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Periodic Orbits of Tonelli Lagrangian Systems

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To Ana

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Preface

The aim of this thesis, as the title suggests, is the study of periodic orbits in Lagrangian dynamics. The class of Lagrangian systems that we shall consider is given by the so-called Tonelli Lagrangian functions on closed (i.e. compact and with empty boundary) configuration spaces. These are Lagrangian functions that, when restricted to the fibers of the tangent bundle of the configuration space, have positive definite Hessian and superlinear growth. The Tonelli class is particularly important in Lagrangian dynamics: in fact, whenever a Lagrangian function is fiberwise convex, its Legendre transform defines a diffeomorphism between the tangent and cotangent bundles of the configuration space if and only if the Lagrangian function belongs to the Tonelli class. In other words, the Tonelli Lagrangians constitute the broadest family of fiberwise convex Lagrangian functions for which the Lagrangian-Hamiltonian duality, given by the Legendre transform, occurs. Furthermore, the Tonelli assumptions imply existence and regularity results for action minimizing orbits joining two given points on the configuration space. Most of the Lagrangian functions that are of interest in mathematical physics turn out to be Tonelli.

By the Lagrangian-Hamiltonian duality, all the results that we obtain for Tonelli Lagrangian systems can be rephrased in the Hamiltonian formulation as results on Tonelli Hamiltonian systems on the cotangent bundles of closed configuration spaces. Here, a Hamiltonian is called Tonelli when it is the dual of a Tonelli Lagrangian, and it turns out that this is equivalent to asking that the Hamiltonian, when restricted to the fibers of the cotangent bundle, has a positive definite Hessian and superlinear growth, in complete analogy with the Lagrangian case.

The motivation of our work comes from a celebrated conjecture formulated in the 1980s by Charles Cameron Conley. This conjecture, in its original form, states that every 1-periodic Hamiltonian system on the standard symplectic torus has infinitely many integer periodic orbits. The conjecture was soon confirmed, in 1984 by Conley and Zehnder [CZ], for Hamiltonian systems with only non-degenerate orbits. The significantly more complicated general case was proved by Hingston [Hi] in 2004. The existence of infinitely many periodic orbits has also been established for Hamiltonian systems on closed symplectically aspherical manifolds (i.e. closed symplectic manifolds whose symplectic form and first Chern class are cohomologically annihilated by pull-back with elements of the second homotopy group of the manifold). This was done in 1992 by Salamon and Zehnder [SZ] under a non-degeneracy assumption on the periodic orbits, and in 2006 by Ginzburg [Gi] in full generality. Other related results are contained in [FS, GG, Gü, Sc2, Vi2]. Nowadays, people refer to the statement about the existence of infinitely many periodic orbits of Hamiltonian systems in a certain setting as the *Conley conjecture* in that setting.

The main result of this thesis (theorem V.9) confirms the Conley conjecture for Tonelli Hamiltonian systems on the cotangent bundle of closed configuration spaces. It might be more appropriate, due to our approach, to describe the result in the Lagrangian formulation: we shall prove that the Euler-Lagrange system of a smooth 1-periodic Tonelli Lagrangian, over a closed configuration space, admits infinitely many integer periodic solutions with a priori bounded mean action. Moreover, if the system under consideration admits only finitely many 1-periodic solutions, then it must admit periodic orbits of arbitrarily high period (and a priori bounded mean action as before). Here, the mean action which we are referring to is the usual Lagrangian action divided by the period of the involved periodic orbit.

The existence of infinitely many integer periodic orbits, in the particular case of fiberwise quadratic Lagrangian functions with the configuration space given by a torus, was proved, in 2000, by Long [Lo]. More recently, in 2008, Lu [Lu] extended Long's result in many directions, in particular to a general closed configuration space. Other related results are contained in [LL2, LW1, LW2]. These proofs are based on the Morse theory for the Lagrangian action functional, combined with the iteration theory for the Conley-Zehnder index, a symplectic invariant of periodic orbits known to coincide with the Morse index. This approach is far from being immediately extendable to the general Tonelli case. In fact, a functional setting in which the action functional of Tonelli Lagrangians is both regular and satisfies the Palais-Smale condition, the minimum requirements for a critical-point theoretical treatment, is not known.

In 2006, Abbondandolo and Figalli [AF] showed how to apply Morse theory to the Tonelli action functional by means of convex quadratic modifications of the involved Lagrangian. Basically, their idea consists of modifying the Tonelli Lagrangian outside a given neighborhood of the zero section of the tangent bundle, making the modified Lagrangian fiberwise quadratic there. With a suitable a-priori estimate, they proved that the two Lagrangian functions, the original Tonelli one and the modified one, share the same periodic orbits on a given action sublevel, provided the modification was performed sufficiently far from the zero section. This allows the application of Morse theoretic methods to the action functional of the modified Lagrangian, for which a suitable functional setting is well-known, in order to prove the existence of periodic orbits of the Lagrangian system associated to the original Tonelli Lagrangian. In particular, if the loop space of the configuration space has rich homology (for instance whenever the configuration space is simply connected), this analysis allows to assert the existence of an infinite sequence of 1-periodic orbits with action that goes to infinity.

As for Long's one, the approach of convex quadratic modifications is not immediately applicable to give a proof of the Conley conjecture: in fact, the above mentioned a priori estimate and consequently all the forthcoming strategy work in a prescribed period, while, in order to prove the existence of infinitely many periodic orbits, we need to look for orbits with different, possibly arbitrarily high, periods. Our recipe for the proof of the Conley conjecture is a combination of these two approaches.

In a first part of the thesis, **chapter III**, we consider the class of Lagrangian functions on close configuration spaces that have uniformly positive definite fiberwise Hessian and fiberwise quadratic growth (throughout the manuscript, we informally refer to this class as to the class of convex quadratic growth Lagrangians). These Lagrangians are in particular Tonelli, and their action functional is well known to be C^1 on the space of $W^{1,2}$ loops and to satisfy the Palais-Smale condition there. This is sufficient to apply most of the results of Morse theory, for instance the Morse inequalities. However, it is not enough to obtain information on the local homology groups of the periodic orbits (that are, with the terminology of Chang [Ch], the critical groups of the functional). In fact, a characterization of the local homology groups is provided by the Morse lemma together with the theory developed by Gromoll and Meyer [GM], but both require to deal with a C^2 functional. Motivated by this fact, we develop a discretization technique for the Lagrangian action that is a generalization of the classical broken geodesics approximation of the path space, described for instance in [Mi, section 16] or [Kl, section A.1]. Roughly speaking, what we do consists in restricting the action functional on the spaces of continuous loops that are broken smooth solutions of the Euler-Lagrange system of the involved Lagrangian. As the number of portions in which the loop is broken (i.e. the discretization pass) grows, we obtain bigger and bigger loop spaces that, in some sense, approximate the $W^{1,2}$ loop space. We prove that each of these spaces is a finite dimensional manifold over which the action functional is smooth, and its closed sublevels are compact (provided the discretization pass is big enough), implying the Palais-Smale condition. Moreover, we show that the critical points of the unrestricted action functional (namely the periodic orbits) correspond to the critical points of the restricted action functional and have the same local homology groups (again, up to choosing a sufficiently big discretization pass). On the one hand, this allows us to recover all the abstract results on the local homology groups that require C^2 regularity. On the other hand, the machinery that we have developed is general, and hopefully may have other applications.

In view of the Conley conjecture, a property that needs to be established is the persistence of the local homology groups of periodic orbits (for systems defined by a convex quadratic growth Lagrangian) under iteration, whenever the iteration leaves the Morse index and nullity pair of the orbit unchanged. In **chapter IV**, this result is obtained as a particular case of a general abstract principle: for an arbitrary C^2 functional defined on an open set of a Hilbert space with a Fredholm Hessian at a given critical point, the local homology groups at the critical point do not change under restriction to a Hilbert subspace provided this subspace is gradient-flow invariant and the Morse index and nullity pair does not change under the restriction. The application to the C^1 Lagrangian action functional is obtained by means of our above discussed discretization technique.

In the last part of the thesis, **chapter V**, we come back to consider the class of Tonelli Lagrangian functions and their convex quadratic modifications. We show that we can still recover a homological invariant for a periodic orbit in a Tonelli system in the following way. The periodic orbit is a critical point of the action functionals associated to all the suitable convex quadratic modifications of the original Tonelli Lagrangian. Each of these action functionals can be discretized as explained above, and it turns out that all the obtained discrete action functionals have the same germ at the considered periodic orbit. Therefore, their local homology at the orbit is an invariant associated to the original Tonelli Lagrangian. Once this preliminary is established, we can prove the Conley conjecture by contradiction: assuming that the Euler-Lagrange system of a Tonelli Lagrangian admits only finitely many integer periodic solutions, we show that there must exist a periodic orbit whose local homology is non-trivial and, satisfying the hypothesis of the theorem proved in chapter IV, persists under iteration, a contradiction to a homological vanishing property that we establish (an extension of a principle for the geodesics action functional, due to Bangert and Klingenberg [BK]).

We have tried to make this thesis accessible even to non-specialists: to this end, we have added two preliminary chapters and an appendix. In **chapter I** we briefly introduce the Lagrangian and Hamiltonian formalisms, trying to emphasize the variational and dynamical systems aspects of these theories. We also define and characterize the Tonelli systems and the index pairs of periodic orbits that we will make use of in the forthcoming chapters. In **chapter II** we introduce the $W^{1,2}$ functional setting for the action functional of convex quadratic growth Lagrangians. In **appendix A** we review the main results of classical Morse theory for functionals defined over infinite dimensional Hilbert manifolds, having possibly degenerate critical points. None of the results contained in this preliminary part are original, and throughout the text we try to provide complete references.

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Chapter I

Lagrangian and Hamiltonian systems

This chapter might be considered as a sort of informal introduction to the subject of Lagrangian and Hamiltonian dynamics, from the point of view that is relevant in this thesis. Everything we will discuss can be found, even if not in this form, in the literature. In section I.1 we give the basic definitions and we briefly review the duality between Lagrangian and Hamiltonian systems. Even though these topics come from mathematical physics, we almost avoid to mention it. We rather insist on the variational and dynamical systems flavors of these theories. In section I.2 we introduce the important class of Tonelli Lagrangians and Tonelli Hamiltonians. The main results about the existence of periodic orbits, that we will prove in chapter V, will be valid for Tonelli systems. Here, we will try to motivate the importance of the Tonelli assumptions, as they naturally give broad families of Lagrangians and Hamiltonians for which the above mentioned duality occurs. In section I.3 we discuss the Conley-Zehnder-Long index pair for periodic orbits of Hamiltonian systems, that is essentially a homotopic invariant for paths in the symplectic group. Our interest in this index pair is due to its relation with the Morse index and nullity of the Lagrangian action, as it will be discussed in section I.4.

I.1 The formalism of classical mechanics

Classical mechanics describes the motion of a mechanical system in a configuration space. In this section we briefly review its language, in particular we introduce the Lagrangian and Hamiltonian formalisms. Of course this note does not attempt to be a comprehensive introduction to the subject. For that purpose, we refer the reader to one of the many textbooks of mathematical physics (e.g. [Ar, AM2]), dynamical systems (e.g. [Fa, HZ]) or symplectic geometry (e.g. [Ca, MS]).

Lagrangian mechanics treats the description of the motion of a mechanical system constrained on a configuration space. For us, the **configuration space** will be a closed manifold M of dimension m. A **Lagrangian** on the configuration space is meant to be a smooth function $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$. In general, the points in the domain $\mathbb{R} \times \mathrm{T}M$ will be denoted by (t, q, v), i.e. $t \in \mathbb{R}$, $v \in \mathrm{T}_q M$. The $\mathbb{R}$-component in the domain of $\mathscr{L}$ must be interpreted as a time dependence, and if the Lagrangian happens to be independent of t it is called **autonomous**. A point (q, v) in the tangent bundle $\mathrm{T}M$ of the configuration space is interpreted in the following way: q gives the position of the mechanical system, while the vector v in the tangent space of q gives the velocity of the mechanical system.

Consider a bounded real interval $[t_0, t_1] \subset \mathbb{R}$. We define the **action functional** $\mathscr{A}^{t_0,t_1}$ associated to the Lagrangian $\mathscr{L}$ as

$$\mathscr{A}^{t_0,t_1}(\gamma) = \int_{t_0}^{t_1} \mathscr{L}(t, \gamma(t), \dot\gamma(t))\, \mathrm{d}t, \tag{I.1}$$

where $\gamma : [t_0, t_1] \to M$. For the moment, we do not discuss a functional setting for the action $\mathscr{A}^{t_0,t_1}$. We just consider it defined on some space of curves γ as above such that the function $t \mapsto \mathscr{L}(t, \gamma(t), \dot\gamma(t))$ is integrable on the interval $[t_0, t_1]$, for instance we might consider $\mathscr{A}^{t_0,t_1}$ defined on the space C^2 curves $\gamma : [t_0, t_1] \to M$. A C^2 map $\Sigma : (-\varepsilon, \varepsilon) \times [t_0, t_1] \to M$ is called a **variation** of γ when $\Sigma(0, \cdot) = \gamma$, $\Sigma(\cdot, t_0) \equiv \gamma(t_0)$ and $\Sigma(\cdot, t_1) \equiv \gamma(t_1)$. The curve γ is a **motion** in the Lagrangian system defined by $\mathscr{L}$ when, for each variation Σ of γ, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathscr{A}^{t_0,t_1}(\Sigma(s, \cdot)) = 0.$$

Namely, a curve on M is a motion when it is an **extremal** of the action functional.

Now, let us fix a finite atlas $\mathfrak{U} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha = 0, ..., u\}$ for the compact manifold M. This has associate atlases for the tangent and cotangent bundle of M, i.e. the atlases $\mathrm{T}\mathfrak{U} = \{\mathrm{T}\phi_\alpha : \mathrm{T}U_\alpha \to \mathbb{R}^m \times \mathbb{R}^m \mid \alpha = 0, ..., u\}$ and $\mathrm{T}^*\mathfrak{U} = \{\mathrm{T}^*\phi_\alpha : \mathrm{T}^*U_\alpha \to \mathbb{R}^m \times (\mathbb{R}^m)^* \mid \alpha = 0, ..., u\}$, where

$$\begin{aligned} \mathrm{T}\phi_\alpha(q, v) &= (\phi_\alpha(q), \mathrm{d}\phi_\alpha(q)v), && \forall q \in M,\ v \in \mathrm{T}_q M,\\ \mathrm{T}^*\phi_\alpha(q, p) &= (\phi_\alpha(q), p \circ \mathrm{d}\phi_\alpha^{-1}(\phi_\alpha(q))), && \forall q \in M,\ p \in \mathrm{T}^*_q M. \end{aligned}$$

We denote the components of the introduced charts by

$$\begin{aligned} \phi_\alpha &= (q^1_\alpha, ..., q^m_\alpha),\\ \mathrm{T}\phi_\alpha &= (q^1_\alpha, ..., q^m_\alpha, v^1_\alpha, ..., v^m_\alpha),\\ \mathrm{T}^*\phi_\alpha &= (q^1_\alpha, ..., q^m_\alpha, p_{\alpha,1}, ..., p_{\alpha,m}). \end{aligned}$$

We define the **fiberwise derivative** of the Lagrangian $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$ at (t, q, v) as the covector $\partial_v \mathscr{L}(t, q, v) \in \mathrm{T}^*_q M$ given in local coordinates as

$$\partial_v \mathscr{L}(t, q, v) = \sum_{j=1}^{m} \frac{\partial \mathscr{L}}{\partial v^j_\alpha}(t, q, v)\, \mathrm{d}q^j_\alpha.$$

Notice that this definition does not depend of the chosen local coordinates, in fact, if $q \in U_\alpha \cap U_\beta$, then

$$\begin{aligned}
&\sum_{j=1}^{m} \frac{\partial \mathscr{L}}{\partial v_\alpha^j}(t,q,v)\, \mathrm{d}q_\alpha^j(q) \\
&\quad = \sum_{j=1}^{m}\sum_{h=1}^{m}\Bigg(\frac{\partial \mathscr{L}}{\partial v_\beta^h}(t,q,v) \underbrace{\frac{\partial v_\beta^h}{\partial v_\alpha^j}(q,v)}_{=\frac{\partial q_\beta^h}{\partial q_\alpha^j}(q)} \mathrm{d}q_\alpha^j(q) + \frac{\partial \mathscr{L}}{\partial q_\beta^h}(t,q,v) \underbrace{\frac{\partial q_\beta^h}{\partial v_\alpha^j}(q,v)}_{=0} \mathrm{d}q_\alpha^j(q)\Bigg) \\
&\quad = \sum_{h=1}^{m} \frac{\partial \mathscr{L}}{\partial v_\beta^h}(t,q,v)\, \mathrm{d}q_\beta^h(q).
\end{aligned}$$

Extremal curves of the action functional can be characterized as follows. Assume $\gamma : [t_0, t_1] \to M$ is a C^2 extremal curve, and consider a subdivision $t_0 = r_0 < r_1 < \ldots < r_n = t_1$ such that the support $\gamma([r_k, r_{k+1}])$ is contained in some coordinate domain U_{α_k} of the atlas $\mathfrak{U}$, for each $k = 0, \ldots, n-1$. Then, for each variation Σ of γ, if we denote by σ the section of $\gamma^*\mathrm{T}M$ given by

$$\sigma(t) = \frac{\partial \Sigma}{\partial s}(0,t), \qquad \forall t \in [t_0, t_1],$$

we have

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \mathscr{A}^{t_0,t_1}(\Sigma(s,\cdot)) \\
&= \sum_{k=0}^{n-1}\sum_{j=1}^{m} \int_{r_k}^{r_{k+1}} \left(\frac{\partial \mathscr{L}}{\partial q_{\alpha_k}^j}(t,\gamma,\dot\gamma)\, \sigma_{\alpha_k}^j(t) + \frac{\partial \mathscr{L}}{\partial v_{\alpha_k}^j}(t,\gamma,\dot\gamma)\, \dot\sigma_{\alpha_k}^j(t) \right) \mathrm{d}t \\
&= \sum_{k=0}^{n-1}\sum_{j=1}^{m} \int_{r_k}^{r_{k+1}} \left(\frac{\partial \mathscr{L}}{\partial q_{\alpha_k}^j}(t,\gamma,\dot\gamma) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial v_{\alpha_k}^j}(t,\gamma,\dot\gamma) \right) \sigma_{\alpha_k}^j(t)\mathrm{d}t \\
&\quad + \sum_{k=0}^{n-1}\Big(\partial_v \mathscr{L}(r_{k+1}, \gamma(r_{k+1}), \dot\gamma(r_{k+1}))\, \sigma(r_{k+1}) - \partial_v \mathscr{L}(r_k, \gamma(r_k), \dot\gamma(r_k))\, \sigma(r_k) \Big) \\
&= \sum_{k=0}^{n-1}\sum_{j=1}^{m} \int_{r_k}^{r_{k+1}} \left(\frac{\partial \mathscr{L}}{\partial q_{\alpha_k}^j}(t,\gamma,\dot\gamma) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial v_{\alpha_k}^j}(t,\gamma,\dot\gamma) \right) \sigma_{\alpha_k}^j(t)\mathrm{d}t,
\end{aligned}$$

where we have adopted the common notation $\sigma_\alpha(t) := \mathrm{d}\phi_\alpha(\gamma(t))\sigma(t)$. By the fundamental lemma of the calculus of variations, the above expression is zero for each variation Σ of the curve γ (that is, for each C^1 section σ of the vector bundle $\gamma^*\mathrm{T}M$) if and only if γ satisfies in local coordinates

$$\text{(I.2)} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial v^j}(t,\gamma,\dot\gamma) - \frac{\partial \mathscr{L}}{\partial q^j}(t,\gamma,\dot\gamma) = 0, \qquad \forall j = 1, \ldots, m.$$

The above system of second order ordinary differential equations is known as the **Euler-Lagrange system** associated to the Lagrangian $\mathscr{L}$. Hence, we have a second characterization of motion curves as solutions of the Euler-Lagrange system (I.2).

Now, for each $(t,q,v)\in\mathbb{R}\times \mathrm{T}M$, consider the quadratic form on T_qM given by

$$w\mapsto \sum_{j,h=1}^{m}\frac{\partial^2\mathscr{L}}{\partial v^j\partial v^h}(t,q,v)w^jw^h,\qquad \forall w=\sum_{j=1}^{m}w^j\frac{\partial}{\partial q^j}\in \mathrm{T}_qM.$$

Notice this quadratic form is independent of the local coordinates used to define it (by the same argument that we gave to show that the fiberwise derivative is intrinsically defined). We say that the Lagrangian $\mathscr{L}$ is **non-degenerate** when the above quadratic form is non-degenerate on the whole domain of $\mathscr{L}$, i.e. for each $(t,q,v)\in\mathbb{R}\times \mathrm{T}M$ and for each nonzero $w\in \mathrm{T}_qM$ there exists $z\in \mathrm{T}_qM$ such that

$$\sum_{j,h=1}^{m}\frac{\partial^2\mathscr{L}}{\partial v^j\partial v^h}(t,q,v)w^jz^h\neq 0.$$

Equivalently, the Lagrangian $\mathscr{L}$ is non-degenerate when the $m\times m$ real matrix

$$\left(\frac{\partial^2\mathscr{L}}{\partial v^j\partial v^h}(t,q,v)\right)_{j,h=1,\dots,m}$$

is invertible for each $(t,q,v)\in\mathbb{R}\times \mathrm{T}M$. If this condition is fulfilled we can put the Euler-Lagrange system (I.2) in normal form as

$$\ddot\gamma^j(t)=\sum_{h=1}^{m}\left[\frac{\partial^2\mathscr{L}}{\partial v\,\partial v}(t,\gamma(t),\dot\gamma(t))\right]^{-1}_{j,h}\left(\frac{\partial\mathscr{L}}{\partial q^h}(t,\gamma(t),\dot\gamma(t))-\sum_{l=1}^{m}\frac{\partial^2\mathscr{L}}{\partial q^l\partial v^h}(t,\gamma(t),\dot\gamma(t))\dot\gamma^l(t)\right),\qquad \forall j=1,\dots,m.$$

In other words, the non-degeneracy condition allows us to define a smooth time-dependent vector field $X_{\mathscr{L}}$ on $\mathrm{T}M$ as

$$X_{\mathscr{L}}(t,q,v)=\sum_{j=1}^{m}\left(v^j\frac{\partial}{\partial q^j}+\sum_{h=1}^{m}\left[\frac{\partial^2\mathscr{L}}{\partial v\,\partial v}(t,q,v)\right]^{-1}_{j,h}\left(\frac{\partial\mathscr{L}}{\partial q^h}(t,q,v)-\sum_{l=1}^{m}\frac{\partial^2\mathscr{L}}{\partial q^l\partial v^h}(t,q,v)v^l\right)\frac{\partial}{\partial v^j}\right),$$

for each $(t,q,v)\in\mathbb{R}\times \mathrm{T}M$. This vector field is called **Euler-Lagrange vector field** associated to $\mathscr{L}$, and its integral curve are precisely the solutions of the Euler-Lagrange system associated to $\mathscr{L}$. By the Cauchy-Lipschitz theorem, $X_{\mathscr{L}}$ can be locally uniquely integrated. In other words, there exists a continuous function $\varepsilon:\mathbb{R}\times \mathrm{T}M\to(0,\infty)$ such that, for every $(t_0,q_0,v_0)\in\mathbb{R}\times \mathrm{T}M$ and for every $t_1\in\mathbb{R}$ with $|t_1-t_0|<\varepsilon(t_0,q_0,v_0)$, there exists a unique smooth solution $\gamma:[t_0,t_1]\to M$ of the Euler-Lagrange system (I.2) with $\gamma(t_0)=q_0$ and $\dot\gamma(t_0)=v_0$. This defines a partial flow $\Phi_{\mathscr{L}}$ on $\mathrm{T}M$, the **Euler-Lagrange flow** associated to $\mathscr{L}$, as

$$\Phi_{\mathscr{L}}^{t_1,t_0}(\gamma(t_0),\dot\gamma(t_0))=(\gamma(t_1),\dot\gamma(t_1)),$$

where $\gamma : [t_0, t_1] \to M$ is a solution of the Euler-Lagrange system. We say that $\Phi_{\mathscr{L}}$ is **global** when, for each $t_0 \in \mathbb{R}$, it defines a function

$$\Phi_{\mathscr{L}}^{\cdot,t_0} : \mathbb{R} \times \mathrm{T}M \to \mathrm{T}M, \qquad \forall t_0 \in \mathbb{R},$$
$$(t,q,v) \mapsto \Phi_{\mathscr{L}}^{t,t_0}(q,v).$$

So far we have recalled the language of the Lagrangian formulation of classical mechanics. As we already stated at the beginning of this section, there is another point of view for describing classical mechanics that is given by the Hamiltonian formulation. Quoting Vladimir Arnold [Ar, page 161], Hamiltonian mechanics is *"geometry in phase space"*. The **phase space** is the ambient of the considered mechanical system, and it has the structure of a symplectic manifold. We will only consider phase spaces that are cotangent bundles over a smooth closed manifold M. In these cases, the Hamiltonian formulation is, in some sense, dual to the Lagrangian one.

A **Hamiltonian** on the cotangent bundle T^*M is meant to be a smooth function $\mathscr{H} : \mathbb{R} \times \mathrm{T}^*M \to \mathbb{R}$. In general, the points in the domain $\mathbb{R} \times \mathrm{T}^*M$ will be denoted by (t,q,p), i.e. $t \in \mathbb{R}$, $p \in \mathrm{T}^*_qM$. As for the Lagrangian case, the $\mathbb{R}$-component in the domain of $\mathscr{H}$ must be interpreted as a time dependence, and if this dependence is missing the Hamiltonian is called **autonomous**. A point (q,p) in the cotangent bundle T^*M is interpreted in the following way: q still gives the position of the mechanical system, while the covector p in the cotangent space of q gives the momentum of the mechanical system. The cotangent bundle T^*M has a **canonical symplectic structure**, that is a two form ω over T^*M that is closed (i.e. $\mathrm{d}\omega = 0$) and nondegenerate (i.e. $\omega(v,\cdot) \neq 0$ for each non-zero $v \in \mathrm{T}(\mathrm{T}^*M)$). This two-form ω can be defined as follows. At first, we define the **Liouville form** of T^*M, that is a one-form λ on T^*M given in local coordinates by

$$\lambda = \sum_{j=1}^{m} p_j \mathrm{d}q^j.$$

It is easy to verify that the above expression gives an intrinsic definition for λ (i.e. a definition that is independent of the chosen local coordinates). Moreover, the Liouville form can also be characterized as the unique one-form λ on T^*M such that, for each one-form μ on M, we have $\mu^*\lambda = \mu$. Then the canonical symplectic form of T^*M is defined as $\omega = -\mathrm{d}\lambda$. In local coordinates we have

$$\omega = \sum_{j=1}^{m} \mathrm{d}q^j \wedge \mathrm{d}p_j,$$

and it is immediate to verify that the above expression gives a non-degenerate two-form, that is clearly closed (being exact by its definition). A Hamiltonian $\mathscr{H}$ as above defines a smooth time-dependent vector field $X_{\mathscr{H}}$ on T^*M given by

$$X_{\mathscr{H}}(t,q,p) \lrcorner\, \omega = \mathrm{d}\left(\mathscr{H}(t,\cdot)\right)(q,p), \qquad \forall (t,q,p) \in \mathbb{R} \times \mathrm{T}^*M,$$

where "$\lrcorner$" stands for the interior product between vectors and forms, i.e. $X_{\mathscr{H}} \lrcorner \omega = \omega(X_{\mathscr{H}}, \cdot)$. In local coordinates we have

$$X_{\mathscr{H}}(t,q,p) = \frac{\partial \mathscr{H}}{\partial p_j}(t,q,p)\,\frac{\partial}{\partial q^j} - \frac{\partial \mathscr{H}}{\partial q^j}(t,q,p)\,\frac{\partial}{\partial p_j}, \qquad \forall (t,q,p) \in \mathbb{R} \times \mathrm{T}^*M.$$

By the Cauchy-Lipschitz theorem, this vector field can be locally integrated around any point of $\mathbb{R} \times \mathrm{T}^*M$, and therefore it defines a partial flow $\Phi_{\mathscr{H}}$ on T^*M as

$$\Phi_{\mathscr{H}}^{t_1,t_0}(\Gamma(t_0)) = (\Gamma(t_1)),$$

where $\Gamma : [t_0, t_1] \to \mathrm{T}^*M$ is an integral curve of $X_{\mathscr{H}}$. If we write Γ as (γ, ρ), where $\gamma : [t_0, t_1] \to M$ and ρ is a section of $\gamma^*\mathrm{T}^*M$, then in local coordinates (γ, ρ) satisfies the **Hamilton system**

$$\text{(I.3)} \qquad \dot{\gamma}^j(t) = \frac{\partial \mathscr{H}}{\partial p_j}(t, \gamma(t), \rho(t)), \qquad \dot{\rho}^j(t) = -\frac{\partial \mathscr{H}}{\partial q^j}(t, \gamma(t), \rho(t)), \qquad \forall j = 1, ..., m.$$

We call $X_{\mathscr{H}}$ the **Hamiltonian vector field** associated to $\mathscr{H}$, and $\Phi_{\mathscr{H}}$ the correspondent **Hamiltonian flow**. The integral curves of $X_{\mathscr{H}}$, or rather their projection onto the base manifold M, are the motions of the Hamiltonian mechanical system defined by $\mathscr{H}$.

