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On Quasilinear Parabolic Systems

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0. Introduction

In this paper we prove some results on local existence of continuously differentiable solutions $u=(u^1,\dots,u^N)$ of quasilinear parabolic systems under general nonlinear boundary conditions. Such results were announced, without proof (but with mistakes!) in [1]; here we correct the mistakes, and give some improvements concerning continuity of solutions with respect to the initial data.

For the sake of simplicity we just consider second order systems; as a model we take the following problem:

$$\begin{cases} u_t - \sum_{ij=1}^{n} A_{ij}(t,x,u,Du)\cdot D_iD_ju = f(t,x,u,Du),\ (t,x)\in[t_0,T]\times\bar{\Omega}, \\ u(t_0,x)=\phi(x),\ x\in\bar{\Omega}, \\ \sum_{i=1}^{n} B_i(t,x,u)\cdot D_iu = g(t,x,u),\ (t,x)\in[t_0,T]\times\partial\Omega, \end{cases} \tag{0.1}$$

where $T>t_0\geqq 0$ and Ω is a bounded open set of $\mathbb{R}^n$ with C^2 boundary.

We assume the following hypotheses:

(0.2) *Ellipticity.* The pair

$$\left\{\sum_{ij=1}^{n} A_{ij}(t,\cdot,u,p)\cdot D_iD_j,\ \sum_{i=1}^{n} B_i(t,\cdot,u)\cdot D_i\right\}$$

is elliptic in the sense of [4, 7], uniformly in (t,u,p) on bounded subsets of $[0,T]\times\mathbb{C}^n\times\mathbb{C}^{Nn}$. More precisely, the $N\times N$ matrices

$$A(\theta;t,x,u,p;\xi,\varrho):=\sum_{sj=1}^{n} A_{sj}(t,x,u,p)\xi_s\xi_j+e^{i\theta}\varrho^2 I,$$

$$B(t,x,u;\xi):=\sum_{j=1}^{n} B_j(t,x,u)\xi_j,$$

where $\theta \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\varrho \in \mathbb{R}$, must satisfy, for each $M>0$ and provided $t \in [0,T]$, $|u|+|p| \leqq M$, the following conditions:

(i) there exist $\theta_M \in]\frac{\pi}{2}, \pi[$, $C_M>0$ such that

$$|\det A(\theta; t, x, u, p; \xi, \varrho)| \geqq C_M(|\xi|^2+\varrho^2)^N$$
$$\forall x \in \bar{\Omega},\ \forall \theta \in [-\theta_M, \theta_M],\ \forall \xi \in \mathbb{R}^n,\ \forall \varrho \in \mathbb{R};$$

(ii) for each $x \in \partial\Omega$, $\theta \in [-\theta_M, \theta_M]$, $\xi \in \mathbb{R}^n$, $\varrho \in \mathbb{R}$ with $|\xi|^2+\varrho^2>0$ and $\xi \cdot v(x)=0$, the polynomial

$$\tau \to \det A(\theta; t, x, u, p; \xi + \tau v(x), \varrho)$$

has precisely N roots $\tau_j^+(\theta; t, x, u, p; \xi, \varrho)$ with positive imaginary part. Here $v(x)$ is the unit outward normal vector at x.

(0.3) *Complementarity.* For each $M>0$, if $t \in [0,T]$, $x \in \partial\Omega$, $|u|+|p| \leqq M$, $\theta \in [-\theta_M, \theta_M]$, $\xi \in \mathbb{R}^n$, $\varrho \in \mathbb{R}$ with $|\xi|^2+\varrho^2>0$ and $\xi \cdot v(x)=0$, the rows of the matrix

$$B(t,x,u;\xi+\tau v(x)) \cdot [A(\theta; t, x, u, p; \xi+\tau v(x), \varrho)]^*$$

are linearly independent modulo the polynomial

$$\tau \to \prod_{j=1}^{n} (\tau - \tau_j^+(\theta; t, x, u, p; \xi, \varrho)).$$

We denote here by M^* the algebraic adjoint of the matrix M.

(0.4) *Regularity.* For $h,k,m=1,\dots,N$, $i,j=1,\dots,n$ the functions A_{ij}^{hk}, f^h, B_i^{hk}, g^h, $\frac{\partial B_i^{hk}}{\partial x_j}, \frac{\partial B_i^{hk}}{\partial u^m}, \frac{\partial g^h}{\partial x_j}, \frac{\partial g^h}{\partial u^k}$ are of class C^α in t, continuous in x, locally Lipschitz continuous in (u,p); the functions B_i^{hk}, g^h are also of class $C^{\alpha+\frac{1}{2}}$ in t. Here α is any exponent from $]0, 1/2[$.

(0.5) *Compatibility.* $\phi \in C^1(\bar{\Omega}, \mathbb{C}^N)$ and

$$\sum_{i=1}^{n} B_i(t_0, x, \phi(x)) \cdot D_i\phi(x) = g(t_0, x, \phi(x)) \quad \forall x \in \partial\Omega.$$

It is not restrictive to assume that the functions B_i^{hk} and g^h are defined on the whole $\bar{\Omega}$; this will simplify our notations. Moreover when no confusion can arise we will just write $L^p, C, W^{1,p}, \dots$, instead of $L^p(\Omega, \mathbb{C}^N)$, $C(\bar{\Omega}, \mathbb{C}^N)$, $W^{1,p}(\bar{\Omega}, \mathbb{C}^N), \dots$.

1. Main Result

Fix any $p>n$ and let ϕ_0 be a fixed element of $W^{2,p}(\Omega, \mathbb{C}^N)$. For $t_0 \in [0, T[$, $r_0>0$, $N_0>0$ we set:

$$\begin{aligned} B(\phi_0, N_0, r_0, t_0) := \{&\phi \in W^{2,p}(\Omega, \mathbb{C}^N) : \|\phi - \phi_0\|_{W^{2,p}} \leqq r_0, \\ &P(t_0, \phi) = 0, \quad Q(t_0, \phi) \in B_\infty^{2\alpha,p}(\Omega, \mathbb{C}^N) \text{ and } \|Q(t_0,\phi)\|_{B_\infty^{2\alpha,p}} \leqq N_0\}, \end{aligned} \tag{1.1}$$

where $B_\infty^{2\alpha,p}$ is the Besov-Nikolskij space and

$$P(t_0, \phi) := \sum_{i=1}^{n} B_i(t_0, \cdot, \phi) \cdot D_i\phi - g(t_0, \cdot, \phi), \qquad x \in \partial\Omega, \tag{1.2}$$

$$Q(t_0,\phi):=\sum_{j=1}^{n} A_{ij}(t_0,\cdot,\phi,D\phi)\cdot D_iD_j\phi+f(t_0,\cdot,\phi,D\phi), \qquad x\in\Omega. \tag{1.3}$$

We note that $B(\phi_0,N_0,r_0,t_0)$ is a closed subset of $W^{2,p}(\Omega,\mathbb{C}^N)$ as an easy check shows.

Our goal is the following result:

Theorem 1.1. *Assume* (0.2), ..., (0.5). *There exists* $\tau\in]t_0,T]$ *such that for each* $\phi\in B(\phi_0,N_0,r_0,t_0)$ *problem* (0.1) *has a unique solution* $u=(u^1,\dots,u^N)$ *in* $[t_0,\tau]$, *which satisfies*

$$u\in C^{1+\alpha}([t_0,\tau],L^p(\Omega,\mathbb{C}^N))\cap C^{\alpha}([t_0,\tau],W^{2,p}(\Omega,\mathbb{C}^N)); \tag{1.4}$$

moreover the map $\phi\to u$ *is continuous in the following sense: denoting by* u_ϕ, u_ψ *the solutions corresponding to the initial data* $\phi,\psi\in B(\phi_0,N_0,r_0,t_0)$, *we have:*

$$\begin{aligned}\|u_\phi-u_\psi\|_{C^{1+\delta}(L^p)}+\|u_\phi-u_\psi\|_{C^{\delta}(W^{2,p})}&\leqq C(p,\alpha,\delta,N_0,\phi_0,r_0)\{\|\phi-\psi\|_{W^{2,p}}\\ &+\|Q(t_0,\varphi)-Q(t_0,\psi)\|_{B^{2\delta,p}_{\infty}}\}\quad \forall\delta\in]0,\alpha].\end{aligned} \tag{1.5}$$

If, in addition, $\phi\in C^2(\bar\Omega,\mathbb{C}^N)$ *and* $Q(t_0,\phi)\in C^{2\alpha}(\bar\Omega,\mathbb{C}^N)$, *then*

$$u_t,\ \sum_{ij=1}^{n} A_{ij}(\cdot,\cdot,u,Du)\cdot D_iD_ju\in C^{\delta}([t_0,\tau],C(\bar\Omega,\mathbb{C}^N)),\ \textit{for each } \delta\in]0,\alpha[. \tag{1.6}$$

The proof will be given in the next sections.

Remark 1.2. The compatibility conditions concerning $P(t_0,\phi)$ and $Q(t_0,\phi)$ are necessary for the validity of (1.4), so that this result is optimal. On the other hand, in (1.5) we are not able to replace δ by α: this is due to the "bad" behaviour of the space $C(\bar\Omega,\mathbb{C}^N)$ with respect to maximal regularity properties in parabolic evolution problems (see also [2, Remark 6.4]).

Remark 1.3. We believe that a similar result holds as well for quasilinear parabolic systems of arbitrary order, with the elliptic part satisfying the assumptions of [4] and [7].

Remark 1.4. If one is only interested to (1.4), then the dependence of the right members f and g on x may be slightly relaxed: namely, to prove (1.4) we just need that the functions f^h, g^h, $\frac{\partial g^h}{\partial x_j}$, $\frac{\partial g^h}{\partial u^k}$ are L^p in x.

Remark 1.5. Theorem 1.1 is a local existence result, but it is clear that the usual standard machinery allows to construct the maximal solution starting at time t_0 from the point ϕ; it will be defined in a maximal interval $[t_0,T(\phi)[$.

