)^{1-\alpha}U(T,t)^*(\lambda_0 - A(T)^*)^{-\beta}Q_T(\lambda_0 - A(T))^{-\beta}U(T,t)|_{\mathcal{L}(H)} \\
&\qquad + \left| \int_t^T (\lambda_0 - A(t)^*)^{1-\alpha}U(r,t)^*C(r)^*C(r)U(r,t)\,dr \right|_{\mathcal{L}(H)} \\
&\qquad + \left| \int_t^T [(\lambda_0 - A(t)^*)^{1-\alpha}U(r,t)^*P(r)(\lambda_0 - A(r))^{1-\alpha}] \right. \\
&\qquad\quad \left. \times K(r)[(\lambda_0 - A(r)^*)^{1-\alpha}P(r)]U(r,t)\,dr \right|_{\mathcal{L}(H)} \\
&\quad \le c\rho_0[1 + (T-t)^{\beta+\alpha-1}] + c(T-t)^\alpha \\
&\qquad + c\rho^2 \int_t^T (T-t)^\gamma[1 + (T-r)^{(\alpha+\beta-1)}]^2(r-t)^{\alpha-1}\,dr \\
&\quad \le c[\rho_0 + 1 + \rho^2(T-t)^{\min\{\gamma+\alpha,\gamma+\beta+2\alpha-1\}}][1 + (T-t)^{\alpha+\beta-1}] \quad \forall t \in [T_0, T[.
\end{aligned}$$

On the other hand, for $T_0 \le s < t < T$ we have by (A.4), (1.6), and (A.1(ii))–(A.1(iii))

$$\begin{aligned}
&|(\lambda_0 - A(t)^*)^{1-\alpha}[\Gamma(P)](t)U(t,s)(\lambda_0 - A(s))^\beta|_{\mathcal{L}(H)} \\
&\quad \le |(\lambda_0 - A(t)^*)^{1-\alpha}U(T,t)^*(\lambda_0 - A(T)^*)^{-\beta} \\
&\qquad \times Q_T(\lambda_0 - A(T))^{-\beta}U(T,s)(\lambda_0 - A(s))^\beta|_{\mathcal{L}(H)} \\
&\qquad + \left| \int_t^T (\lambda_0 - A(t)^*)^{1-\alpha}U(r,t)^*C(r)^*C(r)U(r,s)(\lambda_0 - A(s))^\beta\,dr \right|_{\mathcal{L}(H)} \\
&\qquad + \left| \int_t^T [(\lambda_0 - A(t)^*)^{1-\alpha}U(r,t)^*P(r)(\lambda_0 - A(r))^{1-\alpha}] \right. \\
&\qquad\quad \left. \times K(r)[(\lambda_0 - A(r)^*)^{1-\alpha}P(r)]U(r,s)(\lambda_0 - A(s))^\beta\,dr \right|_{\mathcal{L}(H)} \\
&\quad \le c\rho_0[1 + (T-t)^{\beta+\alpha-1}] + c(T-t)^\alpha(t-s)^{-\beta} \\
&\qquad + c\rho^2 \int_t^T (T-t)^\gamma[1 + (T-r)^{\beta+\alpha-1}]^2(r-t)^{\alpha-1}(T-s)^\gamma(r-s)^{-\beta}\,dr \\
&\quad \le c[\rho_0[1 + (T-t)^{\beta+\alpha-1}] + (T-t)^\alpha(t-s)^{-\beta} \\
&\qquad + \rho^2[(T-t)^{\gamma+\alpha} + (T-t)^{\gamma+2\beta+3\alpha-2}](T-s)^\gamma(t-s)^{-\beta}] \\
&\quad \le c(T-s)^\gamma[1 + (T-t)^{\beta+\alpha-1}](t-s)^{-\beta}\,[\rho_0 + 1 + \rho^2(T-t)^{\min\{\gamma+\alpha,\gamma+\beta+2\alpha-1\}}].
\end{aligned}$$

The above estimates show that

$$\|\Gamma(P)\|_{X(T_0,T)} \le c[\rho_0 + 1 + \rho^2(T-t)^{\min\{\gamma+\alpha,\gamma+\beta+2\alpha-1\}}].$$

As $\gamma = \min\{\beta, 1-\alpha-\beta\}$ and $\beta > 1/2 - \alpha$, we have in any case $\gamma + \beta + 2\alpha - 1 > 0$. Hence we can find a large ρ and a T_0 sufficiently close to T such that

$$\Gamma(P) \in B(\rho) \quad \forall P \in B(\rho). \tag{A.7}$$

Now we have to prove that the map Γ is a contraction in $B(\rho)$. Indeed, if $P, Q \in B(\rho)$ we can estimate the $X(T_0,T)$-norm of $\Gamma(P) - \Gamma(Q)$ exactly as before (and the calculation is even simpler); the result is

$$\|\Gamma(P) - \Gamma(Q)\|_{X(T_0,T)} \le c\rho\,\|P - Q\|_{X(T_0,T)}(T - T_0)^{\min\{\gamma+\alpha,\gamma+\beta+2\alpha-1\}}. \tag{A.8}$$

Hence we can find a large ρ and a T_0 sufficiently close to T such that both (A.7) and (A.8) hold, and the result follows by the contraction principle. □

REFERENCES

[A1] P. Acquistapace, *Abstract linear non-autonomous parabolic equations: A survey*, in Differential Equations in Banach Spaces, Proceedings Bologna 1991, G. Dore, A. Favini, E. Obrecht, and A. Venni, eds., Lecture Notes in Pure and Appl. Math. 148, M. Dekker, New York, 1993, pp. 1–19.