Now, consider a Lagrangian function $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$ as before. We define the **Legendre transform** given by $\mathscr{L}$ as the map

$$\text{(I.4)} \qquad \mathrm{Leg}_{\mathscr{L}} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R} \times \mathrm{T}^*M, \qquad \mathrm{Leg}_{\mathscr{L}}(t,q,v) = (t, q, \partial_v \mathscr{L}(t,q,v)).$$

Let us assume that, for the considered Lagrangian $\mathscr{L}$, the Legendre transform is a diffeomorphism of $\mathbb{R} \times \mathrm{T}M$ onto $\mathbb{R} \times \mathrm{T}^*M$ (conditions on $\mathscr{L}$ under which this is true will be discussed in the next section). Notice that this condition implies that $\mathscr{L}$ is non-degenerate. Then we can define the Hamiltonian $\mathscr{H} : \mathbb{R} \times \mathrm{T}^*M \to \mathbb{R}$ **Legendre dual** to the Lagrangian $\mathscr{L}$ as

$$\mathscr{H} \circ \mathrm{Leg}_{\mathscr{L}}(t,q,v) := \partial_v \mathscr{L}(t,q,v)v - \mathscr{L}(t,q,v), \qquad \forall (t,q,v) \in \mathbb{R} \times \mathrm{T}M.$$

Actually, this sets up a duality between the Lagrangian system of $\mathscr{L}$ and the Hamiltonian system of $\mathscr{H}$. In fact, an easy computation shows that

$$\mathrm{d}(\pi_2 \circ \mathrm{Leg}_{\mathscr{L}})(t,q,v)\ (X_{\mathscr{L}}(t,q,v)) = X_{\mathscr{H}} \circ \mathrm{Leg}_{\mathscr{L}}(t,q,v), \qquad \forall (t,q,v) \in \mathbb{R} \times \mathrm{T}M,$$

where $\pi_2 : \mathbb{R} \times \mathrm{T}^*M \to \mathrm{T}^*M$ is the projection onto the second factor of $\mathbb{R} \times \mathrm{T}^*M$. Therefore the Lagrangian and Hamiltonian flows are conjugated by the Legendre transform. In other words, a curve $\gamma : [t_0, t_1] \to M$ is a solution of the Euler-Lagrange system (I.2) if and only if the curve $(\gamma, \rho) : [t_0, t_1] \to \mathrm{T}^*M$, where $\rho(t) := \partial_v \mathscr{L}(t, \gamma(t), \dot{\gamma}(t))$, is a solution of the Hamilton system (I.3).

As for the Lagrangian case, the motion curves of the Hamiltonian system associated to a smooth $\mathscr{H} : \mathbb{R} \times \mathrm{T}^*M \to \mathbb{R}$ admit a variational characterization. If $\Gamma : [t_0, t_1] \to \mathrm{T}^*M$ is a C^2 curve, we define its **Hamiltonian action** as

$$\mathscr{A}^{t_0,t_1}(\Gamma) = \int_{t_0}^{t_1} \Big(\Gamma^*\lambda - \mathscr{H}(t, \Gamma(t)) \Big)\, \mathrm{d}t.$$

An easy computation shows that the solutions of the Hamilton system of $\mathscr{H}$ are precisely the extremal curves of $\mathscr{A}^{t_0,t_1}$. Moreover, this Hamiltonian action is related to the Lagrangian one of equation (I.1) in the following way: if $\mathscr{H}$ and $\mathscr{L}$ are Legendre dual and $\Gamma = (\gamma, \rho) : [t_0, t_1] \to \mathrm{T}^*M$ is a solution of the Hamilton system of $\mathscr{H}$ (so that $\gamma : [t_0, t_1] \to M$ is a solution of the Euler-Lagrange system of $\mathscr{L}$), then the Hamiltonian action of Γ coincides with the Lagrangian action of γ, for

$$\begin{aligned} \int_{t_0}^{t_1} \Big(\Gamma^*\lambda - \mathscr{H}(t, \Gamma(t)) \Big)\, \mathrm{d}t &= \int_{t_0}^{t_1} \Big(\rho(t)[\dot\gamma(t)] - \mathscr{H}(t, \gamma(t), \rho(t)) \Big)\, \mathrm{d}t \\ &= \int_{t_0}^{t_1} \mathscr{L}(t, \gamma(t), \dot\gamma(t))\, \mathrm{d}t. \end{aligned}$$

I.2 Tonelli systems

Let us fix, once for all, a Riemannian metric $\langle \cdot, \cdot \rangle_\cdot$ on the closed manifold M, i.e. for each $q \in M$ we will denote by $\langle \cdot, \cdot \rangle_q$ the Riemannian inner product on T_qM and by $|\cdot|_q$ the corresponding norm. We say that a smooth Lagrangian $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$ is **Tonelli** when

(T1) the fiberwise Hessian of $\mathscr{L}$ is positive definite, i.e.

$$\sum_{j,h=1}^{m} \frac{\partial^2 \mathscr{L}}{\partial v^j \partial v^h}(t, q, v) w^j w^h > 0,$$

for all $(t, q, v) \in \mathbb{R} \times \mathrm{T}M$ and $w = \sum_{j=1}^m w^j \frac{\partial}{\partial q^j} \in \mathrm{T}_qM$ with $w \neq 0$;

(T2) $\mathscr{L}$ is fiberwise superlinear, i.e.

$$\lim_{|v|_q \to \infty} \frac{\mathscr{L}(t, q, v)}{|v|_q} = \infty,$$

for all $(t, q) \in \mathbb{R} \times M$.

Notice that the convexity condition **(T1)** implies that each absolutely continuous[1] curve $\zeta : [t_0, t_1] \to M$ has a well defined action

$$\mathscr{A}^{t_0,t_1}(\zeta) = \int_{t_0}^{t_1} \mathscr{L}(t, \zeta(t), \dot\zeta(t))\, \mathrm{d}t \in \mathbb{R} \cup \{+\infty\}.$$

[1] We recall that absolutely continuous curves can be characterized as those curves having integrable weak derivative.

An absolutely continuous curve $\gamma : [t_0, t_1] \to M$ is an **action minimizer** with respect to the Lagrangian $\mathscr{L}$ when every other absolutely continuous curve $\zeta : [t_0, t_1] \to M$ with the same endpoints of γ satisfies $\mathscr{A}^{t_0,t_1}(\gamma) \leq \mathscr{A}^{t_0,t_1}(\zeta)$. The Tonelli assumptions guarantee the existence and regularity of action minimizers, as given by the following statement that is essentially due to Tonelli (see [Fa, theorem 3.7.2] or [BGH] for a modern treatment).

Theorem I.1 (Existence and regularity of action minimizers). *Let $\mathscr{L}$ be a smooth Tonelli Lagrangian.*

(i) *For each real interval $[t_0, t_1] \subset \mathbb{R}$ and for all $q_0, q_1 \in M$ there exists an action minimizer (with respect to $\mathscr{L}$) $\gamma : [t_0, t_1] \to M$ with $\gamma(t_0) = q_0$ and $\gamma(t_1) = q_1$.*

(ii) *Every action minimizer (with respect to $\mathscr{L}$) is a smooth solution of the Euler-Lagrange system of $\mathscr{L}$.*

Now, let us consider a fiberwise convex (but not necessarily Tonelli) smooth Lagrangian $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$. We want to show how the geometric requirement of the Legendre transform $\mathrm{Leg}_{\mathscr{L}}$ being a diffeomorphism can be translated into fiberwise requirements for $\mathscr{L}$. First of all, notice that the Legendre transform $\mathrm{Leg}_{\mathscr{L}}$ is a fiber preserving smooth map between $\mathbb{R} \times \mathrm{T}M$ and $\mathbb{R} \times \mathrm{T}^*M$. Hence, asking that it is a diffeomorphism is equivalent to ask that, for each $(t, q) \in \mathbb{R} \times M$, its fiberwise restriction

$$\partial_v \mathscr{L}(t, q, \cdot) = \mathrm{d}(\mathscr{L}|_{\{t\} \times \mathrm{T}_q M}) : \mathrm{T}_q M \to \mathrm{T}^*_q M$$

is a diffeomorphism. Hence, all we have to do is characterize convex functions on $\mathbb{R}^m$ whose differential is a diffeomorphism onto $(\mathbb{R}^m)^*$.

Proposition I.2. *Let $L : \mathbb{R}^m \to \mathbb{R}$ be a convex smooth function. Then $\mathrm{d}L : \mathbb{R}^m \to (\mathbb{R}^m)^*$ is a diffeomorphism if and only if L is superlinear, meaning*

$$\lim_{|v| \to \infty} \frac{L(v)}{|v|} = \infty,$$

and its Hessian is positive definite.

Proof. Assume that L is superlinear with positive definite Hessian. Consider an arbitrary $p_0 \in (\mathbb{R}^m)^*$ and define $L^{p_0} : \mathbb{R}^m \to \mathbb{R}$ as $L^{p_0}(v) = L(v) - p_0(v)$. This function is superlinear, as well as L, hence it reaches its minimum at some $v_0 \in \mathbb{R}^m$. In particular $\mathrm{d}L^{p_0}(v_0) = 0$, and $\mathrm{d}L(v_0) = p_0$. This shows that $\mathrm{d}L$ is surjective. Moreover, every $v \in \mathbb{R}^m$ such that $\mathrm{d}L(v) = p_0$ must be a critical point of L^{p_0}. Since the hessian of L^{p_0} is positive definite, the function is strictly convex and it cannot have critical points other than v_0, and therefore $\mathrm{d}L$ is bijective. By the positivity of the Hessian, we can apply the inverse function theorem to assert that $\mathrm{d}L$ is a bijective local diffeomorphism, i.e. a global diffeomorphism.

Conversely, assume that L is convex and $\mathrm{d}L$ is a diffeomorphism. Hence the Hessian of L must be non-degenerate and, by the convexity assumption on L, even positive definite. Moreover, since L is convex, we have

$$L(v) - L(v_0) \geq \mathrm{d}L(v_0)[v - v_0], \qquad \forall v_0, v \in \mathbb{R}^m. \tag{I.5}$$

For each real constant $k > 0$ we define the compact set $S_k := \{v \in \mathbb{R}^m \mid |\mathrm{d}L(v)| = k\}$. For each $v \in \mathbb{R}^m$, there exists a unique $v_0 = v_0(v) \in \mathbb{R}^m$ such that

$$\mathrm{d}L(v_0) = \frac{k}{|v|} \langle v, \cdot \rangle \,,$$

where we have denoted by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbb{R}^m$. Notice that $\mathrm{d}L(v_0) \in S_k$ and $\mathrm{d}L(v_0)v = k|v|$. Hence, by (I.5), for each $v \in \mathbb{R}^m$ we have

$$L(v) \geq \mathrm{d}L(v_0(v))v + L(v_0(v)) - \mathrm{d}L(v_0(v))v_0(v) \geq k|v| + \inf_{w \in S_k} \{L(w) - \mathrm{d}L(w)w\} \,.$$

This shows that L is superlinear. ■

For a smooth function L as in the above statement we can define a Legendre-dual smooth function $H : (\mathbb{R}^m)^* \to \mathbb{R}$ by

$$H \circ \mathrm{d}L(v) = \mathrm{d}L(v)v - L(v).$$

Proposition I.3. *Consider a smooth function $L : \mathbb{R}^m \to \mathbb{R}$ that is superlinear with positive definite Hessian, and its Legendre dual function $H : (\mathbb{R}^m)^* \to \mathbb{R}$. Then*

(i) *L and H satisfy the* **Fenchel relation** *$L(v) + H(p) \geq p(v)$, and the equality holds true if and only if $p = \mathrm{d}L(v)$;*

(ii) *$\mathrm{d}H = (\mathrm{d}L)^{-1} : (\mathbb{R}^m)^* \to \mathbb{R}^m \simeq (\mathbb{R}^m)^{**}$;*

(iii) *H is superlinear with positive definite Hessian.*

Proof. Take arbitrary $v \in \mathbb{R}^m$ and $p \in (\mathbb{R}^m)^*$. By the assumptions on L, $\mathrm{d}L$ is a diffeomorphism and in particular $p = \mathrm{d}L(w)$ for some $w \in \mathbb{R}^m$. Therefore

$$\begin{aligned} L(v) + H(p) - p(v) &= L(v) - H(\mathrm{d}L(w)) - \mathrm{d}L(w)v \\ &= L(v) - L(w) - \mathrm{d}L(w)(v - w) \geq 0, \end{aligned}$$

where the last inequality follows by the convexity of L. Moreover, since L is strictly convex, equality holds if and only if $v = w$, that is if and only if $p = \mathrm{d}L(v)$. This proves (i). By the Fenchel relation, for each $p_0, p_1 \in (\mathbb{R}^m)^*$, we get

$$\begin{aligned} H((1-\lambda)p_0 + \lambda p_1) &= \max_{v \in \mathbb{R}^m} \{(1-\lambda)p_0(v) + \lambda p_1(v) - L(v)\} \\ &\leq \max_{v \in \mathbb{R}^m} \{(1-\lambda)p_0(v) - (1-\lambda)L(v)\} + \max_{v \in \mathbb{R}^m} \{\lambda p_1(v) - \lambda L(v)\} \\ &\leq (1-\lambda)H(p_0) + \lambda H(p_1), \end{aligned}$$

therefore H is convex. Now, let us fix an arbitrary $p_0 \in (\mathbb{R}^m)^*$. Then, for $v_0 = (\mathrm{d}L)^{-1}(p_0)$, we have $p_0(v_0) = H(p_0) + L(v_0)$. By the Fenchel relation, for any $p \in (\mathbb{R}^m)^*$, we have

$$p(v_0) \leq H(p) + L(v_0) = H(p) - H(p_0) + p_0(v_0).$$

Hence, for any $p \in (\mathbb{R}^m)^*$ we have $(p - p_0)v_0 \leq H(p) - H(p_0)$, and this is possible if and only if $v_0 = \mathrm{d}H(p_0)$ (we are making the canonical identification between $(\mathbb{R}^m)^{**}$ and $\mathbb{R}^m$). This proves (ii), and in particular that $\mathrm{d}H$ is a diffeomorphism. Applying proposition I.2 to H in place of L we obtain that H is superlinear with positive definite Hessian, hence (iii) holds. ■

If $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$ is a fiberwise convex Lagrangian, by proposition I.2, the Legendre transform $\mathrm{Leg}_{\mathscr{L}}$ is a diffeomorphism if and only if $\mathscr{L}$ is a Tonelli Lagrangian. Now, we say that a Hamiltonian $\mathscr{H} : \mathbb{R} \times \mathrm{T}^*M \to \mathbb{R}$ is **Tonelli** when

(T1') the fiberwise Hessian of $\mathscr{H}$ is positive definite, i.e.

$$\sum_{j,h=1}^{m} \frac{\partial^2 \mathscr{H}}{\partial p_j \partial p_h}(t,q,p) r_j r_h > 0,$$

for all $(t,q,p) \in \mathbb{R} \times \mathrm{T}^*M$ and $r = \sum_{j=1}^m r_j \mathrm{d}q^j \in \mathrm{T}^*_q M$ with $r \neq 0$;

(T2') $\mathscr{H}$ is fiberwise superlinear, i.e.

$$\lim_{|p|_q \to \infty} \frac{\mathscr{H}(t,q,p)}{|p|_q} = \infty,$$

for all $(t,q) \in \mathbb{R} \times M$ (here, by a common abuse of notation, we write $|\cdot|_q$ also for the norm in $\mathrm{T}^*_q M$ induced by the Riemannian metric of M).

By proposition I.3, the Tonelli Hamiltonians are precisely the Hamiltonian functions that are dual to Tonelli Lagrangians. Namely, the Legendre duality sets up a one to one correspondence

$$\left\{ \begin{array}{c} \mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R} \\ \text{Tonelli} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathscr{H} : \mathbb{R} \times \mathrm{T}^*M \to \mathbb{R} \\ \text{Tonelli} \end{array} \right\}.$$

This discussion should have motivated the importance of the Tonelli class in the study of Lagrangian and Hamiltonian systems.

Remark I.1 (Uniform fiberwise superlinearity). It turns out that the Tonelli assumptions, both for Lagrangians and Hamiltonians, imply that the fiberwise superlinearity of conditions **(T2)** and **(T2')** is uniform[2], which means that the limits in **(T2)** and **(T2')** are satisfied uniformly in $(t,q) \in \mathbb{R} \times M$. In fact, if $\mathscr{L}$ is a

[2]For a non-compact configuration space M, we would only get locally uniform fiberwise superlinearity.

Tonelli Lagrangian with dual Tonelli Hamiltonian $\mathscr{H}$, by the Fenchel relation (cf. proposition I.3(i)) we have

$$\begin{aligned}\mathscr{L}(t,q,v) &\geq \max_{|p|_q\leq k}\{p(v)-\mathscr{H}(t,q,p)\}\\ &\geq \max_{|p|_q\leq k}\{p(v)\}-\max_{|p|_q\leq k}\{\mathscr{H}(t,q,p)\}\\ &\geq k|v|_q-\max\left\{\mathscr{H}(t',q',p')\,|\,(t',q',p')\in\mathbb{R}\times\mathrm{T}^*M,\ |p'|_{q'}\leq k\right\},\end{aligned}$$

for each $k\in\mathbb{N}$ and $(t,q,v)\in\mathbb{R}\times\mathrm{T}M$. Analogously

$$\mathscr{H}(t,q,p)\geq k|p|_q-\max\left\{\mathscr{L}(t',q',v')\,|\,(t',q',v')\in\mathbb{R}\times\mathrm{T}M,\ |v'|_{q'}\leq k\right\},$$

for each $k\in\mathbb{N}$ and $(t,q,p)\in\mathbb{R}\times\mathrm{T}^*M$. ■

I.3 The Conley-Zehnder-Long index pair

The Hamiltonian flow $\Phi_{\mathscr{H}}$ defines symplectic transformations of T^*M. In fact, assume that there exists an open set $(t_0,t_1)\times U\subset\mathbb{R}\times\mathrm{T}^*M$ such that $\Phi_{\mathscr{H}}^{t,t'}|_U$ is a well defined diffeomorphism onto its image, for each $t,t'\in(t_0,t_1)$. Then, this diffeomorphism is **symplectic**, meaning

$$(\Phi_{\mathscr{H}}^{t,t'})^*\omega=\omega.$$

This follows elementarily by the Cartan formula, since $(\Phi_{\mathscr{H}}^{t',t'})^*\omega=(\mathrm{id}_U)^*\omega=\omega$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_{\mathscr{H}}^{t,t'})^*\omega=(\Phi_{\mathscr{H}}^{t,t'})^*\,\mathrm{Lie}_{X_{\mathscr{H}}}\omega=(\Phi_{\mathscr{H}}^{t,t'})^*\big(\mathrm{d}(\underbrace{X_{\mathscr{H}}\lrcorner\omega}_{=\mathrm{d}\mathscr{H}})+X_{\mathscr{H}}\lrcorner\underbrace{\mathrm{d}\omega}_{=0}\big)=0,$$

where $\mathrm{Lie}_{X_{\mathscr{H}}}$ denotes the Lie derivative with respect to $X_{\mathscr{H}}$. The symplectic diffeomorphisms that are constructed by means of a Hamiltonian flow as above are called **Hamiltonian diffeomorphisms**.

Now, assume that the Hamiltonian $\mathscr{H}$ is **1-periodic in time**, namely that it is a function of the form $\mathscr{H}:\mathbb{R}/\mathbb{Z}\times\mathrm{T}^*M\to\mathbb{R}$. For $n\in\mathbb{N}$, we consider a periodic orbit $\Gamma:\mathbb{R}/n\mathbb{Z}\to\mathrm{T}^*M$ of the Hamiltonian system of $\mathscr{H}$, i.e. $\Gamma(t)=\Phi_{\mathscr{H}}^{t,0}(\Gamma(0))=\Gamma(t+n)$ for each $t\in\mathbb{R}$. Then the differential $\mathrm{d}\Phi_{\mathscr{H}}^{n,0}(\Gamma(0))$ is a linear symplectic automorphism of the symplectic vector space $(\mathrm{T}_{\Gamma(0)}\mathrm{T}^*M,\omega)$. Moreover, by means of a symplectic trivialization of the pull-back vector bundle $\Gamma^*\mathrm{T}\mathrm{T}^*M$, we can view the map $t\mapsto\mathrm{d}\Phi_{\mathscr{H}}^{t,0}(\Gamma(0))$ as a path in the symplectic group $\mathrm{Sp}(2m)$, which is the group of linear symplectic transformations of $\mathbb{R}^{2m}$. In this section we want to recall the definition and the properties of a classical invariant for paths in the symplectic group, and then we show how to associate this invariant to periodic orbits Γ independently of the chosen symplectic trivialization of $\Gamma^*\mathrm{T}\mathrm{T}^*M$.

We recall that the **standard symplectic structure** of $\mathbb{R}^{2m}$ is given by the skew-symmetric bilinear form $\omega_0:\mathbb{R}^{2m}\wedge\mathbb{R}^{2m}\to\mathbb{R}$ defined by

$$\omega_0(v,w)=\langle\mathrm{J}_0v,w\rangle\,,\qquad\forall v,w\in\mathbb{R}^{2m}.$$

Here, $\langle\cdot,\cdot\rangle$ is the standard inner product of $\mathbb{R}^{2m}$, while J_0 is the **standard complex structure** of $\mathbb{R}^{2m}$, given in matrix form by

$$\mathrm{J}_0 = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}.$$

The **symplectic group** $\mathrm{Sp}(2m)$ is defined as the subgroup of $\mathrm{GL}(2m)$ given by the automorphisms A such that $A^*\omega = \omega$, or equivalently

$$\mathrm{Sp}(2m) = \{A \in \mathrm{GL}(2m) \mid A^*\mathrm{J}_0 A = \mathrm{J}_0\},$$

where A^* denote the adjoint of A (i.e. the transpose of the matrix defining A). We denote by $\mathrm{Sp}^0(2m)$ the subgroup of $\mathrm{Sp}(2m)$ consisting of automorphisms having 1 as an eigenvalue, and by $\mathrm{Sp}^*(2m)$ the complementary subspace of $\mathrm{Sp}(2m)$, i.e.

$$\mathrm{Sp}^0(2m) = \{A \in \mathrm{Sp}(2m) \mid \det(A - I) = 0\}, \qquad \mathrm{Sp}^*(2m) = \mathrm{Sp}(2m) \setminus \mathrm{Sp}^0(2m).$$

The space $\mathrm{Sp}^*(2m)$ is the disjoint union of the two connected components $\mathrm{Sp}^+(2m)$ and $\mathrm{Sp}^-(2m)$ given by

$$\mathrm{Sp}^\pm(2m) = \{A \in \mathrm{Sp}(2m) \mid \pm\det(A - I) > 0\}.$$

Every symplectic automorphism A admits a unique **polar decomposition** $A = PQ$, where $P = (AA^*)^{1/2}$ is a symmetric and positive definite automorphism in $\mathrm{Sp}(2m)$, while $Q = (AA^*)^{-1/2}A \in \mathrm{Sp}(2m) \cap \mathrm{O}(2m)$. Moreover, the map $A \mapsto (AA^*)^{-1/2}A$ is a retraction $r : \mathrm{Sp}(2m) \to \mathrm{Sp}(2m) \cap \mathrm{O}(2m)$ coming from the deformation retraction $r_t : \mathrm{Sp}(2m) \to \mathrm{Sp}(2m)$ given by

$$r_t(A) = (AA^*)^{-t/2}A. \tag{I.6}$$

In particular r is a homotopy equivalence. Moreover, if we consider the general linear group of $\mathbb{C}^m$ as a subgroup of $\mathrm{GL}(2m)$ via the embedding

$$X + iY \longmapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}, \qquad \forall X + iY \in \mathrm{GL}(m, \mathbb{C}),$$

then $\mathrm{Sp}(2m) \cap \mathrm{O}(2m)$ is identified with the unitary group $\mathrm{U}(m)$. We denote by $\det_{\mathbb{C}}(M) : \mathrm{Sp}(2m) \cap \mathrm{O}(2m) \to S^1 \subset \mathbb{C}$ the complex determinant function, that is

$$\det_{\mathbb{C}}\left(\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right) := \det(X + iY), \qquad \forall M = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} \in \mathrm{Sp}(2m) \cap \mathrm{O}(2m).$$

Let $\mathscr{P}$ be the space of continuous paths $\Psi : [0,1] \to \mathrm{Sp}(2m)$ such that $\Psi(0) = \mathrm{I}$. This space is the disjoint union of the subspaces $\mathscr{P}^*$ and $\mathscr{P}^0$ given by the Ψ's such that $\Psi(1) \in \mathrm{Sp}^*(2m)$ and $\Psi(1) \in \mathrm{Sp}^0(2m)$ respectively. We want to introduce an index for paths that belongs to $\mathscr{P}$. At first, we do it for paths $\Psi \in \mathscr{P}^*$. Since

$\mathrm{Sp}^+(2m)$ and $\mathrm{Sp}^-(2m)$ are path connected, there exist a (non unique) path $\tilde\Psi : [1,2] \to \mathrm{Sp}^*(2m)$ such that $\tilde\Psi(1) = \Psi(1)$ and $\tilde\Psi(2) \in \{W^+, W^-\}$, where

$$W^+ = -I \in \mathrm{Sp}^+(2m),$$

$$W^- = \left[\begin{array}{cccc|cccc} 2 & & & & & & & \\ & -1 & & & & & & \\ & & \ddots & & & & & \\ & & & -1 & & & & \\ \hline & & & & 1/2 & & & \\ & & & & & -1 & & \\ & & & & & & \ddots & \\ & & & & & & & -1 \end{array}\right] \in \mathrm{Sp}^-(2m).$$

We denote by $\Psi \bullet \tilde\Psi : [0,2] \to \mathrm{Sp}(2m)$ the concatenation of the paths Ψ and $\tilde\Psi$, i.e.

$$(\Psi \bullet \tilde\Psi)(t) = \begin{cases} \Psi(t), & t \in [0,1], \\ \tilde\Psi(t), & t \in [1,2]. \end{cases}$$

Since

$$\det\nolimits_{\mathbb{C}}^2 \circ r \circ (\Psi \bullet \tilde\Psi)(0) = \det\nolimits_{\mathbb{C}}^2 \circ r(\mathrm{I}) = 1 = \det\nolimits_{\mathbb{C}}^2 \circ r(W^\pm) = \det\nolimits_{\mathbb{C}}^2 \circ r \circ (\Psi \bullet \tilde\Psi)(2),$$

the composition $\det_{\mathbb{C}}^2 \circ r \circ (\Psi \bullet \tilde\Psi)$ can be considered as a map of the form $\mathbb{R}/2\mathbb{Z} \to S^1$, namely a map between two topological circles. We define the **Conley-Zehnder index** $\iota(\Psi)$ of the path $\Psi \in \mathscr{P}^*$ as the Brouwer degree of the above composition, i.e.

$$\iota(\Psi) := \deg\left(\det\nolimits_{\mathbb{C}}^2 \circ r \circ (\Psi \bullet \tilde\Psi)\right) \in \mathbb{Z}.$$

It turns out that the integer computed by the above formula is independent of the chosen extension $\tilde\Psi$ of the path Ψ, hence $\iota(\Psi)$ is well defined. We refer the reader to the celebrated paper[3] of Salamon and Zehnder [SZ, section 3] for the proofs of this fact and of the following statement.

