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On *BMO* Regularity for Linear Elliptic Systems (*) (**).

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Summary. – *We prove a refinement of Campanato's result on local and global (under Dirichlet boundary conditions) BMO regularity for the gradient of solutions of linear elliptic systems of second order in divergence form: we just need that the coefficients are «small multipliers of $BMO(\Omega)$», a class neither containing, nor contained in $C^0(\overline{\Omega})$. We also prove local and global L^p regularity: this result neither implies, nor follows by the classical one by Agmon, Douglis and Nirenberg.*

0. – Introduction.

This paper contains a refinement of some results of Campanato concerning local and global (under Dirichlet boundary conditions) regularity for the gradient of solutions $u \in H^1(\Omega, \mathbf{R}^N)$ of second order linear strongly elliptic systems of the form

$$(0.1) \qquad \sum_{ij=1}^{n} \int_{\Omega} (A_{ij}(x) \cdot D_j u | D_i \phi)\, dx = \sum_{i=1}^{n} \int_{\Omega} (f_i(x) | D_i \phi)\, dx \qquad \forall \phi \in C_0^\infty(\Omega, \mathbf{R}^N),$$

in the «limit» case $f \in BMO(\Omega, \mathbf{R}^{nN})$ (here BMO is the John-Nirenberg space). It is well known [6, Ch. II, Th. 5.I] that $Du \in BMO$ provided the coefficients A_{ij} are Hölder continuous in $\overline{\Omega}$; we show here that a revisitation of Campanato's proof yields the same result when the coefficients just belong to the class of «small multipliers of $BMO(\Omega)$», which turns out to be optimal (in a sense) and is exactly characterized [9]: a function g is a multiplier of $BMO(\Omega)$ if and only if g is essentially bounded in Ω and in addition its mean oscillation over cubes

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Q_σ of edge σ behaves like $|\log \sigma|^{-1}$; the attribute «small» means that, in addition,

$$\sup\left\{|\lg \sigma| \fint_{Q_\sigma} |g - g_{Q_\sigma}|\, dx\colon 0 < \sigma \leqslant r\right\}$$

tends to 0 as $r \downarrow 0$.

Now it turns out that such class neither contains, nor is contained in $C^0(\bar{\Omega})$: on one hand, this forces us to impose strong ellipticity rather than ellipticity, in order to have existence of solutions; on the other hand, because of this fact our result implies some relevant consequences.

Firstly, by Stampacchia's interpolation theorem, we get the L^p regularity theory for a class of linear systems having discontinuous coefficients (but not for all systems with continuous coefficients): thus from this point of view Campanato's approach is at least as powerful as potential theory [1], and independent of it. Secondly, from L^p theory we deduce an extension of De Giorgi's regularity theorem to a class of linear systems with discontinuous coefficients. Unfortunately, the class of «small multipliers of *BMO*» does not seem to be handy enough to obtain similar results for nonlinear systems.

We also remark that our result is nearly sharp, since in the case $n = 1$ it is easy to verify (see Theorem 5.1 below) that *BMO* regularity is true if and only if the coefficients are multipliers of *BMO* (not necessarily «small»).

If one considers only the subclass of continuous small multipliers of *BMO*, then our result applies to (not necessarily strongly) elliptic systems: as the functions of such subclass are not Dini continuous in general, our result does not follow by the well known ones concerning Dini regularity [2].

For the sake of simplicity, only second order systems with no lower order terms are considered here, but this restriction might be easily dropped; similarly, the method applies, «mutatis mutandis», to higher order systems (under Dirichlet boundary conditions). However we believe that our approach works in the case of Neumann boundary conditions as well.

Let us sketch our method of proof. We start from the Dirichlet problem for a system with smooth coefficients, for which the *BMO* regularity is provided by Campanato's result: our main task consists in obtaining a sharp estimate for the *BMO* norm of Du, which does not involve the Hölder norms of the coefficients, but just their norm in the space of multipliers of *BMO*. Once we have this estimate, we consider a system whose coefficients are small multipliers of *BMO*, and in order to get our result we just need to approximate suitably (not uniformly) our coefficients by smooth ones: in this step we cannot replace «small multipliers» simply by «multipliers». In this way, we get global *BMO* regularity for the solution of the Dirichlet problem for the system (0.1).

Next, we prove a local *BMO* regularity result, which however does not follow in a standard way by the global one. The difficulty is that the usual localization argument does not work here, essentially because we do not have «lower regularity results», i.e. the regularity theory in the Morrey spaces $L^{2,\lambda}$, $0<\lambda<n$; this is due to the fact that our coefficients are discontinuous in general.

We overcome the above difficulty by using the global L^p theory deduced from Stampacchia's interpolation theorem: in the L^p setting the localization argument does work, and from local L^p results we are able to deduce the local *BMO* result.

It is to be noted that our starting point is the *BMO* theory with smooth coefficients, so that Campanato's *BMO* theory is not replaced by our paper but, on the contrary, our arguments are based on it. Similarly, the L^p regularity is not merely a corollary of our result since it is a basic tool in order to get a complete *BMO* theory.

The paper is organized in the following way: Section 1 is devoted to the study of $\mathcal{L}_\Phi$ spaces, i.e. the sets of functions whose mean oscillation over cubes Q_σ behaves like $\Phi(\sigma)$; the properties of these spaces are crucial in revisiting Campanato's argument.

Section 2 concerns global *BMO* regularity; Section 3 deals with L^p theory, whereas in Section 4 we study local *BMO* regularity. Finally Section 5 contains some improvements, counter examples and further remarks.

We end this section by introducing some notations.

If $x_0 \in \boldsymbol{R}^n$ and $\sigma>0$ we set

$$Q(x_0,\sigma):=\{x\in\boldsymbol{R}^n\colon |x_i-x_{i0}|\leqslant\sigma,\ 1\leqslant i\leqslant n\}\,,\qquad B(x_0,\sigma):=\{x\in\boldsymbol{R}^n\colon |x-x_0|\leqslant\sigma\}\,;$$

if x_0 lies in the «plane» $x_n=0$, we set

$$Q^+(x_0,\sigma):=Q(x_0,\sigma)\cap\{x_n\geqslant 0\}\,,\qquad B^+(x_0,\sigma):=B(x_0,\sigma)\cap\{x_n\geqslant 0\}\,,$$

$$\Gamma(x_0,\sigma):=Q(x_0,\sigma)\cap\{x_n=0\}\,.$$

When no confusion can arise, we will simply write $Q_\sigma, B_\sigma, Q_\sigma^+, B_\sigma^+, \Gamma_\sigma$. If A is a measurable subset of $\boldsymbol{R}^n$ with positive measure, and f is an integrable function defined on A, we set

$$f_A\equiv \fint_A f(x)\,dx:=\frac{1}{m(A)}\int_A f(x)\,dx\,.$$

We will use the sum convention on repeated indices, so that $a_i b_i$ means $\sum_{i=1}^n a_i b_i$. The inner product in $\boldsymbol{R}^N$ will be denoted by $(x|\,y)_N$.

Next, if $X(\Omega,\boldsymbol{R}^N)$ is a Banach space of R^N-valued functions defined in Ω, we will denote the norm of $X(\Omega,\boldsymbol{R}^N)$ simply by $\|\cdot\|_{X(\Omega)}$.

Finally if $X(\Omega)$ is a Banach space of scalar functions defined in Ω, we denote by $M(X(\Omega))$ the space of multipliers of $X(\Omega)$, i.e. the space of functions

g such that $f \cdot g \in X(\Omega)$ for each $f \in X(\Omega)$, endowed with the norm

$$\|g\|_{M(X(\Omega))} = \sup\{\|fg\|_{X(\Omega)} : f \in X(\Omega),\ \|f\|_{X(\Omega)} \leq 1\}.$$

1. – $\mathcal{L}_\Phi$ spaces.

Throughout this section we assume that

(1.1) $\Phi: [0, d] \to [0, \infty[$ is a continuous, non-decreasing function such that $\sigma \to \Phi(\sigma)/\sigma$ is almost decreasing, i.e. there exists $K_\Phi \geq 1$ such that

$$K_\Phi \frac{\Phi(t)}{t} \geq \frac{\Phi(s)}{s} \qquad \forall 0 \leq t < s \leq d.$$

For instance the functions σ^α, $|\lg \sigma|^{-\beta}$, $\exp(\sigma^\gamma) - 1$ ($\alpha, \beta \in [0, 1]; \gamma \geq 1$) satisfy the above assumption (in suitable intervals $[0, d]$).

DEFINITION 1.1. – Let Ω be an open set of $\boldsymbol{R}^n$, $n \geq 1$. We denote by $\mathcal{L}_\Phi(\Omega)$ the set of all functions $f \in L^2(\Omega)$ for which the quantity

$$(1.2) \qquad [f]_{\mathcal{L}_\Phi(\Omega)} := \sup\left\{\Phi(\sigma)^{-1}\left[\ \fint_{Q(x_0,\sigma)\cap\Omega} |f(y) - f_{Q(x_0,\sigma)\cap\Omega}|^2\, dy\right]^{1/2} : x \in \Omega, \sigma \in]0, d]\right\}$$

is finite. We denote by $l_\Phi(\Omega)$ the subspace of all $f \in \mathcal{L}_\Phi(\Omega)$ such that

$$(1.3) \qquad [f]_{\Phi,\Omega,r} := \sup\left\{\Phi(\sigma)^{-1}\left[\ \fint_{Q(x_0,\sigma)\cap\Omega} |f(y) - f_{Q(x_0,\sigma)\cap\Omega}|^2\, dy\right]^{1/2} : x \in \Omega, \sigma \in]0, r]\right\} = $$
$$= o(1) \text{ as } r \downarrow 0.$$

$\mathcal{L}_\Phi(\Omega)$ is a Banach space with norm

$$\|f\|_{\mathcal{L}_\Phi(\Omega)} := \|f\|_{L^2(\Omega)} + [f]_{\mathcal{L}_\Phi(\Omega)}.$$

The $\mathcal{L}_\Phi$ classes, introduced by SPANNE [12], generalize (among others) Campanato's $\mathcal{L}^{p,\lambda}$ spaces [3], which are defined for $p \in [1, \infty[$ and $\lambda \in [0, n+2]$ by:

$$\mathcal{L}^{p,\lambda}(\Omega) := \left\{f \in L^p(\Omega) : \|f\|^p_{\mathcal{L}^{p,\lambda}(\Omega)} := \sup\left\{\sigma^{-\lambda} \int_{Q(x_0,\sigma)\cap\Omega} |f - f_{Q(x_0,\sigma)\cap\Omega}|^p\, dx : x_0 \in \Omega, \sigma > 0\right\} < \infty\right\}.$$

We recall that by [6, Ch. I, Th. 2.I] we have:

$$(a) \qquad \mathcal{L}^{p,\lambda}(\Omega) = L^{p,\lambda}(\Omega) \qquad \forall p \in [1, \infty[,\ \forall \lambda \in [0, n[,$$

the Morrey class $L^{p,\lambda}(\Omega)$ being defined by

$$(1.4)\quad \mathcal{L}^{p,\lambda}(\Omega) := \left\{ f \in L^p(\Omega) \colon \|f\|^p_{\mathcal{L}^{p,\lambda}(\Omega)} := \sup \left\{ \sigma^{-\lambda} \int\limits_{Q(x_0,\sigma)\cap\Omega} |f|^p\, dx \colon x_0 \in \Omega, \sigma > 0 \right\} < \infty \right\};$$

$$(b)\qquad \mathcal{L}^{p,\lambda}(\Omega) = C^{0,(\lambda-n)/p}(\overline{\Omega}) \qquad \forall p \in [1,\infty[,\ \ \forall \lambda \in]n, n+2],$$

where $C^{0,\alpha}(\overline{\Omega})$, $\alpha \in [0,1]$, is the Hölder-Lipschitz space:

$$\mathcal{L}^{p,n}(\Omega) = \mathcal{L}^{2,n}(\Omega) = BMO(\Omega) \qquad \forall p \in [1,\infty[$$

($BMO(\Omega)$ is the John-Nirenberg class, see [10]).

Now if $\Phi(\sigma) := \sigma^{2\alpha}$, $\alpha \in]0,1]$, we find $\mathcal{L}_\Phi(\Omega) = \mathcal{L}^{2,n+2\alpha}(\Omega) = C^{0,\alpha}(\overline{\Omega})$, whereas if $\Phi(\sigma) \equiv 1$ we get $\mathcal{L}_\Phi(\Omega) = BMO(\Omega)$; if moreover $\Phi(\sigma) = |\lg \sigma|^{-\beta}$, $\beta \in]0,1]$, we obtain the Orlicz class defined by the function $M(s) := \exp(|s|^{1/(1-\beta)}) - 1$ (see [12]).

REMARK 1.2. – It is worth to recall the trivial but basic property

$$(1.5)\qquad \int\limits_A |f - f_A|^2\, dx = \min_{c\in R} \int\limits_A |f - c|^2\, dx \qquad \forall f \in L^2(A),$$

whose role in the whole paper is crucial. This property will be systematically used throughout, often without explicit reference.

The subspace $l_\Phi(\Omega)$ is obviously closed in $\mathcal{L}_\Phi(\Omega)$; moreover we have:

PROPOSITION 1.2. – Let Ω be a bounded open set of $\boldsymbol{R}^n$ with $\partial\Omega \in \mathrm{Lip}$. If $\lim\limits_{\sigma\downarrow 0}(\sigma/\Phi(\sigma)) = 0$, then $l_\Phi(\Omega)$ coincides with the closure of $C^\infty(\overline{\Omega})$ in $\mathcal{L}_\Phi(\Omega)$.

PROOF. – It is easy to verify that $C^\infty(\overline{\Omega}) \subset l_\Phi(\Omega)$ provided $\lim\limits_{\sigma\downarrow 0}(\sigma/\Phi(\sigma)) = 0$; as $l_\Phi(\Omega)$ is a closed subspace, we also have $\overline{C^\infty(\overline{\Omega})} \subset l_\Phi(\Omega)$.

The proof of the converse needs an extension lemma for functions in $l_\Phi(\Omega)$:

LEMMA 1.4. – Under the assumptions of Proposition 1.3, there exists an extension operator $E\colon \mathcal{L}_\Phi(\Omega) \to \mathcal{L}_\Phi(\boldsymbol{R}^n)$ such that $Ef \in l_\Phi(\boldsymbol{R}^n)$ $\forall f \in l_\Phi(\boldsymbol{R}^n)$ and

$$(1.6)\qquad [Ef]_{\mathcal{L}_\Phi(\boldsymbol{R}^n)} \leq c(n,\Omega,K_\Phi)[f]_{\mathcal{L}_\Phi(\Omega)}.$$

PROOF. – Firstly, we claim that if $f \in \mathcal{L}_\Phi(B^+)$ (resp. $l_\Phi(B^+)$), then the function

$$F(x) := \begin{cases} f(x) & \text{if } x_n \geq 0, \\ f(x_1, \dots, x_{n-1}, -x_n) & \text{if } x_n < 0, \end{cases}$$

belongs to $\mathcal{L}_\Phi(B)$ (resp. $l_\Phi(B)$); here $B^+ := B^+(0,1)$, $B := B(0,1)$.

Indeed, let $x_0 \in B$; if $Q(x_0, \sigma)$ does not intersect $\Gamma := \Gamma(0,1)$, setting $z_0 := (x_{01}, \dots, x_{0,n-1}, |x_{0n}|)$, we get

$$\Phi(\sigma)^{-2} \fint_{Q(x_0,\sigma)\cap B} |F - F_{Q(x_0,\sigma)\cap B}|^2\, dx = \Phi(\sigma)^{-2} \fint_{Q(z_0,\sigma)\cap B^+} |f - f_{Q(z_0,\sigma)\cap B^+}|^2\, dx \leq [f]^2_{\mathcal{L}_\Phi(B^+)},$$

whereas if $Q(x_0, \sigma) \cap \Gamma \neq \emptyset$, setting $y_0 := (x_{01}, \dots, x_{0,n-1}, 0)$ we obtain by (1.5)

$$\Phi(\sigma)^{-2} \fint_{Q(x_0,\sigma)\cap B} |F - F_{Q(x_0,\sigma)\cap B}|^2\, dx \leq c(n, K_\Phi)\, \Phi(2\sigma)^{-2} \fint_{Q(y_0,2\sigma)\cap B} |F - F_{Q(y_0,2\sigma)\cap B}|^2\, dx =$$

$$= c(n, K_\Phi)\, \Phi(2\sigma)^{-2} \fint_{Q(y_0,2\sigma)\cap B^+} |f - f_{Q(y_0,2\sigma)\cap B^+}|^2\, dx \leq c(n, K_\Phi)[f]^2_{\mathcal{L}_\Phi(B^+)};$$

this clearly implies our claim.