Remark 1.6. Results of local existence for general parabolic systems in variational form were obtained by [8] in the second order case; the variational case was also previously treated in [6] for slightly less general systems (or arbitrary order) with a completely different technique.

Our proof relies on the usual method of linearization and use of the contraction principle, with in addition a suitable regularization technique. It consists of four steps.

Step 1. The linear autonomous case: existence, representation and estimates for solutions in the class

$$C^{1+\alpha}([t_0,T], \quad L^p(\Omega,\mathbb{C}^N))\cap C^\alpha([t_0,T], W^{2,p}(\Omega,\mathbb{C}^N))$$

with $p\in]n,\infty[$.

Step 2. The quasilinear case: local existence of solutions in

$$C^{1+\delta}([t_0,T], L^p(\Omega,\mathbb{C}^N))\cap C^\delta([t_0,T], W^{2,p}(\Omega,\mathbb{C}^N))$$

with $\delta\in]0,\alpha[$ and $p\in\left]\frac{n}{1-2(\alpha-\delta)},\infty\right[$ $\Big($ that is to say $p>n$ and $\delta\in\left]\left(\alpha-\frac{1}{2}\left(1-\frac{n}{p}\right)\right)\wedge 0,\alpha\right[$: the reason of this restriction will be clear in Sect. 5 below$\Big)$.

Step 3. The linear non-autonomous case: global existence of solutions in

$$C^{1+\alpha}([t_0,T], L^p(\Omega,\mathbb{C}^N))\cap C^\alpha([t_0,T], W^{2,p}(\Omega,\mathbb{C}^N)),$$

$p>n$, by use of a suitable integral equation.

Step 4. The quasilinear case: regularization of the local solution and conclusion of the proof.

2. The Linear Autonomous Problem

The starting point of our proof is a basic elliptic estimate. Set for $u\in W^{2,p}(\Omega,\mathbb{C}^N)$

$$A(x,D)u:=\sum_{ij=1}^{n} A_{ij}(x)\cdot D_iD_ju\,,\qquad x\in\bar{\Omega}\,, \tag{2.1}$$

$$B(x,D):=\sum_{i=1}^{n} B_i(x)\cdot D_iu\,,\qquad x\in\partial\Omega\,, \tag{2.2}$$

the coefficients $\{A_{ij}\}, \{B_i\}$ satisfying (0.2)–(0.3)–(0.4). Then the linear problem

$$\left.\begin{aligned}\lambda u-A(x,D)u&=f\in L^p\\ B(x,D)u&=g\in W^{1,p}\end{aligned}\right\} \tag{2.3}$$

has a unique solution $u\in W^{2,p}$ which satisfies the spectral estimate

$$|\lambda|\,\|u\|_{L^p}+|\lambda|^{1/2}\|Du\|_{L^p}+\|D^2u\|_{L^p}\leqq C_p\{\|f\|_{L^p}+|\lambda|^{1/2}\|g\|_{L^p}+\|Dg\|_{L^p}\}\,, \tag{2.4}$$

provided λ belongs to the sector ($\omega_p>0$, $\theta_p\in]\pi/2,\pi[$)

$$S_{\theta_p,\omega_p}:=\{z\in\mathbb{C}:|\arg(z-\omega_p)|<\theta_p\}\,.$$

This is the classical Agmon's estimate (see [5, 7]). Define now for $\lambda\in S_{\theta_p,\omega_p}$ the operators $R(\lambda):L^p\to W^{2,p}$, $N(\lambda):W^{1,p}\to W^{2,p}$ by:

$$u=R(\lambda)f\Leftrightarrow\begin{cases}\lambda u-A(x,D)u=f & \text{in }\Omega\,,\\ B(x,D)u=0 & \text{on }\partial\Omega\,,\end{cases} \tag{2.5}$$

$$u=N(\lambda)g \Leftrightarrow \begin{cases} \lambda u - A(x,D)u = 0 & \text{in } \Omega, \\ B(x,D)u = g & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

As a consequence of (2.4) we get for $k=0,1,2$ (see [10, (2.2) and (2.8)]):

$$\|R(\lambda)f\|_{W^{k,p}} \leqq C_p |\lambda|^{\frac{k}{2}-1} \|f\|_{L^p}, \tag{2.7}$$

$$\|N(\lambda)g\|_{W^{k,p}} \leqq C_p \inf\left\{ |\lambda|^{\frac{k}{2}-\frac{1}{2}} \|\psi\|_{L^p} + |\lambda|^{\frac{k}{2}-1} \|D\psi\|_{L^p} : \psi \in W^{1,p}, \psi = g \text{ on } \partial\Omega \right\}. \tag{2.8}$$

Consider now the linear autonomous version of (0.1):

$$\begin{cases} u_t - A(x,D)u = f(t,x), \ (t,x) \in [t_0, T] \times \bar{\Omega}, \\ u(t_0, x) = \phi(x) \in \bar{\Omega}, \\ B(x,D)u = g(t,x), \ (t,x) \in [t_0, T] \times \partial\Omega, \end{cases} \tag{2.9}$$

where $A(x,D)$, $B(x,D)$ are defined in (2.1), (2.2) and their coefficients satisfy (0.2), (0.3), (0.4).

The following result is proved in [10]:

Proposition 2.1. *Fix $p>n$, and assume that $\phi \in W^{2,p}$, $f \in C^\alpha([t_0,T], L^p)$,*

$$g \in C^\alpha([t_0,T], W^{1,p}) \cap C^{\alpha+1/2}([t_0,T], L^p),$$

with the compatibility conditions

$$B(\cdot, D)\phi = g(t_0, \cdot) \text{ on } \partial\Omega, \qquad A(\cdot, D)\phi + f(t_0, \cdot) \in B^{2\alpha,p}_\infty. \tag{2.10}$$

Then problem (2.9) has a unique global solution

$$u \in C^{1+\alpha}([t_0,T], L^p) \cap C^\alpha([t_0,T], W^{2,p});$$

it can be represented by

$$u(t,\cdot) = \fint_\gamma e^{(t-t_0)\lambda}\phi \, d\lambda + \int_{t_0}^t \fint_\gamma e^{(t-s)\lambda} R(\lambda) f(s,\cdot) \, d\lambda \, ds$$
$$+ \int_{t_0}^t \fint_\gamma e^{(t-s)\lambda} N(\lambda) g(s,\cdot) \, d\lambda \, ds, \tag{2.11}$$

where $\fint_\gamma$ means $\frac{1}{2\pi i}\int_\gamma$ and γ is a smooth curve joining $+\infty e^{-i\theta}$ and $+\infty e^{i\theta}$ ($\theta \in]\pi/2, \theta_p[$), and lying in S_{θ_p, ω_p}. Moreover we have the estimate ($\varepsilon \in]0, 1/2p[\cap]0, \alpha]$):

$$\|u_t\|_{C(L^p)} + \|u\|_{C(W^{2,p})} \leqq C_0(p,\varepsilon)\{\|\phi\|_{W^{2,p}} + \|f(t_0,\cdot)\|_{L^p}$$
$$+ (T-t_0)^\varepsilon [[f]_{C^\varepsilon(L^p)} + [g]_{C^\varepsilon(W^{1,p})} + [g]_{C^{\varepsilon+1/2}(L^p)}]\}, \tag{2.12}$$

$$[u_t]_{C^\alpha(L^p)} + [u]_{C^\alpha(W^{2,p})} \leqq C_1(p,\alpha)\{\|A(\cdot,D)\phi + f(t_0,\cdot)\|_{B^{2\alpha,p}}$$
$$+ [f]_{C^\alpha(L^p)} + [g]_{C^\alpha(W^{1,p})} + [g]_{C^{\alpha+1/2}(L^p)}\}. \tag{2.13}$$

Proof. For the case $t_0 = 0$, see [10, Theorems 3.1 and 5.1]; of course the general case is quite similar. □

3. Linearization

We go back to problem (0.1) and assume (0.2), ..., (0.5). For fixed $t_0 \in [0, T[$, $\delta \in]0, \alpha[$, and $p \in \left]\frac{n}{1-2(\alpha-\delta)}, \infty\right[$, consider the Banach space

$$E_{\delta,p}(t_0,\tau) := C^{1+\delta}([t_0,\tau], L^p) \cap C^\delta([t_0,\tau], W^{2,p}), \tag{3.1}$$

with its obvious norm. We also introduce

$$[u]_{E_{\delta,p}(t_0,\tau)} := [u']_{C^\delta(L^p)} + [D^2u]_{C^\delta(L^p)}. \tag{3.1 bis}$$

By interpolation it is clear that

$$E_{\delta,p}(t_0,\tau) \hookrightarrow C^{\delta+1/2}([t_0,\tau], W^{1,p}). \tag{3.2}$$

For each $\phi \in B(\phi_0, N_0, r_0, t_0)$ we define:

$$B_{M,\delta,p,t_0,\tau,\phi} := \{v \in E_{\delta,p}(t_0,\tau) : \|v-\phi\|_{E_{\delta,p}(t_0,\tau)} \leqq M,\ v(t_0,\cdot) = \phi\}. \tag{3.3}$$

Next, we linearize problem (0.1) by considering, for any fixed $v \in B_{M,\delta,p,t_0,\tau,\phi}$, the linear autonomous problem

$$\left.\begin{aligned}
&u_t - \sum_{ij=1}^n A_{ij}(t_0, x, \phi, D\phi) \cdot D_iD_ju = f(t,x,v,Dv) \\
&\qquad - \sum_{ij=1}^n [A_{ij}(t_0x, \phi, D\phi) - A_{ij}(t,x,v,Dv)] \cdot D_iD_jv \\
&\quad =: F_{v,\phi}(t,x), (t,x) \in [t_0,\tau] \times \bar{\Omega} \\
&u(t_0,x) = \phi(x),\ x \in \bar{\Omega}, \\
&\sum_{i=1}^n B_i(t_0,x,\phi) \cdot D_i\phi = g(t,x,v) + \sum_{i=1}^n [B_i(t_0,x,\phi) \\
&\qquad - B_i(t,x,v)] \cdot D_iv =: G_{v,\phi}(t,x),\ (t,x) \in [t_0,\tau] \times \partial\Omega
\end{aligned}\right\} \tag{3.4}$$

Lemma 3.1. *We have*

$$F_{v,\phi} \in C^\delta([t_0,\tau], L^p), G_{v,\phi} \in C^\delta([t_0,\tau], W^{1,p}) \cap C^{\delta+1/2}([t_0,\tau], L^p)$$

and

$$\|F_{v,\phi}\|_{C(L^p)} + \|G_{v,\phi}\|_{C(W^{1,p})} \leqq C_2(p, M, \phi_0, r_0) \tag{3.5}$$

$$\begin{aligned}&[F_{v,\phi}]_{C^\delta(L^p)} + [G_{v,\phi}]_{C^\delta(W^{1,p})} + [G_{v,\phi}]_{C^{\delta+1/2}(L^p)} \\ &\quad \leqq C_3(p,\alpha,M,\phi_0,r_0)\omega_{p,\alpha,\delta}(\tau - t_0)\end{aligned} \tag{3.6}$$

where $\omega_{p,\alpha,\delta}(\cdot)$ is a continuous, increasing function of $t \in [0,T]$, vanishing at $t=0$.