Proposition I.4. *The Conley-Zehnder index satisfies the following properties:*

(Naturality) *For each path $\Psi \in \mathscr{P}^*$ and for each symplectic automorphism $A \in \mathrm{Sp}(2m)$ we have $\iota(\Psi) = \iota(A^{-1}\Psi A)$.*

(Homotopy) *If $\Psi : [0,1]\times[0,1] \to \mathrm{Sp}(2m)$ is a homotopy in $\mathscr{P}^*$, i.e. Ψ is continuous and $\Psi(s,\cdot) \in \mathscr{P}^*$ for each $s \in [0,1]$, then $\iota(\Psi(s,\cdot)) = \iota(\Psi(0,\cdot))$ for each $s \in [0,1]$.*

[3] In [SZ], Salamon and Zehnder call *"Maslov index"* the Conley-Zehnder index of a path in the symplectic group. In the classical literature, people call Maslov index an integer invariant for loops in the symplectic group (and also for paths in the Lagrangian Grassmannian of $(\mathbb{R}^{2m}, \omega_0)$).

Following Long [Lo2, page 143], The Conley-Zehnder index of paths in $\mathscr{P}^0$ is defined by lower semicontinuity extension as

$$\iota(\Psi) := \liminf_{\substack{\Psi' \to \Psi \\ \Psi' \in \mathscr{P}^*}} \iota(\Psi'), \qquad \forall \Psi \in \mathscr{P}^0. \tag{I.7}$$

For each $\Psi \in \mathscr{P}$, we denote by $\nu(\Psi)$ the algebraic multiplicity of 1 as an eigenvalue of $\Psi(1)$, i.e.

$$\nu(\Psi) := \dim \ker(\Psi(1) - \mathrm{I}).$$

Notice that the paths Ψ with non-zero $\nu(\Psi)$ are precisely the ones in $\mathscr{P}^0$. We refer to the pair of integers $(\iota(\Psi), \nu(\Psi))$ as to the **Conley-Zehnder-Long index pair** of $\Psi \in \mathscr{P}$. By (I.7), it is immediate that the Conley Zehnder index satisfies the naturality property of proposition I.4 even for paths $\Psi \in \mathscr{P}^0$. The general homotopy invariance property is the following (see [Lo2, page 145] for a proof).

Proposition I.5 (Homotopy). *Let $\Psi : [0,1] \times [0,1] \to \mathrm{Sp}(2m)$ be a continuous map such that $\Psi(s, \cdot) \in \mathscr{P}$ and $\nu(\Psi(s, \cdot)) = \nu(\Psi(0, \cdot))$ for each $s \in [0,1]$. Then $\iota(\Psi(s, \cdot)) = \iota(\Psi(0, \cdot))$ for each $s \in [0,1]$.*

Remark I.2. So far we have introduced the Conley-Zehnder-Long index pair for paths parametrized on $[0,1]$. This invariant, by its definition, is independent of the parametrization of the involved path. Hence, we can also define the Conley-Zehnder-Long index pair of the path $\Psi : [a,b] \to \mathrm{Sp}(2m)$, with $\Psi(a) = \mathrm{I}$, as the index pair of a reparametrization of Ψ on the interval $[0,1]$. ■

Before going back to consider periodic orbits of a Hamiltonian system, we want to discuss the behavior of the Conley-Zehnder-Long index pair under "iteration" of a path. If $\Psi \in \mathscr{P}$, we define its n^{th} **iteration** as the path $\Psi^{[n]} : [0,n] \to \mathrm{Sp}(2m)$ defined by

$$\Psi^{[n]}(j+t) := \Psi(t) \underbrace{\Psi(1)...\Psi(1)}_{j \text{ times}}, \qquad \forall j \in \{0, ..., n-1\}, \ t \in [0,1].$$

The following statement, due to Liu and Long (see [LL, theorem 1.1] for a proof), shows that the the Conley-Zehnder index grows linearly with respect to the iteration exponent $n \in \mathbb{N}$.

Theorem I.6 (Iteration inequality). *For each $\Psi \in \mathscr{P}$, the following limit exists and it is finite:*

$$\widehat{\iota}(\Psi) = \lim_{n \to \infty} \frac{\iota(\Psi^{[n]})}{n}. \tag{I.8}$$

Moreover, the following inequality holds:

$$n\,\widehat{\iota}(\Psi) - m \le \iota(\Psi^{[n]}) \le n\,\widehat{\iota}(\Psi) + m - \nu(\Psi^{[n]}), \qquad \forall n \in \mathbb{N}.$$

The real number $\widehat{\iota}(\Psi)$ given by (I.8) is called **mean Conley-Zehnder index** of Ψ.

Now, let us consider a 1-periodic Hamiltonian $\mathscr{H} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}^*M \to \mathbb{R}$. Following Abbondandolo and Schwarz [AS], we want to show how to assign a Conley-Zehder-Long index pair to a periodic orbit $\Gamma : \mathbb{R}/n\mathbb{Z} \to \mathrm{T}^*M$ of the Hamiltonian system of $\mathscr{H}$. We will assume that the pull-back bundle $\gamma^*\mathrm{T}M$ is trivial, where $\gamma = \tau^* \circ \Gamma : \mathbb{R}/n\mathbb{Z} \to M$ and $\tau^* : \mathrm{T}^*M \to M$ is the projection of the cotangent bundle onto the base manifold. This assumption is verified if the orbit γ is contractible, or whenever the manifold M is orientable (in fact, in this latter case, $\gamma^*\mathrm{T}M$ would be an oriented vector bundle over the circle and therefore it must be trivial). For the general case, that is not relevant for our purposes, we refer the reader to Weber [We].

In order to simplify the notation, let us assume that $n = 1$, so that Γ is a 1-periodic orbit (then everything will go through word-by-word in any integer period, see remark I.2). We denote by $\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M$ the vertical subbundle of $\mathrm{T}\mathrm{T}^*M$, i.e. $\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M = \ker(\mathrm{T}\tau^*)$. It is straightforward to verify that this vector bundle is isomorphic to the pull-back of $\mathrm{T}M$ by the map τ^*. Therefore, the pull-back bundle $\Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M$ is trivial, being

$$\Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M \simeq \Gamma^*(\tau^*)^*\mathrm{T}M \simeq \gamma^*\mathrm{T}M.$$

Consider an almost complex structure J on T^*M **compatible** with the canonical symplectic structure ω of T^*M. This means precisely that $\omega(\cdot, \mathrm{J}\cdot)$ is a Riemannian metric on T^*M. With respect to this metric, we can fix an orthogonal trivialization

$$\tilde{\phi} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^m \xrightarrow{\simeq} \Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M,$$

and then we can extend $\tilde{\phi}$ to a trivialization

$$\phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m} \xrightarrow{\simeq} \Gamma^*\mathrm{T}\mathrm{T}^*M,$$

as

$$\phi(t, \cdot) = (-\mathrm{J} \circ \tilde{\phi}(t, \cdot) \circ \mathrm{J}_0) \oplus \tilde{\phi}(t, \cdot), \qquad \forall t \in \mathbb{R}/\mathbb{Z},$$

where J_0 is the standard complex structure on $\mathbb{R}^{2m}$ introduced previously. By construction, the obtained trivialization ϕ sends the vertical Lagrangian subspace $\mathbb{V}^m := \{\mathbf{0}\} \times \mathbb{R}^m \subset \mathbb{R}^{2m}$ diffeomorphically onto the vertical subbundle $\Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M$, i.e.

$$\phi|_{\mathbb{R}/\mathbb{Z}\times\mathbb{V}^m} : \mathbb{R}/\mathbb{Z} \times \mathbb{V}^m \xrightarrow{\simeq} \Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M. \tag{I.9}$$

Let $e_1, ..., e_m, f_1, ..., f_m$ be the standard symplectic basis of $\mathbb{R}^{2m}$, which means that $e_1, ..., e_m$ is an orthonormal base of $\mathbb{R}^m \times \{0\}$ and $f_1, ..., f_m$ is an orthonormal base of $\mathbb{V}^m = \{0\} \times \mathbb{R}^m$ with $f_j = \mathrm{J}_0 e_j$ for each $j \in \{1, ..., m\}$. Then, for each $t \in \mathbb{R}/\mathbb{Z}$, if we put

$$\tilde{f}_j := \phi(t, f_j) = \tilde{\phi}(t, f_j), \qquad \tilde{e}_j := \phi(t, e_j) = -J \circ \tilde{\phi}(t, \cdot) \circ J_0 e_j = -J\tilde{f}_j, \qquad \forall j \in \{1, ..., m\},$$

it is straightforward to verify that $\tilde{e}_1, ..., \tilde{e}_m, \tilde{f}_1, ..., \tilde{f}_m$ is a symplectic basis of the tangent space $\mathrm{T}_{\Gamma(t)}\mathrm{T}^*M$, which means

$$\omega(\tilde{e}_j, \tilde{e}_h) = \omega(\tilde{f}_j, \tilde{f}_h) = 0, \qquad \omega(\tilde{e}_j, \tilde{f}_h) = \begin{cases} 1 & j = h, \\ 0 & j \neq h, \end{cases}$$
$$\forall j, h \in \{1, ..., m\}.$$

This shows that the trivialization ϕ is **symplectic**, in the sense that

$$\phi(t, \cdot)^* \omega = \omega_0, \qquad \forall t \in \mathbb{R}/\mathbb{Z}, \tag{I.10}$$

and therefore the differential of the Hamiltonian flow along Γ defines a path $\Gamma_\phi : [0,1] \to \mathrm{Sp}(2m)$ as

$$\Gamma_\phi(t) := \phi(t, \cdot)^{-1} \circ \mathrm{d}\Phi^{t,0}_{\mathscr{H}}(\Gamma(0)) \circ \phi(0, \cdot), \qquad \forall t \in [0,1].$$

Notice that $\Gamma_\phi(0) = \mathrm{I}$, hence $\Gamma_\phi \in \mathscr{P}$. We will use this path to define the Conley-Zender index pair of the periodic orbit Γ. At first, we need some preliminaries.

We denote by $\mathrm{Sp}(2m, \mathbb{V}^m)$ the subgroup of $\mathrm{Sp}(2m)$ consisting of those automorphisms that preserve the vertical Lagrangian subspace $\mathbb{V}^m \subset \mathbb{R}^{2m}$, i.e.

$$\begin{aligned} \mathrm{Sp}(2m, \mathbb{V}^m) &= \{A \in \mathrm{Sp}(2m) \mid A\mathbb{V}^m = \mathbb{V}^m\} \\ &= \left\{ \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \;\middle|\; A_1^* A_3 = \mathrm{I},\ A_1^* A_2 = A_2^* A_1 \right\} \end{aligned}$$

Lemma I.7. *Consider the inclusion* $j : \mathrm{Sp}(2m, \mathbb{V}^m) \hookrightarrow \mathrm{Sp}(2m)$ *and the induced fundamental group homomorphism* $\pi_1(j) : \pi_1(\mathrm{Sp}(2m, \mathbb{V}^m)) \to \pi_1(\mathrm{Sp}(2m))$*. Then* $\pi_1(j)$ *is the zero homomorphism.*

Proof. The deformation retraction of equation (I.6) restricts to a deformation retraction of $\mathrm{Sp}(2m, \mathbb{V}^m)$ onto

$$\mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m) = \left\{ \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \;\middle|\; R \in \mathrm{O}(m) \right\}.$$

Therefore, we know that in the following diagram of inclusions

$$\begin{array}{ccc} \mathrm{Sp}(2m, \mathbb{V}^m) & \xhookrightarrow{\quad j \quad} & \mathrm{Sp}(2m) \\ \sim \uparrow & & \sim \uparrow \\ \mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m) & \xhookrightarrow{\quad h \quad} & \mathrm{U}(m) \end{array}$$

the vertical arrows are homotopy equivalences, and in order to conclude we just need to show that h induces the zero homomorphism between fundamental groups. The complex determinant $\det_{\mathbb{C}}$ induces a fundamental group isomorphism

$$\pi_1(\det_{\mathbb{C}}) : \pi_1(U(m)) \xrightarrow{\simeq} \pi_1(S^1) \simeq \mathbb{Z}.$$

Then, consider an arbitrary $[\vartheta] \in \pi_1(\mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m))$, i.e.

$$\vartheta : (\mathbb{R}/\mathbb{Z}, 0) \to (\mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m), \mathrm{I}).$$

Since $\det_{\mathbb{C}} \circ \vartheta \equiv 1$, we have $\pi_1(\det_{\mathbb{C}}) \circ \pi_1(h)[\vartheta] = [\det_{\mathbb{C}} \circ \vartheta] = 0$, and we conclude $\pi_1(h)[\vartheta] = 0$. ■

Lemma I.8. *The Conley-Zehnder-Long index pair* $(\iota(\Gamma_\phi), \nu(\Gamma_\phi))$ *is independent of the trivialization* ϕ*, provided* ϕ *satisfies* (I.9) *and* (I.10).

Proof. Let $\psi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m} \xrightarrow{\simeq} \Gamma^*\mathrm{T}\mathrm{T}^*M$ be another symplectic trivialization that satisfies (I.9). Consider the loop $\vartheta : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2m, \mathbb{V}^m)$ defined by

$$\vartheta(t) = \phi(t, \cdot)^{-1} \circ \psi(t, \cdot), \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

Notice that

$$\Gamma_\psi(t) = \vartheta(t)^{-1} \circ \Gamma_\phi(t) \circ \vartheta(0), \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

By lemma I.7 there exists a homotopy $\Theta : [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2m)$ such that $\Theta(0, \cdot) = \vartheta$, $\Theta(s, 0) = \vartheta(0)$ for each $s \in [0,1]$, and $\Theta(1, \cdot) \equiv \vartheta(0)$. Hence we can build a homotopy $\Omega : [0,1] \times [0,1] \to \mathrm{Sp}(2m)$ as

$$\Omega(s, t) = \Theta(s, t)^{-1} \circ \Gamma_\phi(t) \circ \vartheta(0), \qquad \forall (s, t) \in [0,1] \times [0,1],$$

such that $\Omega(0, \cdot) = \Gamma_\psi$, $\Omega(1, \cdot) = \vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)$, $\Omega(s, 0) = \Gamma_\psi(0)$ and $\Omega(s, 1) = \Gamma_\psi(1)$ for each $s \in [0,1]$. Since the homotopy Ω fixes the endpoints of the homotoped path, we have that $\nu(\Omega(s, \cdot)) = \nu(\Gamma_\psi)$ for each $s \in [0,1]$, hence by the homotopy invariance of the Conley-Zehnder index (proposition I.5) we conclude

$$\iota(\Gamma_\psi) = \iota(\Omega(s, \cdot)), \qquad \nu(\Gamma_\psi) = \nu(\Omega(s, \cdot)), \qquad \forall s \in [0,1],$$

and in particular

$$\iota(\Gamma_\psi) = \iota\left(\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)\right), \qquad \nu(\Gamma_\psi) = \nu\left(\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)\right).$$

Finally, since $\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)$ and Γ_ϕ are conjugated paths we immediately get

$$\nu(\Gamma_\psi) = \nu\left(\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)\right) = \nu(\Gamma_\phi),$$

and, by the naturality property of the Conley-Zehnder index, we conclude

$$\iota(\Gamma_\psi) = \iota\left(\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)\right) = \iota(\Gamma_\phi).$$ ■

By the above lemma we can define the **Conley-Zehnder-Long index pair** of the periodic orbit Γ as the integers $(\iota(\mathscr{H}, \Gamma), \nu(\mathscr{H}, \Gamma)) := (\iota(\Gamma_\phi), \nu(\Gamma_\phi))$, where ϕ is any trivialization satisfying (I.9) and (I.10).

As we did before for paths in the symplectic group, we define the n^{th} **iteration** of the periodic orbit $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathrm{T}^*M$ as the closed curve $\Gamma^{[n]} : \mathbb{R}/n\mathbb{Z} \to \mathrm{T}^*M$ defined by the composition of Γ with the n-fold covering map of the circle $\mathbb{R}/n\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, i.e.

$$\Gamma^{[n]}(j+t) = \Gamma(t), \qquad \forall j \in \{0,...,n-1\}\,,\ t \in \mathbb{R}/\mathbb{Z}.$$

If ϕ is any trivialization satisfying (I.10) and (I.9), we denote by $\Gamma_\phi^{[n]} : [0,n] \to \mathrm{Sp}(2m)$ the path given by

$$\Gamma_\phi^{[n]}(t) = \phi(t,\cdot)^{-1} \circ \mathrm{d}\Phi_{\mathscr{H}}^{t,0} \circ \phi(0,\cdot), \qquad \forall t \in [0,n].$$

With this notation, the Conley-Zehnder-Long index pair of $\Gamma^{[n]}$ is

$$(\iota(\mathscr{H},\Gamma^{[n]}), \nu(\mathscr{H},\Gamma^{[n]})) = (\iota(\Gamma_\phi^{[n]}), \nu(\Gamma_\phi^{[n]})).$$

Since $\mathscr{H}$ is 1-periodic, we have $\Phi_{\mathscr{H}}^{t_1,t_0} = \Phi_{\mathscr{H}}^{t_1-1,t_0-1}$ for each $t_0, t_1 \in \mathbb{R}$ such that the left hand side is defined and, for each $j \in \mathbb{N}$ and $t \in \mathbb{R}/\mathbb{Z}$, we have

$$\begin{aligned}\mathrm{d}\Phi_{\mathscr{H}}^{j+t,0}(\Gamma(0)) &= \mathrm{d}\Phi_{\mathscr{H}}^{j+t,j}(\Gamma(0))\,\mathrm{d}\Phi_{\mathscr{H}}^{j,j-1}(\Gamma(0))\,\mathrm{d}\Phi_{\mathscr{H}}^{j-1,j-2}(\Gamma(0))...\mathrm{d}\Phi_{\mathscr{H}}^{1,0}(\Gamma(0))\\ &= \mathrm{d}\Phi_{\mathscr{H}}^{t,0}(\Gamma(0))\,\underbrace{\mathrm{d}\Phi_{\mathscr{H}}^{1,0}(\Gamma(0))...\mathrm{d}\Phi_{\mathscr{H}}^{1,0}(\Gamma(0))}_{j\ \text{times}}.\end{aligned}$$

Therefore, for each $j \in \{0,...,n-1\}$ and $t \in [0,1]$, we have $\Gamma_\phi^{[n]}(j+t) = \Gamma_\phi(t)\Gamma_\phi(1)^n$. By theorem I.6 we readily obtain the following **iteration inequality** for the Conley-Zehnder-Long index pair of a periodic orbit Γ:

$$n\,\widehat{\iota}(\mathscr{H},\Gamma) - m \le \iota(\mathscr{H},\Gamma^{[n]}) \le n\,\widehat{\iota}(\mathscr{H},\Gamma) + m - \nu(\mathscr{H},\Gamma^{[n]}), \qquad \forall n \in \mathbb{N}, \tag{I.11}$$

where $\widehat{\iota}(\Gamma) := \widehat{\iota}(\Gamma_\phi) \in [0,\infty)$ is called the **mean Conley-Zehnder index** of Γ.

I.4 The Morse index and nullity pair

Let us consider a 1-periodic Tonelli Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ with associated action $\mathscr{A}$ defined on some space of closed curves $\gamma : \mathbb{R}/\mathbb{Z} \to M$ by

$$\mathscr{A}(\gamma) = \int_0^1 \mathscr{L}(t,\gamma(t),\dot\gamma(t))\,\mathrm{d}t.$$

Assume furthermore that a closed curve $\gamma : \mathbb{R}/\mathbb{Z} \to M$ is an extremal of this functional, namely it is a smooth periodic solution of the Euler-Lagrange system of $\mathscr{L}$. We want to define a quadratic form that, euristically, represents the second variation of the functional $\mathscr{A}$ at the extremal γ. Hence, let us consider a smooth section σ of the pull-back bundle $\gamma^*\mathrm{T}M$. Notice that σ is 1-periodic, being a map of the form $\sigma : \mathbb{R}/\mathbb{Z} \to \gamma^*\mathrm{T}M$. We can use this section to define a homotopy $\Sigma : (-\varepsilon,\varepsilon) \times \mathbb{R}/\mathbb{Z} \to M$ of γ as

$$\Sigma(s,t) := \exp_{\gamma(t)}(s\,\sigma(t)), \qquad \forall (s,t) \in (-\varepsilon,\varepsilon) \times \mathbb{R}/\mathbb{Z},$$

where $\varepsilon > 0$ is a sufficiently small real constant and exp is the exponential map associated to an arbitrary Riemannian metric on M. Then, the section σ is obtained by differentiating the homotopy Σ in the s direction at $s = 0$, for

$$\frac{\partial}{\partial s}\bigg|_{s=0} \Sigma(s,t) = \mathrm{d}\exp_{\gamma(t)}(0)(\sigma(t)) = \sigma(t) \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

If we consider a finite atlas $\mathfrak{U} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha = 0, ..., u\}$ for M, as in section I.1, we can find a subdivision $0 = r_0 < r_1 < ... < r_n = 1$ such that the support $\gamma([r_k, r_{k+1}])$ is contained in some coordinate domain U_{α_k}, for each $k = 0, ..., n-1$. We define

$$\begin{aligned}
\mathscr{B}_\gamma(\sigma) :=& \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} \mathscr{A}(\Sigma(s,t)) \\
=& \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \sum_{k=0}^{n-1}\sum_{j=1}^{m} \int_{r_k}^{r_{k+1}} \left[\frac{\partial \mathscr{L}}{\partial v^j_{\alpha_k}} \left(t, \Sigma(s,t), \frac{\partial \Sigma}{\partial t}(s,t)\right) \frac{\partial^2 \Sigma^j_{\alpha_k}}{\partial s\, \partial t}(s,t) \right. \\
&+ \left. \frac{\partial \mathscr{L}}{\partial q^j_{\alpha_k}} \left(t, \Sigma(s,t), \frac{\partial \Sigma}{\partial t}(s,t)\right) \frac{\partial \Sigma^j_{\alpha_k}}{\partial s}(s,t) \right] \mathrm{d}t \\
=& \sum_{k=0}^{n-1}\sum_{j,h=1}^{m} \int_{r_k}^{r_{k+1}} \left[\frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial v^j_{\alpha_k}}(t,\gamma,\dot\gamma)\dot\sigma^j_{\alpha_k}\dot\sigma^h_{\alpha_k} \right. \\
&+ \left. 2\frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma)\sigma^j_{\alpha_k}\dot\sigma^h_{\alpha_k} + \frac{\partial^2 \mathscr{L}}{\partial q^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma)\sigma^j_{\alpha_k}\sigma^h_{\alpha_k} \right] \mathrm{d}t \in \mathbb{R}.
\end{aligned}$$

Notice that $\mathscr{B}_\gamma(\sigma)$ is independent of the particular choice of the homotopy Σ, and for each $r \in \mathbb{R}$ we have $\mathscr{B}_\gamma(r\sigma) = r^2\mathscr{B}_\gamma(\sigma)$. This shows that $\mathscr{B}_\gamma$ is a well defined quadratic form, and by polarization we can define the symmetric bilinear form

$$\begin{aligned}
\text{(I.12)}\quad \mathscr{B}_\gamma(\sigma,\xi) :=& \frac{1}{2}[\mathscr{B}_\gamma(\sigma+\xi) - \mathscr{B}_\gamma(\sigma-\xi)] \\
=& \sum_{k=0}^{n-1}\sum_{j,h=1}^{m} \int_{r_k}^{r_{k+1}} \left[\frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial v^j_{\alpha_k}}(t,\gamma,\dot\gamma)\dot\sigma^j_{\alpha_k}\dot\xi^h_{\alpha_k} + \frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma)\sigma^j_{\alpha_k}\dot\xi^h_{\alpha_k} \right. \\
&+ \left. \frac{\partial^2 \mathscr{L}}{\partial q^h_{\alpha_k} \partial v^j_{\alpha_k}}(t,\gamma,\dot\gamma)\dot\sigma^j_{\alpha_k}\xi^h_{\alpha_k} + \frac{\partial^2 \mathscr{L}}{\partial q^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma)\sigma^j_{\alpha_k}\xi^h_{\alpha_k} \right] \mathrm{d}t,
\end{aligned}$$

where σ and ξ are smooth sections of $\gamma^*\mathrm{T}M$. The above expression still makes sense if we only require that σ and ξ have $W^{1,2}$ regularity[4], and actually $\mathscr{B}_\gamma$ extends to a continuous symmetric bilinear form

$$\mathscr{B}_\gamma : W^{1,2}(\gamma^*\mathrm{T}M) \otimes W^{1,2}(\gamma^*\mathrm{T}M) \to \mathbb{R},$$

where we denoted by $W^{1,2}(\gamma^*\mathrm{T}M)$ the Hilbert space of $W^{1,2}$ sections of $\gamma^*\mathrm{T}M$.

[4]See section II.1 for the background on $W^{1,2}$ sections of the pull-back bundle $\gamma^*\mathrm{T}M$.