Next, arguing as in [5, Appendix I, Theorem V], we see that if $T: \overline{\Omega}' \to \overline{\Omega}$ is a Lipschitz homeomorphism, then $f \circ T \in \mathcal{L}_\Phi(\Omega')$ (resp. $l_\Phi(\Omega')$) for each $f \in \mathcal{L}_\Phi(\Omega)$ (resp. $l_\Phi(\Omega)$), and

$$[f \circ T]_{\mathcal{L}_\Phi(\Omega')} \leq c(\Omega, \Omega', K_\Phi)[f]_{\mathcal{L}_\Phi(\Omega)}. \tag{1.7}$$

Now, as $\partial\Omega \in \mathrm{Lip}$, there exists a finite covering $\{\Omega_i\}_{1 \leq i \leq m}$ of $\partial\Omega$, and a family $\{T_i\}_{1 \leq i \leq m}$ of Lipschitz homeomorphisms $T_i: \overline{\Omega}_i \to \overline{B}$, such that $T_i(\overline{\Omega}_i \cap \Omega) = B^+$, $T_i(\overline{\Omega}_i \cap \partial\Omega) = \Gamma$. Let $\Omega_0 \subset\subset \Omega$ be such that $\Omega \subset \bigcup_{i=0}^m \Omega_i =: \Omega'$, and let $\{\psi_i\}_{0 \leq i \leq m}$ be a C^∞ partition of unity associated to $\{\Omega_i\}_{0 \leq i \leq m}$. For $f \in \mathcal{L}_\Phi(\Omega)$ set:

$$F_i(y) := \begin{cases} (f\psi_i) \circ T_i^{-1}(y) & \text{if } y \in B \text{ and } y_n \geq 0, \\ (f\psi_i) \circ T_i^{-1}(y_1, \dots, y_{n-1}, -y_n) & \text{if } y \in B \text{ and } y_n < 0, \end{cases} \quad 1 \leq i \leq m,$$

$$f_i(x) := \begin{cases} 0 & \text{if } x \in \Omega' \setminus \Omega_i, \\ F_i \circ T_i(x) & \text{if } x \in \Omega_i, \end{cases} \quad 1 \leq i \leq m,$$

$$f_0(x) := \begin{cases} 0 & \text{if } x \in \Omega' \setminus \Omega_0, \\ f(x)\psi_0(x) & \text{if } x \in \Omega_0. \end{cases}$$

Then by the above assertion we have

$$[f_i]_{\mathcal{L}_\Phi(\Omega')} \leqslant c(n, \Omega, K_\Phi)[f]_{\mathcal{L}_\Phi(\Omega)}, \qquad 0 \leqslant i \leqslant m;$$

finally, setting

$$Ef(x) := \begin{cases} 0 & \text{if } x \notin \Omega' \\ \sum_{i=0}^{m} f_i(x) & \text{if } x \in \Omega', \end{cases}$$

we immediately get (1.6). On the other hand the above argument shows also that $Ef \in l_\Phi(\boldsymbol{R}^n)$ if $f \in l_\Phi(\Omega)$. The proof of Lemma 1.4 is complete. ■

Fix now $f \in l_\Phi(\Omega)$ and extend it outside Ω via Lemma 1.4, and consider the convolutions

$$f_k(x) := \int_{\boldsymbol{R}^n} Ef\left(x - \frac{1}{k} z\right) \theta(z)\, dz, \qquad k \in \boldsymbol{N}^+, \quad x \in \overline{\Omega},$$

where $\theta \in C_0^\infty(\boldsymbol{R}^n)$, $\theta \geqslant 0$, $\theta \equiv 0$ outside B, $\int_B \theta(z)\, dz = 1$.

As $l_\Phi(\Omega) \subset BMO(\Omega) \subset \bigcap_{1 \leqslant p < \infty} L^p(\Omega)$, we clearly have

$$f_k \to f \quad \text{as } k \to \infty \text{ in } L^p(\Omega) \qquad \forall p \in [1, \infty[. \tag{1.8}$$

Let us show that

$$f_k \to f \quad \text{as } k \to \infty \text{ in } \mathcal{L}_\Phi(\Omega); \tag{1.9}$$

this will complete the proof of Proposition 1.3.

Fix $\varepsilon > 0$. As $f \in l_\Phi(\Omega)$, we have $Ef \in l_\Phi(\boldsymbol{R}^n)$, so that there exists $\sigma_\varepsilon > 0$ such that

$$\Phi(\sigma)^{-2} \fint_{Q(x_0,\sigma)} |Ef - (Ef)_{Q(x_0,\sigma)}|^2 dx < \varepsilon \qquad \forall x_0 \in \boldsymbol{R}^n, \quad \forall \sigma \in]0, \sigma_\varepsilon]. \tag{1.10}$$

Now if $x_0 \in \Omega$ and $\sigma > \sigma_\varepsilon$ we have by (1.5) and (1.8)

$$\Phi(\sigma)^{-2} \fint_{Q(x_0,\sigma)\cap\Omega} |f - f_k - (f - f_k)_{Q(x_0,\sigma)\cap\Omega}|^2 dx \leqslant c(n)\, \Phi(\sigma)^{-2} \sigma^{-n} \int_\Omega |f - f_k|^2 dx \leqslant$$

$$\leqslant c(n)\, \Phi(\sigma_\varepsilon)^{-2} \sigma_\varepsilon^{-n} \|f - f_k\|^2_{L^2(\Omega)} < \varepsilon,$$

provided that k is larger than a suitable k_ε; otherwise if $x_0 \in \Omega$ and $\sigma \in]0, \sigma_\varepsilon]$ we get by (1.10)

$$\Phi(\sigma)^2 \fint_{Q(x_0,\sigma)\cap\Omega} |f-f_k-(f-f_k)_{Q(x_0,\sigma)\cap\Omega}|^2\,dx \leqslant c(n)\,\Phi(\sigma)^{-2} \fint_{Q(x_0,\sigma)} |Ef-f_k-(Ef-f_k)_{Q(x_0,\sigma)}|^2 =$$

$$= c(n)\,\Phi(\sigma)^{-2} \fint_{Q(x_0,\sigma)} \left| Ef-(Ef)_{Q(x_0,\sigma)} - \int_{R^n} \theta(z)\left[Ef\left(x-\frac{1}{k}z\right) - \fint_{Q(x_0,\sigma)} Ef\left(y-\frac{1}{k}z\right)dy\right]dz\right|^2 dx \leqslant$$

$$\leqslant c(n)\Phi(\sigma)^{-2}\left\{\fint_{Q(x_0,\sigma)} |Ef-(Ef)_{Q(x_0,\sigma)}|^2\,dx + \int_{R^n}\theta(z) \fint_{Q(x_0-(1/k)z,\sigma)} |Ef(\xi)-(Ef)_{Q(x_0-(1/k)z,\sigma)}|^2\,d\xi\,dz\right\} \leqslant$$

$$\leqslant c(n)[Ef]^2_{\Phi,R^n,\sigma_\varepsilon} < c(n,f)\,\varepsilon \qquad \forall k \in N^+.$$

Hence if $k \geqslant k_\varepsilon$ we get

$$[f-f_k]^2_{\mathcal{L}_\Phi(\Omega)} < c(n,f)\,\varepsilon\,,$$

and (1.9) is proved. ■

REMARK 1.5. – We have in fact proved that under the assumptions of Proposition 1.3, if $f \in l_\Phi(\Omega)$ there exists a sequence $\{f_k\}_{k\in N} \subset C^\infty(\bar\Omega)$ such that:

$$\text{(1.11)} \qquad \lim_{k\to\infty} (\|f_k-f\|_{\mathcal{L}_\Phi(\Omega)} + \|f_k-f\|_{L^p(\Omega)}) = 0 \qquad \forall p \in [1,\infty[\,,$$

$$\text{(1.12)} \qquad \lim_{\sigma\downarrow 0}\,[f_k]_{\Phi,\Omega,\sigma} = 0 \qquad \text{uniformly in } k \in N.$$

If, moreover, f belongs to $L^\infty(\Omega)$, too, the sequence $\{f_k\}$ satisfies

$$\text{(1.13)} \qquad \|f_k\|_{L^\infty(\Omega)} \leqslant \|f\|_{L^\infty(\Omega)}\,.$$

The following result, due to JANSON [9], shows that the $\mathcal{L}_\Phi$ classes occur as spaces of multipliers.

PROPOSITION 1.6. – $M(\mathcal{L}_\Phi(\Omega)) = L^\infty(\Omega) \cap \mathcal{L}_\psi(\Omega)$, where $\psi(\sigma) := \left[\int_\sigma^d \frac{\Phi(r)}{r}\,dr\right]^{-1}$. ■

In particular we have (with $d := e$):

$$\text{(1.14)} \qquad M(BMO(\Omega)) = L^\infty(\Omega) \cap \mathcal{L}_\phi(\Omega)\,, \qquad \phi(\sigma) := (1+|\lg\sigma|)^{-1};$$

from now on, $\phi(\sigma)$ will always mean $(1+|\lg\sigma|)^{-1}$, $\sigma \in [0,1]$.

REMARK 1.7. – The result (1.14) is not surprising since it is well known that if $f \in BMO(\Omega)$ and $Q_\sigma \subset \Omega$ we have

$$|f_{Q_\sigma}| = O(1+|\lg\sigma|) \qquad \text{as } \sigma\downarrow 0\,,$$

(see [11, § 3.10], [9]). A more precise estimate will be proved in Propositions 1.15, 1.16 below.

We want to prove now that the space of multipliers of $BMO(\Omega)$ neither contains, nor is contained in $C^0(\overline{\Omega})$. Indeed we have:

PROPOSITION 1.9. – Let $\phi(\sigma) := (1+|\lg \sigma|)^{-1}$. Then $C^0(\overline{\Omega}) \setminus \mathcal{L}_\phi(\Omega)$ and $[L^\infty(\Omega) \cap \cap l_\phi(\Omega)] \setminus C^0(\overline{\Omega})$ are not empty.

PROOF. – Set

$$\chi(\sigma) := \left(1 + \frac{2}{3}|\lg \sigma|\right)^{-1/2}, \qquad \sigma \in [0,1]:$$

then $\chi \in C^0([0,1])$, $\chi(0)=0$, $\chi(1)=1$, χ is strictly increasing and concave and $\chi(\sigma)/\sigma$ is almost decreasing; moreover

$$\lim_{\sigma \downarrow 0} \chi(\sigma)|\lg \sigma| = +\infty .$$

Next, consider in $\Omega :=]-1,1[$ the function

$$g(x) := \chi(|x|)\operatorname{sgn} x, \qquad x \in [-1,1].$$

Clearly, $g \in C^0([-1,1])$ and by the concavity of χ

$$\fint_{-\sigma}^{\sigma} |g(y) - g_{[-\sigma,\sigma]}|^2 dy = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} |g(y)|^2 dy = \frac{1}{\sigma}\int_0^{\sigma} [\chi(y)]^2 dy \geq$$

$$\geq \left[\frac{1}{\sigma}\int_0^{\sigma} \chi(y)\,dy\right]^2 \geq \left[\frac{1}{\sigma}\int_0^{\sigma} \frac{\chi(\sigma)}{\sigma} y\,dy\right]^2 = \frac{[\chi(\sigma)]^2}{4},$$

which implies

$$\lim_{\sigma\downarrow 0} [g]_{\Omega,\phi,\sigma} \geq \lim_{\sigma \downarrow 0} \phi(\sigma)^{-1}\left[\fint_{-\sigma}^{\sigma} |g - g_{[-\sigma,\sigma]}|^2 dy\right]^{1/2} \geq \frac{1}{2}\lim_{\sigma\downarrow 0} |\lg\sigma|\,\chi(\sigma) = +\infty .$$

Thus $g \notin \mathcal{L}_\Phi(\Omega)$ and this proves the first assertion.

Next, set $\Omega :=]-1,1[$ and define:

(1.15) $$\psi(\sigma) := [(1+|\lg\sigma|)[1+\lg(1+|\lg\sigma|)]]^{-1}, \qquad \sigma\in[0,1],$$

(1.16) $$\eta(\sigma) := \int_\sigma^1 \frac{\psi(r)}{r}\,dr = \lg[1+\lg(1+|\lg\sigma|)], \qquad \sigma \in [0,1],$$

(1.17) $$g(x) := \eta(|x|), \qquad x \in \overline{\Omega}.$$

It is known [12] that $g \in \mathcal{L}_\psi(\Omega)$ and consequently $g \in l_\phi(\Omega)$ since $\lim_{\sigma \downarrow 0} (\psi(\sigma))/(\phi(\sigma)) = 0$. On the other hand we have $g \notin L^\infty(\Omega)$ because

$$\lim_{x \to 0} g(x) = +\infty . \tag{1.18}$$

Now consider the function

$$f(x) := \sin g(x), \qquad x \in \overline{\Omega}; \tag{1.19}$$

as $t \to \sin t$ is Lipschitz continuous and bounded, it is clear that $f \in l_\phi(\Omega) \cap L^\infty(\Omega)$. In addition by (1.18) we have

$$\limsup_{x \to 0} f(x) = 1, \qquad \liminf_{x \to 0} f(x) = -1, \tag{1.20}$$

so that $f \notin C^0(\overline{\Omega})$. Proposition 1.9 is completely proved. ∎

PROPOSITION 1.10. – The class of Dini continuous functions in $\overline{\Omega}$ is strictly contained in $C^0(\overline{\Omega}) \cap l_\phi(\Omega)$.

PROOF. – Firstly it is easy to construct functions belonging to $C^0(\overline{\Omega}) \cap l_\phi(\Omega)$ which are not Dini continuous. An example in $\Omega :=]-1, 1[$ is $f(x) := \psi(x) \cdot \operatorname{sgn} x$, where ψ is the function (1.15): arguing as in the proof of Proposition 1.9, we see that $f \in \mathcal{L}_\psi(\Omega) \subset l_\phi(\Omega)$, whereas the oscillation of f in $]-\sigma, \sigma[$ is $\omega(\sigma) := 2\psi(\sigma/2)$ and clearly the Dini condition $\int_0^1 (\omega(\sigma)/\sigma)\, d\sigma < \infty$ is not fulfilled.

Next, let $f: \overline{\Omega} \to \boldsymbol{R}$ be Dini continuous and let $\omega: [0, \operatorname{diam} \Omega] \to \boldsymbol{R}^+$ be its oscillation: thus ω is continuous, concave, non-decreasing, such that $\omega(0) = 0$ and $\int_0^1 (\omega(\sigma)/\sigma)\, d\sigma < \infty$. The estimate

$$\fint_{Q(x,\sigma)\cap\Omega} |f - f_{Q(x,\sigma)\cap\Omega}|^2 dy \leq \fint_{Q(x,\sigma)\cap\Omega} |f - f(x)|^2 dy \leq [\omega(\sigma\sqrt{n})]^2 =: [\overline{\omega}(\sigma)]^2,$$

shows that $f \in \mathcal{L}_{\overline{\omega}}(\Omega)$; thus it is sufficient to show that (rewriting ω in place of $\overline{\omega}$) $\mathcal{L}_\omega(\Omega) \subset l_\phi(\Omega)$, i.e. that

$$\lim_{\sigma \to 0^+} \frac{\omega(\sigma)}{\phi(\sigma)} = 0 \qquad (\phi(\sigma) := (1 + |\lg \sigma|)^{-1}).$$

Indeed, assume by contradiction that $\limsup_{\sigma \to 0^+} (\omega(\sigma)/\phi(\sigma)) > 0$: then we may suppose (possibly replacing ω by $c \cdot \omega$) that there is a sequence $\{t_k\}_{k \in N} \subset]0, 1[$ such that

$$\sqrt{t_{k+1}} < t_k, \qquad \frac{\omega(t_k)}{\phi(t_k)} > 1 \qquad \forall k \in \boldsymbol{N}.$$

Denote by $r(\sigma)$ the broken line joining all points $(t_k, \phi(t_k))$, i.e.

$$r(\sigma) := \left[\frac{\phi(t_k) - \phi(t_{k+1})}{t_k - t_{k+1}}\right]\sigma + \left[\phi(t_k) - t_k\left(\frac{\phi(t_k) - \phi(t_{k+1})}{t_k - t_{k+1}}\right)\right], \qquad \sigma \in [t_{k+1}, t_k],\ k \in \boldsymbol{N};$$

by the concavity of ω we have $\omega \geq r$ in $[0, t_0]$. Hence

$$\int_0^{t_0} \frac{\omega(\sigma)}{\sigma} d\sigma \geq \int_0^{t_0} \frac{r(\sigma)}{\sigma} d\sigma = \sum_{k=0}^{\infty}\left\{[\phi(t_k) - \phi(t_{k+1})] + \left[\phi(t_k) - t_k\left(\frac{\phi(t_k) - \phi(t_{k+1})}{t_k - t_{k+1}}\right)\right]\log\frac{t_k}{t_{k+1}}\right\},$$

and after standard manipulations we get

$$\int_0^{t_0} \frac{\omega(\sigma)}{\sigma} d\sigma \geq \phi(t_0) + \sum_{k=0}^{\infty} \lg\frac{t_k}{t_{k+1}} \frac{\dfrac{\phi(t_{k+1})}{t_{k+1}} - \dfrac{\phi(t_k)}{t_k}}{\dfrac{1}{t_{k+1}} - \dfrac{1}{t_k}} =$$

$$= \phi(t_0) + \sum_{k=0}^{\infty} \lg\frac{t_k}{t_{k+1}} \frac{\displaystyle\int_{t_{k+1}}^{t_k} -\left[\frac{d}{d\sigma}\frac{\phi(\sigma)}{\sigma}\right] d\sigma}{\dfrac{1}{t_{k+1}} - \dfrac{1}{t_k}}.$$

As

$$-\frac{d}{d\sigma}\frac{\phi(\sigma)}{\sigma} = \frac{-\phi'(\sigma)\cdot\sigma + \phi(\sigma)}{\sigma^2} = \frac{\phi(\sigma) - [\phi(\sigma)]^2}{\sigma^2},$$

we easily obtain

$$\int_0^{t_0} \frac{\omega(\sigma)}{\sigma} d\sigma \geq \phi(t_0) + \sum_{k=0}^{\infty} \lg\frac{t_k}{t_{k+1}} \phi(t_{k+1})(1 - \phi(t_k))$$

and since $t_k > \sqrt{t_{k+1}}$, recalling also that $\phi(t_k) \downarrow 0$, we conclude that

$$\int_0^{t_0} \frac{\omega(\sigma)}{\sigma} d\sigma \geq \phi(t_0) + c\sum_{k=0}^{\infty} \phi(t_{k+1}) \lg\frac{1}{\sqrt{t_{k+1}}} = \phi(t_0) + \frac{c}{2}\sum_{k=0}^{\infty} \frac{\lg\dfrac{1}{t_{k+1}}}{1 + \lg\dfrac{1}{t_{k+1}}} = +\infty.$$

This contradicts the Dini continuity assumption of f. The proof is complete. ∎

Let again Φ be a function satisfying (1.1).