Proof. We just prove the results concerning $F_{v,\phi}$ since the others are analogous. For each $t \in [t_0,\tau]$ and $x \in \bar{\Omega}$ we have:

$$|v(t,x)| + |Dv(t,x)| \leqq \|v-\phi\|_{E_{\delta,p}(t_0,\tau)} + \|\phi-\phi_0\|_{W^{2,p}} + \|\phi_0\|_{W^{2,p}} \leqq M + r_0 + \|\phi_0\|_{W^{2,p}},$$

hence if we set

$$A := \{(t,x,u,p) : t \in [0,T],\ x \in \bar{\Omega},\ |u| + |p| \leqq M + r_0 + \|\phi_0\|_{W^{2,p}}\},$$

we can find a constant K which bounds the sup and Hölder norms, for $(t,x,u,p) \in A$, of f, g, $\sum_{ij=1}^n |A_{ij}|$ and $\sum_{i=1}^n |B_i|$ and their derivatives appearing in (0.4).

Consequently, it is easy to see that

$$\|F_{v,\phi}(t,\cdot)\|_{L^p} \leqq C(p,K,M,\phi_0,r_0) \leqq C_2(p,M,\phi_0,r_0) \quad \forall t\in[t_0,\tau].$$

Next, we remark that if $t, r\in[t_0,\tau]$

$$\begin{aligned}\|v(t,\cdot)-v(r,\cdot)\|_C &\leqq \|v(t,\cdot)-v(r,\cdot)\|_{W^{1,p}} \leqq \|v\|_{E_{\delta,p}(t_0,\tau)}(t-r)^{\delta+1/2}\\ &\leqq (M+r_0+\|\phi_0\|_{W^{2,p}})(t-r)^{\delta}\omega(\tau-t_0),\end{aligned}$$

whereas, choosing $\theta\in\left]\frac{n}{p},1\right[$ and using interpolation,

$$\begin{aligned}\|Dv(t,\cdot)-Dv(r,\cdot)\|_C &\leqq \|Dv(t,\cdot)-Dv(r,\cdot)\|_{B^{\theta,p}}\\ &\leqq \|Dv(t,\cdot)-Dv(r,\cdot)\|_{L^p}^{1-\theta}\|Dv(t,\cdot)-Dv(t,\cdot)\|_{W^{1,p}}^{\theta}\\ &\leqq \|v\|_{E_{\delta,p}(t_0,\tau)}(t-r)^{\delta+\frac{1}{2}-\frac{\theta}{2}}\\ &\leqq (M+r_0+\|\phi_0\|_{W^{2,p}})(t-r)^{\delta}\omega_p(\tau-t_0).\end{aligned}$$

Hence it is just a tedious routine to verify that

$$\begin{aligned}\|F_{v,\phi}(t,\cdot)-F_{v,\phi}(r,\cdot)\|_{L^p} &\leqq C(p,\alpha,K,M,\phi_0,r_0)(t-r)^{\delta}\omega_{p,\alpha,\delta}(\tau-t_0)\\ &\leqq C_3(p,\alpha,M,\phi_0,r_0)(t-r)^{\delta}\omega_{p,\alpha,\delta}(\tau-t_0). \quad \square\end{aligned}$$

We now invoke Proposition 2.1 and obtain a unique solution $u:=S(v)\in E_{\delta,p}(t_0,\tau)$ of problem (3.4). Moreover $u-\phi\in E_{\delta,p}(t_0,\tau)$ and solves:

$$\left.\begin{aligned}&(u-\phi)_t-\sum_{ij=1}^{n}A_{ij}(t_0,\cdot,\phi,D\phi)\cdot D_iD_j(u-\phi)\\ &\quad=F_{v,\phi}+\sum_{ij=1}^{n}A_{ij}(t_0,\cdot,\phi,D\phi)\cdot D_iD_j\phi && \text{in } [t_0,\tau]\times\bar{\Omega},\\ &(u-\phi)(t_0,\cdot)=0 && \text{in } \bar{\Omega},\\ &\sum_{i=1}^{n}B_i(t_0,\cdot,\phi)\cdot D_i(u-\phi)=G_{v,\phi}-\sum_{i=1}^{n}B_i(t_0,\cdot,\phi)\cdot D_i\phi && \text{in } [t_0,\tau]\times\partial\Omega.\end{aligned}\right\} \tag{3.7}$$

Note that the compatibility conditions (2.10) are satisfied in this problem.

Hence, combining (2.12), (2.13) and (3.5), (3.6) we obtain the following estimate:

$$\begin{aligned}\|u-\phi\|_{E_{\delta,p}(t_0,\tau)} &\leqq C_4(p,\delta)\{\|Q(t_0,\tau)\|_{B^{2\delta,p}}\\ &\qquad+C_5(p,\alpha,M,\phi_0,r_0)\omega_{p,\alpha,\delta}(\tau-t_0)\}\\ &\leqq C_4(p,\delta)\{N_0+C_5(p,\alpha,M,\phi_0,r_0)\omega_{p,\alpha,\delta}(\tau-t_0)\}\end{aligned} \tag{3.8}$$

where $Q(t_0,\phi)$ is defined in (1.3).

Next, if v, w are fixed elements of $B_{M,\delta,p,t_0,\tau,\phi}$, we consider the function $z:=S(v)-S(w)$. It solves

$$\left.\begin{aligned}&z_t-\sum_{ij=1}^{n}A_{ij}(t_0,\cdot,\phi,D\phi)\cdot D_iD_jz=F_{v,\phi}(t,\cdot)-F_{w,\phi}(t,\cdot) && \text{in } [t_0,\tau]\times\bar{\Omega},\\ &z(t_0,\cdot)=0 && \text{in } \bar{\Omega}\\ &\sum_{i=1}^{n}B_i(t_0,\cdot,\phi)\cdot D_iz=G_{v,\phi}(t,\cdot)-G_{w,\phi}(t,\cdot) && \text{in } [t_0,\tau]\times\partial\Omega;\end{aligned}\right\} \tag{3.9}$$

as

$$F_{v,\phi}(t_0,\cdot)-F_{w,\phi}(t_0,\cdot)=0\,, \qquad G_{v,\phi}(t_0,\cdot)-G_{w,\phi}(t_0,\cdot)=0\,, \tag{3.10}$$

the compatibility conditions (2.10) obviously hold. Now concerning $F_{v,\phi}-F_{w,\phi}$ and $G_{v,\phi}-G_{w,\phi}$ we have the following estimate, which is stated with more generality for further purposes:

Lemma 3.2. *Let $\phi,\psi\in B(\phi_0,N_0,r_0,t_0)$ and v,w in $B_{M,\delta,p,t_0,\tau,\phi}$ and $B_{M,\delta,p,t_0,\tau,\psi}$ respectively. The following estimates hold:*

$$\begin{aligned}&\|F_{v,\phi}(t_0,\cdot)-F_{w,\psi}(t_0,\cdot)\|_{L^p}+\|G_{v,\phi}(t_0,\cdot)-G_{w,\psi}(t_0,\cdot)\|_{W^{1,p}}\\&\quad\le C_6(p,M,\phi_0,r_0)\,\|\phi-\psi\|_{W^{2,p}},\end{aligned} \tag{3.11}$$

$$\begin{aligned}&[F_{v,\phi}-F_{w,\phi}]_{C^\delta(L^p)}+[G_{v,\phi}-G_{w,\psi}]_{C^\delta(W^{1,p})}+[G_{v,\phi}-G_{w,\psi}]_{C^{\delta+1/2}(L^p)}\\&\quad\le C_7(p,\alpha,M,\psi_0,r_0)\,[v-w]_{E_{\delta,p}(t_0,\tau)}\omega_{p,\alpha,\delta}(\tau-t_0)\,,\end{aligned} \tag{3.12}$$

where $\omega_{p,\alpha,\delta}(t)\downarrow 0$ as $t\downarrow 0$.

