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Nonlinear Analysis and Applications

Edited by

V. Lakshmikantham
University of Texas at Arlington
Arlington, Texas

MARCEL DEKKER, INC. New York and Basel

NEW RESULTS ON LOCAL EXISTENCE FOR QUASILINEAR PARABOLIC SYSTEMS

Paolo Acquistapace

Scuola Normale Superiore

Pisa, Italy

We look for local existence of continuously differentiable solutions $u = (u^1,\dots,u^N)$ of quasilinear parabolic systems under nonlinear boundary conditions; as a model we take:

$$(1)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{ij=1}^{n} A_{ij}(t,x,u,Du)\cdot D_i D_j u = f(t,x,u,Du), & (t,x)\in[0,T]\times\overline{\Omega}\ (T>0), \\ u(0,x) = \phi(x),\ x\in\overline{\Omega}, & \\ \displaystyle\sum_{i=1}^{n} b_i(t,x,u)D_i u = g(t,x,u), & (t,x)\in[0,T]\times\partial\Omega \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open set with C^2 boundary. We assume:

(2) <u>Ellipticity</u>: $\displaystyle\sum_{hk=1}^{N}\sum_{ij=1}^{n} \mathrm{Re}\, A_{ij}^{hk}(t,x,u,p)\xi_i\xi_j\eta^h\eta^k \geq \nu|\xi|^2|\eta|^2 \quad (\nu>0)$

$\forall\xi\in\mathbb{R}^n$, $\forall\eta\in\mathbb{R}^N$, $\forall(t,x,u,p)\in[0,T]\times\overline{\Omega}\times\mathbb{C}^N\times\mathbb{C}^{nN}$.

(3) <u>Non-tangentiality</u> : $\mathrm{Im}\, b_i(t,x,u)=0$, $\displaystyle\sum_{i=1}^{n} b_i(t,x,u)\nu_i(x)\neq 0$

$\forall(t,x,u)\in[0,T]\times\partial\Omega\times\mathbb{C}^N$

($\nu(x)$ is the unit normal vector at $x\in\partial\Omega$).

(4) <u>Regularity</u>: A_{ij}^{hk}, f^h, b_i, $\dfrac{\partial b_i}{\partial x_j}$, $\dfrac{\partial b_i}{\partial u^k}$, $\dfrac{\partial g^h}{\partial x_j}$, $\dfrac{\partial g^h}{\partial u^k}$ are C^α in t, C in x, locally Lipschitz in (u,p); b_i, g^h are $C^{\alpha+1/2}$ in t ($\alpha\in]0,1/2[$).

(5) <u>Compatibility</u>: $\phi\in C^1$ and $\displaystyle\sum_{i=1}^{n} b_i(0,x,\phi(x))D_i\phi(x)=g(0,x,\phi(x))\quad \forall x\in\partial\Omega$.

(For simplicity we write $L^p, C, W^{1,p}$ etc. instead of $L^p(\Omega)$, $C(\bar{\Omega})$, $W^{1,p}(\Omega)$, etc. provided no confusion arises).

Our main result is the following:

Theorem 1. Let (2),...,(5) hold; assume in addition $\phi \in C^2$ and $\psi := \sum_{ij=1}^{n} A_{ij}(0,\cdot,\phi,D\phi)\cdot D_i D_j \phi + f(0,\cdot,\phi,D\phi) \in C^{2\alpha}$. Then there exists $t_0 \in]0,T]$ such that (1) has a unique solution $u \in C^1([0,t_0]\times\bar{\Omega}) \cap C([0,t_0], \bigcap_{p<\infty} W^{2,p}(\Omega))$ which satisfies $\frac{\partial u}{\partial t}, \sum_{ij=1}^{n} A_{ij}(\cdot,\cdot,u,Du)\cdot D_i D_j u \in C^{\alpha}([0,t_0], C(\bar{\Omega}))$.