We are interested to introduce some equivalent seminorms in $\mathcal{L}_\Phi(Q)$, where Q is a cube of $\boldsymbol{R}^n$ whose edges are parallel to the coordinate axes.

We need the following

LEMMA 1.11. – Let $x_0 \in \mathbf{R}^n$, $r_0 \in]0, d/2]$, $Q := Q(x_0, r_0)$. For $f \in \mathcal{L}_\Phi(Q)$, set

$$[f]^*_{\mathcal{L}_\Phi(Q)} := \sup\left\{\Phi(\sigma)^{-1} \fint_{Q(x_0,\sigma)} |f(y) - f_{Q(x,\sigma)}|\, dy \colon Q(x,\sigma) \subset Q\right\}, \tag{1.21}$$

$$\psi_s(\sigma) := \int_\sigma^s \frac{\Phi(r)}{r}\, dr, \qquad \sigma \in [0, s],\ \ s \in]0, d]. \tag{1.22}$$

Then we have:

$$\operatorname{meas}\{y \in Q\colon |f(y) - f_Q| > \sigma\} \begin{cases} \leqslant \left[\psi_{2r_0}^{-1}\left(\dfrac{\sigma}{b_n [f]^*_{\mathcal{L}_\Phi(Q)}}\right)\right]^n & \text{if } \dfrac{\sigma}{b_n [f]^*_{\mathcal{L}_\Phi(Q)}} \in [\psi_{2r_0}(r_0), \psi_{2r_0}(0)[\,, \\ = 0 & \text{if } \dfrac{\sigma}{b_n [f]^*_{\mathcal{L}_\Phi(Q)}} \geqslant \psi_{2r_0}(0), \end{cases}$$

where b_n is a positive constant and $\psi_{2r_0}^{-1}$ is the inverse function of ψ_{2r_0}.

PROOF. – It is essentially contained in [12, proof of (4.3)]. Set $m(\sigma) := \operatorname{meas}\{y \in Q\colon |f(y) - f_Q| > \sigma\}$.

For fixed $j \in \mathbf{N}^+$, divide Q into 2^{jn} parallel subcubes $Q_{jk} := Q(x_{jk}, r_j)$, where $r_j := r_0 \cdot 2^{-j}$.

If $y \in Q_{jk}$ and $|f(y) - f_Q| > \sigma$ we have

$$\begin{aligned} \sigma &< |f(y) - f_{Q_{jk}}| + |f_{Q_{jk}} - f_Q| \leqslant \\ &\leqslant |f(y) - f_{Q_{jk}}| + \sum_{h=0}^{j-1} \sup\left\{\fint_{Q(z, r_{h+1})} |f(y) - f_{Q(x, r_h)}|\, dy\colon Q(z, r_{h+1}) \subset Q(x, r_h) \subset Q\right\} \leqslant \\ &\leqslant |f(y) - f_{Q_{jk}}| + 2^n [f]^*_{\mathcal{L}_\Phi(Q)} \sum_{h=0}^{j-1} \Phi(r_h); \end{aligned}$$

hence if $y \in Q_{jk}$ and $|f(y) - f_Q| > \sigma$ we obtain

$$|f(y) - f_{Q_{jk}}| > \sigma - 2^n [f]^*_{\mathcal{L}_\Phi(Q)} \sum_{h=0}^{j-1} \Phi(r_h);$$

consequently

$$m(\sigma) \leqslant \sum_{k=1}^{2^{nj}} \operatorname{meas}\left\{y \in Q_{jk}\colon |f(y) - f_{Q_{jk}}| > \sigma - 2^n [f]^*_{\mathcal{L}_\Phi(Q)} \sum_{h=0}^{j-1} \Phi(r_h)\right\}. \tag{1.23}$$

Choose in particular

$$\sigma_j := (2^n + n a_n)[f]^*_{\mathcal{L}_\Phi(Q)} \sum_{h=0}^{j-1} \Phi(r_h), \tag{1.24}$$

a_n being a suitable positive constant: as

$$\sigma_j \geq \left[2^n \sum_{h=0}^{j-1} \Phi(r_h) + nja_n \Phi(r_j)\right][f]^*_{\mathcal{L}_\Phi(Q)},$$

we get by (1.23)

$$m(\sigma_j) \leq \sum_{k=1}^{2^{nj}} \text{meas}\,\{y \in Q_{jk} : |f(y) - f_{Q_{jk}}| > nja_n \Phi(r_j)[f]^*_{\mathcal{L}_\Phi(Q)}\}. \tag{1.25}$$

On the other hand by John-Nirenberg's lemma [10, Lemma 1], arguing as in [12, Lemma 4] we easily deduce that there exists $a_n > 0$ such that

$$\text{meas}\,\{y \in Q_{jk} : |f(y) - f_{Q_{jk}}| > nja_n \Phi(r_j)[f]^*_{\mathcal{L}_\Phi(Q)}\} \leq 2^{-nj} r_j^n,$$

so that

$$m(\sigma_j) \leq \sum_{k=1}^{2^{nj}} 2^{-nj} r_j^n = r_j^n. \tag{1.26}$$

Now we observe that

$$\sum_{h=0}^{j-1} \Phi(r_h) = (\log 2)^{-1} \sum_{h=0}^{j-1} \int_{r_0 \cdot 2^{-h}}^{r_0 \cdot 2^{-h+1}} \frac{ds}{s} \cdot \Phi(r_h) \leq$$

$$\leq (\log 2)^{-1} \sum_{h=0}^{j-1} \int_{r_0 \cdot 2^{-h}}^{r_0 \cdot 2^{-h+1}} \frac{\Phi(s)}{s} ds = (\log 2)^{-1} \psi_{2r_0}(r_{j-1}),$$

which implies by (1.24)

$$\sigma_j \leq \frac{2^n + na_n}{\log 2} [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(r_{j-1}) =: b_n [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(r_{j-1}). \tag{1.27}$$

Now let $r \in]0, r_0]$, so that there is a unique $j \in \boldsymbol{N}^+$ for which $r_j \leq r \leq r_{j-1}$. Then by (1.27) and (1.26)

$$m(b_n [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(r)) \leq m(b_n [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(r_{j-1})) \leq m(\sigma_j) \leq r_j^n \leq r^n. \tag{1.28}$$

Hence setting $\sigma := b_n [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(r)$, $r \in]0, r_0]$, we get

$$m(\sigma) \leq r^n = \left[\psi_{2r_0}^{-1}\left(\frac{\sigma}{b_n [f]^*_{\mathcal{L}_\Phi(Q)}}\right)\right]^n \quad \text{if} \quad \frac{\sigma}{b_n [f]^*_{\mathcal{L}_\Phi(Q)}} \in [\psi_{2r_0}(r_0), \psi_{2r_0}(0)[\,;$$

on the other hand if $\sigma \geq b_n [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(0)$ then by (1.28)

$$m(\sigma) \leq m(b_n [f]^*_{\mathcal{L}_\Phi(Q)} \psi_{2r_0}(0)) = 0$$

and the result is proved. ∎

A quite simple similar proof gives:

LEMMA 1.12. – Let $x_0 \in \boldsymbol{R}^{n-1} \times \{0\}$, $r_0 \in]0, d/2]$, $Q^+ := Q^+(x_0, r_0)$. For $f \in \mathcal{L}_\Phi(Q^+)$, set

$$[f]^{**}_{\mathcal{L}_\Phi(Q^+)} := M_1 \vee M_2, \tag{1.29}$$

$$M_1 := \sup\left\{\Phi(\sigma)^{-1} \fint_{Q(x,r)} |f(y) - f_{Q(x,r)}|\,dy \colon Q(x,r) \subset Q^+\right\}, \tag{1.30}$$

$$M_2 := \sup\left\{\Phi(\sigma)^{-1} \fint_{Q^+(x,r)} |f(y) - f_{Q^+(x,r)}|\,dy \colon x \in \Gamma(x_0, r_0),\ Q^+(x,r) \subset Q^+\right\}. \tag{1.31}$$

Then we have

$$\operatorname{meas}\{y \in Q^+ \colon |f(y) - f_{Q^+}| > \sigma\} \begin{cases} \leqslant \left[\psi_{2r_0}^{-1}\left(\dfrac{\sigma}{b_n [f]^{**}_{\mathcal{L}_\Phi(Q^+)}}\right)\right]^n & \text{if } \dfrac{\sigma}{b_n [f]^{**}_{\mathcal{L}_\Phi(Q^+)}} \in [\psi_{2r_0}(r_0), \psi_{2r_0}(0)[, \\ = 0 & \text{if } \dfrac{\sigma}{b_n [f]^{**}_{\mathcal{L}_\Phi(Q^+)}} \geqslant \psi_{2r_0}(0), \end{cases}$$

where ψ_{2r_0} is defined by (1.22) and b_n is a positive constant. ∎

The following result is very important for us. It is related to the well known fact that for *BMO* functions (and, a fortiori, for $\mathcal{L}_\Phi$ functions) all L^p norms with $1 \leqslant p < \infty$ are equivalent.

PROPOSITION 1.13. – Let Ω be a bounded open set of $\boldsymbol{R}^n$. For $p \in [1, \infty[$, $\Omega' \subset\subset \Omega$, $\delta \in\,]0, \min\{d/2, 1/(2\sqrt{n})\operatorname{dist}(\partial\Omega', \partial\Omega)\}]$ and $f \in \mathcal{L}_\Phi(\Omega)$, set

$$N_p(f; \Phi, \Omega', \delta) := \sup\left\{\Phi(\sigma)^{-1}\left[\fint_{Q(x,\sigma)} |f(y) - f_{Q(x,\sigma)}|^p\,dy\right]^{1/p} \colon x \in \Omega',\ \sigma \in]0, \delta]\right\}. \tag{1.32}$$

Then we have for each $f \in \mathcal{L}_\Phi(\Omega)$ (see (1.3))

$$N_1(f; \Phi, \Omega', \delta) \leqslant N_p(f; \Phi, \Omega', \delta) \leqslant C(p, n, K_\Phi)[f]_{\Phi,\Omega,\delta}.$$

PROOF. – The first inequality is obvious. To prove the second one, it is clearly sufficient to take $p = m \in \boldsymbol{N}^+$. We use a modification of the argument of [12, proof of Th. 1(*b*)]. If $x \in \Omega'$ and $\sigma \in]0, \delta]$ (so that $Q(x,\sigma) \subset \Omega$), we have

$$\fint_{Q(x,\sigma)} |f(y) - f_{Q(x,\sigma)}|^m\,dy = (2\sigma)^{-n}\int_0^\infty \operatorname{meas}\{y \in Q(x,\sigma) \colon |f(y) - f_{Q(x,\sigma)}| > t\} \cdot m t^{m-1}\,dt,$$

and by Lemma 1.11, setting $k := b_n[f]^*_{\mathcal{L}_\Phi(Q(x,\sigma))}$, we get after a change of variable:

$$(1.33)\quad \fint_{Q(x,\sigma)} |f(y) - f_{Q(x,\sigma)}|^m dy \leq m\sigma^{-n} \int_0^{k\psi_{2\sigma}(\sigma)} t^{m-1}\sigma^n dt + m\sigma^{-n} \int_{k\psi_{2\sigma}(\sigma)}^{k\psi_{2\sigma}(0)} t^{m-1}\left[\psi_{2\sigma}^{-1}\left(\frac{t}{k}\right)\right]^n dt =$$

$$= k^m\left\{[\psi_{2\sigma}(\sigma)]^m + m\sigma^{-n}\int_0^\sigma [\psi_{2\sigma}(s)]^{m-1}s^{n-1}\Phi(s)\,ds\right\} \leq$$

$$\leq k^m\left\{[\psi_{2\sigma}(\sigma)]^m + m\sigma^{-n}[\Phi(2\sigma)]^m\int_0^\sigma\left[\int_s^{2\sigma}\frac{du}{u}\right]^{m-1} s^{n-1}ds\right\}.$$

The last integral becomes, by repeated integrations by parts:

$$I_m := \int_0^\sigma\left[\int_s^{2\sigma}\frac{du}{u}\right]^{m-1} s^{n-1}ds = \frac{\sigma^n}{n}[\log 2]^{m-1} + I_{m-1}\frac{m-1}{n} = \ldots =$$

$$= \frac{\sigma^n}{n}\sum_{h=1}^{m-1}\frac{[\log 2]^h}{n^{m-1-h}}\frac{(m-1)!}{h!} + I_1\frac{(m-1)!}{n^{m-1}} = \sigma^n\sum_{h=0}^{m-1}\frac{(m-1)!}{h!}\frac{(\log 2)^h}{n^{m-h}};$$

hence by (1.33) we derive (since $\phi(\sigma)$ is non-decreasing and $(\Phi(\sigma))/\sigma$ is almost decreasing):

$$\fint_{Q(x,\sigma)} |f(y) - f_{Q(x,\sigma)}|^m dy \leq k^m\left\{[\psi_{2\sigma}(\sigma)]^m + [\Phi(2\sigma)]^m\sum_{h=0}^{m-1}\frac{m!}{h!}\frac{(\log 2)^h}{n^{m-h}}\right\} \leq$$

$$\leq k^m\sum_{h=0}^{m}\frac{m!}{h!}\frac{(\log 2)^h}{n^{m-h}}[\Phi(2\sigma)]^m \leq [2k\cdot K_\Phi]^m\sum_{h=0}^{m}\frac{m!}{h!}\frac{(\log 2)^h}{n^{m-h}}[\Phi(\sigma)]^m,$$

and the result follows since $k \leq c(n)[f]_{\Phi,\Omega,\delta}$. ∎

A similar proof, using Lemma 1.12 as well as Lemma 1.11, gives:

PROPOSITION 1.14. – Let $x_0 \in \boldsymbol{R}^{n-1}\times\{0\}$, $r_0 > 0$, $B^+ := B^+(x_0, r_0)$. For $p \in [1, \infty[$, $0 < r' < r_0/\sqrt{n}$, $\delta \in]0, \min\{d/2, 1/2(r_0/\sqrt{n} - r')\}$ and $f \in \mathcal{L}_\Phi(Q^+)$, set

$$(1.34)\quad N_p^+(f;\Phi, r', \delta) := N_1 \vee N_2,$$

$$(1.35)\quad N_1 := \sup\left\{\Phi(\sigma)^{-1}\left[\fint_{Q(x,\sigma)}|f(y) - f_{Q(x,\sigma)}|^p dy\right]^{1/p} : x \in Q^+(x_0, r'), \sigma \in]0, \delta], Q(x,\sigma) \subset B^+\right\},$$

(1.36) $$N_2 := \sup\left\{\Phi(\sigma)^{-1}\left[\ \fint_{Q^+(x,\sigma)} |f(y) - f_{Q^+(x,\sigma)}|^p\, dy\right]^{1/p} : x \in \Gamma(x_0, r'), \sigma \in]0, \delta]\right\}.$$

Then we have

$$N_1^+(f;\Phi, r', \delta) \leq N_p^+(f;\Phi, r', \delta) \leq c(p, n, K_\Phi)[f]_{\Phi, B^+, \delta} \qquad \forall f \in \mathcal{L}_\Phi(B^+). \qquad \blacksquare$$

We end this section with some useful inequalities for *BMO* functions.