[A2] ———, *Boundary control for parabolic problems in non-cylindrical domains*, in Boundary Control and Variation, Proceedings Sophia-Antipolis 1992, J. P. Zolésio, ed., Lecture Notes in Pure and Appl. Math. 163, M. Dekker, New York, 1994, pp. 1–12.

[AT1] P. Acquistapace and B. Terreni, *A unified approach to abstract linear non-autonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova, 78 (1987), pp. 47–107.

[AT2] ———, *On fundamental solutions for abstract parabolic equations*, in Differential Equations in Banach Spaces, Proceedings Bologna 1985, A. Favini and E. Obrecht, eds., Lecture Notes in Math. 1223, Springer-Verlag, Berlin, 1986, pp. 1–11.

[AT3] ———, *Regularity properties of the evolution operator for abstract linear parabolic equations*, Differential Integral Equations, 5 (1992), pp. 1151–1184.

[AFT] P. Acquistapace, F. Flandoli, and B. Terreni, *Initial boundary value problems and optimal control for non-autonomous parabolic systems*, SIAM J. Control Optim., 29 (1991), pp. 89–118.

[BDDM] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter, *Representation and Control of Infinite Dimensional Systems*, Vol. II, Birkhäuser-Verlag, Basel, 1993.

[B1] F. Bucci, *A Dirichlet boundary control problem for the strongly damped wave equation*, SIAM J. Control Optim., 30 (1992), pp. 1092–1100.

[B2] ———, *A Boundary Control Problem with Infinite Horizon for the Strongly Damped Wave Equation*, preprint, Dip. Mat., Univ. Pisa, 1990.

[D] R. Datko, *Uniform asymptotic stability of evolutionary process in Banach spaces*, SIAM J. Math. Anal., 3 (1972), pp. 428–445.

[DI1] G. Da Prato and A. Ichikawa, *Bounded solutions on the real line to non-autonomous Riccati equations*, Rend. Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. (8), 79 (1985), pp. 107–112.

[DI2] ———, *Quadratic control for linear periodic systems*, Appl. Math. Optim., 18 (1988), pp. 39–66.

[DI3] ———, *Quadratic control for linear time-varying systems*, SIAM J. Control Optim., 28 (1990), pp. 359–381.

[DZ] G. Da Prato and J. P. Zolésio, *A boundary control problem for a parabolic equation in noncylindrical domain*, in Stability of Flexible Structures, Proceedings Montpellier 1987, A. V. Balakrishnan and J. P. Zolésio, eds., Optimization Software Inc. Publications Division, New York, 1988, pp. 52–61.

[F1] F. Flandoli, *Algebraic Riccati equation arising in boundary control problems*, SIAM J. Control Optim., 25 (1987), pp. 612–636.

[F2] ———, *A new proof of an a priori estimate arising in boundary control theory*, Appl. Math. Lett., 2 (1989), pp. 341–343.

[F3] ———, *On the direct solutions of Riccati equations arising in boundary control theory*, Ann. Mat. Pura Appl. (4), 163 (1993), pp. 93–131.

[F] M. Fuhrman, *Bounded solutions for abstract time-periodic parabolic equations with nonconstant domains*, Differential Integral Equations, 4 (1991), pp. 493–518.

[G] J. S. Gibson, *The Riccati integral equations for optimal control problems on Hilbert spaces*, SIAM J. Control Optim., 17 (1979), pp. 537–565.

[LT1] I. Lasiecka and R. Triggiani, *Dirichlet boundary control problem for parabolic equations with quadratic cost: Analyticity and Riccati's feedback synthesis*, SIAM J. Control Optim., 21 (1983), pp. 41–67.

[LT2] ———, *Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*, Lecture Notes in Control and Inform. Sci. 164, Springer-Verlag, Berlin Heidelberg, 1991.

[LLP] I. Lasiecka, D. Lukes, and L. Pandolfi, *Input dynamics and nonstandard Riccati equations with applications to boundary control of damped wave and plate equations*, preprint, 1993, J. Optim. Theory Appl., 84 (1995), pp. 549–574.

[L1] A. Lunardi, *Bounded solutions of linear periodic abstract parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A, 110 (1988), pp. 135–159.

[L2] ———, *Stabilizability of time periodic parabolic equations*, SIAM J. Control Optim., 29 (1991), pp. 810–828.

[L3] ———, *Neumann boundary stabilization of structurally damped time periodic wave and plate equations*, in Differential Equations with Applications in Biology, Physics and Engineering, Proceedings Leibnitz 1990, J. A. Goldstein, F. Kappel, and W. Schappacher, eds., Lecture Notes in Pure and Appl. Math. 133, M. Dekker, New York, 1991, pp. 241–257.

[T] R. Triggiani, *Optimal Quadratic Boundary Control Problem for Wave- and Plate-like Equations with High Internal Damping: An Abstract Approach*, in Control of Partial Differential Equations, Proceedings Trento 1993, G. Da Prato and L. Tubaro, eds., Lecture Notes in Pure and Appl. Math. 165, M. Dekker, New York, 1994, pp. 215–263.