Proof. Again we just prove the estimates concerning $F_{v,\phi}-F_{w,\psi}$, since the other ones are similar. The proof of (3.11) is very easy, since

$$F_{v,\phi}(t_0,\cdot)-F_{w,\psi}(t_0,\cdot)=f(t_0,\cdot,\phi,D\phi)-f(t_0,\cdot,\psi,D\psi)\,,$$

and we can omit it, too. Concerning (3.12), if $t,r\in[t_0,\tau]$ we can write (deleting for notational simplicity the dependence on x):

$$\begin{aligned}&F_{v,\phi}(t)-F_{w,\psi}(t)-F_{v,\phi}(r)+F_{w,\psi}(r)\\
&=\int_0^1\frac{d}{d\lambda}\{f(t,\lambda v(t)+(1-\lambda)w(t),\lambda Dv(t)+(1-\lambda)Dw(t))\\
&\quad-f(r,\lambda v(r)+(1-\lambda)w(r),\lambda Dv(r)+(1-\lambda)Dw(r))\}d\lambda\\
&\quad+\sum_{ij=1}^{n}\int_0^1\frac{d}{d\lambda}\{A_{ij}(r,\lambda v(r)+(1-\lambda)w(r),\lambda Dv(r)+(1-\lambda)Dw(r))\\
&\quad-A_{ij}(t,\lambda v(t)+(1-\lambda)w(t),\lambda Dv(t)+(1-\lambda)Dw(t))\}\cdot D_iD_jw(t)d\lambda\\
&\quad+\sum_{ij=1}^{n}[A_{ij}(r,w(r),Dw(r))-A_{ij}(t,w(t),Dw(t))]\\
&\quad\times[D_iD_jv(t)-D_iD_jw(t)]\\
&\quad+\sum_{ij=1}^{n}\int_0^1\frac{d}{d\lambda}\{A_{ij}(t_0,\lambda\phi+(1-\lambda)\psi,\lambda D\psi+(1-\lambda)D\psi)\\
&\quad-A_{ij}(r,\lambda v(r)+(1-\lambda)w(r),\lambda Dv(r)+(1-\lambda)Dw(r))\}\\
&\quad\times[D_iD_jv(t)-D_iD_jv(r)]d\lambda\\
&\quad+\sum_{ij=1}^{n}[A_{ij}(t_0,\psi,D\psi)-A_{ij}(r,w(r),Dw(r))]\\
&\quad\times[D_iD_jv(t)-D_iD_jv(r)-D_iD_jw(t)+D_iD_jw(r)]\,.\end{aligned}$$

The desired estimate then follows in a tedious but standard way, by arguing as in the proof of Lemma 3.1. □

By the above lemma and by Proposition 2.1 we easily obtain for the solution z of (3.9) the estimate

$$\|z\|_{E_{\delta,p}(t_0,\tau)} \leqq C_8(p,\alpha,\delta,M,\phi_0,r_0)\, \|v-w\|_{E_{\delta,p}(t_0,\tau)}\omega_{p,\alpha,\delta}(\tau-t_0). \tag{3.13}$$

Now the inequalities (3.8) and (3.13) show that the map S satisfies

$$S(v)\in B_{M,\delta,p,t_0,\tau,\phi} \ \forall v\in B_{M,\delta,p,t_0,\tau,\phi},$$
$$\|S(v)-S(w)\|_{E_{\delta,p}(t_0,\tau)} \leqq \tfrac{1}{2}\|v-w\|_{E_{\delta,p}(t_0,\tau)} \ \forall v,w\in B_{M,\delta,p,t_0,\tau,\phi}$$

provided we fix in advance $M \geqq \frac{1}{2}+C_4N_0$, and choose τ so close to t_0 that

$$\omega_{p,\alpha,\delta}(\tau-t_0) \leqq (2C_8)^{-1}\wedge(2C_4C_5)^{-1}. \tag{3.14}$$

Hence the map S is a contraction on (the complete metric space) $B_{M,\delta,p,t_0,\tau,\phi}$, so that we find a unique $u\in B_{M,\delta,p,t_0,\tau,\phi}$ such that $S(u)=u$, i.e. a unique solution in $[t_0,\tau]$ of problem (0.1).

Note that the time interval length $\tau-t_0$ depends on $p,\alpha,\delta,\phi_0,N_0,r_0$ but neither on $\phi\in B(\phi_0,N_0,r_0,t_0)$, nor on $t_0\in[0,T[$. We have thus shown that under assumptions (0.2), ..., (0.5) there exists a local solution u of problem (0.1), which belongs to

$$C^{1+\delta}([t_0,\tau],L^p)\cap C^{\delta}([t_0,\tau],W^{2,p})\left(\delta\in]0,\alpha[,\ p\in\left]\frac{n}{1-2(\alpha-\delta)},\infty\right[\right).$$

The higher regularity of u will be proved in Step 4 below.

Now fix ϕ, $\psi\in B(\phi_0, N_0, r_0, t_0)$ and let u_ψ, u_ϕ be the solutions of the corresponding quasilinear problems (0.1). Then $v:=u_\phi-u_\psi$ is the solution of:

$$\left.\begin{aligned}
&v_t-\sum_{ij=1}^{n} A_{ij}(t_0,\cdot,\phi,D\phi)\cdot D_iD_jv=F_{u_\phi,\phi}-F_{u_\psi,\psi}\\
&\quad+\sum_{ij=1}^{n}[A_{ij}(t_0,\cdot,\phi,D\phi)-A_{ij}(t_0,\cdot,\psi,D\psi)]\cdot D_iD_ju_\psi=:F^{\phi,\psi} \text{ in } [t_0,\tau]\times\bar\Omega\\
&v(t_0,\cdot)=\phi-\psi \text{ in } \bar\Omega\\
&\sum_{i=1}^{n} B_i(t_0,\cdot,\phi)\cdot D_iv=G_{u_\phi,\phi}-G_{u_\psi,\psi}-\sum_{i=1}^{n}[B_i(t_0,\cdot,\phi)\\
&\quad-B_i(t_0,\cdot,\psi)]\cdot D_iu_\psi=:G^{\phi,\psi} \text{ in } [t_0,\tau]\times\partial\Omega
\end{aligned}\right\} \tag{3.15}$$

It is readily seen that, once again, the compatibility conditions (2.10) are satisfied.

Lemma 3.3. *We have:*

$$\|F^{\phi,\psi}(t_0,\cdot)\|_{L^p}+\|G^{\phi,\psi}(t_0,\cdot)\|_{W^{1,p}} \leqq C_q(p,M,\phi_0,r_0)\,\|\phi-\psi\|_{W^{2,p}}, \tag{3.16}$$

$$\begin{aligned}&[F^{\phi,\psi}]_{C^\delta(L^p)}+[G^{\phi,\psi}]_{C^\delta(W^{1,p})}+[G^{\phi,\psi}]_{C^{\delta+1/2}(L^p)}\\ &\quad\leqq C_{10}(p,\alpha,M,\phi_0,r_0)\{\|\phi-\psi\|_{W^{2,p}}+[u_\phi-u_\psi]_{E_{\delta,p}(t_0,\tau)}\omega_{p,\alpha,\delta}(\tau-t_0)\}\end{aligned} \tag{3.17}$$

where $\omega_{p,\alpha,\delta}(t)\downarrow 0$ as $t\downarrow 0$.

Proof. It is a straightforward consequence of Lemma 3.2 and some standard calculations. □

By (2.12), (2.13), (3.16), and (3.17) we easily get:

$$\|u_\phi - u_\psi\|_{E_{\delta,p}(t_0,\tau)} \leqq C_{11}(p,\alpha,\delta,M,\phi_0,r_0) \{\|\phi-\psi\|_{W^{2,p}} + \|Q(t_0,\phi)-Q(t_0,\psi)\|_{B^{2\delta,p}_\infty} + [u_\phi - u_\psi]_{E_{\delta,p}(t_0,\tau)}\omega_{p,\alpha,\delta}(\tau-t_0)\},$$

so that if we suppose, besides (3.14), that

$$\omega_{p,\alpha,\delta}(\tau-t_0) \leqq (2C_{11})^{-1}$$

then we get

$$\|u_\phi - u_\psi\|_{E_{\delta,p}(t_0,\tau)} \leqq C_{12}(p,\alpha,\delta,M,\phi_0,r_0) \times \{\|\phi-\psi\|_{W^{2,p}} + \|Q(t_0,\phi)-Q(t_0,\psi)\|_{B^{2\delta,p}_\infty}\}, \tag{3.18}$$

which is (1.5). Thus we have shown continuous dependence on ϕ of the solution u_ϕ of problem (0.1).

Summing up, we have proved:

Proposition 3.4. *Assume* (0.2), ..., (0.5), *and fix*

$$t_0 \in [0,T[, \quad \delta \in]0,\alpha[, \quad p \in \left]\frac{n}{1-2(\alpha-\delta)}, \infty\right[.$$

There exists $\tau \in]t_0, T]$ *(depending on* $p,\alpha,\delta,\phi_0,N_0,r_0$*) such that for each* $\phi \in B(\phi_0,N_0,r_0,t_0)$ *problem* (0.1) *has a unique solution* u *in* $[t_0,\tau]$, *which satisfies*

$$u \in C^{1+\delta}([t_0,\tau],L^p) \cap C^\delta([t_0,\tau],W^{2,p}); \tag{3.19}$$

moreover the map $\phi \to u$ *is continuous, in the sense that* (3.18) *holds for any* $\phi,\psi \in B(\phi_0,N_0,r_0,t_0)$. □

4. The Linear Non-Autonomous Problem

First we need some notation. Let $A(t,x,D)$ and $B(t,x,D)$ be defined by:

$$A(t,x,D)u := \sum_{ij=1}^n A_{ij}(t,x)\cdot D_iD_ju, \quad (t,x) \in [0,T]\times\bar\Omega, \tag{4.1}$$

$$B(t,x,D)u := \sum_{i=1}^n B_i(t,x)\cdot D_ju, \quad (t,x) \in [0,T]\times\partial\Omega, \tag{4.2}$$

where

$$A_{ij} \in C^\alpha([0,T],[C(\bar\Omega)]^{N^2}), \quad B_i \in C^\alpha([0,T],[C^1(\bar\Omega)]^{N^2}) \cap C^{\alpha+1/2}([0,T],[C(\bar\Omega)]^{N^2}); \tag{4.3}$$

we also assume that (0.2)–(0.3) are satisfied. Then for each fixed $t \in [0,T]$, we can define the operators $R(\lambda,t)$, $N(\lambda,t)$ as in (2.5), (2.6), and the following estimates [analogous to (2.7), (2.8)] hold for $k=0,1,2$ and $\lambda \in S_{\theta_p,\omega_p}$:

$$\|R(\lambda,t)f\|_{W^{k,p}} \leqq C_p|\lambda|^{k/2-1}\|f\|_{L^p}, \tag{4.4}$$

$$\|N(\lambda,t)g\|_{W^{k,p}} \leqq C_p \inf\{|\lambda|^{k/2-1/2}\|\psi\|_{L^p} + |\lambda|^{k/2-1}\|D\psi\|_{L^p} : \psi \in W^{1,p},\ \psi = g \text{ on } \partial\Omega\}. \tag{4.5}$$

Consider now the linear non-autonomous problem

$$\left.\begin{aligned} &u_t - A(t,x,D)u = f(t,x),\ (t,x)\in[t_0,T]\times\bar{\Omega},\\ &u(t_0,x)=\phi(x),\ x\in\bar{\Omega},\\ &B(t,x,D)u=g(t,x),\ (t,x)\in[t_0,T]\times\partial\Omega,\end{aligned}\right\} \tag{4.6}$$

where $f\in C^\alpha([t_0,T],L^p)$, $\phi\in W^{2,p}$, $g\in C^\alpha([t_0,T],W^{1,p})\cap C^{\alpha+1/2}([t_0,T],L^p)$ and the compatibility condition $B(t_0,\cdot,D)\phi=g(t_0,\cdot)$ $(x\in\partial\Omega)$ holds.