PROPOSITION 1.15. – Let Ω be a bounded open set of $\boldsymbol{R}^n$; let $\Omega' \subset\subset \Omega$ with $\delta := \operatorname{dist}(\partial\Omega', \partial\Omega) \leq 2$. For $x_0 \in \overline{\Omega}'$, $\sigma \in]0, \delta]$ and $f \in BMO(\Omega)$, we have:

(1.37) $$|f_{Q(x_0,\sigma)}| \leq c(n)\{[1 + |\lg \sigma|][f]_{BMO(\Omega)} + \delta^{-n/2}\|f\|_{L^2(\Omega)}\}.$$

PROOF. – This argument is essentially that of [12, proof of Lemma 2 (*a*) and Lemma 5]. Firstly, if $0 < \rho < \sigma \leq \delta$ and $x_0 \in \overline{\Omega}'$ we have

(1.38) $$\fint_{Q(x_0,\rho)} |f - f_{Q(x_0,\rho)}|\, dx \leq \fint_{Q(x_0,\rho)} |f - f_{Q(x_0,\sigma)}|\, dx + |f_{Q(x_0,\sigma)} - f_{Q(x_0,\sigma)}| \leq$$

$$\leq 2 \fint_{Q(x_0,\rho)} |f - f_{Q(x_0,\sigma)}|\, dx \leq 2\left(\frac{\sigma}{\rho}\right)^n \fint_{Q(x_0,\sigma)} |f - f_{Q(x_0,\sigma)}|\, dx.$$

Next, let $x_0 \in \overline{\Omega}'$, $\sigma \in]0, \delta]$. If $\sigma \in [\delta/4, \delta]$ then $2\sigma \geq \delta/2$ so that

$$|f_{Q(x_0,\sigma)}| \leq \left(\frac{2}{\delta}\right)^{-n} \|f\|_{L^2(Q(x_0,\delta))}(2\delta)^{n/2} \leq \left(\frac{8}{\delta}\right)^{n/2} \|f\|_{L^2(\Omega)}.$$

If, otherwise, $\sigma \in]0, \delta/4]$, there exists a unique $k \in N^+$ such that $2^k\sigma < \delta/2 \leq 2^{k+1}\sigma$. Hence

$$|f_{Q(x_0,\sigma)}| \leq \sum_{h=0}^{k-1} |f_{Q(x_0,2^h\sigma)} - f_{Q(x_0,2^{h+1}\sigma)}| + |f_{Q(x_0,2^k\sigma)}| \leq$$

$$\leq \sum_{h=0}^{k-1} \fint_{Q(x_0,2^h\sigma)} |f - f_{Q(x_0,2^{h+1}\sigma)}|\, dx + \left(\frac{2}{\delta}\right)^n \int_{Q(x_0,\delta)} |f|\, dx \leq$$

$$\leq \sum_{h=0}^{k-1} 2^n \fint_{Q(x_0,2^{h+1}\sigma)} |f - f_{Q(x_0,2^{h+1}\sigma)}|\, dx + \left(\frac{4}{\delta}\right)^{n/2} \|f\|_{L^2(\Omega)} =$$

$$= \sum_{h=0}^{k-1} \frac{2^n}{\lg 2}\left[\int_{2^{h+1}\sigma}^{2^{h+2}\sigma} \fint_{Q(x_0,2^{h+1}\sigma)} |f - f_{Q(x_0,2^{h+1}\sigma)}|\, dx \frac{ds}{s}\right] + \left(\frac{4}{\delta}\right)^{n/2} \|f\|_{L^2(\Omega)},$$

so that (1.38) yields

$$|f_{Q(x_0,\sigma)}| \leq \sum_{h=0}^{k-1} \frac{2^n}{\lg 2} \int_{2^{h+1}\sigma}^{2^{h+2}\sigma} 2\left(\frac{s}{2^{h+1}\sigma}\right)^n \fint_{Q(x_0,s)} |f - f_{Q(x_0,s)}|\,dx\,\frac{ds}{s} + \left(\frac{4}{\delta}\right)^{n/2} \|f\|_{L^2(\Omega)} \leq$$

$$\leq \frac{2^{2n+1}}{\lg 2} \int_{2\sigma}^{2^{k+1}\sigma} \frac{ds}{s} N_1(f;1,\Omega',\delta) + \left(\frac{4}{\delta}\right)^{n/2} \|f\|_{L^2(\Omega)} \leq$$

$$\leq \frac{2^{2n+1}}{\lg 2} \lg \frac{\delta}{2\sigma} N_2(f;1,\Omega',\delta) + \left(\frac{4}{\delta}\right)^{n/2} \|f\|_{L^2(\Omega)},$$

and the result follows. ■

PROPOSITION 1.16. – Let $x_0 \in \boldsymbol{R}^{n-1} \times \{0\}$ and fix $r > r' > 0$. For $x \in \Gamma(x_0, r)$, $\sigma \in\,]0, r - r']$ and $f \in BMO(Q^+(x_0, r))$ we have:

$$|f_{Q^+(x,\sigma)}| \leq c(n)\{[1 + |\lg \sigma|][f]_{BMO(Q^+(x_0,r))} + (r - r')^{-n/2} \|f\|_{L^2(Q^+(x_0,r))}\}.$$

PROOF. – Extend f to $Q(x_0, r)$ by setting

$$F(x_1, \dots, x_n) = \begin{cases} f(x_1, \dots, x_n) & \text{if } x_n \geq 0, \\ f(x_1, \dots, x_{n-1}, |x_n|) & \text{if } x_n < 0. \end{cases}$$

Then $F \in BMO(Q(x_0, r))$ with $\|F\|^2_{L^2(Q(x_0,r))} = 2\|f\|^2_{L^2(Q^+(x_0,r))}$ and $[F]_{BMO(Q(x_0,r))} \leq c(n)[f]_{BMO(Q^+(x_0,r))}$ (see the proof of Lemma 1.4); thus the result follows easily by Proposition 1.15. ■

2. – Global *BMO* regularity.

Let Ω be a bounded open set of $\boldsymbol{R}^n$ and consider the operator

$$Eu := -\operatorname{div}(A(x) \cdot Du), \qquad u \in H^1(\Omega, \boldsymbol{R}^N), \tag{2.1}$$

where $A \in L^\infty(\Omega, \boldsymbol{R}^{N^2 n^2}) \cap l_\phi(\Omega, \boldsymbol{R}^{N^2 n^2})$ with $\phi(\sigma) := (1 + |\lg \sigma|)^{-1}$; we assume the strong ellipticity condition

$$(A_{ij}(x) \cdot \xi^j | \xi^i)_N \geq \nu \sum_{i=1}^n |\xi^i|^2 \qquad \forall x \in \Omega,\ \forall \xi^1, \dots, \xi^n \in \boldsymbol{R}^N. \tag{2.2}$$

REMARK 2.1. – (i) In all what follows, we may take $A \in C^0(\overline{\Omega}, \boldsymbol{R}^{N^2 n^2}) \cap l_\phi(\Omega, \boldsymbol{R}^{N^2 n^2})$ under the (weaker) ellipticity assumption

$$(A_{ij}(x) \cdot \eta | \eta)_N \xi_i \xi_j \geq \nu |\xi|^2 |\eta|^2 \qquad \forall x \in \Omega,\ \forall \xi \in \boldsymbol{R}^n,\ \forall \eta \in \boldsymbol{R}^N. \tag{2.3}$$

(ii) To avoid formal complications, we assume $n > 2$. If $n = 1$ or $n = 2$ we need a slight modification in the assumptions on data; see Remark 2.5 below.

We consider the Dirichlet problem associated to the operator E in the variational sense:

$$(2.4)\qquad \begin{cases} u \in H_0^1(\Omega, \mathbf{R}^N), \\ \int_\Omega (A_{ij}(x) \cdot D_j u | D_i \theta)\, dx = \int_\Omega (f_i(x) | D_i \theta)\, dx - \int_\Omega (f_0(x)|\theta)\, dx \quad \forall \theta \in H_0^1(\Omega, \mathbf{R}^N), \end{cases}$$

where $f \in L^2(\Omega, \mathbf{R}^{Nn})$, $f_0 \in L^{2n/(n+2)}(\Omega, \mathbf{R}^N)$. By Lax-Milgram's Theorem, the solution of problem (2.4) always exists and is unique, since by [6, Ch. I, Lemma 4.1] the distribution $f_0 - \operatorname{div} f$ belongs to $H^{-1,2}(\Omega, \mathbf{R}^N) = (H_0^1(\Omega, \mathbf{R}^N))^*$. We want to prove the following result:

THEOREM 2.2. – Let u be the solution of the Dirichlet problem (2.4), with $\partial\Omega \in C^{1+\beta}$ $(\beta > 0)$, $f \in BMO(\Omega, \mathbf{R}^{nN})$, $f_0 \in L^{2n/(n+2),\, n^2/(n+2)}(\Omega, \mathbf{R}^N)$ (see (1.4)), $A \in L^\infty(\Omega, \mathbf{R}^{N^2 n^2}) \cap l_\phi(\Omega, \mathbf{R}^{N^2 n^2})$ $(\phi(\sigma) := (1 + |\lg \sigma|)^{-1})$. Then $Du \in BMO(\Omega, \mathbf{R}^{Nn})$ and

$$(2.5)\quad [Du]_{BMO(\Omega)} \le c(n, \nu, \Omega, \beta, \omega_A, \|A\|_{L^\infty(\Omega)})[1 + [A]_{\mathcal{L}_\phi(\Omega)}]\left[\|f\|_{BMO(\Omega)} + \|f_0\|_{L^{2n/(n+2),\, n^2/(n+2)}(\Omega)}\right],$$

where (see (1.3))

$$(2.6)\qquad \omega_A(\sigma) := [A]_{\phi, \Omega, \sigma}, \qquad \sigma \in]0, 1].$$

PROOF. – Our proof splits in two steps:

Step 1: The estimate (2.5) holds under the stronger assumption $A \in C^\beta(\overline{\Omega}, \mathbf{R}^{N^2 n^2})$, which guarantees «a priori» that $Du \in BMO(\Omega, \mathbf{R}^{Nn})$: see [5, Theorem 16.I] for the case $N = 1$; the extension to $N \ge 1$ is straightforward (compare with [6, Ch. II, Theorem 5.I]).

Step 2: We approximate A by a smooth sequence $\{A_k\}_{k \in \mathbf{N}}$, in such a way that:

(i) the solutions u_k of the approximating systems converge to the solution u of the original problem,

(ii) the sequence $\{A_k\}$ fulfills, uniformly in $k \in \mathbf{N}$, all the relevant properties required for A in Step 1;

(iii) the estimate (2.5), written for u_k and A_k, is preserved when $k \to \infty$, thus yielding the result in its full generality.

To start with, we remark that since $A \in C^\beta(\overline{\Omega}, \mathbf{R}^{N^2 n^2}) \subset l_\phi(\overline{\Omega}, \mathbf{R}^{N^2 n^2})$, we have by (2.6) and (1.3)

$$(2.7)\qquad \lim_{\sigma \downarrow 0} \omega_A(\sigma) = 0.$$

Moreover, as Ω is bounded with $\partial\Omega \in C^{1+\beta}$, there exists a finite number of open sets $\Omega_0, \Omega_1, \ldots \Omega_m$ contained in Ω, such that:

$$(2.8)\quad \begin{cases} \text{(i) for } 1 \leq s \leq m \text{ there exists a } C^{1+\beta} \text{ diffeomorphism } T_s\colon \overline{\Omega}_s \to \overline{B^+(0,1)} \\ \qquad\qquad \text{such that } T_s(\overline{\Omega}_s \cap \partial\Omega) = \Gamma(0,1); \\ \text{(ii) there exist } c_1(\Omega),\ c_2(\Omega) > 0 \text{ such that} \\ \qquad c_1(\Omega) \leq |\det T_s^{-1}(y)| \leq c_2(\Omega) \quad \forall y \in \overline{B^+(0,1)},\ \forall s \in \{1, \ldots, m\}; \\ \text{(iii) } \Omega_0 \subset\subset \Omega \quad \text{and} \quad \Omega = \Omega_0 \cup \left[\bigcup_{s=1}^{m} T_s^{-1}\left(Q^+\left(0, \dfrac{1}{\sqrt{n}}\right)\right)\right]. \end{cases}$$

PROOF OF STEP 1. – Let u be the solution of the Dirichlet problem (2.4), with $f \in BMO(\Omega, \boldsymbol{R}^{Nn})$, $f_0 \in L^{2n/(n+2), n^2/(n+2)}(\Omega, \boldsymbol{R}^N)$ and $A \in C^\beta(\overline{\Omega}, \boldsymbol{R}^{N^2 n^2})$. We will prove:

Step 1A: Estimate for $[Du]_{BMO(\Omega_0)}$;

Step 1B: Estimate for $[DU_s]_{BMO(Q^+(0,R))}$, $0 < R < 1/\sqrt{n}$, where $U_s := u \circ T_s^{-1}$;

Step 1C: Final estimate for $[Du]_{BMO(\Omega)}$.

PROOF OF STEP 1A. – Set $d_0 := \operatorname{dist}(\partial\Omega_0, \partial\Omega)$; it is not restrictive to assume $d_0 \in]0, 1/2]$. Fix a cube $Q(x_0, \sigma)$ with $x_0 \in \Omega_0$ and $\sigma \in]0, 1/(2\sqrt{n}]$, so that $Q(x_0, \sigma) \subset\subset \Omega$. In $Q(x_0, \sigma)$ split $u = v + w$, where w is the unique solution of the Dirichlet problem

$$(2.9)\quad \begin{cases} w \in H_0^1(Q(x_0, \sigma), \boldsymbol{R}^N), \\ \displaystyle\int_{Q(x_0,\sigma)} (A_{Q(x_0,\sigma)} \cdot Dw | D\theta)\, dx = - \int_{Q(x_0,\sigma)} ([A(x) - A_{Q(x_0,\sigma)}] \cdot [Du - (Du)_{Q(x_0,\sigma)}] | D\theta)\, dx - \\ \displaystyle\qquad - \int_{Q(x_0,\sigma)} ([A(x) - A_{Q(x_0,\sigma)}] \cdot (Du)_{Q(x_0,\sigma)} | D\theta)\, dx + \\ \displaystyle\qquad + \int_{Q(x_0,\sigma)} (f(x) - f_{Q(x_0,\sigma)} | D\theta)\, dx - \int_{Q(x_0,\sigma)} (f_0(x) | \theta)\, dx \quad \forall \theta \in H_0^1(Q(x_0, \sigma), \boldsymbol{R}^N); \end{cases}$$

then $v := u - w$ is a solution of the homogeneous system with constant coefficients

$$(2.10)\quad \begin{cases} v \in H^1(Q(x_0, \sigma), \boldsymbol{R}^N), \\ \displaystyle\int_{Q(x_0,\sigma)} (A_{Q(x_0,\sigma)} \cdot Dv | D\theta)\, dx = 0 \quad \forall \theta \in H_0^1(Q(x_0, \sigma), \boldsymbol{R}^N). \end{cases}$$

For the function v we have the fundamental estimate [6, Ch. II, Theorem 3.III]

$$(2.11)\quad \int_{Q(x_0, t\sigma)} |Dv - (Dv)_{Q(x_0,t\sigma)}|^2\, dx \leq c(\nu)\, t^{n+2} \int_{Q(x_0,\sigma)} |Dv - (Dv)|_{Q(x_0,\sigma)}|^2\, dx \quad \forall t \in]0, 1].$$

For the function w we have the variational estimate [6, Ch. II, Theorem 1.III]

$$(2.12)\qquad \int_{Q(x_0,\sigma)} |Dw|^2\,dx \leqslant \frac{1}{\nu^2}\left\{\int_{Q(x_0,\sigma)} [|A - A_{Q(x_0,\sigma)}|^2\,|Du - (Du)_{Q(x_0,\sigma)}|^2 + \right.$$

$$\left. + |A - A_{Q(x_0,\sigma)}|^2\,|(Du)_{Q(x_0,\sigma)}|^2 + |f - f_{Q(x_0,\sigma)}|^2]\,dx + c(n)\left[\int_{Q(x_0,\sigma)} |f_0|^{2n/(n+2)}\,dx\right]^{(n+2)/n}\right\}.$$

Let us estimate the right member of (2.12). As Du $\in BMO(\Omega)$, we have by Proposition 1.13 and (1.3):

$$(2.13)\qquad \int_{Q(x_0,\sigma)} |A - A_{Q(x_0,\sigma)}|^2\,|Du - (Du)_{Q(x_0,\sigma)}|^2 \leqslant$$

$$\leqslant \left[\int_{Q(x_0,\sigma)} |A - A_{Q(x_0,\sigma)}|^4\,dx\right]^{1/2}\left[\int_{Q(x_0,\sigma)} |Du - (Du)_{Q(x_0,\sigma)}|^4\,dx\right]^{1/2} \leqslant$$

$$\leqslant \frac{2^n\sigma^n}{(1+|\lg\sigma|)^2}[N_4(A;\phi,\Omega_0,\sigma)]^2[N_4(Du;1,\Omega_0,\sigma)]^2 \leqslant$$

$$\leqslant c(n)\frac{\sigma^n}{(1+|\lg\sigma|)^2}[A]^2_{\phi,\Omega,\sigma}[Du]^2_{BMO(\Omega)}\,.$$

Next, by Proposition 1.15

$$(2.14)\qquad \int_{Q(x_0,\sigma)} |A - A_{Q(x_0,\sigma)}|^2\,|(Du)_{Q(x_0,\sigma)}|^2\,dx \leqslant$$

$$\leqslant c(n)\frac{\sigma^n}{(1+|\lg\sigma|)^2}[N_2(A;\phi,\Omega_0,\sigma)]^2\left\{(1+|\lg\sigma|)^2[Du]^2_{BMO(\Omega)} + \left(\frac{d_0}{2\sqrt{n}}\right)^{-n}\|Du\|^2_{L^2(\Omega)}\right\} \leqslant$$

$$\leqslant c(n)\,\sigma^n\left\{[A]^2_{\phi,\Omega,\sigma}[Du]^2_{BMO(\Omega)} + \frac{d_0^{-n}}{(1+|\lg\sigma|)^2}[A]^2_{\mathcal{L}_\phi(\Omega)}\|Du\|^2_{L^2(\Omega)}\right\},$$

and finally

$$\text{(2.15)}\qquad \int_{Q(x_0,\sigma)} |f - f_{Q(x_0,\sigma)}|^2 dx + c(n)\left[\int_{Q(x_0,\sigma)} |f_0|^{2n/(n+2)} dx\right]^{(n+2)/n} \leqslant$$

$$\leqslant c(n)\,\sigma^n \{[f]^2_{BMO(\Omega)} + \|f_0\|^2_{L^{2n/(n+2),\,n^2/(n+2)}(\Omega)}\}\,.$$

By (2.12), (2.13), (2.14) and (2.15) we get, recalling (2.6)

$$\text{(2.16)}\qquad \int_{Q(x_0,\sigma)} |Dw|^2 dx \leqslant c(n,\nu,d_0)\,\sigma^n \{[\omega_A(\sigma)]^2 \cdot [Du]^2_{BMO(\Omega)} + \Lambda\}\,,$$

where we have set

$$\text{(2.17)}\qquad \Lambda := \{\|A\|^2_{L^\infty(\Omega)} + [A]^2_{\mathcal{L}_\varphi(\Omega)}\}\|Du\|^2_{L^2(\Omega)} + \|f\|^2_{BMO(\Omega)} + \|f_0\|^2_{L^{2n/(n+2),\,n^2/(n+2)}(\Omega)}\,.$$

(The quantity Λ contains more terms than necessary, but we shall need them all later on).