Remark 2. Condition $\psi \in C^{2\alpha}$ is also necessary in order that $\frac{\partial u}{\partial t} \in C^{\alpha}([0,t_0],C)$.

Remark 3. The Hölder exponent α may vary in $]0,1[-\{1/2\}$ without essential modifications in the proof.

Remark 4. A similar result holds for quasilinear parabolic systems of arbitrary order, with the elliptic part satisfying the assumptions of [2], [5]. See [1] for more details.

Remark 5. The compatibility condition $\psi \in C^{2\alpha}$ may be dropped by replacing in Theorem 1 the space $C^{\alpha}([0,t_0],C)$ by a suitable weighted Hölder space; the proof is analogous but much more technical.

Remark 6. Previous local existence results for general quasilinear parabolic systems in variational form are in [6], [4].

The proof of Theorem 1 relies on the usual method of linearization and use of the contraction principle, with in addition a suitable regularization technique. It consists of four main steps:

Step 1. The linear autonomous case: existence, representation and estimates for solutions in $C^{1+\alpha}([0,T],L^p)$ $(p \in]n,\infty[)$.

Step 2. The quasilinear case: local existence in $C^{1+\delta}([0,t_0],L^p)$ $(\delta \in]0,\alpha[, p \in]n,\infty[)$.

Step 3. The linear non-autonomous case: existence in $C^{1+\alpha}([0,T],C)$ by solving a suitable integral equation.

Step 4. The quasilinear case: conclusion of the proof.

Proof of Step 1: consider the linear problem

$$(6)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{ij=1}^{n} A_{ij}(x)\cdot D_i D_j u = f(t,x) \ , \ (t,x)\in[0,T]\times\overline{\Omega}, \\ u(0,x) = \phi(x) \ , \ x\in\overline{\Omega}, \\ \sum\limits_{i=1}^{n} b_i(x) D_i u = g(t,x) \ , \ (t,x)\in[0,T]\times\partial\Omega . \end{cases}$$

Proposition 7. Let $A_{ij}\in C$, $b_i\in C^1$ satisfy (2), (3). Fix $p\in]n,\infty[$ and assume that $\phi\in W^{2,p}$, $f\in C^{\alpha}([0,T],L^p)$, $g\in C^{\alpha}([0,T],W^{1,p})\cap C^{\alpha+1/2}([0,T],L^p)$, with the compatibility conditions $\sum\limits_{i=1}^{n} b_i(x)D_i\phi(x)=g(0,x)$ on $\partial\Omega$ and $\sum\limits_{ij=1}^{n} A_{ij}\cdot D_i D_j\phi+$ $+ f(0,\cdot)\in B^{2\alpha,p}_{\infty}(\Omega)$ (the Besov-Nikolskij space). Then (6) has a unique global solution $u\in C^{\alpha}([0,T],W^{2,p})\cap C^{1+\alpha}([0,T],L^p)$; moreover we have:

$$(7)\quad \|u-\phi\|_{C^{\alpha}(W^{2,p})} + \|u-\phi\|_{C^{1+\alpha}(L^p)} \le M_p \, \Big\| \sum_{ij=1}^{n} A_{ij}\cdot D_i D_j \phi + f(0,\cdot)\Big\|_{B^{2\alpha,p}_{\infty}} +$$

$$+ \ \omega_p(T)\ \{\|f\|_{C^{\alpha}(L^p)} + \|g\|_{C^{\alpha}(W^{1,p})} + \|g\|_{C^{\alpha+1/2}(L^p)}\}\ ,$$

where $\lim\limits_{T\downarrow 0} \omega_p(T) = 0$.

Sketch of the proof (a complete proof is in [7]): a representation formula for u is constructed by means of suitable Dunford integrals, whose convergence follows by Agmon's spectral estimate (see [3]); by direct inspection, such formula yields the solutions of (6). Estimate (7) can be derived similarly.