We define the **nullity** $\nu(\mathscr{B}_\gamma)$ of $\mathscr{B}_\gamma$ as the dimension of its null-space, i.e.

$$\nu(\mathscr{B}_\gamma) := \dim \left\{ \zeta \in W^{1,2}(\gamma^*\mathrm{T}M) \,|\, \mathscr{B}_\gamma(\zeta, \cdot) = \mathbf{0} \right\}.$$

Then, we define the **Morse index** $\iota(\mathscr{B}_\gamma)$ of $\mathscr{B}_\gamma$ as the supremum of the dimensions of the vector subspaces $\boldsymbol{W} \subseteq W^{1,2}(\gamma^*\mathrm{T}M)$ such that the restriction $\mathscr{B}_\gamma|_{\boldsymbol{W}\otimes\boldsymbol{W}}$ is negative definite. Notice that, for now, we are not claiming that $\iota(\mathscr{B}_\gamma)$ is the Morse index of the functional $\mathscr{A}$ at γ. In fact we have not introduced any functional setting for the functional $\mathscr{A}$, and it does not even make sense to speak about its critical points.

The Morse index $\iota(\mathscr{B}_\gamma)$ and the nullity $\nu(\mathscr{B}_\gamma)$, a priori, might be infinite. However, this is not the case as long as the involved Lagrangian $\mathscr{L}$ is Tonelli, as a consequence of the following fundamental result.

Theorem I.9 (Index theorem). *Let $\mathscr{H} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ be the Hamiltonian that is Legendre-dual to $\mathscr{L}$, and $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathrm{T}^*M$ the Hamiltonian periodic orbit corresponding to γ, i.e. $\Gamma(t) = (\gamma(t), \partial_v\mathscr{L}(t, \gamma(t), \dot\gamma(t)))$ for each $t \in \mathbb{R}/\mathbb{Z}$. If $\gamma^*\mathrm{T}M$ is a trivial bundle, then the Morse index and nullity pair of $\mathscr{B}_\gamma$ concides with the Conley-Zehnder-Long index pair of Γ, i.e.*

$$(\iota(\mathscr{B}_\gamma), \nu(\mathscr{B}_\gamma)) = (\iota(\mathscr{H}, \Gamma), \nu(\mathscr{H}, \Gamma)).$$

The above result, in this generality, is due to Long and An [LA, Lo2], and it is based on previous results by Duistermaat [Du] and Viterbo [Vi] (see also [Ab] for a functional analytic proof). The choice of period 1 that we have made so far is only for syntactic convenience, but all the arguments go through for periodic orbits of any period that is a multiple of the period of the considered Lagrangian function $\mathscr{L}$.

Chapter II

Functional setting for the Lagrangian action

In order to apply the machinery of critical point theory to study periodic solutions of Tonelli Lagrangian systems on a closed manifold M, one is tempted to find a nice functional setting for the Lagrangian action functional: a suitable free loop space on M with a (infinite dimensional) manifold structure, over which the action functional is regular, say at least C^1, and such that its sublevels satisfy some sort of compactness, such as the Palais-Smale condition. For the special case of the geodesic action functional, a suitable free loop space is known to be the Hilbert manifold $W^{1,2}(\mathbb{R}/\mathbb{Z}; M)$, see [Kl, chapter 1]. In the general case of a Tonelli Lagrangian, a functional setting over which the action functional fulfills the above requirements is not known. However, $W^{1,2}(\mathbb{R}/\mathbb{Z}; M)$ is still a good choice for the subclass of Lagrangians that are fiberwise convex with fiberwise quadratic-growth: the fact that a Lagrangian grows at most quadratically guarantees that the action functional is regular on the $W^{1,2}$ free loop space, while the fact that the Lagrangian grows at least quadratically implies the Palais-Smale condition.

In section II.1 we recall how to construct a Hilbert manifold structure on the $W^{1,2}$ free loop space. In section II.2 we briefly discuss some topological properties of this free loop space that will be useful in the forthcoming chapters. Finally, in section II.3 we introduce the class of "convex quadratic-growth" Lagrangians and, following [Ben] and [AS2], we discuss the regularity of the action functional and we prove that it satisfies the Palais-Smale condition.

II.1 A Hilbert manifold structure for the free loop space

Let M be a smooth closed manifold of dimension m. The **free loop space** of M is, loosely speaking, a set of maps from the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ to the manifold M. It may be the set of all the continuous maps $C^0(\mathbb{T}; M)$, as it is common in topology, or the smaller set of all the smooth maps $C^\infty(\mathbb{T};M)$, as it is often considered in differential geometry. Each of these spaces is endowed with the corresponding topology, i.e. the one induced by the uniform convergence for $C^0(\mathbb{T};M)$ and the one induced by the uniform convergence of all the derivatives for $C^\infty(\mathbb{T};M)$. In nonlinear analysis it is rather useful to consider an intermediate loop space, as $W^{1,2}(\mathbb{T};M)$, that has the structure of an infinite dimensional Hilbert manifold.

A possible way to define $W^{1,2}(\mathbb{T};M)$ is the following. By the Whitney embedding theorem, there exists a smooth embedding $M \hookrightarrow \mathbb{R}^{2m+1}$, and we define

$$W^{1,2}(\mathbb{T};M) := \left\{\gamma \in W^{1,2}(\mathbb{T};\mathbb{R}^{2m+1}) \mid \gamma(\mathbb{T}) \subset M\right\},$$

where $W^{1,2}(\mathbb{T};\mathbb{R}^{2m+1})$ is, as usual, the Sobolev space of L^2 maps $\gamma : \mathbb{T} \to \mathbb{R}^{2m+1}$ whose (weak) derivative is in L^2. Notice that the above is a good definition, since by the Sobolev embedding theorem we know that the maps in $W^{1,2}(\mathbb{T};\mathbb{R}^{2m+1})$ are continuous, and therefore it makes sense to talk about the image of these maps.

The maps in $W^{1,2}(\mathbb{T};M)$ can be approximated by smooth maps in the topology of the uniform convergence. In fact, consider a tubular neighborhood $N \subseteq \mathbb{R}^{2m+1}$ of M, with retraction $r : N \to M$. If $\gamma \in W^{1,2}(\mathbb{T};M)$, by the density of $C^\infty(\mathbb{T};\mathbb{R}^{2m+1})$ in $W^{1,2}(\mathbb{T};\mathbb{R}^{2m+1})$ there exists a sequence $\{\gamma_k\} \subset C^\infty(\mathbb{T};\mathbb{R}^{2m+1})$ that converges to γ in $W^{1,2}(\mathbb{T};\mathbb{R}^{2m+1})$ and in particular uniformly. Hence the sequence $\{r \circ \gamma_k\} \subset C^\infty(\mathbb{T};M)$ still converges to γ uniformly.

Now, following [Kl, chapter 1], we want to show that $W^{1,2}(\mathbb{T};M)$ can be endowed with the structure of Hilbert manifold. At first, we put a Riemannian metric $\langle\cdot,\cdot\rangle_\cdot$ on M, for instance the pull-back of the flat metric on $\mathbb{R}^{2m+1}$ via the embedding $M \hookrightarrow \mathbb{R}^{2m+1}$, and we denote by $|\cdot|_\cdot$ the corresponding norm. For each $\gamma \in W^{1,2}(\mathbb{T};M)$, we denote by $W^{1,2}(\gamma^*\mathrm{T}M)$ the space of $W^{1,2}$ sections of $\gamma^*\mathrm{T}M$, namely the space of continuous sections ξ of $\gamma^*\mathrm{T}M$ that are weakly differentiable and such that the following quantity is finite:

$$\int_0^1 \left[|\xi(t)|_{\gamma(t)} + |\nabla_t\xi|_{\gamma(t)}\right] \mathrm{d}t.$$

Here, ∇_t denotes the covariant derivative induced by the Levi-Civita connection of the Riemannian manifold $(M,\langle\cdot,\cdot\rangle_\cdot)$. It turns out that $W^{1,2}(\gamma^*\mathrm{T}M)$ is a separable Hilbert space, with inner product given by

$$\langle\!\langle \xi,\zeta\rangle\!\rangle_\gamma := \int_0^1 \left[\langle\xi(t),\zeta(t)\rangle_{\gamma(t)} + \langle\nabla_t\xi,\nabla_t\zeta\rangle_{\gamma(t)}\right] \mathrm{d}t, \qquad \forall \xi,\zeta \in W^{1,2}(\gamma^*\mathrm{T}M).$$

It is in some sense analogous to the Sobolev space $W^{1,2}(\mathbb{T};\mathbb{R}^m)$, since for each $\xi \in W^{1,2}(\gamma^*\mathrm{T}M)$ and $t \in \mathbb{T}$ the point $\xi(t)$ belongs to the m-dimensional vector space $\mathrm{T}_{\gamma(t)}M$.

Now, let us fix $\gamma \in C^\infty(\mathbb{T};M)$. For each $\varepsilon > 0$ we denote by U_ε the ε-open neighborhood of the zero section of $\mathrm{T}M$, i.e. if $\tau : \mathrm{T}M \to M$ denotes the projection onto the base, $U_\varepsilon = \left\{v \in \mathrm{T}M \mid |v|_{\tau(v)} < \varepsilon\right\}$. Then, we denote by $W^{1,2}(\gamma^*U_\varepsilon)$ the subspace of $W^{1,2}(\gamma^*\mathrm{T}M)$ consisting of those sections that take values in the neighborhood U_ε. If ε is less than the injectivity radius of M, we know that the exponential map $\exp_q : U_\varepsilon \cap \mathrm{T}_q M \to M$ is a diffeomorphism onto its image. By means of the exponential map, we can define an injective map

$$\exp_\gamma : W^{1,2}(\gamma^* U_\varepsilon) \to W^{1,2}(\mathbb{T}; M)$$

as

$$\exp_\gamma(\xi)(t) := \exp_{\gamma(t)}(\xi(t)), \qquad \forall \xi \in W^{1,2}(\gamma^* U_\varepsilon),\ t \in \mathbb{R}/\mathbb{Z}.$$

We denote by $\mathscr{U}_\gamma \subset W^{1,2}(\mathbb{T}; M)$ the image of the map $\exp_\gamma$. It is easy to see that $\mathscr{U}_\gamma$ is an open set in $W^{1,2}(\mathbb{T}; M)$, hence the map $\exp_\gamma$ is a homeomorphism onto it. By the density argument given above, the family $\{\mathscr{U}_\gamma \,|\, \gamma \in C^\infty(\mathbb{T}; M)\}$ is an open cover of the loop space $W^{1,2}(\mathbb{T}; M)$. Moreover, if we consider $\gamma_1, \gamma_2 \in C^\infty$ such that $\mathscr{U}_{\gamma_1} \cap \mathscr{U}_{\gamma_2} \neq \varnothing$, it turns out that the composition

$$\exp_{\gamma_1}^{-1} \circ \exp_{\gamma_2} |_{\exp_{\gamma_2}(\mathscr{U}_{\gamma_1} \cap \mathscr{U}_{\gamma_2})} : \exp_{\gamma_2}(\mathscr{U}_{\gamma_1} \cap \mathscr{U}_{\gamma_2}) \to \exp_{\gamma_1}(\mathscr{U}_{\gamma_1} \cap \mathscr{U}_{\gamma_2})$$

is a diffeomorphism between open subsets of the Hilbert spaces $W^{1,2}(\gamma_2^* \mathrm{T}M)$ and $W^{1,2}(\gamma_1^* \mathrm{T}M)$. With this in mind, we can endow the free loop space $W^{1,2}(\mathbb{T}; M)$ with a Hilbert manifold structure given by the atlas $\left\{(\mathscr{U}_\gamma, \exp_\gamma^{-1}) \,|\, \gamma \in C^\infty(\mathbb{T}; M)\right\}$. The tangent space of $W^{1,2}(\mathbb{T}; M)$ at a point γ is given by $\mathrm{T}_\gamma W^{1,2}(\mathbb{T}; M) = W^{1,2}(\gamma^* \mathrm{T}M)$, and $\langle\langle \cdot, \cdot \rangle\rangle_\cdot$ is a Hilbert-Riemannian metric on $W^{1,2}(\mathbb{T}; M)$. By means of this metric, $W^{1,2}(\mathbb{T}; M)$ turns out to be a **complete** Hilbert-Riemannian manifold (see the definition in section A.3).

Remark II.1. An analogous Hilbert-Riemannian manifold is the **path space**

$$W^{1,2}(\mathbb{I}; M) := \left\{\chi \in W^{1,2}(\mathbb{I}; \mathbb{R}^{2m+1}) \,|\, \chi(\mathbb{I}) \subset M\right\},$$

where $\mathbb{I}$ is the interval $[0,1] \subset \mathbb{R}$. An atlas for the path space is given by charts of the form $(\mathscr{V}_\lambda, \exp_\lambda^{-1})$, where $\mathscr{V}_\lambda \subset W^{1,2}(\mathbb{I}; M)$ is an open neighborhood of $\lambda \in C^\infty(\mathbb{I}; M)$ and $\exp_\lambda : W^{1,2}(\lambda^* U_\varepsilon) \to \mathscr{V}_\lambda$ is still defined by

$$\exp_\lambda(\vartheta)(t) := \exp_{\lambda(t)}(\vartheta(t)), \qquad \forall \vartheta \in W^{1,2}(\lambda^* U_\varepsilon) \subseteq W^{1,2}(\lambda^* \mathrm{T}M),\ t \in \mathbb{I}.$$

Since the interval $\mathbb{I}$ is contractible, the pull-back bundle $\lambda^* \mathrm{T}M$ admits a smooth trivialization $\phi_\lambda : \lambda^* \mathrm{T}M \to \mathbb{I} \times \mathbb{R}^m$. This trivialization induces an isomorphism of Hilbert spaces

$$W^{1,2}(\phi_\lambda) : W^{1,2}(\lambda^* \mathrm{T}M) \xrightarrow{\simeq} W^{1,2}(\mathbb{I}; \mathbb{R}^m)$$

given by

$$W^{1,2}(\phi_\lambda)(\xi)(t) = \pi_2 \circ \phi_\lambda(t, \xi(t)), \qquad \forall \xi \in W^{1,2}(\lambda^* \mathrm{T}M),\ t \in \mathbb{I},$$

where $\pi_2 : \mathbb{I} \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection onto the $\mathbb{R}^m$ factor of $\mathbb{I} \times \mathbb{R}^m$. This shows that the path space $W^{1,2}(\mathbb{I}; M)$ also admits an atlas $\{(\mathscr{V}_\lambda, \Phi_\lambda) \,|\, \lambda \in C^\infty(\mathbb{I}; M)\}$ with charts of the form $\Phi_\lambda : \mathscr{V}_\lambda \to W^{1,2}(\mathbb{I}; \mathbb{R}^m)$ given by

$$\Phi_\lambda = W^{1,2}(\phi_\lambda) \circ \exp_\lambda^{-1}.$$

The free loop space $W^{1,2}(\mathbb{T}; M)$ is a smooth Hilbert submanifold of the path space $W^{1,2}(\mathbb{I}; M)$, as well as $W^{1,2}(\mathbb{T}; \mathbb{R}^m)$ is a Hilbert subspace of $W^{1,2}(\mathbb{I}; \mathbb{R}^m)$. ■

Remark II.2. In computing expressions involving local coordinates, it is often useful to work with other charts of the loop space $W^{1,2}(\mathbb{T};M)$ that are compatible with the ones in the introduced atlas $\{(\mathscr{U}_\gamma, \exp_\gamma^{-1}) \,|\, \gamma \in C^\infty(\mathbb{T};M)\}$. This charts are obtained as follows. Consider $\gamma \in C^\infty(\mathbb{T};M)$, and denote by $\pi : \mathbb{I} \to \mathbb{T}$ the quotient map that sends the interval onto the circle $\mathbb{T} \simeq \mathbb{I}/\{0,1\}$. As we have already discussed in the previous remark, the pull-back bundle $(\gamma\circ\pi)^*\mathrm{T}M$ admits a smooth trivialization $\vartheta_\gamma : (\gamma\circ\pi)^*\mathrm{T}M \to \mathbb{I}\times\mathbb{R}^m$. Now, let $\pi_2 : \mathbb{I}\times\mathbb{R}^m \to \mathbb{R}^m$ denote the projection onto the $\mathbb{R}^m$ factor of $\mathbb{I}\times\mathbb{R}^m$. We introduce the m-dimensional vector subspace $\mathbb{V}_\gamma \subset \mathbb{R}^m\times\mathbb{R}^m$ given by

$$\mathbb{V}_\gamma = \left\{(\pi_2\circ\vartheta_\gamma(0,v), \pi_2\circ\vartheta_\gamma(1,v)) \,|\, v \in \mathrm{T}_{\gamma(0)}M = \mathrm{T}_{\gamma(1)}M\right\},$$

and we denote by $W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I};\mathbb{R}^m)$ the Hilbert subspace of $W^{1,2}(\mathbb{I};\mathbb{R}^m)$ consisting of those loops whose pair of endpoints belongs to $\mathbb{V}_\gamma$, i.e.

$$W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I};\mathbb{R}^m) = \left\{\chi \in W^{1,2}(\mathbb{I};\mathbb{R}^m) \,|\, (\chi(0),\chi(1)) \in \mathbb{V}_\gamma\right\}.$$

The trivialization ϑ_γ induces an isomorphism of Hilbert spaces

$$W^{1,2}(\vartheta_\gamma) : W^{1,2}(\gamma^*\mathrm{T}M) \xrightarrow{\simeq} W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I};\mathbb{R}^m)$$

given by

$$W^{1,2}(\vartheta_\gamma)(\xi)(t) = \pi_2\circ\vartheta_\gamma(t,\xi(t)), \qquad \forall \xi \in W^{1,2}(\gamma^*\mathrm{T}M),\ t\in[0,1].$$

Now, if we put

$$\Theta_\gamma := W^{1,2}(\vartheta_\gamma)\circ\exp_\gamma^{-1} : \mathscr{U}_\gamma \to W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I};\mathbb{R}^m),$$

the family of charts $\{(\mathscr{U}_\gamma,\Theta_\gamma) \,|\, \gamma \in C^\infty(\mathbb{T};M)\}$ is another smooth atlas, compatible with the one introduced above, for the free loop space $W^{1,2}(\mathbb{T};M)$. Notice that, whenever the bundle $\gamma^*\mathrm{T}M$ is trivial (e.g. if γ is contractible), we immediately get a chart $\Theta_\lambda : \mathscr{U}_\gamma \to W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I};\mathbb{R}^m)$ with $\mathbb{V}_\gamma = \{(v,v) \,|\, v\in\mathbb{R}^m\}$, so that $W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I};\mathbb{R}^m) = W^{1,2}(\mathbb{T};\mathbb{R}^m)$. ■

Now, let us consider a non-zero positive integer $n \in \mathbb{N}$. If we denote by $\mathbb{T}^{[n]}$ the n-periodic circle $\mathbb{R}/n\mathbb{Z}$, the **n-periodic free loop space** $W^{1,2}(\mathbb{T}^{[n]};M)$ is diffeomorphic to the 1-periodic one $W^{1,2}(\mathbb{T};M)$ simply by the map $\gamma\mapsto\tilde\gamma$, where $\tilde\gamma(t) = \gamma(nt)$ for each $t\in[0,1]$. On $W^{1,2}(\mathbb{T}^{[n]};M)$ we can put the rescaled Hilbert-Riemannian metric given by

$$\langle\!\langle \xi,\zeta\rangle\!\rangle_\gamma := \frac{1}{n}\int_0^n \left[\langle\xi(t),\zeta(t)\rangle_{\gamma(t)} + \langle\nabla_t\xi,\nabla_t\zeta\rangle_{\gamma(t)}\right]\mathrm{d}t,$$

$$\forall\gamma\in W^{1,2}(\mathbb{T}^{[n]};M),\ \xi,\zeta\in W^{1,2}(\gamma^*\mathrm{T}M).$$

As we already did in section I.3 for loops in cotangent bundles, we introduce the n^{th} **iteration map** $\psi^{[n]} : W^{1,2}(\mathbb{T}; M) \hookrightarrow W^{1,2}(\mathbb{T}^{[n]}; M)$ defined by

$$\psi^{[n]}(\gamma) = \gamma^{[n]}, \qquad \forall \gamma \in W^{1,2}(\mathbb{T}; M),$$

where $\gamma^{[n]}$ is the composition of γ with the n-fold covering map of the circle $\mathbb{T}^{[n]} \to \mathbb{T}$. We can easily show that the iteration map $\psi^{[n]}$ is smooth. In fact, if we consider two correspondent charts $(\mathscr{U}_\gamma, \exp_\gamma^{-1})$ and $(\mathscr{U}_{\gamma^{[n]}}, \exp_{\gamma^{[n]}}^{-1})$ of $W^{1,2}(\mathbb{T}; M)$ and $W^{1,2}(\mathbb{T}^{[n]}; M)$, the composition $\exp_{\gamma^{[n]}}^{-1} \circ \psi^{[n]} \circ \exp_\gamma$ is simply the analogous iteration map between the Hilbert spaces $W^{1,2}(\gamma^*\mathrm{T}M)$ and $W^{1,2}(\gamma^{[n]*}\mathrm{T}M)$. This latter, in turn, is a linear continuous map and in particular it is smooth, hence $\psi^{[n]}$ is smooth as well.

For each $\gamma \in W^{1,2}(\mathbb{T}; M)$, the differential of the iteration map at γ

$$\mathrm{d}\psi^{[n]}(\gamma) : W^{1,2}(\gamma^*\mathrm{T}M) \to W^{1,2}(\gamma^{[n]*}\mathrm{T}M)$$

is still the iteration map between the above Hilbert spaces, i.e. $\mathrm{d}\psi^{[n]}(\gamma)\xi = \xi^{[n]}$ for each $\xi \in W^{1,2}(\gamma^*\mathrm{T}M)$. By our choice of the Hilbert-Riemannian metrics on the loop spaces, we have

$$\langle\!\langle \xi, \xi \rangle\!\rangle_\gamma = \langle\!\langle \xi^{[n]}, \xi^{[n]} \rangle\!\rangle_{\gamma^{[n]}}, \qquad \forall \xi \in W^{1,2}(\gamma^*\mathrm{T}M).$$

This shows that $\psi^{[n]}$ is an isometry, and we can consider $W^{1,2}(\mathbb{T}; M)$ as a submanifold of $W^{1,2}(\mathbb{T}^{[n]}; M)$ with the pulled-back Hilbert-Riemannian metric.

II.2 Topological properties of the free loop space

In terms of homotopy type, the free loop spaces $C^\infty(\mathbb{T}; M)$, $W^{1,2}(\mathbb{T}; M)$ and $C^0(\mathbb{T}; M)$ are indistinguishable. The fact that the inclusion $W^{1,2}(\mathbb{T}; M) \subseteq C^0(\mathbb{T}; M)$ is a homotopy equivalence follows from an abstract theorem of Palais [Pa2, page 5] about dense inclusions of infinite dimensional vector spaces, and can be found in [Kl, page 15]. Here, we give a proof that the inclusion $C^\infty(\mathbb{T}; M) \subseteq W^{1,2}(\mathbb{T}; M)$ is also a homotopy equivalence, by means of a simple convolution technique.

Proposition II.1. *The continuous inclusion $C^\infty(\mathbb{T}; M) \subset W^{1,2}(\mathbb{T}; M)$ is a homotopy equivalence.*

Proof. We will prove the statement building a continuous homotopy

$$\mathscr{J} : [0,1] \times W^{1,2}(\mathbb{T}; M) \to W^{1,2}(\mathbb{T}; M)$$

such that $\mathscr{J}(0, \cdot)$ is the identity on $W^{1,2}(\mathbb{T}; M)$, and $\mathscr{J}(1, \cdot)$ is a map of the form $\mathscr{J}(1, \cdot) : W^{1,2}(\mathbb{T}; M) \to C^\infty(\mathbb{T}; M)$ that is continuous with respect to the topologies of $W^{1,2}(\mathbb{T}; M)$ and $C^\infty(\mathbb{T}; M)$.

At first, we define a smooth function $k : \mathbb{R} \to \mathbb{R}$ as

$$k(t) = \begin{cases} C \exp\left(\frac{1}{t^2-1}\right), & \text{if } |t| < 1, \\ 0, & \text{if } |t| \geq 1, \end{cases}$$

where $C > 0$ is a constant such that

$$\int_{\mathbb{R}} k(t)\, \mathrm{d}t = 1.$$

For each $\varepsilon > 0$ we define a smooth function $k_\varepsilon : \mathbb{R} \to \mathbb{R}$ as $k_\varepsilon(t) = \varepsilon^{-1} k(t\varepsilon^{-1})$. Notice that

$$\int_{\mathbb{R}} k_\varepsilon(t)\, \mathrm{d}t = 1, \qquad \forall \varepsilon > 0,$$

and k_ε tends to the Dirac delta as $\varepsilon \to 0$, in the sense of distributions. The functions in the family $\{k_\varepsilon \,|\, \varepsilon > 0\}$ can be used as convolution kernels to regularize $W^{1,2}$ maps (in the convolution literature, this family is sometimes called **approximate identity**, see for instance [Ru, page 157]). We define

$$\mathscr{K} : (0, \infty) \times W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1}) \to C^\infty(\mathbb{T}; \mathbb{R}^{2m+1}) \tag{II.1}$$

as $\mathscr{K}(\varepsilon, \zeta) = k_\varepsilon * \zeta$ for each $\varepsilon > 0$ and $\zeta \in W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1})$, i.e.

$$\mathscr{K}(\varepsilon, \zeta)(t) = \int_{\mathbb{R}} k_\varepsilon(t-s)\zeta(s)\, \mathrm{d}s \qquad \forall t \in \mathbb{R}.$$

Notice that $\mathscr{K}$ is in fact a map of the form (II.1) (since the convolution of periodic maps is still periodic) and it is a continuous map. Moreover $\mathscr{K}$ can be continuously extended to a continuous map

$$\mathscr{K} : [0, \infty) \times W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1}) \to W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1})$$

by setting $\mathscr{K}(0, \zeta) = \zeta$ for each $\zeta \in W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1})$.

Recall that, by the Whitney embedding theorem, we may assume that M is a closed submanifold of $\mathbb{R}^{2m+1}$. Consider a tubular neighborhood $N \subseteq \mathbb{R}^{2m+1}$ of M, and a corresponding retraction $r : N \to M$. This retraction induces a smooth map

$$W^{1,2}(r) : W^{1,2}(\mathbb{T}; N) \to W^{1,2}(\mathbb{T}; M)$$

given by $W^{1,2}(r)(\zeta) = r \circ \zeta$, for each $\zeta \in W^{1,2}(\mathbb{T}; N)$. Moreover, $W^{1,2}(r)$ restricts to a smooth map

$$C^\infty(r) : C^\infty(\mathbb{T}; N) \to C^\infty(\mathbb{T}; M).$$

Notice that $W^{1,2}(\mathbb{T}; N)$ and $C^\infty(\mathbb{T}; N)$ are open subsets of $W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1})$ and $C^\infty(\mathbb{T}; \mathbb{R}^{2m+1})$ respectively.