By (2.11), (2.16) and (2.6) we easily deduce for each $t \in [0,1]$ and $\sigma \in]0, d_0/2\sqrt{n}]$:

$$\text{(2.18)}\qquad \int_{Q(x_0,t\sigma)} |Du - (Du)_{Q(x_0,t\sigma)}|^2 dx \leqslant$$

$$\leqslant c(\nu)\,t^{n+2} \int_{Q(x_0,\sigma)} |Du - (Du)_{Q(x_0,\sigma)}|^2 dx + c(n,\nu,d_0)\,\sigma^n \{[\omega_A(\sigma)]^2 [Du]^2_{BMO(\Omega)} + \Lambda\}\,.$$

We now invoke a function-theoretic lemma [6, Ch. I, Lemma 1.I] in order to get for each $t \in]0,1]$ and $\sigma \in]0, d_0/2\sqrt{n}]$:

$$\text{(2.19)}\qquad \fint_{Q(x_0,t\sigma)} |Du - (Du)_{Q(x_0,t\sigma)}|^2 dx \leqslant$$

$$\leqslant c(\nu) \fint_{Q(x_0,\sigma)} |Du - (Du)_{Q(x_0,\sigma)}|^2 dx + c(n,\nu,d_0)\{[\omega_A(\sigma)]^2 [Du]^2_{BMO(\Omega)} + \Lambda\}\,;$$

taking into account (2.17) we deduce for each $x_0 \in \Omega_0$ and $0 < r \leqslant \sigma \leqslant d_0/2\sqrt{n}$:

$$\text{(2.20)}\qquad \fint_{Q(x_0,r)} |Du - (Du)_{Q(x_0,r)}|^2 dx \leqslant c(n,\nu,d_0)\{[\omega_A(\sigma)]^2 [Du]^2_{BMO(\Omega)} + \sigma^{-n}\Lambda\}\,,$$

which easily implies:

$$\text{(2.21)}\qquad [Du]^2_{BMO(\Omega_0)} \leqslant c(n,\nu,d_0)\{[\omega_A(\sigma)]^2 [Du]^2_{BMO(\Omega)} + \sigma^{-n}\Lambda\} \qquad \forall \sigma \in]0, d_0/2\sqrt{n}]\,.$$

This concludes the proof of Step 1A.

PROOF OF STEP 1B. – Fix $s \in \{1, \ldots, m\}$ and write simply U, T for U_s, T_s. According with [5, Appendix III], the function U solves in $B^+ := B^+(0,1)$ the following problem:

$$(2.22)\quad \begin{cases} U \in H^1(B^+, \boldsymbol{R}^N), \\ \int\limits_{B^+} (B_{hk}(y) \cdot D_k U | D_h \theta)\, dy = \int\limits_{B^+} (g_h(y) | D_h \theta)\, dy - \int\limits_{B^+} (g_0(y) | \theta)\, dy \quad \forall \theta \in H_0^1(B^+, \boldsymbol{R}^N), \end{cases}$$

where (setting $J^{-1}(y) := \det |DT^{-1}(y)|$):

$$(2.23)\qquad B_{hk}(y) := J^{-1}(y)[(D_j T_h)(T^{-1}(y))][(D_i T_k)(T^{-1}(y))] A_{ij}(T^{-1}(y)),$$

$$(2.24)\qquad g_h(y) := J^{-1}(y)[(D_j T_h)(T^{-1}(y))] f_j(T^{-1}(y)), \qquad g_0(y) := J^{-1}(y) f_0(T^{-1}(y)).$$

It is easily seen that

$$(2.25)\qquad (B_{hk}(y) \cdot \zeta^h | \zeta^k)_N \geq c(\nu, \Omega) \sum_{h=1}^{n} |\zeta^h|^2 \qquad \forall y \in B^+, \forall \zeta^1, \ldots, \zeta^n \in \boldsymbol{R}^N;$$

In addition we have

$$B \in C^\beta(\overline{B^+}, \boldsymbol{R}^{N^2 n^2}), \qquad g \in BMO(B^+, \boldsymbol{R}^{nN}), \qquad g_0 \in L^{2n/(n+2), n^2/(n+2)}(B^+, \boldsymbol{R}^N)$$

and, arguing as in [5, Appendix I, Theorems IV-V], we see that

$$(2.26)\quad [B]_{\phi, B^+, \sigma/c_0} \leq c_1 \left\{ [A]_{\phi, \Omega, \sigma} + \left[\sup_{0 < r \leq \sigma} r^\beta (1 + |\lg r|) \right] \|A\|_{L^\infty(\Omega)} \right\} \qquad \forall \sigma \in]0, 1[,$$

$$(2.27)\qquad [g]_{BMO(B^+)} + \|g_0\|_{L^{2n/(n+2), n^2/(n+2)}(B^+)} \leq c_2 \left\{ [f]_{BMO(\Omega)} + \|f_0\|_{L^{2n/(n+2), n^2/(n+2)}(\Omega)} \right\},$$

$$(2.28)\qquad [Du]_{BMO(\Omega_s)} < c_3 \|Du\|_{BMO(B^+)} \leq c_4 \|Du\|_{BMO(\Omega)},$$

where c_0, c_1, c_2, c_3, c_4 depend only on Ω; it is not restrictive to assume in (2.26) $c_0 \geq 2\sqrt{n}$.

Fix now $R \in]1/\sqrt{n} - 2/c_0, 1/\sqrt{n}[$. Let $x_0 \in Q^+(0, R)$ and $\sigma \in]0, (1/\sqrt{n} - R)/4]$. Two cases can occur:

(I) $(x_0)_n > \sigma$ so that

$$Q(x_0, \sigma) \subset\subset Q^+\left(0, \frac{1}{2}\left(R + \frac{1}{\sqrt{n}}\right)\right) \subset Q^+\left(0, \frac{1}{\sqrt{n}}\right),$$

(II) $(x_0)_n \in]0, \sigma]$, so that

$$Q(x_0, \sigma) \cap B^+ \subset Q^+(y_0, 2\sigma) \subset Q^+\left(0, \frac{1}{2}\left(R + \frac{1}{\sqrt{n}}\right)\right) \subset Q^+\left(0, \frac{1}{\sqrt{n}}\right),$$

where y_0 is the projection of x_0 on the hyperplane $\{x_n = 0\}$.

In case (I) we repeat exactly the argument of Step 1A, using the quantities $N_p^+(B;\phi,R,\sigma)$ and $N_p^+(DU;1,R,\sigma)$ (see (1.34)) instead of $N_p(A;\phi,\Omega_0,\sigma)$ and $N_p(Du;1,\Omega_0,\sigma)$, and analogously to (2.20) we deduce for $0<r\leq\sigma\leq\frac{1}{4}\left(\frac{1}{\sqrt{n}}-R\right)$

$$\fint_{Q(x_0,r)} |DU-(DU)_{Q(x_0,r)}|^2\,dx \leq c(n,\nu,\Omega,R)\{[B]^2_{\phi,B^+,\sigma}[DU]^2_{BMO(B^+)}+\sigma^{-n}\Lambda^+\}\,, \tag{2.29}$$

where we have set

$$\Lambda^+:=\|B\|^2_{\mathcal{L}_\phi(B^+)}\|DU\|^2_{L^2(B^+)}+[g]^2_{BMO(B^+)}+\|g_0\|^2_{L^{2n/(n+2),n^2/(n+2)}(B^+)}\,. \tag{2.30}$$

In case (II) we split in $Q^+(y_0,2\sigma)$ $U=v+w$, where w is the unique solution of the Dirichlet problem

$$\begin{cases} w\in H_0^1(Q^+(y_0,2\sigma),\boldsymbol{R}^N)\,,\\ \displaystyle\int_{Q^+(y_0,2\sigma)}(B_{Q^+(y_0,2\sigma)}\cdot Dw|\,D\theta)\,dy=\\ \displaystyle\qquad =-\int_{Q^+(y_0,2\sigma)}([B(y)-B_{Q^+(y_0,2\sigma)}]\cdot[DU-(DU)_{Q^+(y_0,2\sigma)}]|\,D\theta)\,dy-\\ \displaystyle\qquad -\int_{Q^+(y_0,2\sigma)}([B(y)-B_{Q^+(y_0,2\sigma)}]\cdot(DU)_{Q^+(y_0,2\sigma)}|\,D\theta)\,dy+\\ \displaystyle\qquad +\int_{Q^+(y_0,2\sigma)}(g(y)-g_{Q^+(y_0,2\sigma)}|\,D\theta)\,dy-\int_{Q^+(y_0,2\sigma)}(g_0(y)|\,\theta)\,dy\\ \qquad\qquad\qquad\qquad \forall\theta\in H_0^1(Q^+(y_0,2\sigma),\boldsymbol{R}^N)\,; \end{cases} \tag{2.31}$$

then $v:=U-w$ is a solution of the homogeneous system with constant coefficients

$$\begin{cases} v\in H^1(Q^+(y_0,2\sigma),\boldsymbol{R}^N)\,,\qquad v=0 \text{ on } \Gamma(y_0,2\sigma)\,,\\ \displaystyle\int_{Q^+(y_0,2\sigma)}(B_{Q^+(y_0,2\sigma)}\cdot Dv|\,D\theta)\,dx=0\quad \forall\theta\in H_0^1(Q^+(y_0,2\sigma),\boldsymbol{R}^N)\,. \end{cases} \tag{2.32}$$

For the function v we have the fundamental estimate (see [5, Corollary 11.I and Lemma 11.II] for the case $N=1$ and [14, Lemma 3.5] for the case $N\geq 1$):

$$\begin{aligned} &\sum_{i=1}^{n-1}\int_{Q^+(y_0,2t\sigma)}|D_iv|^2dy+\int_{Q^+(y_0,2t\sigma)}|D_nv-(D_nv)_{Q^+(y_0,2t\sigma)}|^2dy\leq\\ &\leq c(\nu,\Omega)\left\{\sum_{i=1}^{n-1}\int_{Q^+(y_0,2\sigma)}|D_iv|^2dy+\int_{Q^+(y_0,2\sigma)}|D_nv-(D_nv)_{Q^+(y_0,2\sigma)}|^2dy\right\}\quad \forall t\in]0,1]\,. \end{aligned} \tag{2.33}$$

For the function w we have the variational estimate

$$(2.34)\qquad \int_{Q^+(y_0,2\sigma)} |Dw|^2\,dy \leqslant c(\nu,\Omega)\Bigg\{\int_{Q^+(y_0,2\sigma)} [|B-B_{Q^+(y_0,2\sigma)}|^2\,|DU-(DU)_{Q^+(y_0,2\sigma)}|^2\,dy + +|B-B_{Q^+(y_0,2\sigma)}|^2\,|(DU)_{Q^+(y_0,2\sigma)}|^2 + |g-g_{Q^+(y_0,2\sigma)}|^2]\,dy + + c(n)\Bigg[\int_{Q^+(y_0,2\sigma)} |g_0|^{2n/(n+2)}\,dy\Bigg]^{(n+2)/n}\Bigg\}.$$

We estimate the right as in the proof of Step 1A, obtaining, via Propositions 1.14 and 1.16:

$$(2.35)\qquad \int_{Q^+(y_0,2\sigma)} |Dw|^2\,dy \leqslant c(n,\nu,\Omega,R)(2\sigma)^n\,\{[B]^2_{\phi,B^+,2\sigma}[DU]^2_{BMO(B^+)} + \Lambda^+\}.$$

By (2.33), (2.35) and a function-theoretic lemma [6, Ch. I, Lemma 1.I], as in Step 1A we get for $0<r\leqslant 2\sigma\leqslant(1/\sqrt{n}-R)/2$:

$$(2.36)\qquad \sum_{i=1}^{n-1} \fint_{Q^+(y_0,r)} |D_iU|^2\,dy \leqslant \fint_{Q^+(y_0,r)} |D_nU-(D_nU)_{Q^+(y_0,r)}|^2\,dy \leqslant \leqslant c(n,\nu,\Omega,R)\{[B]^2_{\phi,B^+,2\sigma}[DU]^2_{BMO(B^+)} + (2\sigma)^{-n}\Lambda^+\}.$$

By (2.29) and (2.36) it follows that for each $x_0\in Q^+(0,R)$ and $0<r\leqslant\sigma\leqslant \leqslant(1/\sqrt{n}-R)/4$ we have:

$$(2.37)\qquad \fint_{Q(x_0,r)\cap Q^+(0,R)} |DU-(DU)_{Q(x_0,r)\cap Q^+(0,R)}|^2\,dy \leqslant \leqslant c(n,\nu,\Omega,R)\{[B]^2_{\phi,B^+,2\sigma}[DU]^2_{BMO(B^+)} + \sigma^{-n}\Lambda^+\}$$

and consequently

$$(2.38)\qquad [DU]^2_{BMO(Q^+(0,R))} \leqslant c(n,\nu,\Omega,R)\{[B]^2_{\phi,B^+,2\sigma}[DU]^2_{BMO(B^+)} + \sigma^{-n}\Lambda^+\} \qquad \forall\sigma\in\left]0,\frac14\left(\frac{1}{\sqrt{n}}-R\right)\right].$$

Finally, recalling (2.26), (2.27), (2.28) as well as (2.30), (2.17), (2.6), we conclude that

$$(2.39)\qquad [DU_s]^2_{BMO(Q^+(0,R))} \leqslant$$

$$\leqslant c(n,\nu,\Omega,R)\{[[\omega_A(2c_0\sigma)]^2+[\omega_\beta(2c_0\sigma)]^2\|A\|^2_{L^\infty(\bar\Omega)}]\|Du\|^2_{BMO(\Omega)}+\sigma^{-n}\Lambda\}$$

$$\forall R\in\left]\frac{1}{\sqrt{n}}-\frac{2}{c_0},\frac{1}{\sqrt{n}}\right[,\quad \forall\sigma\in\left]0,\frac14\left(\frac{1}{\sqrt{n}}-R\right)\right],\quad \forall s\in\{1,\dots,m\},$$

where

$$(2.40)\qquad \omega_\beta(t):=\sup_{0<r\leqslant t} r^\beta(1+|\lg r|),\qquad t\in]0,1].$$

This concludes the proof of Step 1B.