Assume that a solution $u\in C^1([t_0,\tau],L^p)\cap C([t_0,\tau],W^{2,p})$ of (4.6) exists, and fix $t\in[t_0,\tau]$: for each $s\in[t_0,t]$ and $\lambda\in S_{\theta_p,\omega_p}$ we have the identity

$$N(\lambda,s)B(s,D)u(s)=u(s)-R(\lambda,s)[\lambda-A(s,D)]u(s). \tag{4.7}$$

[Here and from now on we simply write $A(s,D)$, $B(s,D)$ instead of $A(s,\cdot,D)$, $B(s,\cdot,D)$.]

Multiply (4.7) by $e^{(t-s)\lambda}$ and integrate over γ, γ being a smooth curve joining $+\infty e^{-i\theta}$ and $+\infty e^{i\theta}$ $(\theta\in]\pi/2,\theta_p[)$ and lying in S_{θ_p,ω_p}. The result is

$$\fint_\gamma e^{(t-s)\lambda}N(\lambda,s)B(s,D)u(s)d\lambda = -\fint_\gamma e^{(t-s)\lambda}R(\lambda,s)[\lambda-A(s,D)]u(s)d\lambda$$

or

$$\begin{aligned}&\fint_\gamma e^{(t-s)\lambda}[N(\lambda,s)-N(\lambda,t)]B(s,D)u(s)d\lambda\\ &\quad+\fint_\gamma e^{(t-s)\lambda}[R(\lambda,s)-R(\lambda,t)][\lambda-A(s,D)]u(s)d\lambda\\ &= -\fint_\gamma e^{(t-s)\lambda}N(\lambda,t)g(s)d\lambda-\fint_\gamma e^{(t-s)\lambda}R(\lambda,t)[\lambda u(s)-u'(s)+f(s)]d\lambda,\end{aligned} \tag{4.8}$$

where, as usual, $\fint_\gamma$ means $\dfrac{1}{2\pi i}\int_\gamma$.

Lemma 4.1. *We have for* $0\leqq s\leqq t$, $\lambda\in S_{\theta_p,\omega_p}$ *and* $h\in W^{2,p}$:

$$\begin{aligned}&[N(\lambda,s)-N(\lambda,t)]B(s,D)h+[R(\lambda,s)-R(\lambda,t)][\lambda-A(s,D)]h\\ &\quad=R(\lambda,t)[A(s,D)-A(t,D)]h-N(\lambda,t)[B(s,D)-B(t,D)]h.\end{aligned}$$

Proof. Set $v=N(\lambda,t)B(s,D)h+R(\lambda,t)[\lambda-A(s,D)]h$; as

$$h=N(\lambda,s)B(s,D)h+R(\lambda,s)[\lambda-A(s)]h$$

the function $h-v$ solves

$$\begin{cases}[\lambda-A(t,D)](h-v)=[A(s,D)-A(t,D)]h,\\ B(t,D)(h-v)=-[B(s,D)-B(t,D)]h\end{cases}$$

and the result follows. □

By the above lemma and (4.8) we get:

$$\begin{aligned}&\fint_\gamma e^{(t-s)\lambda}\{R(\lambda,t)[A(s,D)-A(t,D)]u(s)-N(\lambda,t)[B(s,D)-B(t,D)]u(s)\}d\lambda\\ &\quad=-\fint_\gamma e^{(t-s)\lambda}\{R(\lambda,t)[\lambda u(s)-u'(s)+f(s)]+N(\lambda,t)g(s)\}d\lambda.\end{aligned} \tag{4.9}$$

Define now

$$K_\lambda(t,s):=R(\lambda,t)[A(s,D)-A(t,D)]-N(\lambda,t)[B(s,D)-B(t,D)]; \tag{4.10}$$

by (4.4) and (4.5) it is easy to check that

$$\|K_\lambda(t,s)h\|_{W^{2,p}}\leqq C_{p,\alpha}\{(t-s)^\alpha\|\psi\|_{W^{2,p}}+|\lambda|^{1/2}(t-s)^{\alpha+1/2}\|\psi\|_{W^{1,p}}\}. \tag{4.11}$$

Hence we can define

$$K(t,s):=\oint_\gamma e^{(t-s)\lambda}K_\lambda(t,s)d\lambda, \tag{4.12}$$

and (4.11) yields

$$\|K(t,s)h\|_{W^{2,p}}\leqq C_{p,\alpha}(t-s)^{\alpha-1}\|h\|_{W^{2,p}}. \tag{4.13}$$

We can also rewrite (4.9) as:

$$K(t,s)u(s)=-\oint_\gamma e^{(t-s)\lambda}\{R(\lambda,t)[\lambda u(s)-u'(s)+f(s)] \\ +N(\lambda,t)g(s)d\lambda. \tag{4.14}$$

We now integrate between t_0 and t; an integration by parts leads to:

$$\int_{t_0}^{t} K(t,s)u(s)ds=\left[\oint_\gamma e^{(t-s)\lambda}R(\lambda,t)u(s)d\lambda\right]_{t_0}^{t} \\ -\int_{t_0}^{t}\oint_\gamma e^{(t-s)\lambda}\{R(\lambda,t)f(s)+N(\lambda,t)g(s)\}d\lambda ds,$$

and by the well-known properties of the semi-group $E(r):=\oint_\gamma e^{r\lambda}R(\lambda,r)d\lambda$ (see e.g. [10, Proposition 2.1 (i)]) we get the integral equation

$$u(t)-\int_{t_0}^{t} K(t,s)u(s)ds=\oint_\gamma e^{(t-t_0)\lambda}R(\lambda,t)\phi d\lambda \\ +\int_{t_0}^{t}\oint_\gamma e^{(t-s)\lambda}\{R(\lambda,t)f(s)+N(\lambda,t)g(s)\}d\lambda ds=:L(\phi,f,g)(t). \tag{4.15}$$

Thus if u is a solution of problem (4.6) on $[t_0,\tau]$, then – at least formally – u satisfies the integral equation (4.15). We will prove now that (4.15) is indeed meaningful in the sense of $C([t_0,\tau],W^{2,p})$, and that such equation must be fulfilled by any solution

$$u\in C^1([t_0,\tau],L^p)\cap C([t_0,\tau],W^{2,p})$$

of (4.6).

It is clear that $L(\phi,f,g)\in C([t_0,T],L^p)$. In order to show stronger regularity properties of $L(\phi,f,g)$, we need the following lemma.

Lemma 4.2. *We have:*

$$\|[R(\lambda,t)-R(\lambda,r)]h\|_{W^{2,p}}\leqq C_{p,\alpha}|t-r|^\alpha\|h\|_{L^p}, \tag{4.16}$$

$$\|[N(\lambda,t)-N(\lambda,r)]h\|_{W^{2,p}}\leqq C_{p,\alpha}|t-r|^\alpha\{\|h\|_{L^p}+|\lambda|^{1/2}\|h\|_{W^{1,p}}\}. \tag{4.17}$$

Proof. Set $v = R(\lambda, t)h$, $w = R(\lambda, r)h$; then $v - w$ solves

$$\left.\begin{aligned} &[\lambda - A(t,D)]\,(v-w) = -[A(r,D) - A(t,D)]w \\ &B(t,D)\,(v-w) = [B(r,D) - B(t,D)]w \end{aligned}\right\} \tag{4.18}$$

so that (4.16) follows easily by (4.4), (4.5). Similarly, if we set $v = N(\lambda, t)h$, $w = N(\lambda, r)h$, then again $v - w$ solves (4.18), and (4.4) and (4.5) imply now (4.17). □

Proposition 4.3. *We have* $L(\phi, f, g) \in C([t_0, T], W^{2,p})$; *in addition,*

$$L(\phi, f, g) \in C^\alpha([t_0, T], W^{2,p})$$

if and only if $A(t_0, D) + f(t_0, \cdot) \in B^{2\alpha, p}_\infty$.