Now, for each $\gamma \in C^\infty(\mathbb{T}; M)$ we choose a real $\varepsilon_\gamma > 0$ and an open neighborhood $\mathscr{W}_\gamma \subset W^{1,2}(\mathbb{T}; M)$ of γ such that $\mathscr{K}([0, \varepsilon_0] \times \mathscr{W}_\gamma) \subset W^{1,2}(\mathbb{T}; N)$. The family

$\mathfrak{W} = \{\mathscr{W}_\gamma \,|\, \gamma \in C^\infty(\mathbb{T}; M)\}$ is an open cover of $W^{1,2}(\mathbb{T}; M)$, and since this latter is a Hilbert manifold (in particular, it is paracompact) there exists a partition of unity $\{\rho_\gamma \,|\, \gamma \in C^\infty(\mathbb{T}; M)\}$ subordinated to the open cover $\mathfrak{W}$. We define a smooth function $\varepsilon : W^{1,2}(\mathbb{T}; M) \to (0, \infty)$ by

$$\varepsilon(\zeta) := \sum_{\gamma \in C^\infty(\mathbb{T}; M)} \varepsilon_\gamma \rho_\gamma(\zeta).$$

Notice that $\mathscr{K}(\varepsilon(\zeta), \zeta) \in C^\infty(\mathbb{T}; N)$ for each $\zeta \in W^{1,2}(\mathbb{T}; M)$. A homotopy $\mathscr{J}$ as claimed at the beginning can be built by setting

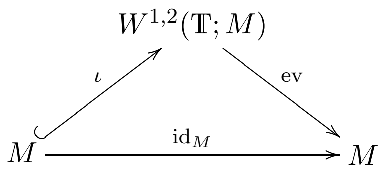
$$\mathscr{J}(s, \zeta) := W^{1,2}(r) \circ \mathscr{K}\,(s\varepsilon(\zeta), \zeta)\,.$$

This concludes the proof. ∎

In literature there are several results concerning the richness of the homology and homotopy of the free loop space under certain conditions on the manifold M, see for instance [VPS]. For our purposes, we only need to remark that the free loop space contains all the homology and homotopy of the manifold M. We denote by $\iota : M \hookrightarrow W^{1,2}(\mathbb{T}; M)$ the map that sends a point to the constant loop at that point, i.e. $\iota(q)(t) = q$ for each $q \in M$ and $t \in \mathbb{T}$.

Proposition II.2. *The homotopy and homology homomorphisms induced by ι are injective.*

Proof. Let $\mathrm{ev} : W^{1,2}(\mathbb{T}; M) \to M$ be the **evaluation map** defined by $\mathrm{ev}(\gamma) = \gamma(0)$ for each $\gamma \in W^{1,2}(\mathbb{T}; M)$. This map is easily seen to be continuous (and even smooth), since it is the restriction of the bounded linear map $\mathrm{ev} : W^{1,2}(\mathbb{T}; \mathbb{R}^{2m+1}) \to \mathbb{R}^{2m+1}$ defined analogously. We obtain the following commutative diagram of continuous maps



The homotopy endomorphism $\pi_*(\mathrm{id}_M) : \pi_*(M) \to \pi_*(M)$ is the identity, and in particular it is group isomorphism. By functoriality, we have $\pi_*(\mathrm{id}_M) = \pi_*(\mathrm{ev}) \circ \pi_*(\iota)$, and therefore $\pi_*(\iota)$ is a monomorphism. The same argument goes through for the homology functor. ∎

II.3 Convex quadratic-growth Lagrangians

Let M be an m-dimensional closed manifold, over which we fix a Riemannian metric $\langle\cdot,\cdot\rangle$ and a finite atlas (so that every expression that involves local coordinates will be implicitly understood with respect to some chart of this atlas). We will consider smooth Lagrangian functions $\mathscr{L} : [0,1] \times \mathrm{T}M \to \mathbb{R}$ that satisfy the following two conditions:

(Q1) there is a positive constant ℓ_0 such that

$$\sum_{i,j=1}^{m} \frac{\partial^2 \mathscr{L}}{\partial v^i \, \partial v^j}(t,q,v)\, w^i w^j \geq \ell_0 \, |w|_q^2 \,,$$

for all $(t,q,v) \in [0,1] \times \mathrm{T}M$ and all $w \in \mathrm{T}_q M$;

(Q2) there is a positive constant ℓ_1 such that

$$\left| \frac{\partial^2 \mathscr{L}}{\partial v^i \, \partial v^j}(t,q,v) \right| \leq \ell_1,$$

$$\left| \frac{\partial^2 \mathscr{L}}{\partial q^i \, \partial v^j}(t,q,v) \right| \leq \ell_1 (1 + |v|_q),$$

$$\left| \frac{\partial^2 \mathscr{L}}{\partial q^i \, \partial q^j}(t,q,v) \right| \leq \ell_1 (1 + |v|_q^2),$$

for all $(t,q,v) \in [0,1] \times \mathrm{T}M$ and $i,j = 1, ..., m$.

If we integrate the inequalities in **(Q2)** along the fibers of $\mathrm{T}M$ we obtain, for some positive constants $\ell_3, \ell_4 > 0$,

$$\left| \frac{\partial \mathscr{L}}{\partial v^i}(t,q,v) \right| \leq \ell_3 (1 + |v|_q), \tag{II.2}$$

$$\left| \frac{\partial \mathscr{L}}{\partial q^i}(t,q,v) \right| \leq \ell_3 (1 + |v|_q^2), \tag{II.3}$$

$$|\mathscr{L}(t,q,v)| \leq \ell_4 (1 + |v|_q^2), \tag{II.4}$$

for all $(t,q,v) \in [0,1] \times \mathrm{T}M$ and $i = 1, ..., m$. Analogously, integrating the inequality in **(Q1)** along the fibers of $\mathrm{T}M$ we get, for some positive constants $\ell_5, \ell_6 > 0$,

$$\left| \frac{\partial \mathscr{L}}{\partial v^i}(t,q,v) \right| \geq \ell_5 (|v|_q - 1),$$

$$\mathscr{L}(t,q,v) \geq \ell_6 (|v|_q^2 - 1),$$

for all $(t,q,v) \in [0,1] \times \mathrm{T}M$ and $i = 1, ..., m$. Notice that the Euler-Lagrange system is unchanged if we add a constant to the Lagrangian $\mathscr{L}$. Therefore, adding ℓ_6 to $\mathscr{L}$ and putting $\underline{\ell} := \ell_6$ and $\overline{\ell} := \ell_4 + \ell_6$ we infer

$$\underline{\ell}\, |v|_q^2 \leq \mathscr{L}(t,q,v) \leq \overline{\ell}\, (|v|_q^2 + 1), \qquad \forall (t,q,v) \in [0,1] \times \mathrm{T}M. \tag{II.5}$$

From these bounds we readily obtain that $\mathscr{L}$ is a Tonelli Lagrangian. In the following, we will informally refer to the class of smooth Lagrangian functions that satisfy **(Q1)** and **(Q2)** as the class of **convex quadratic-growth** Lagrangians.

Apparently, conditions **(Q1)** and **(Q2)** depend on the choice of the Riemannian metric and of the finite atlas on M. However, this is not the case, as stated by the following.

Proposition II.3. *Up to changing the constants ℓ_0 and ℓ_1, conditions* **(Q1)** *and* **(Q2)** *are independent of the choice of the Riemannian metric and of the finite atlas on M.*

Proof. Let $\mathfrak{U} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^m \,|\, \alpha = 0, ..., u\}$ be the previously fixed atlas on M, and consider another finite atlas $\mathfrak{U}' = \{\phi_\beta : U'_\beta \to \mathbb{R}^m \,|\, \beta = 0, ..., u'\}$. As in section I.1, we denote by $\mathrm{T}\mathfrak{U} = \{\mathrm{T}\phi_\alpha : \mathrm{T}U_\alpha \to \mathbb{R}^m \times \mathbb{R}^m \,|\, \alpha = 0, ..., u\}$ and $\mathrm{T}\mathfrak{U}' = \{\mathrm{T}\phi_\beta : \mathrm{T}U'_\beta \to \mathbb{R}^m \times \mathbb{R}^m \,|\, \beta = 0, ..., u'\}$ the associated atlases on the tangent bundle of M. We denote the components of the introduced charts by

$$\phi_\alpha = (q_\alpha^1, ..., q_\alpha^m), \qquad \phi_\beta = (q_\beta^1, ..., q_\beta^m),$$
$$\mathrm{T}\phi_\alpha = (q_\alpha^1, ..., q_\alpha^m, v_\alpha^1, ..., v_\alpha^m), \qquad \mathrm{T}\phi_\beta = (q_\beta^1, ..., q_\beta^m, v_\beta^1, ..., v_\beta^m).$$

Notice that, whenever the domains of two charts $\phi_\alpha \in \mathfrak{U}$ and $\phi_\beta \in \mathfrak{U}'$ intersect, on the intersection we have

$$v_\alpha^i = \sum_{k=1}^m v_\beta^k \frac{\partial q_\alpha^i}{\partial q_\beta^k}, \qquad \frac{\partial v_\alpha^i}{\partial v_\beta^j} = \frac{\partial q_\alpha^i}{\partial q_\beta^j},$$
$$\frac{\partial v_\alpha^i}{\partial q_\beta^j} = \sum_{k=1}^m v_\beta^k \frac{\partial^2 q_\alpha^i}{\partial q_\beta^j \,\partial q_\beta^k}, \qquad \frac{\partial^2 v_\alpha^i}{\partial q_\beta^j \,\partial q_\beta^h} = \sum_{k=1}^m v_\beta^k \frac{\partial^3 q_\alpha^i}{\partial q_\beta^j \,\partial q_\beta^h \,\partial q_\beta^k},$$
$$\forall i, j, h \in \{1, ..., m\}\,.$$

By the compactness of M, there exists a constant $c > 0$ such that, for each $\phi_\alpha \in \mathfrak{U}$ and $\phi_\beta \in \mathfrak{U}'$ with $U_\alpha \cap U'_\beta \neq \varnothing$ and for each $q \in U_\alpha \cap U'_\beta$, $v \in \mathrm{T}_q M$ and $i, j \in \{1, ..., m\}$, we have

$$\left| \frac{\partial q_\alpha^i}{\partial q_\beta^j}(q) \right| \leq c, \qquad \left| \frac{\partial^2 q_\alpha^i}{\partial q_\beta^j \,\partial q_\beta^k}(q) \right| \leq c,$$
$$\left| \frac{\partial v_\alpha^i}{\partial q_\beta^j}(q, v) \right| \leq c\,|v|_q, \qquad \left| \frac{\partial^2 v_\alpha^i}{\partial q_\beta^j \,\partial q_\beta^h}(q, v) \right| \leq c\,|v|_q.$$

This, together with **(Q2)**, implies that

$$\left| \frac{\partial^2 \mathscr{L}}{\partial v_\beta^i \,\partial v_\beta^j}(t, q, v) \right| = \left| \sum_{h,k=1}^m \frac{\partial^2 \mathscr{L}}{\partial v_\alpha^h \,\partial v_\alpha^k}(t, q, v) \,\frac{\partial q_\alpha^h}{\partial q_\beta^i}(q)\, \frac{\partial q_\alpha^k}{\partial q_\beta^j}(q) \right| \leq \ell_1 c^2 m^2,$$

and, by (II.2) and (II.3), we have

$$\begin{aligned}
\left|\frac{\partial^2 \mathscr{L}}{\partial q^i_\beta\,\partial v^j_\beta}(t,q,v)\right| &= \left|\frac{\partial}{\partial q^i_\beta}\left(\sum_{h=1}^m \frac{\partial \mathscr{L}}{\partial v^h_\alpha}(t,q,v)\,\frac{\partial q^h_\alpha}{\partial q^j_\beta}(q)\right)\right|\\
&= \left|\sum_{h=1}^m\left(\frac{\partial \mathscr{L}}{\partial v^h_\alpha}(t,q,v)\,\frac{\partial^2 q^h_\alpha}{\partial q^i_\beta\,\partial q^j_\beta}(q)\right.\right.\\
&\qquad + \sum_{k=1}^m\left(\frac{\partial^2 \mathscr{L}}{\partial q^k_\alpha\,\partial v^h_\alpha}(t,q,v)\,\frac{\partial q^k_\alpha}{\partial q^i_\beta}(q)\,\frac{\partial q^h_\alpha}{\partial q^j_\beta}(q)\right.\\
&\qquad \left.\left.\left. + \frac{\partial^2 \mathscr{L}}{\partial v^k_\alpha\,\partial v^h_\alpha}(t,q,v)\,\frac{\partial v^k_\alpha}{\partial q^i_\beta}(q)\,\frac{\partial q^h_\alpha}{\partial q^j_\beta}(q)\right)\right)\right|\\
&\le \sum_{h=1}^m\left(\ell_3(1+|v|_q)c + \sum_{k=1}^m\big(\ell_1(1+|v|_q)c^2 + \ell_1 c^2|v|_q\big)\right)\\
&\le (m\ell_3 c + 2m^2\ell_1 c^2)(1+|v|_q),\\
\left|\frac{\partial^2 \mathscr{L}}{\partial q^i_\beta\,\partial q^j_\beta}(t,q,v)\right| &= \left|\frac{\partial}{\partial q^i_\beta}\left(\sum_{h=1}^m\left(\frac{\partial \mathscr{L}}{\partial q^h_\alpha}(t,q,v)\,\frac{\partial q^h_\alpha}{\partial q^j_\beta}(q) + \frac{\partial \mathscr{L}}{\partial v^h_\alpha}(t,q,v)\,\frac{\partial v^h_\alpha}{\partial q^j_\beta}(q)\right)\right)\right|\\
&= \left|\sum_{h=1}^m\left(\frac{\partial \mathscr{L}}{\partial q^h_\alpha}(t,q,v)\,\frac{\partial^2 q^h_\alpha}{\partial q^i_\beta\,\partial q^j_\beta}(q) + \frac{\partial \mathscr{L}}{\partial v^h_\alpha}(t,q,v)\,\frac{\partial^2 v^h_\alpha}{\partial q^i_\beta\,\partial q^j_\beta}(q)\right.\right.\\
&\qquad + \sum_{k=1}^m\left(\frac{\partial^2 \mathscr{L}}{\partial q^k_\alpha\,\partial q^h_\alpha}(t,q,v)\,\frac{\partial q^k_\alpha}{\partial q^i_\beta}(q)\,\frac{\partial q^h_\alpha}{\partial q^j_\beta}(q)\right.\\
&\qquad + \frac{\partial^2 \mathscr{L}}{\partial v^k_\alpha\,\partial q^h_\alpha}(t,q,v)\,\frac{\partial v^k_\alpha}{\partial q^i_\beta}(q)\,\frac{\partial q^h_\alpha}{\partial q^j_\beta}(q)\\
&\qquad + \frac{\partial^2 \mathscr{L}}{\partial q^k_\alpha\,\partial v^h_\alpha}(t,q,v)\,\frac{\partial q^k_\alpha}{\partial q^i_\beta}(q)\,\frac{\partial v^h_\alpha}{\partial q^j_\beta}(q)\\
&\qquad \left.\left.\left. + \frac{\partial^2 \mathscr{L}}{\partial v^k_\alpha\,\partial v^h_\alpha}(t,q,v)\,\frac{\partial v^k_\alpha}{\partial q^i_\beta}(q)\,\frac{\partial v^h_\alpha}{\partial q^j_\beta}(q)\right)\right)\right|\\
&\le \sum_{h=1}^m\left(\ell_3(1+|v|_q^2)c + \ell_3(1+|v|_q)c|v|_q\right.\\
&\qquad + \sum_{k=1}^m\big(\ell_1(1+|v|_q^2)c^2 + \ell_1(1+|v|_q)c^2|v|_q\\
&\qquad \left. + \ell_1(1+|v|_q)c^2|v|_q + \ell_1 c^2|v|_q^2\big)\right)\\
&\le (3m\ell_3 c + 6m^2\ell_1 c^2)(1+|v|_q^2).
\end{aligned}$$

These estimates prove that condition **(Q2)** still holds with respect to the atlas $\mathfrak{U}'$, up to replacing the constant ℓ_1 with $(3m\ell_3 c + 6m^2\ell_1 c^2)$. Condition **(Q1)** holds with respect to the atlas $\mathfrak{U}'$, even without changing the constant ℓ_0. In fact, for each $(t,q,v) \in \mathbb{R} \times \mathrm{T}M$ and $w \in \mathrm{T}_q M$, we have

$$\sum_{i,j=1}^{m} \frac{\partial^2 \mathscr{L}}{\partial v^i_\alpha \, \partial v^j_\alpha}(t,q,v)\, w^i_\alpha w^j_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\bigg|_{s=0} \mathscr{L}(t,q,v+sw) = \sum_{i,j=1}^{m} \frac{\partial^2 \mathscr{L}}{\partial v^i_\beta \, \partial v^j_\beta}(t,q,v)\, w^i_\beta w^j_\beta,$$

which implies that condition **(Q1)** is coordinate independent. Finally, the fact that conditions **(Q1)** and **(Q2)** still hold with respect to a different Riemannian metric follows from the compactness of M, together with the fact that all the Riemannian metrics on it are locally equivalent. Namely, if $\langle\!\langle \cdot,\cdot \rangle\!\rangle_\cdot$ is another Riemannian metric on M with correspondent norm $\|\cdot\|_\cdot$, there exists a constant $d \geq 1$ such that

$$d^{-1}|v|_q \leq \|v\|_q \leq d\,|v|_q, \qquad \forall (q,v) \in \mathrm{T}M. \qquad \blacksquare$$

In [Ben], Benci showed that the action functional of convex quadratic-growth Lagrangians is continuously differentiable and satisfies the Palais-Smale condition on $W^{1,2}(\mathbb{T};M)$. In the remaining of this section, we will prove these results following [AS2]. Let us fix a smooth Lagrangian $\mathscr{L} : [0,1] \times \mathrm{T}M \to \mathbb{R}$. If $\mathscr{L}$ satisfies **(Q2)**, the action functional

$$\mathscr{A}(\chi) = \int_0^1 \mathscr{L}(t,\chi(t),\dot\chi(t))\,\mathrm{d}t, \qquad \forall \chi \in W^{1,2}(\mathbb{I};M),$$

is a well defined function $\mathscr{A} : W^{1,2}(\mathbb{I};M) \to \mathbb{R}$ on the path space of M. In fact, the inequality in (II.4) readily implies that every $W^{1,2}$ path has finite action. Moreover, we have the following.

Proposition II.4. *If the smooth Lagrangian $\mathscr{L} : [0,1] \times \mathrm{T}M \to \mathbb{R}$ satisfies* **(Q2)**, *then its action functional $\mathscr{A} : W^{1,2}(\mathbb{I};M) \to \mathbb{R}$ is C^1. Its differential $\mathrm{d}\mathscr{A}$ is Gateaux-differentiable and locally Lipschitz-continuous.*

The proof of this statement makes use of the following elementary fact about convergences in metric spaces.

Lemma II.5. *In a metric space, a sequence $\{x_j\}$ converges to x as $j \to \infty$ if and only if every subsequence $\{x_{j(k)}\}$ has a further subsequence $\{x_{j(k(h))}\}$ that converges to x as $h \to \infty$.*

Proof. The "only if" part is trivial. For the other implication, assume that $\{x_j\}$ does not converge to x as $j \to \infty$. This means that there exists $\varepsilon > 0$ such that, for each $k \in \mathbb{N}$, there exists $j(k) > k$ such that $\mathrm{dist}(x_{j(k)}, x) > \varepsilon$. Hence the subsequence $\{x_{j(k)}\}$ has no subsequences that converge to x. ■

Proof of proposition II.4. Since the statement is of a local nature, we can work in the image of some chart

$$\Phi_\lambda : \mathscr{V}_\lambda \to W^{1,2}(\mathbb{I};\mathbb{R}^m),$$

see section II.1 for the notation. Let U be a sufficiently small neighborhood of the origin in $\mathbb{R}^m$, so that the open set $W^{1,2}(\mathbb{I};U) \subset W^{1,2}(\mathbb{I};\mathbb{R}^m)$ is contained in the image of the chart Φ_λ. We put

$$\Pi_\lambda := \Phi_\lambda^{-1}|_{W^{1,2}(\mathbb{I};U)} : W^{1,2}(\mathbb{I};U) \to \mathscr{V}_\lambda$$

and we define an embedding $\pi_\lambda : \mathbb{I} \times U \times \mathbb{R}^m \hookrightarrow \mathbb{I} \times \mathrm{T}M$ by

$$\pi_\lambda(t,q,v) = \Big(t, \exp_{\lambda(t)} \circ \vartheta_\lambda^{-1}(t,q), \mathrm{d}(\exp_{\lambda(t)} \circ \vartheta_\lambda^{-1})(t,q)(0,v)\Big), \qquad \forall (t,q,v) \in \mathbb{I} \times U \times \mathbb{R}^m.$$

The pulled-back Lagrangian $\mathscr{L} \circ \pi_\lambda : [0,1] \times U \times \mathbb{R}^m \to \mathbb{R}$ is again convex quadratic-growth, since conditions **(Q1)** and **(Q2)** are invariant with respect to coordinate transformations of the form π_λ (up to changing the constants ℓ_0 and ℓ_1 in conditions **(Q1)** and **(Q2)**). The pulled-back functional $\mathscr{A} \circ \Pi_\lambda$ turns out to be the action functional associated to the Lagrangian $\mathscr{L} \circ \pi_\lambda$, i.e.

$$\mathscr{A} \circ \Pi_\lambda(\sigma) = \int_0^1 \mathscr{L} \circ \pi_\lambda(t, \sigma(t), \dot\sigma(t))\, \mathrm{d}t, \qquad \forall \sigma \in W^{1,2}(\mathbb{I};U).$$

Therefore, without loss of generality, we can prove the statement for the action functional $\mathscr{A} : W^{1,2}(\mathbb{I};U) \to \mathbb{R}$ of a convex quadratic-growth Lagrangian $\mathscr{L} : [0,1] \times U \times \mathbb{R}^m \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^m$.

Consider $\lambda \in W^{1,2}(\mathbb{I};U)$, $\sigma \in W^{1,2}(\mathbb{I};\mathbb{R}^m)$ and $\varepsilon \in \mathbb{R}$ with sufficiently small absolute value so that $\lambda + \varepsilon\sigma$ belongs to $W^{1,2}(\mathbb{I};U)$. We have

$$\begin{aligned}&\frac{\mathscr{A}(\lambda+\varepsilon\sigma) - \mathscr{A}(\lambda)}{\varepsilon}\\ &\quad = \int_0^1\!\!\int_0^1 \Big[\langle \partial_q \mathscr{L}(t, \lambda + s\varepsilon\sigma, \dot\lambda + s\varepsilon\dot\sigma), \sigma\rangle + \langle \partial_v \mathscr{L}(t, \lambda + s\varepsilon\sigma, \dot\lambda + s\varepsilon\dot\sigma), \dot\sigma\rangle\Big]\, \mathrm{d}t\, \mathrm{d}s,\end{aligned}$$

where we denoted by $\partial_q\mathscr{L}$ and $\partial_v\mathscr{L}$ the gradients of $\mathscr{L}$ with respect to the q and v variables respectively. By (II.2) and (II.3) we can apply the dominated convergence theorem to assert that, for $\varepsilon \to 0$, the above quantity tends to

$$\mathrm{d}\mathscr{A}(\lambda)\,\sigma := \lim_{\varepsilon\to 0} \frac{\mathscr{A}(\lambda+\varepsilon\sigma) - \mathscr{A}(\lambda)}{\varepsilon} = \int_0^1 \Big[\langle \partial_q \mathscr{L}(t,\lambda,\dot\lambda), \sigma\rangle + \langle \partial_v \mathscr{L}(t,\lambda,\dot\lambda), \dot\sigma\rangle\Big]\, \mathrm{d}t.$$

The functional $\mathrm{d}\mathscr{A}(\lambda) : W^{1,2}(\mathbb{I};\mathbb{R}^m) \to \mathbb{R}$ is the Gateaux differential of the action $\mathscr{A}$ at λ, being a bounded linear functional on $W^{1,2}(\mathbb{I};\mathbb{R}^m)$. Now, we want to prove that the map $\mathrm{d}\mathscr{A} : W^{1,2}(\mathbb{I};U) \to W^{1,2}(\mathbb{I};\mathbb{R}^m)^*$ is continuous, namely we want to prove that, for an arbitrary sequence $\{\lambda_j\} \subset W^{1,2}(\mathbb{I};U)$ converging to some $\lambda \in W^{1,2}(\mathbb{I};U)$, we have

$$\sup_{\|\sigma\|_{W^{1,2}}=1} \{\mathrm{d}\mathscr{A}(\lambda_j)\,\sigma - \mathrm{d}\mathscr{A}(\lambda)\,\sigma\} \underset{j\to\infty}{\longrightarrow} 0.$$

Thus, let us assume that the sequence $\{\lambda_j\}$ converges to λ in $W^{1,2}(\mathbb{I};U)$. This implies that $\{\lambda_j\}$ converges to λ uniformly and $\{\dot\lambda_j\}$ converges to $\dot\lambda$ in L^2, and in particular there exists a curve $f \in L^2(\mathbb{I};U)$ such that $|\dot\lambda_j| < f$ almost everywhere for all j. The L^2 convergence implies that, for every subsequence $\{\dot\lambda_{j(k)}\}$, there exists a further subsequence $\{\dot\lambda_{j(k(h))}\}$ converging to $\dot\lambda$ almost everywhere. Therefore, by (II.2) and (II.3), we can apply the dominated convergence theorem as before to get

$$\partial_q \mathscr{L}(\cdot, \lambda_{j(k(h))}, \dot\lambda_{j(k(h))}) \underset{h\to\infty}{\longrightarrow} \partial_q \mathscr{L}(\cdot,\lambda,\dot\lambda) \text{ in } L^1(\mathbb{I};\mathbb{R}^m),$$
$$\partial_v \mathscr{L}(\cdot, \lambda_{j(k(h))}, \dot\lambda_{j(k(h))}) \underset{h\to\infty}{\longrightarrow} \partial_v \mathscr{L}(\cdot,\lambda,\dot\lambda) \text{ in } L^2(\mathbb{I};\mathbb{R}^m).$$

These convergences, in turn, imply that $\{\mathrm{d}\mathscr{A}(\lambda_{j(k(h))})\}$ converges to $\mathrm{d}\mathscr{A}(\lambda)$ in $W^{1,2}(\mathbb{I};\mathbb{R}^m)^*$ as $h\to\infty$. By lemma II.5 we conclude that the full sequence $\{\mathrm{d}\mathscr{A}(\lambda_j)\}$ converges to $\mathrm{d}\mathscr{A}(\lambda)$ in $W^{1,2}(\mathbb{I};\mathbb{R}^m)^*$ as $h\to\infty$, and by the total differential theorem the functional $\mathscr{A}$ is C^1 with Fréchét differential $\mathrm{d}\mathscr{A}$.