Remark 8. In the case of Dirichlet boundary conditions the datum g has to be taken in $C^{\alpha}([0,T],W^{2,p})\cap C^{\alpha+1}([0,T],L^{p})$, and some more care is required since Agmon's estimate no more guarantees the convergence of the Dunford integrals involved. However the same result follows by writing $u=g+(u-g)$ and representing $u-g$ as above: see [7] for details. The same applies in the situation of Remark 4 when zero order boundary operators occur (see [1]).

Proof of Step 2 : going back to problem (1), the following result holds:

Proposition 9. Suppose (2),...,(5) hold. Fix $p\in]n,\infty[$, $\delta\in]0,\alpha[$; let $\phi\in W^{2,p}$ and assume $\psi\in B_{\infty}^{2\delta,p}$. Then there exists $t_0\in]0,T]$ (depending on δ,p) such that (1) has a unique solution $u\in C^{\delta}([0,t_0],W^{2,p})\cap C^{1+\delta}([0,t_0],L^{p})$.

Sketch of the proof: set $E_{\delta}(T):=C^{\delta}([0,T],W^{2,p})\cap C^{1+\delta}([0,T],L^{p})$; by interpolation $E_{\delta}(T)\subsetneq C^{\delta+1/2}([0,T],W^{1,p})$. Consider also for $M>0$ the ball $B_{M,\delta,T}:=\{v\in E_{\delta}(T):v(0,\cdot)=\phi,\ \|v-\phi\|_{E_{\delta}(T)}\leq M\}$. We linearize problem (1) by taking, for any $v\in B_{M,\delta,T}$, the linear autonomous problem:

$$(8)\quad \begin{cases} \dfrac{\partial u}{\partial t}-\sum\limits_{ij=1}^{n} A_{ij}(0,x,\phi,D\phi)\cdot D_iD_ju=f(t,x,v,Dv)-\sum\limits_{ij=1}^{n}[A_{ij}(0,x,\phi,D\phi)- \\ \qquad\qquad -A_{ij}(t,x,v,Dv)]\cdot D_iD_jv=:F(t,x), \\ \qquad\qquad (t,x)\in[0,T]\times\overline{\Omega}, \\ u(0,x)=\phi(x)\ ,\ x\in\overline{\Omega}, \\ \sum\limits_{i=1}^{n} b_i(0,x,\phi)D_i\phi=g(t,x,v)+\sum\limits_{i=1}^{n}[b_i(0,x,\phi)-b_i(t,x,v)]D_iv \\ \qquad\qquad =:G(t,x),\ (t,x)\in[0,T]\times\partial\Omega. \end{cases}$$

By Prop. 7, there exists a unique $u\in E_{\delta}(T)$ satisfying (8); moreover by (7) we get:

$$(9)\qquad \|u-\phi\|_{E_{\delta}(T)}\leq M_p\|\psi\|_{B_{\infty}^{2\delta,p}}+C(M)\omega_p(T).$$

Denote by S the map $v \to u$, by (9) we see that if $M > M_p \|\psi\|_{B^{2\delta,p}_\infty}$ and $T =: t_0$ is suitably small, then $S(B_{M,\delta,t_0}) \subseteq B_{M,\delta,t_0}$ and again (7) easily implies that S is a contraction. Its fixed point $u \in B_{M,\delta,t_0}$ then solves (8).

Proof of Step 3: consider the linear non-autonomous problem

$$(10)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{ij=1}^{n} A_{ij}(t,x)\cdot D_i D_j u = f(t,x)\ , & (t,x) \in [0,T] \times \overline{\Omega}, \\ u(0,x) = \phi(x)\ , & x \in \overline{\Omega}, \\ \sum\limits_{i=1}^{n} b_i(0,x) D_i u = g(t,x)\ , & (t,x) \in [0,T] \times \partial\Omega . \end{cases}$$