Proof. Using (4.7) and splitting some terms, we can rewrite $L(\phi, f, g)$ as:

$$\begin{aligned} L(\phi, f, g)(t) = \phi &+ \int_\gamma \lambda^{-1} e^{(t-t_0)\lambda}[R(\lambda, t)A(t, D)\phi - N(\lambda, t)B(t, D)\phi]\,d\lambda \\ &+ \int_\gamma \lambda^{-1} e^{(t-t_0)\lambda}[R(\lambda, t)f(t) + N(\lambda, t)g(t)]\,d\lambda \\ &+ \int_{t_0}^{t} \int_\gamma e^{(t-s)\lambda}\{R(\lambda, t)\,[f(s) - f(t)] + N(\lambda, t)\,[g(s) - g(t)]\}\,d\lambda\, ds\,. \end{aligned} \tag{4.19}$$

Hence (4.4) and (4.5) easily yield:

$$\|L(\phi, f, g)(t)\|_{W^{2,p}} \leqq C_{p,\alpha}\{\|\phi\|_{W^{2,p}} + \|f\|_{C^\alpha(L^p)} + \|g\|_{C^\alpha(W^{1,p}) \cap C^{\alpha+1/2}(L^p)}\}\,. \tag{4.20}$$

Moreover if $t_0 \leqq r \leqq t \leqq T$ we have:

$$\begin{aligned} &L(\phi, f, g)(t) - L(\phi, f, g)(r) \\ &\quad = \Big\{\int_\gamma \lambda^{-1} e^{(t-t_0)\lambda} R(\lambda, t)\,[A(t, D) - A(r, D)]\phi\, d\lambda \\ &\qquad + \int_\gamma \lambda^{-1} e^{(t-t_0)\lambda}[R(\lambda, t) - R(\lambda, r)]A(r, D)\phi\, d\lambda \\ &\qquad + \int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda} - e^{(r-t_0)\lambda}]R(\lambda, r)\,[A(r, D) - A(t_0, D)]\phi\, d\lambda \\ &\qquad + \int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda} - e^{(r-t_0)\lambda}]\,[R(\lambda, r) - R(\lambda, t_0)]A(t_0, D)\phi\, d\lambda \\ &\qquad + \int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda} - e^{(r-t_0)\lambda}]R(\lambda, t_0)A(t_0, D)\phi\, d\lambda\Big\} \\ &\qquad + \Big\{-\int_\gamma \lambda^{-1} e^{(t-t_0)\lambda} N(\lambda, t)\,[B(t, D) - B(r, D)]\phi\, d\lambda \\ &\qquad - \int_\gamma \lambda^{-1} e^{(t-t_0)\lambda}[N(\lambda, t) - N(\lambda, r)]\,[B(r, D) - B(t_0, D)]\phi\, d\lambda \\ &\qquad - \int_\gamma \lambda^{-1} e^{(t-t_0)\lambda}[N(\lambda, t) - N(\lambda, r)]B(t_0, D)\phi\, d\lambda \\ &\qquad - \int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda} - e^{(r-t_0)\lambda}]N(\lambda, r)\,[B(r, D) - B(t_0, D)]\phi\, d\lambda \\ &\qquad - \int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda} - e^{(r-t_0)\lambda}]N(\lambda, r)B(t_0, D)\phi\, d\lambda\Big\} \\ &\qquad + \Big\{\int_\gamma \lambda^{-1} e^{(t-t_0)\lambda} R(\lambda, t)\,[f(t) - f(r)]\,d\lambda \end{aligned}$$

$$
\begin{aligned}
&+\int_\gamma \lambda^{-1}e^{(t-t_0)\lambda}[R(\lambda,t)-R(\lambda,r)]f(r)d\lambda \\
&+\int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda}-e^{(r-t_0)\lambda}]R(\lambda,r)[f(r)-f(t_0)]d\lambda \\
&+\int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda}-e^{(r-t_0)\lambda}][R(\lambda,r)-R(\lambda,t_0)]f(t_0)d\lambda \\
&+\int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda}-e^{(r-t_0)\lambda}]R(\lambda,t_0)f(t_0)d\lambda\Big\} \\
&+\Big\{\int_\gamma \lambda^{-1}e^{(t-t_0)\lambda}N(\lambda,t)[g(t)-g(r)]d\lambda \\
&+\int_\gamma \lambda^{-1}e^{(t-t_0)\lambda}[N(\lambda,t)-N(\lambda,r)][g(r)-g(t_0)]d\lambda \\
&+\int_\gamma \lambda^{-1}e^{(t-t_0)\lambda}[N(\lambda,t)-N(\lambda,r)]g(t_0)d\lambda \\
&+\int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda}-e^{(r-t_0)\lambda}]N(\lambda,r)[g(r)-g(t_0)]d\lambda \\
&+\int_\gamma \lambda^{-1}[e^{(t-t_0)\lambda}-e^{(r-t_0)\lambda}]N(\lambda,r)g(t_0)]d\lambda\Big\} \\
&+\Big\{\int_r^t\int_\gamma e^{(t-s)\lambda}R(\lambda,t)[f(s)-f(t)]d\lambda ds \\
&+\int_{t_0}^r\int_\gamma e^{(t-s)\lambda}R(\lambda,t)[f(r)-f(t)]d\lambda ds \\
&+\int_{t_0}^r\int_\gamma e^{(t-s)\lambda}[R(\lambda,t)-R(\lambda,r)][f(s)-f(r)]d\lambda ds \\
&+\int_{t_0}^r\int_\gamma\int_{r-s}^{t-s} \lambda e^{\lambda\sigma}R(\lambda,r)[f(s)-f(r)]d\sigma d\lambda ds\Big\} \\
&+\Big\{\int_r^t\int_\gamma e^{(t-s)\lambda}N(\lambda,t)[g(s)-g(t)]d\lambda ds \\
&+\int_{t_0}^r\int_\gamma e^{(t-s)\lambda}N(\lambda,t)[g(r)-g(t)]d\lambda ds \\
&+\int_{t_0}^r\int_\gamma e^{(t-s)\lambda}[N(\lambda,t)-N(\lambda,r)][g(s)-g(r)]d\lambda ds \\
&+\int_{t_0}^r\int_\gamma\int_{r-s}^{t-s} \lambda e^{\lambda\sigma}N(\lambda,r)[g(s)-g(r)]d\sigma d\lambda ds =: \sum_{i=1}^{28} I_i .
\end{aligned}
$$

Now we clearly have

$$I_8+I_{18}=0, \qquad I_{10}+I_{20}=0,$$

whereas a routine calculation shows that

$$\sum_{i=1}^{4} \|I_i\|_{W^{2,p}}+\sum_{i=6}^{7} \|I_i\|_{W^{2,p}}+\|I_9\|_{W^{2,p}}\leqq C_{\alpha,p}(t-r)^\alpha\|\phi\|_{W^{2,p}}, \tag{4.21}$$

$$\sum_{i=11}^{14} \|I_i\|_{W^{2,p}}+\sum_{i=21}^{24} \|I_i\|_{W^{2,p}}\leqq C_{\alpha,p}(t-r)^\alpha\|f\|_{C^\alpha(L^p)}, \tag{4.22}$$

$$\sum_{i=16}^{17} \|I_i\|_{W^{2,p}} + \|I_{19}\|_{W^{2,p}} + \sum_{i=25}^{28} \|I_i\|_{W^{2,p}} \leqq C_{\alpha,p}(t-r)^\alpha \|g\|_{C^\alpha(W^{1,p}) \cap C^{\alpha+1/2}(L^2)}. \tag{4.23}$$

We still need to estimate I_5 and I_{15}. But

$$I_5 + I_{15} = \int_r^t \oint_\gamma e^{\lambda\sigma} R(\lambda, t_0) [A(t_0, D)\phi + f(t_0)] d\lambda \tag{4.24}$$

so that

$$\|I_5 + I_{15}\|_{W^{2,p}} = \omega_{p,\alpha}(t-r) \{\|\phi\|_{W^{2,p}} + \|f\|_{C(L^p)}\}$$

where $\omega_{p,\alpha}(s)\downarrow 0$ as $s\downarrow 0$. This proves that $L(\phi, f, g) \in C([t_0, T], W^{2,p})$, and by (4.20) we have

$$\|L(\phi, f, g)\|_{C(W^{2,p})} \leqq C_{p,\alpha}\{\|\phi\|_{W^{2,p}} + \|f\|_{C^\alpha(L^p)} + \|g\|_{C^\alpha(W^{1,p}) \cap C^{\alpha+1/2}(L^p)}\}. \tag{4.25}$$

Moreover (4.24) shows that

$$\|I_5 + I_{15}\|_{W^{2,p}} = O((t-r)^\alpha) \quad \text{as} \quad t-r\downarrow 0$$

if and only if $A(t_0, D)\phi + f(t_0) \in B_\infty^{2\alpha,p}$; in this case by (4.21), (4.22) and (4.23) we get $L(\phi, f, g) \in C^\alpha([t_0, T], W^{2,p})$ and

$$[L(\phi, f, g)]_{C^\alpha(W^{2,p})} \leqq C_{p,\alpha}\{\|A(t_0, D)\phi + \mathrm{f}(\mathrm{t}_0)\|_{B_\infty^{2\alpha,p}} + \|f\|_{C^\alpha(L^p)} + \|g\|_{C^\alpha(W^{1,p}) \cap C^{\alpha+1/2}(L^p)}\}. \tag{4.26}$$

The proof is complete. □

Let us now examine the regularity properties of the kernel $K(t,s)$ given by (4.12).

Lemma 4.4. *Let* $K_\lambda(t,s)$ *be defined by* (4.10). *Then*

$$\|[K_\lambda(t,s) - K_\lambda(r,s)]h\|_{W^{2,p}} \leqq C_{p,\alpha}|t-r|^\alpha \|\psi\|_{W^{2,p}}.$$

Proof. Writing

$$\begin{aligned}[K_\lambda(t,s) - K_\lambda(r,s)]h = {} & R(\lambda, t)[A(r,D) - A(t,D)]h \\ & + [R(\lambda,t) - R(\lambda,r)][A(s,D) - A(r,D)]h \\ & - N(\lambda,t)[B(r,D) - B(t,D)]h \\ & - [N(\lambda,t) - N(\lambda,r)][B(s,D) - B(r,D)]h = \sum_{j=1}^4 J_j,\end{aligned}$$

we get by Lemma 4.2

$$\|J_1\|_{W^{2,p}} + \|J_3\|_{W^{2,p}} \leqq C_{p,\alpha}|t-r|^\alpha \|h\|_{W^{2,p}},$$

$$\|J_2\|_{W^{2,p}} + \|J_4\|_{W^{2,p}} \leqq C_{p,\alpha}|t-r|^\alpha |r-s|^\alpha \|h\|_{W^{2,p}},$$

and the result follows. □

Lemma 4.5. *We have for* $0 \leqq s < r \leqq t \leqq T$:

$$\|[K(t,s)-K(r,s)]h\|_{W^{2,p}} \leqq C_{p,\alpha} \frac{(t-r)^\alpha}{(r-s)^{1-\alpha}(t-s)^\alpha} \|h\|_{W^{2,p}}. \tag{4.27}$$