Now, consider $\lambda \in W^{1,2}(\mathbb{I};U)$, $\sigma,\rho \in W^{1,2}(\mathbb{I};\mathbb{R}^m)$ and $\varepsilon\in\mathbb{R}$ with sufficiently small absolute value so that $\lambda+\varepsilon\rho$ belongs to $W^{1,2}(\mathbb{I};U)$. We have

$$\begin{aligned}&\frac{\mathrm{d}\mathscr{A}(\lambda+\varepsilon\rho)\,\sigma - \mathrm{d}\mathscr{A}(\lambda)\,\sigma}{\varepsilon}\\ &\quad= \int_0^1\!\!\int_0^1 \Big[\langle \partial^2_{vv}\mathscr{L}(t,\lambda+s\varepsilon\rho,\dot\lambda+s\varepsilon\dot\rho)\dot\sigma,\dot\rho\rangle + \langle \partial^2_{vq}\mathscr{L}(t,\lambda+s\varepsilon\rho,\dot\lambda+s\varepsilon\dot\rho)\sigma,\dot\rho\rangle\\ &\qquad + \langle \partial^2_{qv}\mathscr{L}(t,\lambda+s\varepsilon\rho,\dot\lambda+s\varepsilon\dot\rho)\dot\sigma,\rho\rangle + \langle \partial^2_{qq}\mathscr{L}(t,\lambda+s\varepsilon\rho,\dot\lambda+s\varepsilon\dot\rho)\sigma,\rho\rangle\Big]\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

By **(Q2)**, we can apply the dominated convergence theorem to assert that the above quantity converges, as $\varepsilon\to 0$, to

$$\begin{aligned}\mathrm{Hess}\mathscr{A}(\lambda)[\rho,\sigma] &:= \lim_{\varepsilon\to 0}\frac{\mathrm{d}\mathscr{A}(\lambda+\varepsilon\rho)\,\sigma - \mathrm{d}\mathscr{A}(\lambda)\,\sigma}{\varepsilon}\\ &= \int_0^1 \Big[\langle \partial^2_{vv}\mathscr{L}(t,\lambda,\dot\lambda)\dot\sigma,\dot\rho\rangle + \langle \partial^2_{vq}\mathscr{L}(t,\lambda,\dot\lambda)\sigma,\dot\rho\rangle\\ &\qquad + \langle \partial^2_{qv}\mathscr{L}(t,\lambda,\dot\lambda)\dot\sigma,\rho\rangle + \langle \partial^2_{qq}\mathscr{L}(t,\lambda,\dot\lambda)\sigma,\rho\rangle\Big]\,\mathrm{d}t.\end{aligned}$$

Since $\mathrm{Hess}\mathscr{A}(\lambda) : W^{1,2}(\mathbb{I};\mathbb{R}^m)\otimes W^{1,2}(\mathbb{I};\mathbb{R}^m)\to\mathbb{R}$ is a bounded symmetric bilinear form on $W^{1,2}(\mathbb{I};\mathbb{R}^m)$, the bounded linear map

$$\mathrm{d}^2\mathscr{A}(\lambda) : W^{1,2}(\mathbb{I};\mathbb{R}^m) \to W^{1,2}(\mathbb{I};\mathbb{R}^m)^*$$

given by

$$(\mathrm{d}^2\mathscr{A}(\lambda)\sigma)\rho := \mathrm{Hess}\mathscr{A}(\lambda)[\sigma,\rho], \qquad \forall \sigma,\rho\in W^{1,2}(\mathbb{I};\mathbb{R}^m),$$

is the Gateaux differential of $\mathrm{d}\mathscr{A}$ at λ. By **(Q2)** there exists a continuous increasing function $\omega : [0,\infty)\to[0,\infty)$ such that

$$\mathrm{d}^2\mathscr{A}(\lambda)[\sigma,\rho] \le \omega(\|\lambda\|_{W^{1,2}})\|\sigma\|_{W^{1,2}}\|\rho\|_{W^{1,2}},$$

For each $R > 0$ and for each $\lambda, \chi \in W^{1,2}(\mathbb{I}; U)$ with $\|\lambda\|_{W^{1,2}}, \|\chi\|_{W^{1,2}} < R$, this bound readily gives

$$\begin{aligned}
\|\mathrm{d}\mathscr{A}(\chi) - \mathrm{d}\mathscr{A}(\lambda)\|_{(W^{1,2})^*} &= \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} \mathrm{d}\mathscr{A}((1-r)\lambda + r\chi)\,\mathrm{d}r \right\|_{(W^{1,2})^*} \\
&= \left\| \int_0^1 \mathrm{d}^2\mathscr{A}((1-r)\lambda + r\chi)(\chi - \lambda)\,\mathrm{d}r \right\|_{(W^{1,2})^*} \\
&\leq \omega(\max\{\|\lambda\|_{W^{1,2}}, \|\chi\|_{W^{1,2}}\})\|\chi - \lambda\|_{W^{1,2}} \\
&\leq \omega(R)\,\|\chi - \lambda\|_{W^{1,2}},
\end{aligned}$$

therefore $\mathrm{d}\mathscr{A}$ is locally Lipschitz-continuous. ■

Remark II.3. It turns out that the action functional $\mathscr{A}$ of a Lagrangian $\mathscr{L}$ that satisfies **(Q2)** is C^2 if and only if the restriction of $\mathscr{L}$ to the fibers $\{t\} \times \mathrm{T}_q M$, for each $(t, q) \in [0, 1] \times M$, is a polynomial of degree at most 2. Namely, if and only if $\mathscr{L}$ has the form

$$\mathscr{L}(t, q, v) = \alpha(t, q)[v, v] + \beta(t, q)v - V(t, q),$$

where α is a time-dependent bilinear form on M, β is a time dependent one-form on M and $V : \mathbb{R} \times M \to \mathbb{R}$ is a function, see [AS2, proposition 2.3]. In this case, $\mathscr{A}$ is even C^∞. In physics, Lagrangians of this form are called **electro-magnetic**: α is the **kinetic tensor** of the system, β is the **magnetic form** and V is the **potential energy**. ■

Let $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ be a 1-periodic smooth Lagrangian that satisfies **(Q2)**. From now on, we will denote by $\mathscr{A}$ the restriction of the action functional of $\mathscr{L}$ to the free loop space $W^{1,2}(\mathbb{T}; M)$. Since this latter space embeds as a Hilbert submanifold in the path space $W^{1,2}(\mathbb{I}; M)$, as a consequence of proposition II.4 we immediately obtain that $\mathscr{A} : W^{1,2}(\mathbb{T}; M) \to \mathbb{R}$ is $C^{1,1}$ and twice Gateaux differentiable. The critical points of $\mathscr{A}$ are extremal closed curves $\gamma : \mathbb{R}/\mathbb{Z} \to M$ and, by our discussion in section I.1, they are smooth periodic solutions of the Euler-Lagrange system of $\mathscr{L}$. We denote by $\iota(\mathscr{A}, \gamma)$ and $\nu(\mathscr{A}, \gamma)$ the **Morse index** and the **nullity** of the action functional $\mathscr{A}$ at the critical point γ (see appendix A for the general definition of Morse index and nullity of a functional). By definition, the Gateaux Hessian of $\mathscr{A}$ at γ coincides with the symmetric bilinear form $\mathscr{B}_\gamma$ defined in (I.12), i.e.

$$\mathrm{Hess}\mathscr{A}(\gamma)[\sigma, \xi] = \mathscr{B}_\gamma(\sigma, \xi), \qquad \forall \sigma, \xi \in W^{1,2}(\gamma^*\mathrm{T}M).$$

In particular the Morse index and nullity pair of $\mathscr{A}$ at γ coincides with the Morse index and nullity pair of the form $\mathscr{B}_\gamma$. Let $\mathscr{H} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}^*M \to \mathbb{R}$ be the Hamiltonian that is Legendre dual to $\mathscr{L}$ and $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathrm{T}^*M$ the Hamiltonian periodic orbit corresponding to γ. By the index theorem (theorem I.9), if the pull-back bundle

$\gamma^*\mathrm{T}M$ is trivial, the Morse index and nullity pair of $\mathscr{A}$ at γ is also equal to the Conley-Zehnder-Long index pair of Γ, i.e.

$$(\text{II.6}) \qquad (\iota(\mathscr{A},\gamma),\nu(\mathscr{A},\gamma)) = (\iota(\mathscr{B}_\gamma),\nu(\mathscr{B}_\gamma)) = (\iota(\mathscr{H},\Gamma),\nu(\mathscr{H},\Gamma)).$$

Now, consider the iteration map $\psi^{[n]} : W^{1,2}(\mathbb{T};M) \hookrightarrow W^{1,2}(\mathbb{T}^{[n]};M)$ introduced in section II.1. By this map we interpret the 1-periodic free loop space $W^{1,2}(\mathbb{T};M)$ as a Hilbert submanifold of the n-periodic free loop space $W^{1,2}(\mathbb{T}^{[n]};M)$. A natural extension of the action functional $\mathscr{A} : W^{1,2}(\mathbb{T};M) \to \mathbb{R}$ to $W^{1,2}(\mathbb{T}^{[n]};M)$ is given by the **mean action functional** $\mathscr{A}^{[n]} : W^{1,2}(\mathbb{T}^{[n]};M) \to \mathbb{R}$ defined by

$$\mathscr{A}^{[n]}(\zeta) = \frac{1}{n}\int_0^n \mathscr{L}(t,\zeta(t),\dot\zeta(t))\,\mathrm{d}t, \qquad \forall\zeta \in W^{1,2}(\mathbb{T}^{[n]};M).$$

This is in fact an extension as claimed above, since $\mathscr{A}^{[n]}\circ\psi^{[n]} = \mathscr{A}$.

The critical points of $\mathscr{A}^{[n]}$ are smooth n-periodic solutions of the Euler-Lagrange system of $\mathscr{L}$. This implies that the n^{th}-iteration of a critical point of $\mathscr{A}$ is still a critical point of $\mathscr{A}^{[n]}$. In other words, the critical points of the restricted functional $\mathscr{A} = \mathscr{A}^{[n]}\circ\psi^{[n]} = \mathscr{A}^{[n]}|_{W^{1,2}(\mathbb{T};M)}$ are also critical points of the unrestricted functional $\mathscr{A}^{[n]}$. Moreover, the Hamiltonian orbit associated to $\gamma^{[n]}$ is $\Gamma^{[n]}$, and by the analogous of (II.6) in period n we get

$$(\text{II.7}) \qquad (\iota(\mathscr{A}^{[n]},\gamma^{[n]}),\nu(\mathscr{A}^{[n]},\gamma^{[n]})) = (\iota(\mathscr{H},\Gamma^{[n]}),\nu(\mathscr{H},\Gamma^{[n]})),$$

provided that $\gamma^*\mathrm{T}M$ is a trivial bundle. We define the **mean Morse index** $\widehat{\iota}(\mathscr{A},\gamma)$ of $\mathscr{A}$ at γ as the mean Conley-Zehnder index of the associated Hamiltonian orbit Γ, i.e.

$$\widehat{\iota}(\mathscr{A},\gamma) := \widehat{\iota}(\mathscr{H},\Gamma) = \lim_{n\to\infty}\frac{\iota(\mathscr{A}^{[n]},\gamma^{[n]})}{n} \in \mathbb{R}.$$

By the above discussion on the regularity of the action functional, equation (II.7) and the iteration inequality (I.11), we conclude the following.

Proposition II.6 (Iteration inequality for the Morse index and nullity). *The mean action functionals $\mathscr{A}^{[n]} : W^{1,2}(\mathbb{T}^{[k]};M) \to \mathbb{R}$, for each $n \in \mathbb{N}$, are C^1 and twice Gateaux differentiable. If $\gamma \in W^{1,2}(\mathbb{T};M)$ is a critical point of $\mathscr{A}$, then $\gamma^{[n]}$ is a critical point of $\mathscr{A}^{[n]}$ for each $n \in \mathbb{N}$. Moreover, if $\gamma^*\mathrm{T}M$ is a trivial bundle, we have*

$$n\,\widehat{\iota}(\mathscr{A},\gamma) - m \le \iota(\mathscr{A}^{[n]},\gamma^{[n]}) \le n\,\widehat{\iota}(\mathscr{A},\gamma) + m - \nu(\mathscr{A}^{[n]},\gamma^{[n]}), \qquad \forall n \in \mathbb{N},$$

where m is the dimension of the manifold M.

In order to apply Morse-theoretic methods in studying the critical points of the action functional $\mathscr{A} : W^{1,2}(\mathbb{T};M) \to \mathbb{R}$, we have to make sure that the sublevels of $\mathscr{A}$ satisfy some sort of compactness. A sufficient requirement, as discussed in appendix A, is given by the **Palais-Smale condition**. Following [AS2], we show that the action functional $\mathscr{A}$ fulfills this condition, provided that the Lagrangian $\mathscr{L}$ satisfies both **(Q1)** and **(Q2)**, i.e. provided that $\mathscr{L}$ is a convex quadratic-growth Lagrangian. Clearly, all the arguments that we give for $\mathscr{A}$ go through for the mean action functionals $\mathscr{A}^{[n]}$, for each $n \in \mathbb{N}$.

Proposition II.7. *Let $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ be a 1-periodic convex quadratic-growth Lagrangian. Then its action functional $\mathscr{A} : W^{1,2}(\mathbb{T}; M) \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. Consider an arbitrary sequence $\{\zeta_j \,|\, j \in \mathbb{N}\} \subseteq W^{1,2}(\mathbb{T}; M)$ such that the sequence of real numbers $\{\mathscr{A}(\zeta_j) \,|\, j \in \mathbb{N}\}$ is bounded and

$$\lim_{j\to\infty} \sup_{\langle\langle \xi,\xi\rangle\rangle_{\zeta_j}=1} \{|\mathrm{d}\mathscr{A}(\zeta_j)\xi|\} = 0. \tag{II.8}$$

In order to conclude we have to prove that there exists a subsequence that converges to some $\zeta \in W^{1,2}(\mathbb{T}; M)$ in the topology of the free loop space $W^{1,2}(\mathbb{T}; M)$.

At first, we recall that we can always assume that the Lagrangian $\mathscr{L}$ satisfies (II.5). Hence, the sequence

$$\left\{ \int_0^1 |\dot\zeta_j(t)|^2_{\zeta_j(t)} \mathrm{d}t \,\middle|\, n \in \mathbb{N} \right\}$$

is bounded, for

$$\int_0^1 |\dot\zeta_j(t)|^2_{\zeta_j(t)} \mathrm{d}t \leq \underline{\ell}^{-1} \int_0^1 \mathscr{L}(t, \zeta_j(t), \dot\zeta_j(t))\, \mathrm{d}t = \underline{\ell}^{-1} \mathscr{A}(\zeta_j).$$

This, in turn, implies that the sequence $\{\zeta_j \,|\, j \in \mathbb{N}\}$ is equi-1/2-Holder continuous, since for every $t_0, t_1 \in \mathbb{R}$ with $0 < t_1 - t_0 < 1$ we have

$$\mathrm{dist}(\zeta_j(t_0), \zeta_j(t_1)) \leq \int_{t_0}^{t_1} |\dot\zeta_j(t)|_{\zeta_j(t)} \mathrm{d}t \leq |t_1 - t_0|^{1/2} \left(\int_0^1 |\dot\zeta_j(t)|^2_{\zeta_j(t)} \mathrm{d}t \right)^{1/2},$$

where $\mathrm{dist} : M \times M \to [0, \infty)$ is the distance induced by the Riemannian metric on M. By the Ascoli-Arzelà theorem, up to the choice of a subsequence, we can assume that $\{\zeta_j \,|\, j \in \mathbb{N}\}$ converges uniformly to some continuous curve $\zeta : \mathbb{T} \to M$. In particular, the curves ζ_j definitely belong to some coordinate open set $\mathscr{U}_\gamma$ of the free loop space $W^{1,2}(\mathbb{T}; M)$, and we can proceed as we did in the proof of proposition II.4, working in the image of the chart $\Theta_\gamma : \mathscr{U}_\gamma \to W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I}; M)$ of the free loop space (see remark II.2 for the notation). Namely, if U is a sufficiently small neighborhood of the origin in $\mathbb{R}^m$, we can assume to deal with a convex quadratic-growth Lagrangian of the form $\mathscr{L} : [0, 1] \times U \times \mathbb{R}^m \to \mathbb{R}$, with associated action functional $\mathscr{A} : W^{1,2}_{\mathbb{V}}(\mathbb{I}; \mathbb{R}^m) \to \mathbb{R}$, where $\mathbb{V}$ is some m-dimensional vector subspace of $\mathbb{R}^m \times \mathbb{R}^m$. Notice that this localization argument allows us to express (II.8) more easily as $\mathrm{d}\mathscr{A}(\zeta_j) \to 0$ in $W^{1,2}_{\mathbb{V}}(\mathbb{I}; \mathbb{R}^m)^*$. In order to conclude, we must show that $\{\zeta_j \,|\, j\in\mathbb{N}\}$ admits a subsequence that converges to ζ in the Hilbert space $W^{1,2}_{\mathbb{V}}(\mathbb{I};\mathbb{R}^m)$.

Since $\{\zeta_j \,|\, j\in\mathbb{N}\}$ is bounded in $W^{1,2}_{\mathbb{V}}(\mathbb{I};\mathbb{R}^m)$, up to the choice of a subsequence we can assume that it converges to some $\tilde\zeta$ weakly in $W^{1,2}_{\mathbb{V}}(\mathbb{I};\mathbb{R}^m)$ and uniformly, and therefore $\zeta=\tilde\zeta\in W^{1,2}_{\mathbb{V}}(\mathbb{I};\mathbb{R}^m)$. Moreover, we have

$$\mathrm{d}\mathscr{A}(\zeta_j)(\zeta_j-\zeta)=\underbrace{\int_0^1\langle\partial_q\mathscr{L}(t,\zeta_j,\dot\zeta_j),\zeta_j-\zeta\rangle\,\mathrm{d}t}_{=:I_j}+\underbrace{\int_0^1\langle\partial_v\mathscr{L}(t,\zeta_j,\dot\zeta_j),\dot\zeta_j-\dot\zeta\rangle\,\mathrm{d}t}_{=:II_j}\xrightarrow[j\to\infty]{}0.$$

By (II.3), the sequence $\{\partial_q\mathscr{L}(\cdot,\zeta_j,\dot\zeta_j)\,|\,j\in\mathbb{N}\}$ is bounded in $L^2(\mathbb{I};\mathbb{R}^m)$ and, since the sequence $\{\zeta_j-\zeta\,|j\in\mathbb{N}\}$ converges to zero uniformly, we conclude that $I_j\to0$ as $j\to\infty$. Hence, we also have $II_j\to0$ as $j\to\infty$. Now, condition **(Q1)** implies that, for almost every $t\in\mathbb{R}/\mathbb{Z}$,

$$\begin{aligned}&\langle\partial_v\mathscr{L}(t,\zeta_j,\dot\zeta_j),\dot\zeta_j-\dot\zeta\rangle-\langle\partial_v\mathscr{L}(t,\zeta_j,\dot\zeta),\dot\zeta_j-\dot\zeta\rangle\\&\qquad=\int_0^1\Big\langle\partial_{vv}\mathscr{L}(t,\zeta_j,\dot\zeta+s(\dot\zeta_j-\dot\zeta))\,(\dot\zeta_j-\dot\zeta),\dot\zeta_j-\dot\zeta\Big\rangle\,\mathrm{d}s\geq\ell_0|\dot\zeta_j(t)-\dot\zeta(t)|^2.\end{aligned}$$

Integrating this inequality between 0 and 1 we get

$$\ell_0\int_0^1|\dot\zeta_j(t)-\dot\zeta(t)|^2\mathrm{d}t\leq\underbrace{\int_0^1\langle\partial_v\mathscr{L}(t,\zeta_j,\dot\zeta_j),\dot\zeta_j-\dot\zeta\rangle\,\mathrm{d}t}_{=:II_j}-\underbrace{\int_0^1\langle\partial_v\mathscr{L}(t,\zeta_j,\dot\zeta),\dot\zeta_j-\dot\zeta\rangle\,\mathrm{d}t}_{=:III_j}.$$

We have already shown that $II_j\to0$ as $j\to\infty$. By the bound in (II.2), we have that $\partial_v\mathscr{L}(\cdot,\zeta_j,\zeta)\to\partial_v\mathscr{L}(\cdot,\zeta,\zeta)$ in $L^2(\mathbb{I};\mathbb{R}^m)$ as $j\to\infty$, and since the sequence $\{\dot\zeta_j\,|\,j\in\mathbb{N}\}$ converges to $\dot\zeta$ weakly in $L^2(\mathbb{I};\mathbb{R}^m)$, we conclude that $III_j\to0$ as $j\to\infty$. This proves that $\dot\zeta_j\to\dot\zeta$ as $j\to\infty$ in $L^2(\mathbb{I};\mathbb{R}^m)$, and therefore $\zeta_j\to\zeta$ as $j\to\infty$ in $W^{1,2}_{\mathbb{V}}(\mathbb{I};\mathbb{R}^m)$. ■

Chapter III

Discretizations

The $W^{1,2}$ functional setting for the action functional $\mathscr{A}$, introduced in chapter II, presents several drawbacks. First of all, the regularity that we can expect for $\mathscr{A}$ is only $C^{1,1}$, at least if we assume to deal with a general convex quadratic-growth Lagrangian. This prevents the applicability of all those abstract results that require more smoothness, for instance the Morse lemma from critical point theory. Moreover, the $W^{1,2}$ topology is sometimes unconfortable to work with. In fact, in several occasions it may be desirable to deal with a topology that is as strong as the C^1 topology, or at least the $W^{1,\infty}$ topology. This would guarantee that the restriction of the action functional $\mathscr{A}$ to a small neighborhood of a loop γ only depends on the values that the Lagrangian assumes on a small neighborhood of the support of the lifted loop $(\gamma, \dot\gamma)$ in the tangent bundle of the configuration space.

In order to overcome these difficulties, in this chapter we develop a discretization technique that is a generalization to Lagrangian systems of the broken geodesics approximation of the path space (see [Mi, section 16] or [Kl, section A.1] for the Riemannian case, and [Ra] for the Finsler case). In section III.1 we prove a uniqueness result for curves in the configuration space that connect sufficiently close given points, minimizing the Lagrangian action. This result, as well as the forthcoming arguments in the remaining of the chapter, will be valid for the class of convex quadratic-growth Lagrangian functions, introduced in the previous chapter. In section III.2 we introduce our discretization technique, that basically consists in reducing our analysis to the spaces Λ_k, for each $k \in \mathbb{N}$, of continuous loops that are k-broken solutions of the Euler-Lagrange system. These loop spaces are finite dimensional submanifolds of $W^{1,2}(\mathbb{T}; M)$, and in particular all the reasonable topologies coincide on them. In section III.3 we define the discrete action functional, that roughly speaking is the restriction of the action functional to the broken Euler-Lagrange loop spaces. In particular, we prove that it is smooth and that it has compact sublevels, showing that it is suitable for a Morse theoretic analysis. Motivated by these facts, in section III.4 we study its critical points, proving that they correspond to critical points of the full action functional. Moreover, we prove that, up to choosing a sufficiently big discretization integer k, the Morse index and nullity pair of corresponding critical points of the action and the discrete action functionals are the same. In section III.5 we show that our discretization technique can be used to build finite dimensional homotopic approximations of the action sublevels and, more importantly, that the action and the discrete action functionals have the same local homology groups (again, up to choosing a sufficiently big discretization integer k). Finally, in section III.6, we remark how to extend all the given arguments to the case of periodic curves of arbitrary integer period.

III.1 Uniqueness of the action minimizers

Throughout this chapter, M will be a fixed m-dimensional smooth closed manifold, endowed with a Riemannian metric $\langle \cdot, \cdot \rangle$. that turns it into a complete metric space with respect to the induced Riemannian distance $\mathrm{dist} : M \times M \to [0, \infty)$.

If $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ is a smooth 1-periodic Tonelli Lagrangian, for every bounded interval $[t_0, t_1] \subset \mathbb{R}$ and for every absolutely continuous curve $\zeta : [t_0, t_1] \to M$, we denote by $\mathscr{A}^{t_0,t_1}(\zeta)$ the usual Lagrangian action of the curve, i.e.

$$\mathscr{A}^{t_0,t_1}(\zeta) = \int_{t_0}^{t_1} \mathscr{L}(t, \zeta(t), \dot{\zeta}(t))\, \mathrm{d}t \in \mathbb{R} \cup \{+\infty\}\,.$$

In section I.2 we have seen that the Tonelli assumptions guarantee that, for each $[t_0, t_1] \subset \mathbb{R}$ and for each $q_0, q_1 \in M$ there exists a (smooth) action minimizer parametrized on $[t_0, t_1]$, with endpoints q_0 and q_1. A more ancient result, that goes back to Weierstrass, states that each sufficiently short minimizer is **unique**, meaning that it is the only curve between its given endpoints that minimizes the action. In modern language, the statement goes as follows (we refer the reader to [Ma, page 175] or [Fa, page 106] for a proof).

Theorem III.1 (Weierstrass). *Let $\mathscr{L}$ be a smooth 1-periodic Tonelli Lagrangian. For each constant $C > 0$ there exist an $\varepsilon_0 > 0$ such that, for each interval $[t_0, t_1] \subset \mathbb{R}$ with $t_1 - t_0 = \varepsilon_0$ and for all $q_0, q_1 \in M$ with $\mathrm{dist}(q_0, q_1) < C\varepsilon_0$, there is a unique action minimizer (with respect to $\mathscr{L}$) $\gamma_{q_0,q_1} : [t_0, t_1] \to M$ with $\gamma_{q_0,q_1}(t_0) = q_0$ and $\gamma_{q_0,q_1}(t_1) = q_1$.*

For our purposes, we will need the following stronger version of this result, that holds only for the smaller class of convex quadratic-growth Lagrangians introduced in section II.3.

Theorem III.2 (Uniqueness of the action minimizers). *Let $\mathscr{L}$ be a smooth 1-periodic convex quadratic-growth Lagrangian. There exist two positive constants $\varepsilon_0 = \varepsilon_0(\mathscr{L}) > 0$ and $\rho_0 = \rho_0(\mathscr{L}) > 0$ such that, for each interval $[t_0, t_1] \subset \mathbb{R}$ with $0 < t_1 - t_0 \leq \varepsilon_0$ and for all $q_0, q_1 \in M$ with $\mathrm{dist}(q_0, q_1) < \rho_0$, there is a unique action minimizer (with respect to $\mathscr{L}$) $\gamma_{q_0,q_1} : [t_0, t_1] \to M$ with $\gamma_{q_0,q_1}(t_0) = q_0$ and $\gamma_{q_0,q_1}(t_1) = q_1$.*

Proof. As we have already shown in section II.3, without loss of generality we can assume that there exist two positive constants $\underline{\ell} < \overline{\ell}$ such that

$$\underline{\ell}\, |v|_q^2 \leq \mathscr{L}(t,q,v) \leq \overline{\ell}(|v|_q^2 + 1), \qquad \forall q \in M, v \in \mathrm{T}_q M. \tag{III.1}$$

Consider two points $q_0, q_1 \in M$ and two real numbers $t_0 < t_1$. We put

$$\rho := \mathrm{dist}(q_0, q_1), \qquad \varepsilon := t_1 - t_0.$$

Since $W^{1,2}([t_0,t_1];M)$ is dense in the space of absolutely continuous maps from $[t_0,t_1]$ to M and since the action minimizers are smooth (cf. theorem I.1(ii)), a curve γ_{q_0,q_1} as in the statement is an action minimizer if and only if it is a global minimum of $\mathscr{A}^{t_0,t_1}$ over the space

$$\mathscr{W}_{q_0,q_1}^{t_0,t_1} = \left\{ \zeta \in W^{1,2}([t_0,t_1];M) \,|\, \zeta(t_0) = q_0,\ \zeta(t_1) = q_1 \right\}.$$

Therefore, all we have to do in order to prove the statement is to show that, for ρ and ε sufficiently small, the functional $\mathscr{A}^{t_0,t_1}|_{\mathscr{W}_{q_0,q_1}^{t_0,t_1}}$ admits a unique global minimum.