PROOF OF STEP 1C. – By (2.8) (iii) it is clear that if $R:=R(\Omega)$ is sufficiently close to $1/\sqrt{n}$, then the family $\{\Omega_0, T_s^{-1}(Q^+(0,R))\}_{1\leqslant s\leqslant m}$ still covers Ω. Moreover, by (2.28) and (2.39) we have for $\sigma\in]0,(1/\sqrt{n}-R)/4]$ and $s=1,\dots,m$:

$$[Du]^2_{BMO(T_s^{-1}(Q^+(0,R)))}\leqslant c(n,\nu,\Omega)\{[[\omega_A(2c_0\sigma)]^2+[\omega_\beta(2c_0\sigma)]^2\cdot\|A\|^2_{L^\infty(\Omega)}]\|Du\|^2_{BMO(\Omega)}+\sigma^{-n}\Lambda\}.$$

Recalling (2.21) we easily deduce

$$(2.41)\qquad [Du]^2_{BMO(\Omega)}\leqslant c(n,\nu,\Omega)\{[[\omega_A(2c_0\sigma)]^2+[\omega_\beta(2c_0\sigma)]^2\cdot\|A\|^2_{L^\infty(\Omega)}]\|Du\|^2_{BMO(\Omega)}+\sigma^{-n}\Lambda\}$$

$$\forall\sigma\in\left]0,\frac14\left(\frac{1}{\sqrt{n}}-R\right)\right]\cap\left]0,\frac{d_0}{2\sqrt{n}}\right].$$

Now taking into account (2.7) and (2.40), there exists $\sigma:=\sigma(n,\nu,\Omega,\beta,\omega_A,\|A\|_{L^\infty(\Omega)})\in$ $\in]0,(1/\sqrt{n}-R)/4]\cap]0,d_0/2\sqrt{n}]$ such that

$$(2.42)\qquad c(n,\nu,\Omega)[[\omega_A(2c_0\sigma)]^2+[\omega_\beta(2c_0\sigma)]^2\|A\|^2_{L^\infty(\Omega)}]\leqslant\frac12;$$

consequently, recalling (2.17), it is clear that (2.41) implies

$$[Du]^2_{BMO(\Omega)}\leqslant$$

$$\leqslant c(n,\nu,\Omega,\beta,\|A\|_{L^\infty(\Omega)},\omega_A)\{[1+[A]^2_{\mathcal{L}_\phi(\Omega)}]\|Du\|^2_{L^2(\Omega)}+\|f\|^2_{BMO(\Omega)}+\|f_0\|^2_{L^{2n/(n+2),n^2/(n+2)}(\Omega)}\}.$$

Finally (2.5) follows recalling the variational estimate [6, Ch. II, Theorem 3.III]

$$\|Du\|_{L^2(\Omega)}\leqslant c(\nu,n)\{\|f\|_{L^2(\Omega)}+\|f_0\|_{L^{2n/(n+2)}(\Omega)}\}.$$

This concludes the proof of Step 1C and hence Step 1 is proved.

PROOF OF STEP 2. – Suppose only that $\partial\Omega\in C^{1+\beta}(\beta>0)$, $f\in BMO(\Omega,\boldsymbol{R}^{Nn})$, $f_0\in$ $\in L^{2n/(n+2),n^2/(n+2)}(\Omega,\boldsymbol{R}^N)$, $A\in L^\infty(\Omega,\boldsymbol{R}^{N^2n^2})\cap l_\phi(\Omega,\boldsymbol{R}^{N^2n^2})$. Let $\{A_k\}_{k\in N}\subset C^\infty(\bar\Omega,\boldsymbol{R}^{N^2n^2})$

be a sequence such that (see Remark 1.5):

$$(2.43)\qquad A_k \to A \quad \text{as } k\to\infty \text{ in } \mathcal{L}_\phi(\Omega, \boldsymbol{R}^{N^2n^2}) \text{ and in } L^p(\Omega, \boldsymbol{R}^{N^2n^2}) \quad \forall p\in[1,\infty[,$$

$$(2.44)\qquad \|A_k\|_{L^\infty(\Omega)} \leq \|A\|_{L^\infty(\Omega)},$$

$$(2.45)\qquad \lim_{\sigma\downarrow 0} \omega_{A_k}(\sigma) = 0 \quad \text{uniformly in } k\in\boldsymbol{N}.$$

We still have (2.42) for some

$$\sigma = \sigma(n,\nu,\Omega,\beta,\omega_A, \|A\|_{L^\infty(\Omega)}) \in \left]0, \frac{1}{4}\left(\frac{1}{\sqrt{n}} - R\right)\right] \cap \left]0, \frac{d_0}{2\sqrt{n}}\right];$$

due to (2.45) we may assume that for the same number σ we also have

$$(2.46)\qquad c(n,\nu,\Omega)[[\omega_{A_k}(2c_0\sigma)]^2 + [\omega_\beta(2c_0\sigma)]^2\|A_k\|^2_{L^\infty(\Omega)}] \leq \frac{1}{2} \quad \forall k\in\boldsymbol{N}.$$

Let u_k be the unique solution of the Dirichlet problem

$$(2.47)\qquad \begin{cases} u_k \in H_0^1(\Omega, \boldsymbol{R}^N), \\ \int_\Omega (A_k(x)\cdot Du_k \mid D\theta)\,dx = \int_\Omega (f(x)\mid D\theta)\,dx - \int_\Omega (f_0(x)\mid\theta)\,dx \quad \forall\theta\in H_0^1(\Omega,\boldsymbol{R}^N). \end{cases}$$

Then, by Step 1 and (2.43), (2.44) (2.46), we deduce that u_k satisfies (2.5) uniformly in k, i.e.

$$(2.48)\qquad [Du_k]^2_{BMO(\Omega)} \leq$$

$$\leq c(n,\nu,\Omega,\beta,\omega_A,\|A\|_{L^\infty(\Omega)})[1 + [A]^2_{\mathcal{L}_\phi(\Omega)}][\|f\|^2_{BMO(\Omega)} + \|f_0\|^2_{L^{2n/(n+2),\, n^2/(n+2)}(\Omega)}] \quad \forall k\in\boldsymbol{N}.$$

On the other hand, $u_k - u$ solves the Dirichlet problem

$$(2.49)\qquad \begin{cases} u_k - u \in H_0^1(\Omega, \boldsymbol{R}^N), \\ \int_\Omega (A_k(x)\cdot[Du_k - Du] \mid D\theta)\,dx = \int_\Omega ([A(x) - A_k(x)]\cdot Du \mid D\theta)\,dx \quad \forall\theta\in H_0^1(\Omega,\boldsymbol{R}^N), \end{cases}$$

and by the variational estimate [6, Ch. II, Theorem 1.III]

$$(2.50)\qquad \|Du_k - Du\|_{L^2(\Omega)} \leq \frac{1}{\nu}\|(A - A_k)\cdot Du\|_{L^2(\Omega)} \quad \forall k\in\boldsymbol{N}^+.$$

Now, passing possibly to a subsequence, by (2.43) and (2.44) we have as $k\to\infty$:

$$[A(x) - A_k(x)]\cdot Du(x) \to 0 \quad \text{a.e. in } \Omega,$$

$$|[A(x) - A_k(x)]\cdot Du(x)|^2 \leq c\|A\|^2_{L^\infty(\Omega)}|Du(x)|^2 \in L^1(\Omega),$$

so that by (2.50), Lebesgue Theorem and Poincaré inequality,

(2.51) $$u_k \to u \quad \text{in } H^1(\Omega, \boldsymbol{R}^N) \text{ as } k \to \infty .$$

Now fix $x_0 \in \Omega$ and $\sigma > 0$: «passing to the limit» in (2.48) we get

$$\fint_{\Omega \cap Q(x_0,\sigma)} |Du - (Du)_{\Omega \cap Q(x_0,\sigma)}|^2 dx =$$

$$= \lim_{k \to \infty} \fint_{\Omega \cap Q(x_0,\sigma)} |Du_k - (Du_k)_{\Omega \cap Q(x_0,\sigma)}|^2 dx \le \liminf_{k \to \infty} [Du_k]^2_{BMO(\Omega)} \le$$

$$\le c(n, \nu, \Omega, \beta, \omega_A, \|A\|_{L^\infty(\Omega)})[1 + [A]^2_{\mathcal{L}_\varphi(\Omega)}]\{\|f\|^2_{BMO(\Omega)} + \|f_0\|^2_{L^{2n/(n+2),\, n^2/(n+2)}(\Omega)}\} .$$

and (2.5) follows at once.

This completes the proof of Step 2. Theorem 2.2 is completely proved. ∎

REMARK 2.3. – The result of Theorem 2.2 holds more generally for «complete» linear systems in divergence form:

$$Eu := - \operatorname{div}(A(x) \cdot Du) - \operatorname{div}(B(x) \cdot u) + C(x) \cdot Du + G(x) \cdot u ,$$

provided we assume $B \in L^\infty(\Omega, \boldsymbol{R}^{N^2 n}) \cap l_\varphi(\Omega, \boldsymbol{R}^{N^2 n})$ and $C \in L^\infty(\Omega, \boldsymbol{R}^{Nn})$, $G \in L^\infty(\Omega, \boldsymbol{R}^N)$. The proof is exactly the same.

Theorem 2.2 can be generalized to the case in which Ω is a cube; the proof is essentially the same and is even easier. Indeed, suppose $\Omega = Q_0 := Q(0,1)$. We use Step 1A in order to estimate the quantities

$$\fint_{Q(x_0,\sigma)} |Du - Du_{Q(x_0,\sigma)}|^2 dx ,$$

when $x_0 \in Q(0,R)$, with fixed $R \in]1/2, 1[$, and $\sigma \in [0, 1-R]$. If, otherwise, $x_0 \in Q_0 \setminus Q(0,R)$ and $\sigma \in [0, 1-R]$, two cases may occur:

(i) $\operatorname{dist}(x_0, \partial Q_0) > \sigma$, so that $Q(x_0,\sigma) \subset\subset Q_0$,

(ii) $\operatorname{dist}(x_0, \partial Q_0) \le \sigma$, so that $Q(x_0,\sigma) \cap Q_0 \subset Q(y_0, 2\sigma) \cap Q_0$, where y_0 is a suitable point of ∂Q_0.

In case (i) we again use Step 1A; in case (ii) we apply Step 1B, remarking that

$$\fint_{Q(x_0,\sigma) \cap Q_0} |Du - Du_{Q(x_0,\sigma) \cap Q_0}|^2 dx \le c(n) \fint_{Q(y_0,2\sigma) \cap Q_0} |Du - Du_{Q(y_0,2\sigma) \cap Q_0}|^2 dx .$$

We do not need Step 1C since we do not need to change the space variables. As a result we can state:

THEOREM 2.4. – Let u be the solution of the Dirichlet problem (2.4) with $\Omega = Q_{\sigma_0}$ ($\sigma_0 > 0$) and $f \in BMO(Q_{\sigma_0}, \boldsymbol{R}^{Nn})$, $f_0 \in L^{2n/(n+2), n^2/(n+2)}(Q_{\sigma_0}, \boldsymbol{R}^N)$ (see (1.4)), $A \in L^\infty(Q_{\sigma_0}, \boldsymbol{R}^{N^2 n^2}) \cap l_\phi(Q_{\sigma_0}, \boldsymbol{R}^{N^2 n^2})$ ($\phi(\sigma) := (1 + |\lg \sigma|)^{-1}$). Then $Du \in BMO(Q_{\sigma_0}, \boldsymbol{R}^{Nn})$ and

$$(2.52) \qquad [Du]_{BMO(Q_{\sigma_0})} \leq c(n, \nu, \sigma_0, \omega_A, \|A\|_{L^\infty(Q_{\sigma_0})})[1 + [A]_{\mathcal{L}_\phi(Q_{\sigma_0})}] \cdot \cdot [\|f\|_{BMO(Q_{\sigma_0})} + \|f_0\|_{L^{2n/(n+2), n^2/(n+2)}(Q_{\sigma_0})}]. \quad \blacksquare$$

REMARK 2.5. – Theorems 2.2 and 2.4 still hold in the cases $n = 1$, $n = 2$, provided:

$$f_0 \in \bigcup_{q \in]1,2]} L^{q, 2-q}(\Omega, \boldsymbol{R}^N) \quad \text{if } n = 2, \text{ and } f_0 \in L^1(\Omega, \boldsymbol{R}^N) \text{ if } n = 1.$$

Indeed, such assumptions guarantee that in the variational estimate (2.12) we can still bound the quantity depending on f_0 by σ^n multiplied by a suitable constant (compare with [6, Ch. I, definitions (4.8)-(4.9)]).

3. – The L^p regularity.

Throughout this section we assume $n \geq 2$ (see Remark 3.8 for the modifications in the case $n = 1$). Consider again the situation described at the beginning of Section 2. Let us first prove the following L^p regularity result on cubes:

THEOREM 3.1. – Suppose $\Omega = Q_{\sigma_0}$, $\sigma_0 \in]0, 1]$, and let u be the solution of the Dirichlet problem (2.4) with $f \in L^p(Q_{\sigma_0}, \boldsymbol{R}^{nN})$, $f_0 \in L^{np/(n+p)}(Q_{\sigma_0}, \boldsymbol{R}^N)$ ($p \in [2, \infty[$ and $A \in L^\infty(Q_{\sigma_0}, \boldsymbol{R}^{n^2 N^2}) \cap l_\phi(Q_{\sigma_0}, \boldsymbol{R}^{n^2 N^2})$ ($\phi(\sigma) := (1 + |\lg \sigma|)^{-1}$). Then $Du \in L^p(Q_{\sigma_0}, \boldsymbol{R}^{nN})$ and

$$(3.1) \qquad [Du]_{L^p(Q_{\sigma_0})} \leq c(n, \nu, p, \omega_A, \|A\|_{L^\infty(Q_{\sigma_0})})[1 + [A]_{\mathcal{L}_\phi(Q_{\sigma_0})}] \cdot \cdot [\|f\|_{L^p(Q_{\sigma_0})} + \|f_0\|_{L^{np/(n+p)}(Q_{\sigma_0})}].$$

PROOF. – We use Stampacchia's interpolation Theorem ([13]; see also [7, Ch. III, Th. 1.4] and [6, Ch. I, Th. 2.II]).

First of all we recall that the distribution $f_0 - \operatorname{div} f$ is in $H^{-1,p}(Q_{\sigma_0}, \boldsymbol{R}^N)$ by [6, Ch. I, Lemma 4.I], and

$$(3.2) \qquad \|f_0 - \operatorname{div} f\|_{H^{-1,p}(Q_{\sigma_0})} \leq c(n, p)\{\|f_0\|_{L^{np/(n+p)}(Q_{\sigma_0})} + \|f\|_{L^p(Q_{\sigma_0})}\};$$

hence there exist $F_0 \in L^p(Q_{\sigma_0}, \boldsymbol{R}^N)$ and $F \in L^p(Q_{\sigma_0}, \boldsymbol{R}^{nN})$ such that $f_0 - \operatorname{div} f = F_0 - \operatorname{div} F$ and [6, Ch. I, (4.5)]

$$(3.3) \qquad \sigma_0 \|F_0\|_{L^p(\Omega)} + \|F\|_{L^p(\Omega)} \leq c(n, p)\|f_0 - \operatorname{div} f\|_{H^{-1,p}(Q_{\sigma_0})}.$$

Thus u solves the Dirichlet problem

$$\begin{cases} Eu = F_0 - \operatorname{div} F & \text{in } Q_{\sigma_0}, \\ u \in H_0^1(Q_{\sigma_0}, \boldsymbol{R}^N). \end{cases} \tag{3.4}$$

Now denote by u_i, $i = 0, 1, \ldots, n$, the solutions of the Dirichlet problems

$$\begin{cases} Eu_0 = F_0 & \text{in } Q_{\sigma_0}, \\ u_0 \in H_0^1(Q_{\sigma_0}, \boldsymbol{R}^N); \end{cases} \qquad \begin{cases} Eu_i = -D_i F_i & \text{in } Q_{\sigma_0} \\ u_i \in H_0^1(Q_{\sigma_0}, \boldsymbol{R}^N); \end{cases} \qquad (i = 1, \ldots, n).$$

By the linearity of problem (3.4), it is clear that $u = \sum_{i=0}^{n} u_i$. The linear operator $T_{ij}: F_i \to D_j u_i$ $(i = 0, 1, \ldots, n;\ j = 1, \ldots, n)$ is bounded from $L^2(Q_{\sigma_0}, \boldsymbol{R}^N)$ into $L^2(Q_{\sigma_0}, \boldsymbol{R}^N)$, by Lax-Milgram Theorem, and from $BMO(Q_{\sigma_0}, \boldsymbol{R}^N)$ into $BMO(Q_{\sigma_0}, \boldsymbol{R}^N)$, by Theorem 2.4, with both norms bounded by

$$c(n, \nu, \sigma_0, \omega_A, \|A\|_{L^\infty(Q_{\sigma_0})})[1 + [A]_{\mathcal{L}_\varphi(Q_{\sigma_0})}].$$