Proposition 10. Let $A_{ij} \in C^\alpha([0,T],C)$, $b_i \in C^\alpha([0,T],C^1) \cap C^{\alpha+1/2}([0,T],C)$ be such that (2), (3) hold. Fix $p \in]n,\infty[$ and assume that $\phi \in W^{2,p}$, $f \in C^\alpha([0,T],L^p)$, $g \in C^\alpha([0,T],W^{1,p}) \cap C^{\alpha+1/2}([0,T],L^p)$ with the compatibility conditions $\sum_{i=1}^{n} b_i(0,x) D_i \phi(x) = g(0,x)$ on $\partial\Omega$ and $\zeta := \sum_{ij=1}^{n} A_{ij}(0,\cdot)\cdot D_i D_j \phi + f(0,\cdot) \in B^{2\alpha,p}_\infty$. Then (10) has a unique global solution $u \in C^\alpha([0,T],W^{2,p}) \cap C^{1+\alpha}([0,T],L^p)$. If, moreover, $\phi \in C^2$, $f \in C^\alpha([0,T],C)$, $g \in C^\alpha([0,T],C^1) \cap C^{\alpha+1/2}([0,T],C)$ and $\zeta \in C^{2\alpha}$, then we get $\frac{\partial u}{\partial t}$, $\sum_{ij=1}^{n} A_{ij}\cdot D_i D_j u \in C^\alpha([0,T],C)$.

Sketch of the proof (a complete proof is in [1]): Prop. 9 yields a unique local solution $u \in C^\delta([0,t_0], W^{2,p}) \cap C^{1+\delta}([0,t_0],L^p)$ of (10) $(\delta \in]0,\alpha[)$. Now for any solution of (10) in some interval $[0,r]$, the following integral equation for $v := \sum_{ij=1}^{n} A_{ij}\cdot D_i D_j u$ must hold:

$$(11)\qquad v(t,\cdot) + \int_0^t K(t,s)v(s,\cdot)ds = L(t,\phi,f,g),\ t \in [0,r],$$

where $K(t,s)$ is a known integrable kernel and $L(\cdot,\phi,f,g)$ is a known function in $C^\alpha([0,r],L^p)$. As a consequence we get an a priori bound for $\|v\|_{C^\alpha(L^p)}$ which leads to global existence. Next, as $v \in C^\alpha(L^p)$ we see by Schauder's

Theorem that $u\in C^{\alpha}([0,T],W^{2,p})\cap C^{1+\alpha}([0,T],L^{p})$. Finally, the further regularity on the data implies $L(\cdot,\phi,f,g)\in C^{\alpha}([0,T],C)$; thus (11) can be solved in $C^{\alpha}([0,r],C)$ and the conclusion follows.

Proof of Step 4: we go back to problem (1). Fix $\delta\in]0,\alpha[$, $p>2n/(1/2-(\alpha-\delta))$; then $\phi\in W^{2,p}$, $\psi\in B^{2\alpha,p}_{\infty}$. Let u be the local solution of (1) (Prop. 9): by interpolation, we get $u\in C^{\alpha}([0,t_0],C^1)\cap C^{\alpha+1/2}([0,t_0],C)$. Hence we can rewrite (1) as a problem of type (10) with $A_{ij}\in C^{\alpha}([0,t_0],C)$, $b_i\in C^{\alpha}([0,t_0],C^1)\cap C^{\alpha+1/2}([0,t_0],C)$, $\phi\in C^2$, $f\in C^{\alpha}([0,t_0],C)$, $g\in C^{\alpha}([0,t_0],C^1)$ $C^{\alpha+1/2}([0,t_0],C)$. As $\zeta\equiv\psi\in C^{2\alpha}$, the last part of Prop. 10 yields $\sum_{ij=1}^{n} A_{ij}(\cdot,\cdot,u,Du)\cdot D_iD_ju\in$ $\in C^{\alpha}([0,t_0],C)$, hence the some is true for $\frac{\partial u}{\partial t}$. The proof of Theorem 1 is complete.

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