Consider an arbitrary real constant $\mu > 1$. By compactness, the manifold M admits a finite atlas $\mathfrak{U} = \{(U_\alpha, \phi_\alpha) \,|\, \alpha = 1, ..., u\}$ such that for all $\alpha \in \{1, ..., u\}$, $q, q' \in U_\alpha$ and $v \in \mathrm{T}_q M$ we have

$$\mu^{-1} \left| \phi_\alpha(q) - \phi_\alpha(q') \right| \leq \mathrm{dist}(q,q') \leq \mu \left| \phi_\alpha(q) - \phi_\alpha(q') \right|, \tag{III.2}$$

$$\mu^{-1} \left| \mathrm{d}\phi_\alpha(q) v \right| \leq |v|_q \leq \mu \left| \mathrm{d}\phi_\alpha(q) v \right|, \tag{III.3}$$

where we denote by $|\cdot|$ the standard norm in $\mathbb{R}^m$ and by $|\cdot|_q$ the Riemannian norm in $\mathrm{T}_q M$ as usual. Moreover, we can further assume that the image $\phi_\alpha(U_\alpha)$ of every chart is a convex subset of $\mathbb{R}^m$ (e.g. a ball). Let $\mathrm{Leb}(\mathfrak{U})$ denote the Lebesgue number[1] of the atlas $\mathfrak{U}$ and consider the two points $q_0, q_1 \in M$ of the beginning with $\mathrm{dist}(q_0,q_1) = \rho$. By definition of Lebesgue number, the Riemannian closed ball

$$\overline{B(q_0, \mathrm{Leb}(\mathfrak{U})/2)} = \{ q \in M \,|\, \mathrm{dist}(q, q_0) \leq \mathrm{Leb}(\mathfrak{U})/2 \}$$

is contained in a coordinate open set U_α for some $\alpha \in \{1, ..., u\}$. Therefore if we require that $\rho \leq \mathrm{Leb}(\mathfrak{U})/2$ the points q_0 and q_1 lie in the same open set U_α.

Let $r : [t_0, t_1] \to U_\alpha$ be the segment from q_0 to q_1 given by

$$r(t) = \phi_\alpha^{-1} \left(\frac{t_1 - t}{\varepsilon}\, \phi_\alpha(q_0) + \frac{t - t_0}{\varepsilon}\, \phi_\alpha(q_1) \right), \qquad \forall t \in [t_0, t_1].$$

[1]We recall that, for every open cover $\mathfrak{U}$ of a compact metric space, there exists a positive number $\mathrm{Leb}(\mathfrak{U}) > 0$, the **Lebesgue number** of $\mathfrak{U}$, such that every subset of the metric space of diameter less than $\mathrm{Leb}(\mathfrak{U})$ is contained in some member of the cover $\mathfrak{U}$.

By (III.1), (III.2) and (III.3) we obtain the following upper bound for the action of the curve ℓ

$$\begin{aligned}\mathscr{A}^{t_0,t_1}(r) &\leq \overline{\ell}\left(\int_{t_0}^{t_1}|\dot r(t)|^2_{r(t)}\mathrm{d}t+\varepsilon\right)\leq \overline{\ell}\left(\varepsilon\max_{t\in[t_0,t_1]}\left\{|\dot r(t)|^2_{r(t)}\right\}+\varepsilon\right)\\ &\leq \overline{\ell}\left(\mu^2\frac{|\phi_\alpha(q_1)-\phi_\alpha(q_0)|^2}{\varepsilon}+\varepsilon\right)\leq \overline{\ell}\left(\mu^4\frac{\mathrm{dist}(q_0,q_1)^2}{\varepsilon}+\varepsilon\right)\\ &\leq \overline{\ell}\mu^4\left(\frac{\rho^2}{\varepsilon}+\varepsilon\right)=C\left(\frac{\rho^2}{\varepsilon}+\varepsilon\right),\end{aligned}$$

where the positive constant $C=\overline{\ell}\mu^4$ does not depend on q_0,q_1 and $[t_0,t_1]$. This estimate, in turn, serves as an upper bound for the action of the minima, i.e.

$$\min_{\zeta\in\mathscr{W}^{t_0,t_1}_{q_0,q_1}}\left\{\mathscr{A}^{t_0,t_1}(\zeta)\right\}\leq C\left(\frac{\rho^2}{\varepsilon}+\varepsilon\right),$$

therefore the action sublevel

$$\text{(III.4)}\qquad \mathscr{U}^{t_0,t_1}_{q_0,q_1}=\mathscr{U}^{t_0,t_1}_{q_0,q_1}(\rho,\varepsilon)=\left\{\zeta\in\mathscr{W}^{t_0,t_1}_{q_0,q_1}\mid \mathscr{A}^{t_0,t_1}(\zeta)\leq C\left(\frac{\rho^2}{\varepsilon}+\varepsilon\right)\right\}$$

is not empty and it must contain a global minimum γ_{q_0,q_1} of the action (the existence of a minimum is guaranteed by theorem I.1(i)). All we have to do in order to conclude is to show that, for ρ and ε sufficiently small, the sublevel $\mathscr{U}^{t_0,t_1}_{q_0,q_1}=\mathscr{U}^{t_0,t_1}_{q_0,q_1}(\rho,\varepsilon)$ cannot contain other minima of the action.

By the first inequality in (III.1) we have

$$\int_{t_0}^{t_1}|\dot\zeta(t)|^2_{\zeta(t)}\mathrm{d}t\leq \underline{\ell}^{-1}\mathscr{A}^{t_0,t_1}(\zeta),\qquad \forall\zeta\in\mathscr{W}^{t_0,t_1}_{q_0,q_1},$$

and this, in turn, gives the following bound for all $\zeta\in\mathscr{U}^{t_0,t_1}_{q_0,q_1}$

$$\begin{aligned}\max_{t\in[t_0,t_1]}\mathrm{dist}(\zeta(t_0),\zeta(t))^2 &\leq \left(\int_{t_0}^{t_1}|\dot\zeta(t)|_{\zeta(t)}\mathrm{d}t\right)^2\leq\varepsilon\int_{t_0}^{t_1}|\dot\zeta(t)|^2_{\zeta(t)}\mathrm{d}t\\ &\leq \varepsilon\underline{\ell}^{-1}\mathscr{A}^{t_0,t_1}(\zeta)\leq C\underline{\ell}^{-1}(\rho^2+\varepsilon^2).\end{aligned}$$

Therefore all the curves $\zeta\in\mathscr{U}_{q_0,q_1}(\rho,\varepsilon)$ have image inside the coordinate open set $U_\alpha\subseteq M$ provided ρ and ε are sufficiently small, more precisely for

$$\text{(III.5)}\qquad \rho^2+\varepsilon^2\leq\frac{\underline{\ell}}{4C}\mathrm{Leb}(\mathfrak{U})^2.$$

This allows us to restrict our attention to the open set U_α. From now on we will briefly identify U_α with $\phi_\alpha(U_\alpha)\subseteq\mathbb{R}^m$, so that

$$q_0\equiv\phi_\alpha(q_0)\in\mathbb{R}^m,\qquad q_1\equiv\phi_\alpha(q_1)\in\mathbb{R}^m.$$

Without loss of generality we can also assume that $q_0 \equiv \phi_\alpha(q_0) = \mathbf{0} \in \mathbb{R}^m$. On the set $U_\alpha \equiv \phi_\alpha(U_\alpha)$ we will consider the standard flat norm $|\cdot|$ of $\mathbb{R}^m$, and the norms $\|\cdot\|_{L^1}$, $\|\cdot\|_{L^2}$ and $\|\cdot\|_{L^\infty}$ will be computed using this norm. We will also consider $\mathscr{L}$ as a convex quadratic-growth Lagrangian of the form

$$\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \phi_\alpha(U_\alpha) \times \mathbb{R}^m \to \mathbb{R}$$

by means of the identification

$$\mathscr{L}(t, q, v) \equiv \mathscr{L}(t, \phi_\alpha^{-1}(q), \mathrm{d}\phi_\alpha^{-1}(\phi_\alpha(q))v).$$

Now, consider the following close convex subset of $W^{1,2}([t_0, t_1]; \mathbb{R}^m)$

$$\text{(III.6)} \quad \mathscr{C}^{t_0,t_1}_{q_0,q_1} = \mathscr{C}^{t_0,t_1}_{q_0,q_1}(\rho, \varepsilon) = \Big\{ \zeta \in W^{1,2}([t_0, t_1]; \mathbb{R}^m) \,\Big|\, \zeta(t_0) = q_0 = \mathbf{0},\ \zeta(t_1) = q_1,\ \|\dot\zeta\|^2_{L^2} \leq \mu C \underline{\ell}^{-1} \left(\frac{\rho^2}{\varepsilon} + \varepsilon \right) \Big\}.$$

Since $\|\zeta\|^2_{L^\infty} \leq \varepsilon \|\dot\zeta\|^2_{L^2}$, for ρ and ε sufficiently small all the curves $\zeta \in \mathscr{C}^{t_0,t_1}_{q_0,q_1}$ have support inside the open set U_α. Moreover, by (III.1), (III.3) and (III.4) we have

$$\|\dot\zeta\|^2_{L^2} \leq \mu \int_{t_0}^{t_1} |\dot\zeta(t)|^2_{\zeta(t)} \mathrm{d}t \leq \mu \underline{\ell}^{-1} \mathscr{A}^{t_0,t_1}(\zeta) \leq \mu C \underline{\ell}^{-1} \left(\frac{\rho^2}{\varepsilon} + \varepsilon \right), \qquad \forall \zeta \in \mathscr{U}^{t_0,t_1}_{q_0,q_1},$$

that implies $\mathscr{U}^{t_0,t_1}_{q_0,q_1} \subseteq \mathscr{C}^{t_0,t_1}_{q_0,q_1}$. Now, since we know that a minimum γ_{q_0,q_1} of the action exists and all the minima lie in the convex closed $\mathscr{C}^{t_0,t_1}_{q_0,q_1} \subseteq W^{1,2}([t_0, t_1]; \mathbb{R}^m)$, in order to conclude that γ_{q_0,q_1} is the unique minimum we only need to show that the Hessian of the action is positive definite on $\mathscr{C}^{t_0,t_1}_{q_0,q_1}$ provided ρ and ε are sufficiently small, i.e. we need to show that there exist $\rho_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\rho \in (0, \rho_0)$ and $\varepsilon \in (0, \varepsilon_0]$, we have

$$\text{(III.7)} \qquad \mathrm{Hess}\mathscr{A}^{t_0,t_1}(\zeta)[\sigma, \sigma] > 0, \qquad \forall \zeta \in \mathscr{C}^{t_0,t_1}_{q_0,q_1} = \mathscr{C}^{t_0,t_1}_{q_0,q_1}(\rho, \varepsilon),\ \sigma \in W^{1,2}_0([t_0, t_1]; \mathbb{R}^m).$$

Notice that the above Hessian is well defined, since $\mathscr{A}^{t_0,t_1}$ is C^1 and twice Gateaux differentiable (see proposition II.4). In (III.7) we denoted by $W^{1,2}_0([t_0, t_1]; \mathbb{R}^m)$ the tangent space of $\mathscr{C}^{t_0,t_1}_{q_0,q_1}$ at ζ, i.e.

$$W^{1,2}_0([t_0, t_1]; \mathbb{R}^m) = \left\{ \sigma \in W^{1,2}([t_0, t_1]; \mathbb{R}^m) \,|\, \sigma(t_0) = \sigma(t_1) = \mathbf{0} \right\},$$

Consider arbitrary $\zeta \in \mathscr{C}^{t_0,t_1}_{q_0,q_1}$ and $\sigma \in W^{1,2}_0([t_0, t_1]; \mathbb{R}^m)$. Then, we have

$$\begin{aligned}
&\mathrm{Hess}\mathscr{A}^{t_0,t_1}(\zeta)[\sigma, \sigma] \\
&\quad = \int_{t_0}^{t_1} \Big(\langle \partial^2_{vv}\mathscr{L}(t, \zeta, \dot\zeta)\dot\sigma, \dot\sigma \rangle + 2 \langle \partial^2_{vq}\mathscr{L}(t, \zeta, \dot\zeta)\sigma, \dot\sigma \rangle + \langle \partial^2_{qq}\mathscr{L}(t, \zeta, \dot\zeta)\sigma, \sigma \rangle \Big) \mathrm{d}t \\
&\quad \geq \int_{t_0}^{t_1} \ell_0 \, |\dot\sigma|^2 \, \mathrm{d}t - \underbrace{\int_{t_0}^{t_1} 2\ell_1 (1 + \mu|\dot\zeta|) \, |\sigma| \, |\dot\sigma| \, \mathrm{d}t}_{=: I_1} - \underbrace{\int_{t_0}^{t_1} \ell_1 (1 + \mu^2 |\dot\zeta|^2) \, |\sigma|^2 \, \mathrm{d}t}_{=: I_2},
\end{aligned}$$

where ℓ_0 and ℓ_1 are the positive constants that appear in **(Q1)** and **(Q2)** with respect to the atlas $\mathfrak{U}$. Now, the quantities I_1 and I_2 can be estimated from above as follows

$$\begin{aligned} I_1 &\leq 2\ell_1\mu\|\sigma\|_{L^\infty}\left(\|\dot\sigma\|_{L^1} + \left\||\dot\zeta|\cdot|\dot\sigma|\right\|_{L^1}\right)\\ &\leq 2\ell_1\mu\sqrt{\varepsilon}\|\dot\sigma\|_{L^2}\left(\sqrt{\varepsilon}\|\dot\sigma\|_{L^2} + \|\dot\zeta\|_{L^2}\|\dot\sigma\|_{L^2}\right)\\ &= 2\ell_1\mu\|\dot\sigma\|^2_{L^2}\left(\varepsilon + \sqrt{\varepsilon}\|\dot\zeta\|_{L^2}\right),\\ I_2 &\leq \ell_1\mu^2\left(\|\sigma\|^2_{L^2} + \|\sigma\|^2_{L^\infty}\|\dot\zeta\|^2_{L^2}\right) \leq \ell_1\mu^2\|\dot\sigma\|^2_{L^2}\left(\varepsilon^2 + \varepsilon\|\dot\zeta\|^2_{L^2}\right), \end{aligned}$$

and since by (III.6) we have

$$\|\dot\zeta\|^2_{L^2} \leq \mu C\underline{\ell}^{-1}\left(\frac{\rho^2}{\varepsilon} + \varepsilon\right),$$

we conclude

$$\begin{aligned} &\mathrm{Hess}\mathscr{A}^{t_0,t_1}(\zeta)[\sigma,\sigma]\\ &\quad\geq \ell_0\|\dot\sigma\|^2_{L^2} - I_1 - I_2\\ &\quad\geq \|\dot\sigma\|^2_{L^2}\underbrace{\left(\ell_0 - 2\ell_1\mu\left(\sqrt{\mu C\underline{\ell}^{-1}} + 1\right)(\rho+\varepsilon) - \ell_1\mu^2\left(\mu C\underline{\ell}^{-1} + 1\right)(\rho^2+\varepsilon^2)\right)}_{=:\,F(\rho,\varepsilon)}. \end{aligned}$$

Notice that the quantity $F(\rho,\varepsilon)$ is independent of the specific choice of the points q_0, q_1 and of the interval $[t_0,t_1]$, but depends only on $\rho = \mathrm{dist}(q_0,q_1)$ and $\varepsilon = t_1 - t_0$. Moreover, there exist $\rho_0 > 0$ and $\varepsilon_0 > 0$ small enough so that for all $\rho \in (0,\rho_0)$ and $\varepsilon \in (0,\varepsilon_0]$ the quantity $F(\rho,\varepsilon)$ is positive. This proves (III.7). ■

Now, we want to remark that the unique action minimizers γ_{q_0,q_1}, whose existence is asserted by the previous theorem, depend smoothly on their endpoints q_0 and q_1. If ρ_0 is the constant given by theorem III.2, we denote by $\Delta(\rho_0)$ the open neighborhood of the diagonal submanifold of $M\times M$ given by

$$\Delta(\rho_0) = \{(q_0,q_1)\in M\times M \,|\, \mathrm{dist}(q_0,q_1) < \rho_0\}.$$

Theorem III.3 (Smooth dependence on endpoints). *With the notation of theorem* III.2, *for each real interval* $[t_0,t_1]\subset\mathbb{R}$ *with* $0 < t_1 - t_0 \leq \varepsilon_0$ *the assignment*

$$(q_0,q_1)\mapsto\gamma_{q_0,q_1}:[t_0,t_1]\to M \tag{III.8}$$

defines a smooth map $\Delta(\rho_0)\to C^\infty([t_0,t_1];M)$.

Proof. By theorem I.1(ii), every action minimizer is smooth. Therefore, (III.8) defines a map of the form $\Delta(\rho_0)\to C^\infty([t_0,t_1];M)$, and we just need to show that the dependence of γ_{q_0,q_1} from (q_0, q_1) is smooth. For $\varepsilon := t_1 - t_0 \in (0, \varepsilon_0]$ and $(q_0, q_1) \in \Delta(\rho_0)$, in the proof of theorem III.2 we have already shown that the minimizer $\gamma_{q_0,q_1} : [t_0, t_1] \to M$ has image contained in a coordinate neighborhood $U_\alpha \subseteq M$ that we can identify with an open set of $\mathbb{R}^m$. The curve γ_{q_0,q_1} is a smooth solution of the Euler-Lagrange system of $\mathscr{L}$. Therefore,

$$\Phi^{t,t_0}_{\mathscr{L}}(q_0, v_0) = (\gamma_{q_0,q_1}(t), \dot\gamma_{q_0,q_1}(t)), \qquad \forall t \in [t_0, t_1],$$

where $v_0 = \dot\gamma_{q_0,q_1}(t_0)$ and $\Phi^{t,t_0}_{\mathscr{L}}$ is the Euler-Lagrange flow associated to $\mathscr{L}$ (see section I.1). We define

$$Q^{t,t_0}_{\mathscr{L}} := \pi \circ \Phi^{t,t_0}_{\mathscr{L}} : U'_\alpha \times V'_\alpha \to U_\alpha, \qquad \forall t \in [t_0, t_1],$$

where $U'_\alpha \subset U_\alpha$ is a small neighborhood of q_0, $V'_\alpha \subset \mathbb{R}^m$ is a small neighborhood of v_0, and $\pi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection onto the first m components, i.e. $\pi(q, v) = q$ for all $(q, v) \in \mathbb{R}^m \times \mathbb{R}^m$. We claim that

$$\text{(III.9)} \qquad \mathrm{d}Q^{t_1,t_0}_{\mathscr{L}}(q_0, v_0)(\{\mathbf{0}\} \times \mathbb{R}^m) = \mathbb{R}^m.$$

In fact, assume by contradiction that (III.9) does not hold. Then there exists a nonzero vector $v \in \mathbb{R}^m$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} Q^{t_1,t_0}_{\mathscr{L}}(q_0, v_0 + sv) = \mathbf{0}.$$

If we define the curve $\sigma : [t_0, t_1] \to \mathbb{R}^m$ by

$$\sigma(t) := \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} Q^{t,t_0}_{\mathscr{L}}(q_0, v_0 + sv),$$

then $\sigma(t_0) = \sigma(t_1) = \mathbf{0}$, and it is readily seen that σ is a solution of the linearized Euler-Lagrange system

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\partial^2_{vv}\mathscr{L}(t, \gamma_{q_0,q_1}, \dot\gamma_{q_0,q_1})\dot\sigma + \partial^2_{vq}\mathscr{L}(t, \gamma_{q_0,q_1}, \dot\gamma_{q_0,q_1})\sigma\big) \\ - \partial^2_{qv}\mathscr{L}(t, \gamma_{q_0,q_1}, \dot\gamma_{q_0,q_1})\dot\sigma - \partial^2_{qq}\mathscr{L}(t, \gamma_{q_0,q_1}, \dot\gamma_{q_0,q_1})\sigma = 0,$$

hence

$$\mathrm{Hess}\mathscr{A}^{t_0,t_1}(\gamma_{q_0,q_1})[\sigma, \sigma] = 0.$$

This contradicts the positive definitiveness of the Hessian of $\mathscr{A}^{t_0,t_1}$ on γ_{q_0,q_1} (see (III.7) in the proof of theorem III.2), and therefore (III.9) must hold.

By the implicit function theorem we obtain a neighborhood $U_{q_0,q_1} \subset \mathbb{R}^m \times \mathbb{R}^m$ of (q_0, q_1), a neighborhood $U_{v_0} \subset \mathbb{R}^m$ of v_0 and a smooth map $V_0 : U_{q_0,q_1} \to U_{v_0}$ such that, for each $(q'_0, q'_1, v'_0) \in U_{q_0,q_1} \times U_{v_0}$, we have $Q^{t_1,t_0}_{\mathscr{L}}(q'_0, v'_0) = q'_1$ if and only if $v'_0 = V_0(q'_0, q'_1)$. Then, we can define a smooth map from U_{q_0,q_1} to $C^\infty([t_0, t_1]; U_\alpha)$ given by

$$\text{(III.10)} \qquad (q'_0, q'_1) \mapsto \zeta_{q'_0,q'_1},$$

where for each $t \in [t_0, t_1]$ we have

$$\zeta_{q'_0,q'_1}(t) = Q^{t,t_0}_{\mathscr{L}}(q'_0, V_0(q'_0, q'_1)).$$

In order to conclude we only have to show that the map in (III.10) coincides with the one in (III.8) on U_{q_0,q_1} provided this latter neighborhood is small enough, i.e. we have to show that $\zeta_{q'_0,q'_1}$ is a unique action minimizer for all (q'_0, q'_1) in a sufficiently small neighborhood U_{q_0,q_1} of (q_0, q_1). This is easily seen as follows. By construction, the curves $\zeta_{q'_0,q'_1}$ are critical points of the action $\mathscr{A}^{t_0,t_1}$ over the space $W^{1,2}_{q'_0,q'_1}$, being solutions of the Euler-Lagrange system of $\mathscr{L}$. By the arguments in the proof of theorem III.2, each of these curves $\zeta_{q'_0,q'_1}$ is a unique action minimizer if and only if it lies in the convex set $\mathscr{C}^{t_0,t_1}_{q'_0,q'_1}$ defined in (III.6). We already know that $\zeta_{q_0,q_1} = \gamma_{q_0,q_1} \in \mathscr{C}^{t_0,t_1}_{q_0,q_1}$. Since the map in (III.10) is smooth, for (q'_0, q'_1) close to (q_0, q_1) we obtain that the curve $\zeta_{q'_0,q'_1}$ is C^1-close to $\zeta_{q_0,q_1} = \gamma_{q_0,q_1}$, and therefore $\zeta_{q'_0,q'_1} \in \mathscr{C}^{t_0,t_1}_{q'_0,q'_1}$. ■

III.2 Broken Euler-Lagrange loop space

Let $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ be a smooth 1-periodic convex quadratic-growth Lagrangian. We consider the positive constants $\varepsilon_0 = \varepsilon_0(\mathscr{L})$ and $\rho_0 = \rho_0(\mathscr{L})$ given by theorem III.2 and, for each $k \in \mathbb{N}$, we denote by $\Delta_k = \Delta_k(\rho_0(\mathscr{L}))$ the neighborhood of the diagonal submanifold of the k-fold product $M \times ... \times M$ given by

$$\Delta_k := \{(q_0, ..., q_{k-1}) \in M \times ... \times M \,|\, \mathrm{dist}(q_j, q_{j+1}) < \rho_0 \ \forall j \in \mathbb{Z}_k\},$$

where $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ is the cyclic group of order k.

Now, we consider the space $C^\infty_k(\mathbb{T}; M)$ of **continuous and k-broken smooth loops**. Namely, this space consists of continuous loops $\zeta : \mathbb{T} \to M$ such that, for each $j \in \mathbb{Z}_k$, the restriction $\zeta|_{[j/k,(j+1)/k]}$ is smooth. We can endow $C^\infty_k(\mathbb{T}; M)$ with a topology that turns it into a Fréchet manifold: an easy way to do it is to view $C^\infty_k(\mathbb{T}; M)$ as a closed subset of the following product of Fréchet manifolds (see for instance [Ham, section I.4])

$$C^\infty([0, \tfrac{1}{k}]; M) \times C^\infty([\tfrac{1}{k}, \tfrac{2}{k}]; M) \times ... \times C^\infty([\tfrac{k-1}{k}, 1]; M).$$

Notice that $C^\infty_k(\mathbb{T}; M)$ continuously embeds into the Hilbert manifold $W^{1,2}(\mathbb{T}; M)$. By theorem III.3, for each $k \geq 1/\varepsilon_0(\mathscr{L})$, we can define a smooth embedding

$$\text{(III.11)} \qquad \lambda_k = \lambda_{k,\mathscr{L}} : \Delta_k \hookrightarrow C^\infty_k(\mathbb{T}; M)$$

in the following way: for each $\boldsymbol{q} = (q_0, ..., q_{k-1}) \in \Delta_k$ we put $\lambda_k(\boldsymbol{q}) := \gamma_{\boldsymbol{q}}$, where $\gamma_{\boldsymbol{q}}$ is the loop whose restrictions $\gamma_{\boldsymbol{q}}|_{[j/k,(j+1)/k]}$ are the unique action minimizers (with respect to $\mathscr{L}$) with endpoints q_j and q_{j+1}, for each $j \in \mathbb{Z}_k$. We define the **k-broken Euler-Lagrange loop space** (with respect to $\mathscr{L}$) as the image

$$\Lambda_k = \Lambda_{k,\mathscr{L}} := \lambda_k(\Delta_k)$$

of this embedding. Therefore Λ_k is a smooth submanifold of $C^\infty_k(\mathbb{T}; M)$ (and of $W^{1,2}(\mathbb{T}; M)$) with finite dimension km, where m is the dimension of M.

III.3 The discrete action functional

Let $\mathscr{A} : W^{1,2}(\mathbb{T};M) \to \mathbb{R}$ be the Lagrangian action functional associated to the Lagrangian $\mathscr{L}$ of the previous section, i.e.

$$\mathscr{A}(\zeta) = \int_0^1 \mathscr{L}(t, \zeta(t), \dot{\zeta}(t))\,\mathrm{d}t, \qquad \forall \zeta \in W^{1,2}(\mathbb{T};M).$$

Restricting $\mathscr{A}$ to the finite dimensional k-broken Euler-Lagrange loop space Λ_k, we obtain a sort of discrete-time version[2] of the Lagrangian action functional. We define the **discrete action functional** $\mathscr{A}_k : \Delta_k \to \mathbb{R}$ as the composition $\mathscr{A} \circ \lambda_k$, i.e. with the notations introduced in the previous section

$$\mathscr{A}_k(\boldsymbol{q}) = \int_0^1 \mathscr{L}(t, \gamma_{\boldsymbol{q}}(t), \dot{\gamma}_{\boldsymbol{q}}(t))\,\mathrm{d}t, \qquad \forall \boldsymbol{q} \in \Delta_k.$$

In the following we will only consider the connected components of Δ_k given by the $\boldsymbol{q}$'s such that the pull-back bundle $\lambda_k(\boldsymbol{q})^*\mathrm{T}M = \gamma_{\boldsymbol{q}}^*\mathrm{T}M$ is trivial. In view of the application of this discretization technique to prove the existence of infinitely many periodic orbits (cf. chapter V), this is not restrictive: in fact we will only look for contractible periodic orbits. We also remark that $\gamma_{\boldsymbol{q}}^*\mathrm{T}M$ is always a trivial bundle if the manifold M is orientable: in fact, if this is verified, $\gamma_{\boldsymbol{q}}^*\mathrm{T}M$ is an orientable vector bundle over the circle, and therefore it is trivial. From now on, we will just denote by Δ_k the above mentioned connected components, i.e.

$$\Delta_k := \left\{\boldsymbol{q} = (q_0, ..., q_{k-1}) \in M \times ... \times M \,|\, \mathrm{dist}(q_j, q_{j+1}) < \rho_0 \ \forall j \in \mathbb{Z}_k,\ \gamma_{\boldsymbol{q}}^*\mathrm{T}M \text{ trivial}\right\},$$

and we will denote by Λ_k the image $\lambda_k(\Delta_k)$.