By Stampacchia's Theorem we deduce that T_{ij} is also bounded from $L^p(Q_{\sigma_0}, \boldsymbol{R}^N)$ into $L^p(Q_{\sigma_0}, \boldsymbol{R}^N)$ more precisely we get for $p \in [2, \infty]$ [6, Ch. I, Theorem 2.II]:

$$\|D_j u_i\|_{L^p(Q_{\sigma_0})} \leq c(n, \nu, p, \sigma_0, \omega_A, \|A\|_{L^\infty(Q_{\sigma_0})})\{1 + [A]_{\mathcal{L}_\varphi(Q_{\sigma_0})}\}\|F_i\|_{L^p(Q_{\sigma_0})},$$

$$1 \leq j \leq n, \quad 0 \leq i \leq n.$$

Summing with respect to j and i we get

$$\|Du\|_{L^p(Q_{\sigma_0})} \leq c(n, \nu, p, \sigma_0, \omega_A, \|A\|_{L^\infty(Q_{\sigma_0})})\{1 + [A]_{\mathcal{L}_\varphi(Q_{\sigma_0})}\}[\|F_0\|_{L^p(Q_{\sigma_0})} + \|F\|_{L^p(Q_{\sigma_0})}];$$

a simple homothetical argument then gives

$$\|Du\|_{L^p(Q_{\sigma_0})} \leq c(n, \nu, p, \omega_A, \|A\|_{L^\infty(Q_{\sigma_0})})\{1 + [A]_{\mathcal{L}_\varphi(Q_{\sigma_0})}\}[\sigma_0\|F_0\|_{L^p(Q_{\sigma_0})} + \|F\|_{L^p(Q_{\sigma_0})}],$$

and the result follows by (3.3) and (3.2). ■

We now want to prove a local L^p-regularity result for solutions of

$$\begin{cases} u \in H^1(\Omega, \boldsymbol{R}^N), \\ Eu = f_0 - \operatorname{div} f & \text{in } \Omega. \end{cases} \tag{3.5}$$

THEOREM 3.2. – Let Ω be a bounded open set of $\boldsymbol{R}^n$. If u solves (3.5) with $f \in L^p(\Omega, \boldsymbol{R}^{nN})$, $f_0 \in L^{np/(n+p)}(\Omega, \boldsymbol{R}^N)$ $(p \in [2, \infty[$ and $A \in L^\infty(\Omega, \boldsymbol{R}^{n^2N^2}) \cap l_\varphi(\Omega, \boldsymbol{R}^{n^2N^2})$

$(\phi(\sigma) := (1 + |\lg \sigma|)^{-1})$, then $Du \in L^p_{\mathrm{loc}}(\Omega, \mathbf{R}^{nN})$ and for each cube $Q_\sigma \subset Q_{2\sigma} \subset \Omega$

$$\|Du\|_{L^p(Q_\sigma)} \leq c(n, \nu, p, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \cdot \{\sigma^{-n(1/2-1/p)}\|Du\|_{L^2(Q_{2\sigma})} + \|f\|_{L^p(Q_{2\sigma})} + \|f_0\|_{L^{np/(n+p)}(Q_{2\sigma})}\}. \tag{3.6}$$

PROOF. – We use the argument of [6, Ch. II, proof of Theorem 9.II].

Fix a cube $Q_\sigma = Q(x_0, \sigma)$ with $\sigma \in]0, (1/2) \wedge \operatorname{dist}(x_0, \partial\Omega)/(4\sqrt{n})]$, and let $\eta \in C_0^\infty(Q_{2\sigma})$ be such that

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } Q_\sigma, |D\eta| \leq \frac{c(n)}{\sigma}. \tag{3.7}$$

The function $v(x) := [u(x) - u_{Q_{2\sigma}}] \cdot \eta(x)$ solves the following Dirichlet problem:

$$\begin{cases} v \in H_0^1(Q_{2\sigma}, \mathbf{R}^N), \\ \int_{Q_{2\sigma}} (A_{ij}(x) \cdot D_j v | D_i \theta)\, dx = \int_{Q_{2\sigma}} \{(\eta f_i + D_j \eta A_{ij} \cdot (u - u_{Q_{2\sigma}}) | D_i \theta) + \\ \qquad + (D_i \eta \cdot f_i - \eta f_0 - D_i \eta \cdot A_{ij} \cdot D_j u | \theta)\}\, dx \qquad \forall \theta \in H_0^1(Q_{2\sigma}, \mathbf{R}^N). \end{cases} \tag{3.8}$$

Suppose first $p \in [2, 2n/(n-2)] (p \in [2, \infty[$ if $n = 2)$. Then by Sobolev-Poincaré inequality

$$\|u - u_{Q_{2\sigma}}\|_{L^p(Q_{2\sigma})} \leq c(n)\, \sigma^{1-n(1/2-1/p)} \|Du\|_{L^2(Q_{2\sigma})}, \tag{3.9}$$

and, since $np/(n+p) \leq 2$,

$$\|Du\|_{L^{np/(n+p)}(Q_{2\sigma})} \leq c(n)\, \sigma^{1-n(1/2-1/p)} \|Du\|_{L^2(Q_{2\sigma})}. \tag{3.10}$$

Hence by Theorem 3.1 we easily get

$$\|Dv\|_{L^p(Q_{2\sigma})} \leq c(n, \nu, p, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \cdot \{\|f\|_{L^p(Q_{2\sigma})} + \|f_0\|_{L^{np/(n+p)}(Q_{2\sigma})} + \|A\|_{L^\infty(\Omega)} \|Du\|_{L^2(Q_{2\sigma})} \sigma^{-n(1/2-1/p)}\} \tag{3.11}$$

and (3.6) is proved for $p \in [2, 2n/(n-2)]$, since $Dv \equiv Du$ in Q_σ.

If $p > 2n/(n-2)$ (and $n > 2$, of course), then there exists $k \in \mathbf{N}^+ \cap [1, n/2[$ such that $p \in]2n/(n-2k), 2n/(n-2(k+1))]$. In this case we iterate the above argument: suppose that the function η satisfies (instead of (3.7))

$$\eta \in C_0^\infty(Q_{2\sigma}), \qquad 0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } Q_{2t\sigma}, \qquad |D\eta| \leq \frac{c(n)}{(1-t)\sigma},$$

where $t := 2^{-1/(k+1)}$. Then $v := (u - u_{Q_{2\sigma}})\eta$ solves (3.8) and as above we obtain

$$\|Du\|_{L^{2n/(n-2)}(Q_{2t\sigma})} \leq c(n, \nu, p, t, \sigma, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \cdot \{\|f\|_{L^p(Q_{2\sigma})} + \|f_0\|_{L^{np/(n+p)}(Q_{2\sigma})} + \|Du\|_{L^2(Q_{2\sigma})}\},$$

which is the first step. Next, assume that for some h, $1 \leq h \leq k$, the following estimate holds:

$$\|Du\|_{L^{2n/(n-2h)}(Q_{2t^h\sigma})} \leq c(n, \nu, p, t, \sigma, h, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \cdot \{\|f\|_{L^p(Q_{2\sigma})} + \|f_0\|_{L^{np/(n+p)}(Q_{2\sigma})} + \|Du\|_{L^2(Q_{2\sigma})}\}; \tag{3.12}$$

then, choosing $\eta \in C_0^\infty(Q_{2t^h\sigma})$ such that

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \text{ in } Q_{2t^{h+1}\sigma}, \qquad |D\eta| \leq \frac{c(n)}{t^h(1-t)\sigma},$$

we find that $v := (u - u_{Q_{2t^h\sigma}}) \cdot \eta$ solves problem (3.8) with $Q_{2\sigma}$ replaced by $Q_{2t^h\sigma}$. Hence by Theorem 3.1 and Sobolev-Poincaré inequality we get as before

$$\|Du\|_{L^{p\wedge 2n/(n-2(h+1))}(Q_{2t^{h+1}\sigma})} \leq c(n, \nu, p, t, \sigma, h, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \cdot \{\|f\|_{L^p(Q_{2t^h\sigma})} + \|f_0\|_{L^{np/(n+p)}(Q_{2t^h\sigma})} + \|Du\|_{L^{2n/(n-2h)}(Q_{2t^h\sigma})}\}, \tag{3.13}$$

and using (3.12) we get again (3.12) with h replaced by $h+1$.

In particular when $h = k$ we have (since $2t^{h+1} = 1$ and k depends only on n, p)

$$\|Du\|_{L^p(Q_\sigma)} \leq c(n, \nu, p, \sigma, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)})(\|f\|_{L^p(Q_{2\sigma})} + \|f_0\|_{L^{np/(n+p)}(Q_{2\sigma})} + \|Du\|_{L^2(Q_{2\sigma})}\},$$

and finally a simple homothetical argument leads to (3.6) for general $p \in [2, \infty[$. ■

A quite similar proof leads to the following boundary result: consider the cubes

$$C_\sigma := \{x \in \boldsymbol{R}^n : |x_i| \leq \sigma/2,\ 1 \leq i \leq n-1;\ 0 \leq x_n \leq \sigma\} \qquad (\sigma > 0)$$

and set $\Lambda_\sigma := C_\sigma \cap \{x_n = 0\}$. Then we have:

THEOREM 3.3. – Let U be a solution of

$$\begin{cases} EU = F_0 - \operatorname{div} F & \text{in } C_{\sigma_0}, \\ U = 0 & \text{in } \Lambda_{\sigma_0}, \\ U \in H^1(C_{\sigma_0}, \boldsymbol{R}^N) \end{cases}$$

where $F \in L^p(C_{\sigma_0}, \boldsymbol{R}^{nN})$, $F_0 \in L^{np/(n+p)}(C_{\sigma_0}, \boldsymbol{R}^N)$ ($p \in [2, \infty[$), $A \in L^\infty(C_{\sigma_0}, \boldsymbol{R}^{n^2N^2}) \cap \cap l_\phi(C_{\sigma_0}, \boldsymbol{R}^{n^2N^2})$ ($\phi(\sigma) := (1 + |\lg \sigma|)^{-1}$); then $U \in H^{1,p}(C_r, \boldsymbol{R}^N)$ $\forall r \in]0, \sigma_0[$, and if $0 < 2\sigma < \sigma_0$ we have

$$\|DU\|_{L^p(C_\sigma)} \leq c(n, \nu, p, \omega_A, \|A\|_{L^\infty(C_{\sigma_0})}, [A]_{\mathcal{L}_\phi(C_{\sigma_0})}) \cdot \cdot \{\|F_0\|_{L^{np/(n+p)}(C_{2\sigma})} + \|F\|_{L^p(C_{2\sigma})} + \sigma^{-n(1/2-1/p)}\|DU\|_{L^2(C_{2\sigma})}\}. \tag{3.14}$$

As a consequence we can state the following global L^p-regularity result whose proof follows by Theorem 3.2 and 3.3 in a standard way:

THEOREM 3.4. – Let Ω be a bounded open set of $\boldsymbol{R}^n$ with $\partial\Omega \in C^{1+\beta}$, $\beta > 0$. Let u be the solution of the Dirichlet problem

$$\begin{cases} u \in H_0^1(\Omega, \boldsymbol{R}^N), \\ Eu = f_0 - \operatorname{div} f \quad \text{in } \Omega, \end{cases}$$

where $f \in L^p(\Omega, \boldsymbol{R}^{nN})$, $f_0 \in L^{np/(n+p)}(\Omega, \boldsymbol{R}^N)$ ($p \in [2, \infty[$), $A \in L^\infty(\Omega, \boldsymbol{R}^{n^2N^2}) \cap \cap l_\phi(\Omega, \boldsymbol{R}^{n^2N^2})$ ($\phi(\sigma) := (1 + |\lg \sigma|)^{-1}$). Then $u \in H^{1,p}(\Omega, \boldsymbol{R}^N)$ and

$$\|Du\|_{L^p(\Omega)} \le c(n, \nu, p, \Omega, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \{\|f\|_{L^p(\Omega)} + \|f_0\|_{L^{np/(n+p)}(\Omega)}\}. \quad \blacksquare$$

REMARK 3.5. – Due to Remark 2.4, similar results hold for complete linear systems, i.e. systems containing also lower order terms.

REMARK 3.6. – As already remarked in the Introduction, our L^p results neither imply nor follow by the classical theory of [1]: indeed Proposition 1.9 and Remark 1.10 show that the class of our coefficients A_{ij} neither contains nor is contained in the class of continuous coefficients of [1].

REMARK 3.7. – By Theorem 3.2 and Sobolev theorem we see that the solutions of linear strongly elliptic systems, whose coefficients are «small multipliers of *BMO*», are locally Hölder continuous provided the right member is an element of $H^{-1,p}(\Omega, \boldsymbol{R}^N)$ with $p > n$. Thus we have a class of elliptic systems with discontinuous coefficients for which De Giorgi's regularity theorem is true.

REMARK 3.8. – If $n = 1$, the results of this Section still hold if we replace the assumption $f_0 \in L^{np/(n+p)}(\Omega, \boldsymbol{R}^N)$ (or $F_0 \in L^{np/(n+p)}(C_{\sigma_0}, \boldsymbol{R}^N)$ in Theorem 3.3) by $f_0 \in \in L^1(\Omega, \boldsymbol{R}^N)$ (or $F_0 \in L^1(C_{\sigma_0}, \boldsymbol{R}^N)$). We note that in (3.6) and (3.14), for homogeneity reasons, the role played by $\|f_0\|_{L^{np/(n+p)}(Q_\sigma)}$ (resp. $\|F_0\|_{L^{np/(n+p)}(C_\sigma)}$) is played when $n = 1$ by $\sigma^{-1/p}\|f_0\|_{L^1(Q_\sigma)}$ (resp. $\sigma^{-1/p}\|F_0\|_{L^1(C_\sigma)}$).

4. – Local *BMO* regularity.

Consider again the situation described at the beginning of Section 2 and let u be a solution of

$$\text{(4.1)} \qquad \begin{cases} u \in H^1(\Omega, \boldsymbol{R}^N), \\ Eu = -\operatorname{div} f + f_0 \quad \text{in } \Omega, \end{cases}$$

under the strong ellipticity assumption (2.2) (or (2.3): see Remark 2.1). We want to prove the following result (written for the case $n>2$: easy modifications have to be done for $n=1$ and $n=2$, compare with Remark 2.5):

THEOREM 4.1. – Let u be a solution of (4.1) with $f \in BMO(\Omega, \boldsymbol{R}^{Nn})$, $f_0 \in L^{2n/(n+2), n^2/(n+2)}(\Omega, \boldsymbol{R}^N)$, $A \in L^\infty(\Omega, \boldsymbol{R}^{N^2n^2}) \cap l_\phi(\Omega, \boldsymbol{R}^{N^2n^2})$ $(\phi(\sigma) := (1+|\lg\sigma|)^{-1})$ and no assumption on $\partial\Omega$. Then $Du \in BMO_{\rm loc}(\Omega, \boldsymbol{R}^{Nn})$ and for each cube $Q(x_0, \sigma) =: Q_\sigma \subset Q_{2\sigma} \subset \subset \Omega$ we have

$$(4.2)\qquad [Du]_{BMO(Q_\sigma)} \leqslant c(n, \nu, \omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}) \cdot \cdot \{\sigma^{-n/2}\|Du\|_{L^2(Q_{2\sigma})} + \|f\|_{BMO(Q_{2\sigma})} + \|f_0\|_{L^{2n/(n+2), n^2/(n+2)}(Q_{2\sigma})}\}.$$

PROOF. – In $Q_{2\sigma}$ we split $u = z + w$ where w solves the Dirichlet problem

$$(4.3)\qquad \begin{cases} w \in H_0^1(Q_{2\sigma}), \\ Ew = -\operatorname{div} f + f_0 \quad \text{in } Q_{2\sigma}, \end{cases}$$

whereas $z := u - w$ solves the homogeneous system

$$(4.4)\qquad \begin{cases} z \in H^1(Q_{2\sigma}), \\ Ez = 0. \end{cases}$$

By Theorem 2.4 we have for w the following estimate:

$$(4.5)\qquad [Dw]_{BMO(Q_{2\sigma})} \leqslant c(n, \nu, \sigma, \omega_A, \|A\|_{L^\infty(Q_{2\sigma})}, [A]_{\mathcal{L}_\phi(Q_{2\sigma})}) \cdot \cdot [\|f\|_{BMO(Q_{2\sigma})} + \|f\|_{L^{2n/(n+2), n^2/(n+2)}(Q_{2\sigma})}].$$

Our goal now is an estimate for z. By Theorem 3.2, we know that $Dz \in L^p_{\rm loc}(Q_{2\sigma}, \boldsymbol{R}^{Nn})$ for each $p < \infty$ and we have an estimate like (3.6):

$$(4.6)\qquad \|Dz\|_{L^p(Q_{(3/2)\sigma})} \leqslant c(n, p, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}, \sigma)\|Dz\|_{L^2(Q_{2\sigma})};$$

consequently, by Hölder inequality, choosing in particular $p = n$,

$$(4.7)\qquad \|Dz\|_{L^{2n/(n+2), n^2/(n+2)}(Q_{(3/2)\sigma})} \leqslant \|Dz\|_{L^n(Q_{(3/2)\sigma})} \leqslant c(n, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\phi(\Omega)}, \sigma)\|Dz\|_{L^2(Q_{2\sigma})}.$$

Let now $\eta \in C_0^\infty(Q_{(3/2)\sigma})$ be such that

$$(4.8)\qquad 0 \leqslant \eta \leqslant 1, \quad \eta \equiv 1 \text{ in } Q_\sigma, \quad |D\eta| \leqslant \frac{c(n)}{\sigma}.$$