Proof. Write

$$\begin{aligned}[K(t,s)-K(r,s)] &= \oint_\gamma e^{(t-s)\lambda}[K_\lambda(t,s)-K_\lambda(r,s)]h\,d\lambda \\ &\quad + \int_{r-s}^{t-s} \oint_\gamma \lambda e^{\lambda\sigma} K_\lambda(r,s) h\,d\lambda\,d\sigma \\ &=: A_1 + A_2;\end{aligned}$$

now by Lemma 4.4

$$\|A_1\|_{W^{2,p}} \leqq C_{p,\alpha} \frac{(t-r)^\alpha}{t-s} \|h\|_{W^{2,p}}$$

whereas by (4.11)

$$\begin{aligned}\|A_2\|_{W^{2,p}} &\leqq C_{p,\alpha} \int_{r-s}^{t-s} \left[\frac{(r-s)^\alpha}{\sigma^2} + \frac{(r-s)^{\alpha+1/2}}{\sigma^{5/2}}\right] d\sigma \|h\|_{W^{2,p}} \\ &\leqq C_{p,\alpha}(r-s)^\alpha \left[\frac{1}{r-s} - \frac{1}{t-s}\right] \|h\|_{W^{2,p}} \\ &= C_{p,\alpha} \frac{t-r}{(r-s)^{1-\alpha}(t-s)} \|h\|_{W^{2,p}},\end{aligned}$$

and this implies the result. □

Introduce the linear integral operator

$$[K_{t_0}h](t) = \int_{t_0}^t K(t,s)h(s)\,ds, \qquad h \in W^{2,p}. \tag{4.28}$$

Proposition 4.6. *Let the operator* K_{t_0} *be defined by* (4.28). *Then:*

(i) $K_{t_0} \in \mathscr{L}(C([t_0,T], W^{2,p}))$ *and* $1 - K_{t_0}$ *is invertible;*

(ii) *if* $h \in C([t_0,T], W^{2,p})$, *then* $K_{t_0}h \in C^\delta([t_0,T], W^{2,p})$ $\forall \delta \in]0,\alpha[$;

(iii) *if* $h \in C^\varepsilon([t_0,T], W^{2,p})$, $\varepsilon \in]0,1]$, *then* $K_{t_0}h \in C^\alpha([t_0,T], W^{2,p})$.

Proof. (i) It is a standard property of Volterra integral operators satisfying (4.13) and (4.28) (see e.g. [3, Proposition 2.4]). (ii) We can write for $t_0 \leqq r < t \leqq T$:

$$\begin{aligned}K_{t_0}h(t) - K_{t_0}h(r) &= \int_r^t K(t,s)h(s)\,ds \\ &\quad + \int_{t_0}^t [K(t,s)-K(r,s)]h(s)\,ds =: S_1 + S_2;\end{aligned} \tag{4.29}$$

on the other hand (4.13) and (4.28) give:

$$\|S_1\|_{W^{2,p}} \leqq C_{p,\alpha}(t-r)^{\alpha}\|h\|_{C(W^{2,p})},$$

$$\|S_2\|_{W^{2,p}} \leqq C_{p,\alpha}\int_{t_0}^{r}\frac{(t-r)^{\alpha}}{(r-s)^{1-\alpha}(t-s)^{\alpha}}\|h\|_{C(W^{2,p})}$$

$$\leqq C_{p,\alpha,\delta}(t-r)^{\delta}\|h\|_{C(W^{2,p})}.$$

(iii) Instead of (4.29), recalling (4.12) we write:

$$\begin{aligned} K_{t_0}h(t)-K_{t_0}h(r) &= \int_r^t K(t,s)h(s)ds + \int_{t_0}^{r}[K(t,s)-K(r,s)]\,[h(s)-h(r)]ds \\ &\quad + \int_{t_0}^{r}\oint_{\gamma} e^{\lambda(t-s)}[K_\lambda(t,s)-K_\lambda(r,s)]h(r)d\lambda ds \\ &\quad + \int_{t_0}^{r}\int_{r-s}^{t-s}\oint_{\gamma}\lambda e^{\lambda\sigma}K_\lambda(r,s)h(r)d\lambda d\sigma ds \\ &=: \sum_{k=1}^{4} K_k. \end{aligned}$$

As before we have

$$\|K_1\|_{W^{2,p}} \leqq C_{p,\alpha}(t-r)^{\alpha}\|h\|_{C(W^{2,p})},$$

whereas by (4.28)

$$\|K_2\|_{W^{2,p}} \leqq C_{p,\alpha}(t-r)^{\alpha}\int_{t_0}^{r}(r-s)^{\varepsilon-1}ds\|h\|_{C^{\varepsilon}(W^{2,p})}$$

$$\leqq C_{p,\alpha,\varepsilon}(t-r)^{\alpha}\|h\|_{C^{\varepsilon}(W^{2,p})},$$

and, by (4.11),

$$\|K_4\|_{W^{2,p}} \leqq C_{p,\alpha}\int_{t_0}^{r}\int_{r-s}^{t-s}\left[\frac{(r-s)^{\alpha}}{(\sigma-t_0)^2}+\frac{(r-s)^{\alpha+1/2}}{(\sigma-t_0)^{5/2}}\right]d\sigma ds\|h\|_{C(W^{2,p})}$$

$$\leqq C_{p,\alpha}(t-r)^{\alpha}\|h\|_{C(W^{2,p})}.$$

Finally, using (4.10) we can evaluate K_3 exactly:

$$\begin{aligned} K_3 &= \oint_{\gamma}\lambda^{-1}[e^{(t-t_0)\lambda}-e^{(t-r)\lambda}]R(\lambda,t)\,[A(r,D)-A(t,D)]h(r)d\lambda \\ &\quad + \int_{t_0}^{r}\oint_{\gamma} e^{(t-s)\lambda}[R(\lambda,t)-R(\lambda,r)]\,[A(s,D)-A(r,D)]h(r)d\lambda ds \\ &\quad - \oint_{\gamma}\lambda^{-1}[e^{(t-t_0)\lambda}-e^{(t-r)\lambda}]N(\lambda,t)\,[B(r,D)-B(t,D)]h(r)d\lambda \\ &\quad - \int_{t_0}^{r}\oint_{\gamma} e^{\lambda(t-s)}[N(\lambda,t)-N(\lambda,r)]\,[B(s,D)-B(r,D)]h(r)d\lambda ds \\ &=: \sum_{h=1}^{4} H_h. \end{aligned}$$

It is easy now, using (4.4), (4.5), (4.16), and (4.17), to show that

$$\|K_3\|_{W^{2,p}} \leqq \sum_{h=1}^{4} \|H_h\|_{W^{2,p}} \leqq C_{p,\alpha}(t-r)^{\alpha}\|h\|_{C(W^{2,p})}.$$

Summing up, we have shown that

$$[K_{t_0}h]_{C^\alpha(W^{2,p})} \leqq C_{p,\alpha,\delta}\|h\|_{C^\alpha(W^{2,p})},$$

which proves the result. □

We are now ready to state the main result of this section.

Theorem 4.7. *Under assumptions* (4.1), (4.2), (4.3), (0.2), (0.3) *consider problem* (4.6) *with* $\phi \in W^{2,p}$, $f \in C^\alpha([t_0, T], L^p)$,

$$g \in C^\alpha([t_0, T], W^{1,p}) \cap C^{\alpha+1/2}([t_0, T], L^p)$$

and the compatibility condition $B(t_0, D)\phi = g(t_0, \cdot)$ *on* $\partial\Omega$. *Then we have:*

(i) *If* $u \in C^1([t_0, \tau], L^p) \cap C([t_0, \tau], W^{2,p})$ *is a solution of* (4.6) *in* $[t_0, \tau]$, *then* u *solves the integral equation* (4.15) *in the sense of* $C([t_0, \tau], W^{2,p})$ *and, in particular,*

$$\begin{aligned}&\|u\|_{C^1(L^p)} + \|u\|_{C(W^{2,p})}\\ &\leqq C_{13}(p,\alpha)\{\|\phi\|_{W^{2,p}} + \|f\|_{C^\alpha(L^p)} + \|g\|_{C^\alpha(W^{1,p})} + \|g\|_{C^{\alpha+1/2}(L^p)}\};\end{aligned} \tag{4.30}$$

(ii) *u is a global solution of* (4.6), *i.e.* $u \in C^1([t_0, T], L^p) \cap C([t_0, T], W^{2,p})$ *and solves* (4.6) *in* $[t_0, T]$;

(iii) $u \in C^{1+\alpha}([t_0, T], L^p) \cap C^\alpha([t_0, T], W^{2,p})$ *if and only if*

$$A(t_0, D)\phi + f(t_0, \cdot) \in B_\infty^{2\alpha, p};$$

in this case we have

$$\begin{aligned}&\|u\|_{C^{1+\alpha}(L^p)} + \|u\|_{C^\alpha(W^{2,p})} \leqq C_{14}(p,\alpha)\{\|A(t_0, D)\phi + f(t_0, \cdot)\|_{B^{2\alpha,p}}\\ &\quad + \|\phi\|_{W^{2,p}} + \|f\|_{C^\alpha(L^p)} + \|g\|_{C^\alpha(W^{1,p})} + \|g\|_{C^{\alpha+1/2}(L^p)}\}.\end{aligned} \tag{4.31}$$

Proof. Part (i) has been proved before.

(ii) Inequality (4.30) is an a-priori estimate for local solutions of (4.6): thus for any such solution we necessarily have $T(\phi) = T$ (see Remark 1.5).

(iii) Equation (4.15) holds now in $[t_0, T]$ and the result follows by (4.15) and Propositions 4.3, 4.6. □

5. Regularization

Go back once again to problem (0.1). Assume (0.2), ..., (0.5) and fix $t_0 \in [0, T[$, $\delta \in]0, \alpha[$, $p \in \left]\dfrac{n}{1-2(\alpha-\delta)}, \infty\right[$.