Then the function $v(x) := [z(x) - z_{Q_{(3/2)\sigma}}]\cdot\eta(x)$ solves a Dirichlet problem like (3.8):

$$(4.9)\quad \begin{cases} v \in H_0^1(Q_{(3/2)\sigma}, \boldsymbol{R}^N), \\ \int\limits_{Q_{(3/2)\sigma}} (A_{ij}(x)\cdot D_j v | D_i\theta)\,dx = \int\limits_{Q_{(3/2)\sigma}} \{(\eta f_i + D_j\eta A_{ij}\cdot(z - z_{Q_{(3/2)\sigma}}) | D_i\theta) + \\ \qquad + (D_i\eta\cdot f_i - \eta f_0 - D_i\eta A_{ij}\cdot D_j z | \theta\}\,dx \qquad \forall\theta\in H_0^1(Q_{(3/2)\sigma}, \boldsymbol{R}^N). \end{cases}$$

Now it is easy to see that, setting $G_i := \eta f_i + D_j\eta A_{ij}\cdot(z - z_{Q_{(3/2)\sigma}})$, $G_0 := D_i\eta f_i - \eta f_0 - - D_i\eta A_{ij}\cdot D_j z$, we have

$$(4.10)\quad \|G\|_{BMO(Q_{(3/2)\sigma})} \leqslant c(n,\sigma)\{\|f\|_{BMO(Q_{(3/2)\sigma})} + [\|A\|_{L^\infty(\Omega)} + [A]_{\mathcal{L}_\varphi(\Omega)}]\|Dz\|_{L^{2n/(n+2),\, n^2/(n+2)}(Q_{(3/2)\sigma})}\},$$

$$(4.11)\quad \|G_0\|_{L^{2n/(n+2),\, n^2/(n+2)}(Q_{(3/2)\sigma})} \leqslant$$
$$\leqslant c(n,\sigma)\{\|f\|_{BMO(Q_{(3/2)\sigma})} + \|f_0\|_{L^{2n/(n+2),\, n^2/(n+2)}(Q_{(3/2)\sigma})} + \|A\|_{L^\infty(\Omega)}\|Dz\|_{L^{2n/(n+2),\, n^2/(n+2)}(Q_{(3/2)\sigma})}\};$$

hence by Theorem 2.4 and (4.10), (4.11), (4.7) we deduce (since $Dv \equiv Dz$ in Q_σ):

$$(4.12)\quad [Dv]_{BMO(Q_\sigma)} \leqslant c(n)[Dz]_{BMO(Q_{(3/2)\sigma})} \leqslant$$
$$\leqslant c(n,\nu,\omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\varphi(\Omega)},\sigma)\{\|Dz\|_{L^2(Q_{2\sigma})} + \|f\|_{BMO(Q_{2\sigma})} + \|f_0\|_{L^{2n/(n+2),\, n^2/(n+2)}(Q_{2\sigma})}\}.$$

Finally we couple (4.5) and (4.12), obtaining for $u = w + z$:

$$(4.13)\quad [Du]_{BMO(Q_\sigma)} \leqslant c(n,\nu,\sigma,\omega_A, \|A\|_{L^\infty(\Omega)}, [A]_{\mathcal{L}_\varphi(\Omega)})\cdot$$
$$\cdot\{\|Du\|_{L^2(Q_{2\sigma})} + \|f\|_{BMO(Q_{2\sigma})} + \|f_0\|_{L^{2n/(n+2),\, n^2/(n+2)}(Q_{2\sigma})}\};$$

and a simple homothetical argument leads to (4.2). This proves Theorem 4.1. ∎

5. – Improvements and remarks.

This final Section is denoted to a few remarks.

Firstly, we want to improve the result of Theorem 2.2 in the special case of one space variable, i.e. $n = 1$.

Thus, assume that $\Omega =]a, b[$, and consider the operator

$$Eu := -(A(x)\cdot u')', \qquad u \in H^1(]a,b[, \boldsymbol{R}^N),$$

where $A \in L^\infty(]a,b[, \boldsymbol{R}^{N^2})$ and

$$(5.1)\qquad (A(x)\xi|\xi)_N \geqslant \nu|\xi|^2 \qquad \forall x\in]a,b[,\ \forall\xi\in\boldsymbol{R}^N.$$

If $f \in BMO(]a, b[, \boldsymbol{R}^N)$, $f_0 \in L^1(]a, b[, \boldsymbol{R}^N)$, the Dirichlet problem

$$\begin{cases} Eu = f_0 - f' & \text{in }]a, b[\,, \\ u \in H_0^1(]a, b[, \boldsymbol{R}^N)\,, \end{cases} \tag{5.2}$$

is obviously equivalent, by (5.1), to

$$\begin{cases} u'(x) = A(x)^{-1} \cdot \left[f(x) - \int_a^x f_0(t)\, dt + c \right], & \text{a.e. in }]a, b[\,, \\ u(a) = u(b) = 0\,, \end{cases} \tag{5.3}$$

where $c = (c^1, \dots, c^N)$ is an arbitrary constant vector. We want to prove the following result:

THEOREM 5.1. – Let $u \in H^1(]a, b[, \boldsymbol{R}^N)$ be the solution of problem (5.2), with $A \in L^\infty(]a, b[, \boldsymbol{R}^{N^2})$ satisfying (5.1) and $f \in BMO(]a, b[, \boldsymbol{R}^N)$, $f_0 \in L^1(]a, b[, \boldsymbol{R}^N)$. Then $u' \in BMO(]a, b[, \boldsymbol{R}^N)$ if and only if $A \in L^\infty(]a, b[, \boldsymbol{R}^{N^2}) \cap L_\phi(]a, b[, \boldsymbol{R}^{N^2})$ (with $\phi(\sigma) := [1 + |\lg \sigma|]^{-1})$); if this is the case we have the estimate

$$[u']_{BMO(a,b)} \leqslant c(\nu, \|A\|_{L^\infty(a,b)}, [A]_{L_\phi(a,b)}) \{\|f\|_{BMO(a,b)} + \|f_0\|_{L^1(a,b)}\}\,. \tag{5.4}$$

PROOF. – We need the following

LEMMA 5.2. – Let $A \in L^\infty(]a, b[, \boldsymbol{R}^{N^2})$ satisfy (5.1). If $\Phi: [0, d] \to \overline{\boldsymbol{R}^+}$ is any function such that (1.1) holds, then $A \in \mathcal{L}_\Phi(]a, b[, \boldsymbol{R}^{N^2})$ if and only if $A^{-1} \in \mathcal{L}_\Phi(]a, b[, \boldsymbol{R}^{N^2})$.

PROOF. – Suppose that $A \in \mathcal{L}_\Phi(]a, b[, \boldsymbol{R}^{N^2})$. Then, setting

$$I(x_0, \sigma) :=]x_0 - \sigma, x_0 + \sigma[\cap]a, b[\,; \qquad A_\sigma := \fint_{I(x_0,\sigma)} A(x)\, dx\,, \qquad M := \|A\|_{L^\infty(a,b)}\,,$$

we have by (5.1)

$$|A_\sigma| \leqslant M\,, \qquad |A_\sigma^{-1}| \leqslant \frac{1}{\nu}\,,$$

and consequently

$$\fint_{I(x_0,\sigma)} |A(x)^{-1} - (A^{-1})_\sigma|^2\, dx \leqslant \fint_{I(x_0,\sigma)} |A(x)^{-1} - A_\sigma^{-1}|^2\, dx =$$

$$= \fint_{I(x_0,\sigma)} |A(x)^{-1}[A_\sigma - A(x)] A_\sigma^{-1}|^2\, dx \leqslant \frac{1}{\nu^4} \fint_{I(x_0,\sigma)} |A(x) - A_\sigma|^2\, dx \leqslant \frac{1}{\nu^4} [\Phi(\sigma)]^2 [A]^2_{\mathcal{L}_\phi(a,b)}\,.$$

This shows that $A^{-1} \in \mathcal{L}_\Phi(]a, b[, \boldsymbol{R}^{N^2})$ and $[A^{-1}]_{\mathcal{L}_\Phi(a,b)} \leqslant (1/\nu^2)[A]_{\mathcal{L}_\Phi(a,b)}$.

Suppose conversely $A^{-1} \in \mathcal{L}_\Phi(]a,b[, \boldsymbol{R}^{N^2})$ then by (5.1)

$$((A^{-1})_\sigma \xi | \xi)_N = \fint_{I(x_0,\sigma)} (A(x)^{-1}\xi | \xi)_N \, dx \geq \nu |A(x)^{-1}\xi|^2 \geq \frac{\nu}{M^2}|\xi|^2 \qquad \forall \xi \in \boldsymbol{R}^N;$$

hence $[(A^{-1})\sigma]^{-1}$ exists and we can write

$$\fint_{I(x_0,\sigma)} |A(x) - A_\sigma|^2 dx \leq \fint_{I(x_0,\sigma)} |A(x) - [(A^{-1})_\sigma]^{-1}|^2 dx =$$

$$= \fint_{I(x_0,\sigma)} |A(x)[(A^{-1})_\sigma - A(x)^{-1}][(A^{-1})_\sigma]^{-1}|^2 dx \leq$$

$$\leq \frac{M^4}{\nu^2} \fint_{I(x_0,\sigma)} |A(x)^{-1} - (A^{-1})_\sigma|^2 dx \leq \frac{M^4}{\nu^2}[\Phi(\sigma)]^2 [A^{-1}]^2_{\mathcal{L}_\Phi(a,b)},$$

that is $A \in \mathcal{L}_\Phi(]a,b[, \boldsymbol{R}^{N^2})$ and $[A]_{\mathcal{L}_\Phi(a,b)} \leq (M^2/\nu)[A^{-1}]_{\mathcal{L}_\Phi(a,b)}$. ■

Now assume that $A \in L^\infty(]a,b[, \boldsymbol{R}^{N^2}) \cap \mathcal{L}_\phi(]a,b[, \boldsymbol{R}^{N^2})$, with $\phi(\sigma) := (1 + |\lg \sigma|)^{-1}$, and suppose that (5.1) holds. Then by Lemma 5.2 and (1.14) we get that A^{-1} is a multiplier of *BMO* (in the sense that $A^{hk} \in M(BMO(]a,b[))$ for $h,k = 1,\ldots,N$. As

$$x \to f(x) - \int_a^x f_0(t)\,dt + c \in BMO(]a,b[, \boldsymbol{R}^N),$$

by (5.3) we readily obtain $u' \in BMO(]a,b[, \boldsymbol{R}^N)$ and (5.4) follows easily.

Conversely, suppose that $A \notin \mathcal{L}_\phi(]a,b[, \boldsymbol{R}^{N^2})$; then also $A^{-1} \notin \mathcal{L}_\phi(]a,b[, \boldsymbol{R}^{N^2})$ (by Lemma 5.2). This implies, by (1.14), that, there exist $h, k \in \{1,\ldots,N\}$ such that $(A^{-1})^{hk} \notin M(BMO)(]a,b[)$. As a consequence, we can find a scalar function $g \in BMO(]a,b[)$ such that $(A^{-1})^{hk} g \notin BMO(]a,b[)$.

Choose now

$$f := \{f^r\}_{r=1,\ldots,N}, \qquad f^r := \begin{cases} 0 & \text{if } r \neq k, \\ g & \text{if } r = k. \end{cases}$$

Then it is clear that $f \in BMO(]a,b[, \boldsymbol{R}^N)$, but $A^{-1}\cdot f \notin BMO(]a,b[, \boldsymbol{R}^N)$.

Choosing also $f_0 := 0$, we easily see that the solution u of problem (5.3) is such that $u' = A^{-1}(f+c) \notin BMO(]a,b[, \boldsymbol{R}^N)$. The proof of Theorem 5.1 is complete. ■

REMARK 5.3. – If $n \geq 1$ we can find an open set Ω and two functions $A \in L^\infty(\Omega, \boldsymbol{R}^{N^2 n^2}) \setminus \mathcal{L}_\phi(\Omega, \boldsymbol{R}^{n^2 N^2})$ (in fact, $A \in C^0(\overline{\Omega}, \boldsymbol{R}^{N^2 n^2}) \setminus \mathcal{L}_\phi(\Omega, \boldsymbol{R}^{N^2 n^2})$) and $f \in$

$\in BMO(\Omega, \boldsymbol{R}^{Nn})$ such that the gradient of the solution of

$$\begin{cases} Eu = -\operatorname{div} f & \text{in } \Omega, \\ u \in H_0^1(\Omega, \boldsymbol{R}^N), \end{cases} \tag{5.5}$$

does not belong to $BMO(\Omega, \boldsymbol{R}^{Nn})$. Indeed, fix $\Omega :=]0,1[^n$, and $A = \{A_{ij}\}_{i,j=1,\dots,n}$, $A_{ij}(x) := a(x_1)I$, $a \in C^0([0,1]) \setminus \mathcal{L}_\phi(]0,1[)$, $a(x) \geq \nu$ in $[0,1]$; then $1/a$ also belongs to $C^0([0,1]) \setminus \mathcal{L}_\phi(]0,1[)$, so that we can select

$$f := \{f_i\}_{1 \leq i \leq n}, \qquad f_i(x) := (g(x_1), 0, \dots, 0) \in BMO(\Omega, \boldsymbol{R}^N)$$

with $g \in BMO(]0,1[)$ such that $g/a \notin BMO(]0,1[)$. Then the solution of (5.5) is

$$U(x) = (u(x_1), 0 \dots, 0), \qquad u(x_1) = \int_0^{x_1} \frac{g(t)}{a(t)} dt - \frac{\int_0^1 \frac{g(s)}{a(s)} ds}{\int_0^1 \frac{ds}{a(s)}} \int_0^x \frac{dt}{a(t)},$$

and the gradient of U is not in $BMO(\Omega, \boldsymbol{R}^{Nn})$ since

$$D_1 U^1(x) = u'(x_1) = \frac{g(x_1)}{a(x_1)} - \frac{c}{a(x_1)}, \qquad c = \frac{\int_0^1 \frac{g(s)}{a(s)} ds}{\int_0^1 \frac{ds}{a(s)}}.$$

This shows in particular that the BMO regularity for elliptic systems like (5.5), whose coefficients are merely continuous, is false.

REMARK 5.4. – Arguing as in [5, Appendix I, Th. III] we see that $u \in H_0^1(\Omega, \boldsymbol{R}^N)$, $Du \in BMO(\Omega, \boldsymbol{R}^{nN}) \Rightarrow u^h \in \mathcal{L}_1^{2,n+2}(\Omega)$, $h = 1, \dots, N$ (i.e. $u \in \mathcal{L}_1^{2,n+2}(\Omega, \boldsymbol{R}^N)$), where

$\mathcal{L}_1^{2,n+2}(\Omega) :=$

$$:= \left\{ f \in L^2(\Omega) \colon [f]_{\mathcal{L}_1^{2,n+2}(\Omega)} := \sup_{x_0 \in \bar{\Omega}, \sigma > 0} \left[\sigma^{-2} \inf_{a \in \boldsymbol{R}^n, b \in \boldsymbol{R}} \left\{ \fint_{\Omega \cap Q(x_0,\sigma)} |f(x) - a \cdot x - b|^2 dx \right\} \right] < \infty \right\};$$

In particular [4, Th. 6.I], u is Hölder continuous with any exponent $\alpha \in]0,1[$.

But we can be more precise: by a result of GREVHOLM [8], we have

$$\mathcal{L}_1^{2,n+2}(\Omega, \boldsymbol{R}^N) = \Lambda^1(\bar{\Omega}, \boldsymbol{R}^N),$$

where $\Lambda^1(\bar{\Omega}, \boldsymbol{R}^N)$ is the Zygmund class, i.e.

$$\Lambda^1(\bar{\Omega}, \boldsymbol{R}^N) := \left\{ f \in C^0(\bar{\Omega}, \boldsymbol{R}^N) \colon [f]_{\Lambda^1(\bar{\Omega})} := \sup_{x, y, (x+y)/2 \in \bar{\Omega}} \frac{|f(x) + f(y) - 2f((x+y)/2)|}{|x-y|} < \infty \right\}.$$

It is well known that $\bigcap_{0<\alpha<1} C^{0,\alpha}(\bar{\Omega}, \boldsymbol{R}^N) \supset \Lambda^1(\bar{\Omega}, \boldsymbol{R}^N) \supset \mathrm{Lip}(\bar{\Omega}, \boldsymbol{R}^N)$ (with proper inclusions).

Thus if $Du \in BMO(\Omega, \boldsymbol{R}^{nN})$ we obtain that u is Zygmund continuous in $\bar{\Omega}$.

For the solutions of elliptic systems under the assumptions of Theorem 2.2 this regularity result is optimal, since u cannot be Lipschitz continuous in general: indeed $u(x) := x \lg x$ solves

$$\begin{cases} \Delta u = (\lg x)' & \text{in }]0,1[\,, \\ u(0) = u(1) = 0\,, \end{cases}$$

and $\lg x \in BMO(0,1)$, $x \lg x \in \Lambda^1([0,1]) / \mathrm{Lip}([0,1])$.

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