Lemma 5.1. *We have*

$$C^\delta([t_0, \tau], W^{2,p}) \cap C^{\delta+1/2}([t_0, \tau], W^{1,p}) \hookrightarrow C^\alpha([t_0, \tau], C^1) \cap C^{\alpha+1/2}([t_0, \tau], C).$$

Proof. By Sobolev's Theorem we have

$$B_\infty^{\theta,p} \hookrightarrow C \quad \text{if} \quad \theta > n/p,$$

$$B_\infty^{\theta,p} \hookrightarrow C^1 \quad \text{if} \quad \theta > n/p + 1.$$

Hence if $t, s \in [t_0, \tau]$ we have (deleting for brevity the dependence on x):

$$\|u(t)-u(s)\|_{C^1} \leqq C_{\varepsilon,p}\|u(t)-u(s)\|_{B_\infty^{1+\varepsilon+\frac{n}{p},p}}$$
$$\leqq C_{\varepsilon,p}\|u(t)-u(s)\|_{W^{1,p}}^{1-\frac{n}{p}-\varepsilon}\|u(t)-u(s)\|_{W^{2,p}}^{\frac{n}{p}+\varepsilon}$$
$$\leqq C_{\varepsilon,p}\|u\|_{E_{\delta,p(t_0,\tau)}}|t-s|^{\delta+\frac{1}{2}\left(1-\frac{n}{p}-\varepsilon\right)},$$
$$\left\|u(t)+u(s)-2u\left(\frac{t+s}{2}\right)\right\|_C \leqq C_{\varepsilon,p}\left\|u(t)+u(s)-2u\left(\frac{t+s}{2}\right)\right\|_{B_\infty^{\frac{n}{p}+\varepsilon,p}}$$
$$\leqq C_{\varepsilon,p}\left\|u(t)+u(s)-2u\left(\frac{t+s}{2}\right)\right\|_{L^p}^{1-\frac{n}{p}-\varepsilon}\left\|u(t)+u(s)-2u\left(\frac{t+s}{2}\right)\right\|_{W^{1,p}}^{\frac{n}{2}+\varepsilon}$$
$$\leqq C_{\varepsilon,p}\|u\|_{E_{\delta,p(t_0,\tau)}}|t-s|^{\frac{1}{2}+\delta+\frac{1}{2}\left(1-\frac{n}{p}-\varepsilon\right)}.$$

As $\delta+\frac{1}{2}\left(1-\dfrac{n}{p}\right)>\alpha$, for sufficiently small ε we get the result. □

Consider the solution u of (0.1), given by Proposition 3.4: by (3.19) and Lemma 5.1 we have

$$u \in C^\alpha([t_0,\tau], C^1) \cap C^{\alpha+1/2}([t_0,\tau], C),$$

and consequently it is easy to see that

$$F(t,\cdot):=f(t,\cdot,u(t,\cdot),\ Du(t,\cdot)) \in C^\alpha([t_0,\tau],C),$$
$$G(t,\cdot):=g(t,\cdot,u(t,\cdot) \in C^\alpha([t_0,\tau],C^1) \cap C^{\alpha+1/2}([t_0,\tau],C),$$
$$A_{ij}(t,\cdot):=A_{ij}(t,\cdot u(t,\cdot),\ Du(t,\cdot)) \in C^\alpha([t_0,\tau],C),$$
$$B_i(t,\cdot):=B_i(t,\cdot,u(t,\cdot)) \in C^\alpha([t_0,\tau],C^1) \cap C^{\alpha+1/2}([t_0,\tau],C).$$

Hence u solves a linear non-autonomous problem of type (4.1), and all assumptions of Theorem 4.7 are fulfilled: thus we get

$$u \in C^{1+\alpha}([t_0,\tau],L^p) \cap C^\alpha([t_0,\tau],L^p).$$

The estimate (1.5) in the case $\delta=\alpha$ can be obtained by arguing as in the proof of (3.18), provided possibly that $\tau-t_0$ is chosen smaller.

Suppose finally that $\phi \in C^2(\bar{\Omega}, \mathbb{C}^N)$ and $Q(t_0,\phi) \in C^{2\alpha}(\bar{\Omega}, \mathbb{C}^N)$; then for each $p>n$ we can apply the above theory, obtaining a local solution u of (0.1) which belongs to $E_{\alpha,p}(t_0,\tau)$, where τ depends on p. If we fix any $\delta \in]0,\alpha[$, and choose $p>\dfrac{1}{2(\alpha-\delta)}$, then we have the continuous inclusion

$$C^{1+\alpha}(L^p) \cap C^{\frac{1}{2}+\alpha}(W^{1,p}) \cap C^\alpha(W^{2,p}) \hookrightarrow C^{1+\delta}(B_\infty^{2(\alpha-\delta),p})$$
$$\cap C^{\frac{1}{2}+\delta}(B_\infty^{1+2(\alpha-\delta),p}) \hookrightarrow C^{1+\delta}(C) \cap C^{\frac{1}{2}+\delta}(C^1). \tag{5.1}$$

Hence $u_t \in C^\delta([t_0,\tau],C)$ and $f(\cdot,\cdot,u,Du) \in C^\delta([t_0,\tau],C)$; thus by (0.1) we get

$$t \to \sum_{ij=1}^{n} A_{ij}(t,\cdot,u(t,\cdot),Du(t,\cdot)) \cdot D_iD_ju(t,\cdot) \in C^\delta([t_0,\tau],C).$$

This concludes the proof of Theorem 1.1. □

Remark 5.2. If we increase the smoothness of data, we can obtain Hölder continuity results for the solution of (0.1) which are very close to those given in [9, Chap. VI, Theorems 4.1–4.2] for quasilinear equations (see also [9, Chap. VII, Theorem 7.1] for quasilinear systems of special form). Namely, replace (0.4) by the following assumption:

The functions $A_{ij}^{hk}, f^h, B_i^{hk}, g^h, \frac{\partial B_i^{hk}}{\partial x_j}, \frac{\partial B_i^{hk}}{\partial u^m}, \frac{\partial g^h}{\partial x_j}, \frac{\partial g^h}{\partial u^k}$ are of class C^α in t, $C^{2\alpha}$ in x, locally Lipschitz continuous in (u,p); moreover the functions B_i^{hk}, g^h are also of class $C^{\alpha+1/2}$ in t. (5.2)

Then we can show that the solution u of (0.1) satisfies:

$$u_t, D_iD_ju \in C([t_0,\tau], C^{2\delta}(\bar{\Omega}, \mathbb{C}^N)) \ \forall \delta \in]0,\alpha[. \tag{5.3}$$

Indeed, similarly to (5.1), we can write

$$C^{1+\alpha}(L^p) \cap C^{\frac{1}{2}+\alpha}(W^{1,p}) \cap C^\alpha(W^{2,p}) \hookrightarrow C^{1+\alpha-\theta/2}(B_\infty^{\theta,p})$$
$$\cap C^{\frac{1}{2}+\alpha-\theta/2}(B_\infty^{1+\theta,p}) \hookrightarrow C^1\left(C^{\theta-\frac{n}{p}}\right) \cap C^{1/2}\left(C^{1+\theta-\frac{n}{p}}\right),$$

provided $p > \frac{n}{2\alpha}$ and $\theta \in \left]\frac{n}{p}, 2\alpha\right[$. Now, fix $\delta \in]0,\alpha[$, select any $\sigma \in]\delta,\alpha[$, and choose $\theta = \alpha+\sigma$, $p = \frac{n}{\alpha-\sigma}$, so that $\theta - \frac{n}{p} = 2\sigma$: then we have

$$u \in C^1([t_0,\tau], C^{2\sigma}),$$
$$F := f(\cdot,\cdot,u,Du), \qquad \bar{A}_{ij} := A_{ij}(\cdot,\cdot,u,Du) \in C([t_0,\tau], C^{2\sigma}),$$
$$G := g(\cdot,\cdot,u), \bar{B}_i := B_i(\cdot,\cdot,u) \in C([t_0,\tau], C^{1+2\sigma});$$

hence, for fixed $t \in [t_0,\tau]$, $u(t,\cdot)$ solves a linear elliptic problem of the following kind:

$$\begin{cases} \sum_{ij=1}^n \bar{A}_{ij}(t,\cdot) \cdot D_iD_ju(t,\cdot) = F(t,\cdot) - u_t(t,\cdot) & \text{in } \bar{\Omega}, \\ \sum_{i=1}^n \bar{B}_i(t,\cdot) \cdot D_iu(t,\cdot) = G(t,\cdot) & \text{on } \partial\Omega. \end{cases}$$

By Schauder's estimate, we easily get $D_iD_ju \in L^\infty(t_0,\tau, C^{2\sigma}(\bar{\Omega}, \mathbb{C}^N))$. Now, as

$$u \in L^\infty(t_0,\tau, C^{2+2\sigma}) \cap C^1(t_0,\tau, C^{2\sigma})$$

we readily obtain, by interpolation, $u \in C([t_0,\tau], C^{2+2\delta})$. This proves (5.3).

Remark 5.3. Due to the presence of the compatibility conditions

$$P(t_0,\phi) = 0, \qquad Q(t_0,\phi) \in B_\infty^{2\alpha,p}(\Omega, \mathbb{C}^N) \tag{5.4}$$

we are not able to improve our result concerning continuous dependence on the initial datum in order to get that our solution is a local semiflow in the sense of [5]. One can avoid conditions (5.4) by replacing the space $C^{1+\alpha}(L^p) \cap C^\alpha(W^{2,p})$ by a suitable weighted Hölder space, introduced in [3], and in this way one can show that solutions of (0.1) in this class indeed generate a local semiflow. This will be done in a forthcoming paper.

Remark 5.4. If in problem (0.1) the boundary conditions are of Dirichlet type, i.e. $B_i \equiv 0$ and $g(t, x, u) \equiv u$, we can apply the same argument, but several changes are necessary since the situation in the basic linear autonomous case is considerably different (see [10]). More generally (and with more technicalities) one can consider the case in which the boundary operators are divided in two sets, the first containing only first-order boundary operators, the second containing zero-order operators. See [10] for details relative to the linear autonomous case.

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