Remark III.1 (Localization). In several arguments of a local nature, it will be useful to adopt suitable local coordinates on Δ_k, analogously to what we did in section II.3 (see in particular the proof of proposition II.4). For the reader convenience, we explain the details in this remark.

For each $\boldsymbol{q'} \in \Delta_k$, we know from section II.1 that the loop $\gamma_{\boldsymbol{q'}}$ belongs to the domain of some chart $\Theta : \mathscr{U} \to W^{1,2}(\mathbb{T};\mathbb{R}^m)$ of $W^{1,2}(\mathbb{T};M)$ given by

$$\Theta(\zeta)(t) = \exp_\gamma^{-1}(\zeta)(t) = \exp_{\gamma(t)}^{-1}(\zeta(t)), \qquad \forall \zeta \in \mathscr{U},\ t \in \mathbb{T},$$

where $\gamma : \mathbb{T} \to M$ is a smooth loop that is C^0-close to $\gamma_{\boldsymbol{q'}}$ and we are identifying $\gamma^*\mathrm{T}M$ with the trivial bundle $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^m$. Namely we are assuming that, for each $t \in \mathbb{R}/\mathbb{Z}$, the exponential map $\exp_{\gamma(t)}$ is a diffeomorphism from an open neighborhood $U \subset \mathbb{R}^m$ of the origin onto an open neighborhood of $\gamma(t)$ in M. Up to restricting the neighborhood $\mathscr{U}$ of $\gamma_{\boldsymbol{q'}}$, we can assume that $\Theta(\mathscr{U}) = W^{1,2}(\mathbb{T};U) \subset W^{1,2}(\mathbb{T};\mathbb{R}^m)$.

[2]In 1984, Chaperon [Ch2] introduced a discrete-time version of the Hamiltonian action functional. His approach was based on generating functions, and had several applications in symplectic topology. See the book of McDuff and Salamon [MS, section 9.2] and the bibliography therein.

We define $U_k := \lambda_k^{-1}(\mathscr{U})$. Notice that this latter is an open neighborhood of $\boldsymbol{q}'$ in Δ_k. On this open set we can build a chart $\vartheta_k : U_k \to \mathbb{R}^m \times ... \times \mathbb{R}^m$ for Δ_k by

$$\vartheta_k(\boldsymbol{q}) = \Big(\exp^{-1}_{\gamma(0)}(q_0), \exp^{-1}_{\gamma(1/k)}(q_1), ..., \exp^{-1}_{\gamma((k-1)/k)}(q_{k-1})\Big)\,, \qquad \forall \boldsymbol{q} = (q_0, ..., q_{k-1}) \in U_k.$$

If we put $W_k := \vartheta_k(U_k)$, we univocally obtain an embedding $\tilde{\lambda}_k$ such that the following diagram commutes.

$$\text{(III.12)} \qquad \begin{array}{ccccccc} \Delta_k & \hookleftarrow & U_k & \xrightarrow[\simeq]{\vartheta_k} & W_k & \hookrightarrow & \mathbb{R}^m \times ... \times \mathbb{R}^m \\ & & \Big\downarrow \lambda_k & & \Big\downarrow \tilde{\lambda}_k & & \\ W^{1,2}(\mathbb{T}; M) & \hookleftarrow & \mathscr{U} & \xrightarrow[\simeq]{\Theta} & W^{1,2}(\mathbb{T}; U) & \hookrightarrow & W^{1,2}(\mathbb{T}; \mathbb{R}^m) \end{array}$$

Now, we put $\Pi := \Theta^{-1}$ and we define the smooth embedding $\pi : \mathbb{R}/\mathbb{Z} \times U \times \mathbb{R}^m \hookrightarrow \mathbb{R}/\mathbb{Z} \times \mathrm{T}M$ by

$$\pi(t, q, v) = \Big(t, \exp_{\gamma(t)}(q), \mathrm{d}\exp_{\gamma(t)}(q)(v)\Big)\,, \qquad \forall (t, q, v) \in \mathbb{R}/\mathbb{Z} \times U \times \mathbb{R}^m,$$

so that $\mathscr{A} \circ \Pi$ turns out to be the Lagrangian action functional associated to the convex quadratic-growth Lagrangian $\mathscr{L} \circ \pi : \mathbb{R}/\mathbb{Z} \times U \times \mathbb{R}^m \to \mathbb{R}$, i.e.

$$\mathscr{A} \circ \Pi(\sigma) = \int_0^1 \mathscr{L} \circ \pi(t, \zeta(t), \dot{\zeta}(t))\,\mathrm{d}t, \qquad \forall \zeta \in W^{1,2}(\mathbb{T}; U).$$

Then, with the notation of (III.11), the embedding $\tilde{\lambda}_k$ in diagram (III.12) is given by $\lambda_{k,\mathscr{L}\circ\pi}$. Namely, for each $\boldsymbol{q} = (q_0, ..., q_{k-1}) \in W_k$, $\gamma_{\boldsymbol{q}} = \tilde{\lambda}_k(\boldsymbol{q})$ is the unique loop such that $\gamma_{\boldsymbol{q}}|_{[j/k,(j+1)/k]}$ is a unique action minimizer (with respect to $\mathscr{L} \circ \pi$) with endpoints q_j and q_{j+1}, for each $j \in \mathbb{Z}_k$.

Summing up, whenever we work locally near some point $\boldsymbol{q}' \in \Delta_k$ we can assume that $\boldsymbol{q}'$ belongs to some open neighborhood of the origin in the k-fold product $\mathbb{R}^m \times ... \times \mathbb{R}^m$. Moreover, we can assume that the involved convex quadratic-growth Lagrangian is a function of the form $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times U \times \mathbb{R}^m \to \mathbb{R}$, where $U \subset \mathbb{R}^m$ is some open neighborhood of the origin. Under these assumptions, the action functional $\mathscr{A}$ is a functional of the form $\mathscr{A} : W^{1,2}(\mathbb{T}; U) \to \mathbb{R}$ and the discrete action has the form $\mathscr{A}_k = \mathscr{A} \circ \lambda_k : W_k \to \mathbb{R}$, where $W_k = \lambda_k^{-1}(W^{1,2}(\mathbb{T}; U))$ is an open neighborhood of $\boldsymbol{q}'$ in the k-fold product $\mathbb{R}^m \times ... \times \mathbb{R}^m$ and λ_k is defined as usual by $\lambda_k(\boldsymbol{q}) = \gamma_{\boldsymbol{q}}$ for each $\boldsymbol{q} \in W_k$. ■

Proposition III.4. *The discrete action functional $\mathscr{A}_k : \Delta_k \to \mathbb{R}$ is smooth.*

Proof. By the localization argument of remark (III.1), we just need to prove that $\mathscr{A}_k$ is smooth on an open set $W_k \subset \mathbb{R}^m \times ... \times \mathbb{R}^m$. The result can be easily proved by induction. By proposition II.4 we know that $\mathscr{A}$ is C^1 and twice Gateaux differentiable, and so must be $\mathscr{A}_k$. Now, let us assume that $\mathscr{A}_k$ is C^{p-1}, for some integer $p \geq 2$. A straightforward computation shows that the p^{th} Gateaux differential of $\mathscr{A}_k$, seen as a symmetric multilinear form, is given by

$$\begin{aligned}&\mathrm{d}^p \mathscr{A}_k(\boldsymbol{q})\,[\boldsymbol{v}', \boldsymbol{v}'', ..., \boldsymbol{v}^{(p)}]\\ &\qquad = \sum_{j_1,...,j_p=1}^{m} \int_0^1 \left[\partial_{j_1}...\partial_{j_p}\mathscr{L}(t, \gamma_{\boldsymbol{q}}(t), \dot\gamma_{\boldsymbol{q}}(t))\right] \xi^{j_1}_{\boldsymbol{q},\boldsymbol{v}'}(t) ... \xi^{j_p}_{\boldsymbol{q},\boldsymbol{v}^{(p)}}(t)\mathrm{d}t,\\ &\qquad\qquad \forall (\boldsymbol{q}, \boldsymbol{v}', \boldsymbol{v}'', ..., \boldsymbol{v}^{(p)}) \in W_k \times \mathbb{R}^m \times ... \times \mathbb{R}^m,\end{aligned}$$

where $\xi_{\boldsymbol{q},\boldsymbol{v}^{(i)}} : \mathbb{T} \to \mathbb{R}^m \times \mathbb{R}^m$ is defined by

$$\xi_{\boldsymbol{q},\boldsymbol{v}^{(i)}}(t) := \left(\left(\mathrm{d}\lambda_k(\boldsymbol{q})\boldsymbol{v}^{(i)} \right)(t), \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{d}\lambda_k(\boldsymbol{q})\boldsymbol{v}^{(i)} \right)(t) \right), \qquad \forall t \in \mathbb{T},$$

and we have adopted the shortcut notation

$$\partial_j = \begin{cases} \partial_{q^j}, & j \in \{1, ..., m\}, \\ \partial_{v^{j-m}}, & j \in \{m+1, ..., 2m\}. \end{cases}$$

Since $\lambda_k : W_k \hookrightarrow C^\infty_k(\mathbb{T}; M)$ is smooth, if $\boldsymbol{q}_n \to \boldsymbol{q}$ we have that $\gamma_{\boldsymbol{q}_n} \to \gamma_{\boldsymbol{q}}$ in $C^\infty_k(\mathbb{T}; \mathbb{R}^m)$ and $\xi_{\boldsymbol{q}_n,\boldsymbol{v}^{(i)}} \to \xi_{\boldsymbol{q},\boldsymbol{v}^{(i)}}$ in $C^\infty_k(\mathbb{T}; \mathbb{R}^m \times \mathbb{R}^m)$ for each $i \in \{1, ..., p\}$ and $\boldsymbol{v}^{(i)} \in \mathbb{R}^m$. Hence

$$[\partial_{j_1}...\partial_{j_p}\mathscr{L}(\cdot, \gamma_{\boldsymbol{q}_n}, \dot\gamma_{\boldsymbol{q}_n})]\xi^{j_1}_{\boldsymbol{q}_n,\boldsymbol{v}'}...\xi^{j_p}_{\boldsymbol{q}_n,\boldsymbol{v}^{(p)}} \to [\partial_{j_1}...\partial_{j_p}\mathscr{L}(\cdot, \gamma_{\boldsymbol{q}}, \dot\gamma_{\boldsymbol{q}})]\xi^{j_1}_{\boldsymbol{q},\boldsymbol{v}'}...\xi^{j_p}_{\boldsymbol{q},\boldsymbol{v}^{(p)}}$$

uniformly in $t \in \mathbb{T}$, and we conclude that

$$\mathrm{d}^p\mathscr{A}_k(\boldsymbol{q}_n)[\boldsymbol{v}', ..., \boldsymbol{v}^{(p)}] \to \mathrm{d}^p\mathscr{A}_k(\boldsymbol{q})[\boldsymbol{v}', ..., \boldsymbol{v}^{(p)}], \qquad \forall \boldsymbol{v}', ..., \boldsymbol{v}^{(p)} \in \mathbb{R}^m.$$

By the total differential theorem we conclude that $\mathscr{A}_k$ is C^p, and by induction we obtain the claim. ∎

To conclude the section, we want to remark that the discrete action functional $\mathscr{A}_k$ is suitable for Morse theory. In fact, every closed action sublevel is a compact subset of Δ_k, provided the discretization pass $k \in \mathbb{N}$ is big enough.

Proposition III.5. *For each $c \in \mathbb{R}$ there exists $\bar k = \bar k(c) \in \mathbb{N}$ such that, for each $k \geq \bar k$, the closed sublevel $\mathscr{A}_k^{-1}(-\infty, c]$ is compact.*

Proof. Consider the compact subset of Δ_k defined by

$$C_k := \{(q_0, ..., q_{k-1}) \in \Delta_k \,|\, \mathrm{dist}(q_j, q_{j+1}) \leq \rho_0/2 \ \forall j \in \mathbb{Z}_k\}.$$

In order to prove the statement, we just need to show that

$$\lim_{k\to\infty} \min \{\mathscr{A}_k(\boldsymbol{q}) \,|\, \boldsymbol{q} \in \partial C_k\} = \infty.$$

As explained in section II.3, without loss of generality we can assume that there exists a constant $\underline{\ell} > 0$ such that

$$\mathscr{L}(t,q,v) \geq \underline{\ell}\,|v|_q^2, \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M.$$

Then, notice that, for each $\boldsymbol{q} = (q_1, ..., q_k)$ that belongs to the boundary of C_k, we have that $\mathrm{dist}(q_j, q_{j+1}) = \rho_0/2$ for some $j \in \mathbb{Z}_k$ and therefore we obtain the desired estimate

$$\begin{aligned}
\mathscr{A}_k(\boldsymbol{q}) &\geq \int_{j/k}^{(j+1)/k} \mathscr{L}(t, \gamma_{\boldsymbol{q}}(t), \dot\gamma_{\boldsymbol{q}}(t))\,\mathrm{d}t \geq \int_{j/k}^{(j+1)/k} \underline{\ell}\,|\dot\gamma_{\boldsymbol{q}}(t)|^2_{\gamma_{\boldsymbol{q}}(t)}\,\mathrm{d}t \\
&\geq k\,\underline{\ell} \left(\int_{j/k}^{(j+1)/k} |\dot\gamma_{\boldsymbol{q}}(t)|_{\gamma_{\boldsymbol{q}}(t)}\,\mathrm{d}t \right)^2 \geq k\,\underline{\ell}\,\mathrm{dist}(q_j, q_{j+1})^2 \\
&\geq k\,\underline{\ell}\,(\rho_0/2)^2.
\end{aligned}$$

■

III.4 Critical points of the discrete action

We have already seen in the previous chapters that every critical point $\gamma \in W^{1,2}(\mathbb{T}; M)$ of the action functional $\mathscr{A}$ is a smooth solution of the Euler Lagrange system of the convex quadratic-growth Lagrangian $\mathscr{L}$. This implies that, for each $k \in \mathbb{N}$ sufficiently big, the loop γ belongs to the k-broken Euler-Lagrange loop space Λ_k, and therefore the corresponding point $\boldsymbol{q} = \lambda_k^{-1}(\gamma) \in \Delta_k$ is a critical point of the discrete action $\mathscr{A}_k = \mathscr{A} \circ \lambda_k$. Now, we want to study the converse implication, namely that the critical points of the discrete action functional correspond to critical points of the full action functional. In the following, let us definitely assume to work in a coordinate open set of Δ_k that we identify with an open set W_k of the k-fold product $\mathbb{R}^m \times ... \times \mathbb{R}^m$, in such a way that we can adopt the localization argument of remark III.1.

Proposition III.6. *For each $\boldsymbol{q} \in W_k$ and $\boldsymbol{v} = (v_1, ..., v_k) \in \mathbb{R}^m \times ... \times \mathbb{R}^m$ we have*

$$\mathrm{d}\mathscr{A}_k(\boldsymbol{q})\,\boldsymbol{v} = \sum_{j=0}^{k-1} \left\langle \partial_v \mathscr{L}(\tfrac{j}{k}, \gamma_{\boldsymbol{q}}(\tfrac{j}{k}), \dot\gamma_{\boldsymbol{q}}(\tfrac{j^-}{k})) - \partial_v \mathscr{L}(\tfrac{j}{k}, \gamma_{\boldsymbol{q}}(\tfrac{j}{k}), \dot\gamma_{\boldsymbol{q}}(\tfrac{j^+}{k})), v_j \right\rangle,$$

where $\gamma_{\boldsymbol{q}} = \lambda_k(\boldsymbol{q})$.

Proof. For each $\boldsymbol{v} \in \mathbb{R}^m \times ... \times \mathbb{R}^m$, let $\zeta_{\boldsymbol{q},\boldsymbol{v}} := \mathrm{d}\lambda_k(\boldsymbol{q})\boldsymbol{v} \in C_k^\infty(\mathbb{T}; \mathbb{R}^m)$. Since $\gamma_{\boldsymbol{q}}$ is a solution of the Euler-Lagrange system of $\mathscr{L}$ on each interval $[\frac{j}{k}, \frac{j+1}{k}]$, integration by parts gives

$$
\begin{aligned}
\mathrm{d}\mathscr{A}_k(\boldsymbol{q})\,\boldsymbol{v} &= \sum_{j=0}^{k-1}\int_{j/k}^{(j+1)/k}\big(\langle\partial_v\mathscr{L}(t,\gamma_{\boldsymbol{q}},\dot\gamma_{\boldsymbol{q}}),\dot\zeta_{\boldsymbol{q},\boldsymbol{v}}\rangle + \langle\partial_q\mathscr{L}(t,\gamma_{\boldsymbol{q}},\dot\gamma_{\boldsymbol{q}}),\zeta_{\boldsymbol{q},\boldsymbol{v}}\rangle\big)\mathrm{d}t\\
&= \sum_{j=0}^{k-1}\Big[\Big\langle\partial_v\mathscr{L}(\tfrac{j+1}{k},\gamma_{\boldsymbol{q}}(\tfrac{j+1}{k}),\dot\gamma_{\boldsymbol{q}}(\tfrac{j+1^-}{k})),\zeta_{\boldsymbol{q},\boldsymbol{v}}(\tfrac{j+1}{k})\Big\rangle\\
&\qquad - \Big\langle\partial_v\mathscr{L}(\tfrac{j}{k},\gamma_{\boldsymbol{q}}(\tfrac{j}{k}),\dot\gamma_{\boldsymbol{q}}(\tfrac{j^+}{k})),\zeta_{\boldsymbol{q},\boldsymbol{v}}(\tfrac{j}{k})\Big\rangle\\
&\qquad + \int_{j/k}^{(j+1)/k}\langle\underbrace{-\frac{\mathrm{d}}{\mathrm{d}t}\partial_v\mathscr{L}(t,\gamma_{\boldsymbol{q}},\dot\gamma_{\boldsymbol{q}}) + \partial_q\mathscr{L}(t,\gamma_{\boldsymbol{q}},\dot\gamma_{\boldsymbol{q}})}_{=0},\zeta_{\boldsymbol{q},\boldsymbol{v}}\rangle\mathrm{d}t\Big]\\
&= \sum_{j=0}^{k-1}\Big\langle\partial_v\mathscr{L}(\tfrac{j}{k},\gamma_{\boldsymbol{q}}(\tfrac{j}{k}),\dot\gamma_{\boldsymbol{q}}(\tfrac{j^-}{k})) - \partial_v\mathscr{L}(\tfrac{j}{k},\gamma_{\boldsymbol{q}}(\tfrac{j}{k}),\dot\gamma_{\boldsymbol{q}}(\tfrac{j^+}{k})),\zeta_{\boldsymbol{q},\boldsymbol{v}}(\tfrac{j}{k})\Big\rangle\,.
\end{aligned}
$$

Then, by definition of the embedding λ_k, we have

$$
\zeta_{\boldsymbol{q},\boldsymbol{v}}(\tfrac{j}{k}) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\gamma_{\boldsymbol{q}+s\boldsymbol{v}}(\tfrac{j}{k}) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(q_j + sv_j) = v_j, \qquad \forall j\in\mathbb{Z}_k,
$$

and the claim follows. ∎

Corollary III.7. *If $\boldsymbol{q}\in W_k$ is a critical point of $\mathscr{A}_k$, then the corresponding loop $\gamma_{\boldsymbol{q}} = \lambda_k(\boldsymbol{q})$ is a smooth solution of the Euler-Lagrange system of $\mathscr{L}$, and in particular it is a critical point of the action functional $\mathscr{A}$.*

Proof. By proposition III.6, $\boldsymbol{q}$ is a critical point of $\mathscr{A}_k$ if and only if

$$
\text{(III.13)}\qquad \partial_v\mathscr{L}(\tfrac{j}{k},\gamma_{\boldsymbol{q}}(\tfrac{j}{k}),\dot\gamma_{\boldsymbol{q}}(\tfrac{j^-}{k})) - \partial_v\mathscr{L}(\tfrac{j}{k},\gamma_{\boldsymbol{q}}(\tfrac{j}{k}),\dot\gamma_{\boldsymbol{q}}(\tfrac{j^+}{k})) = 0, \qquad \forall j\in\mathbb{Z}_k.
$$

Assumption **(Q1)** (see section II.3) implies that the maps

$$
\partial_v\mathscr{L}(\tfrac{j}{k},\gamma_{\boldsymbol{q}}(\tfrac{j}{k}),\cdot):\mathbb{R}^m\to\mathbb{R}^m, \qquad \forall j\in\mathbb{Z}_k
$$

are diffeomorphisms, as we have proved in section I.2. Hence the equality in (III.13) holds if and only if $\dot\gamma_{\boldsymbol{q}}(\tfrac{j^-}{k}) = \dot\gamma_{\boldsymbol{q}}(\tfrac{j^+}{k})$ for each $j\in\mathbb{Z}_k$, that is if and only if $\gamma_{\boldsymbol{q}}$ is C^1. This implies that $\gamma_{\boldsymbol{q}}:\mathbb{T}\to\mathbb{R}^m$ satisfies the Euler-Lagrange system of $\mathscr{L}$ on the whole $\mathbb{T}$, and therefore it is smooth. ∎

Now, let us fix a critical point $\boldsymbol{q'}\in W_k$ of the discrete action functional $\mathscr{A}_k$. The tangent space of Λ_k at the smooth loop $\gamma_{\boldsymbol{q'}}$, that is the image of the differential

$$
\mathrm{d}\lambda_k(\boldsymbol{q'}):\mathrm{T}_{\boldsymbol{q'}}\Delta_k \xrightarrow{\simeq} \mathrm{T}_{\gamma_{\boldsymbol{q'}}}\Lambda_k,
$$

can be characterized as follows. We define a Lagrangian $L : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ by

$$L(t,q,v) = \frac{1}{2}\langle a(t)v,v\rangle + \langle b(t)q,v\rangle + \frac{1}{2}\langle c(t)q,q\rangle\,, \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^m \times \mathbb{R}^m, \tag{III.14}$$

where, for each $t \in \mathbb{R}/\mathbb{Z}$, $a(t)$, $b(t)$ and $c(t)$ are the $m \times m$ matrices given by

$$\begin{aligned} a_{ij}(t) &= \frac{\partial^2 \mathscr{L}}{\partial v^i\,\partial v^j}(t,\gamma_{\boldsymbol{q}'}(t),\dot\gamma_{\boldsymbol{q}'}(t)),\\ b_{ij}(t) &= \frac{\partial^2 \mathscr{L}}{\partial v^i\,\partial q^j}(t,\gamma_{\boldsymbol{q}'}(t),\dot\gamma_{\boldsymbol{q}'}(t)),\\ c_{ij}(t) &= \frac{\partial^2 \mathscr{L}}{\partial q^i\,\partial q^j}(t,\gamma_{\boldsymbol{q}'}(t),\dot\gamma_{\boldsymbol{q}'}(t)). \end{aligned}$$

A straightforward computation shows that the Euler-Lagrange system associated to L is given by the following linear system of ordinary differential equations for curves σ on $\mathbb{R}^m$

$$a\,\ddot\sigma + (b + \dot a - b^T)\,\dot\sigma + (\dot b - c)\,\sigma = 0. \tag{III.15}$$

This is precisely the linearization of the Euler-Lagrange system of $\mathscr{L}$ along the periodic solution $\gamma_{\boldsymbol{q}'}$. The 1-periodic solutions $\sigma : \mathbb{T} \to \mathbb{R}^m$ of (III.15) are precisely the critical points of the action functional $A : W^{1,2}(\mathbb{T};\mathbb{R}^m) \to \mathbb{R}$ associated to L, given as usual by

$$A(\xi) = \int_0^1 L(t,\xi(t),\dot\xi(t))\,\mathrm{d}t, \qquad \forall \xi \in W^{1,2}(\mathbb{T};\mathbb{R}^m).$$

Lemma III.8. *The tangent space* $\mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ *is the space of continuous and piecewise smooth loops* $\sigma : \mathbb{T} \to \mathbb{R}^m$ *such that, for each* $j \in \mathbb{Z}_k$, *the restriction* $\sigma|_{[j/k,(j+1)/k]}$ *is a solution of the Euler-Lagrange system* (III.15).

Proof. By definition of tangent space, $\mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ consists of those continuous loops $\sigma : \mathbb{T} \to \mathbb{R}^m$ given by

$$\sigma(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \Sigma(s,t), \qquad \forall t \in \mathbb{T}, \tag{III.16}$$

for some continuous $\Sigma : (-\varepsilon,\varepsilon) \times \mathbb{T} \to \mathbb{R}^m$ such that:

- the restriction $\Sigma|_{(-\varepsilon,\varepsilon)\times[j/k,(j+1)/k]}$ is smooth for all $j \in \mathbb{Z}_k$,
- $\Sigma(s,\cdot) \in \Lambda_k$ for all $s \in (-\varepsilon,\varepsilon)$,
- $\Sigma(0,\cdot) = \gamma_{\boldsymbol{q}'}$.

Namely, Σ is a piecewise smooth variation of $\gamma_{\boldsymbol{q}'}$ such that the loops $\Sigma_s = \Sigma(s,\cdot)$ satisfy the Euler-Lagrange equations associated to $\mathscr{L}$ on the intervals $[\frac{j}{k}, \frac{j+1}{k}]$ for all $j \in \mathbb{Z}_k$, i.e.

$$\partial^2_{vv}\mathscr{L}(t,\Sigma_s,\dot\Sigma_s)\,\ddot\Sigma_s + \partial^2_{vq}\mathscr{L}(t,\Sigma_s,\dot\Sigma_s)\,\dot\Sigma_s + \partial^2_{vt}\mathscr{L}(t,\Sigma_s,\dot\Sigma_s) - \partial_q\mathscr{L}(t,\Sigma_s,\dot\Sigma_s) = 0, \quad \forall t \in [j/k,(j+1)/k].$$

Differentiating the above equation with respect to s in $s=0$ we obtain the Euler-Lagrange system (III.15) for the loop σ (as before, satisfied on the intervals $[\frac{j}{k}, \frac{j+1}{k}]$ for all $j \in \mathbb{Z}_k$). Viceversa, a continuous loop $\sigma : \mathbb{T} \to \mathbb{R}^m$ whose restrictions $\sigma|_{[j/k,(j+1)/k]}$ satisfy (III.15) is of the form (III.16) for some Σ as above, and therefore it is an element of $\mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$. ■

Now, we want to investigate the relationship between the Morse index and nullity pair of the functionals $\mathscr{A}_k$ and $\mathscr{A}$ at the corresponding critical points $\boldsymbol{q}'$ and $\gamma_{\boldsymbol{q}'}$. We begin by characterizing the null-space of the Hessian of $\mathscr{A}$ at $\gamma_{\boldsymbol{q}'}$.

Lemma III.9. *The null-space of* $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ *consists of smooth loops* $\sigma : \mathbb{T} \to \mathbb{R}^m$ *that are solutions of the Euler-Lagrange system* (III.15) *on the whole* $\mathbb{T}$.

Proof. For every $\sigma, \xi \in W^{1,2}(\mathbb{T};\mathbb{R}^m)$ we have

$$\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})[\sigma,\xi] = \int_0^1 \Big(\langle a\,\dot\sigma, \dot\xi\rangle + \langle b\,\sigma, \dot\xi\rangle + \langle b^T\,\dot\sigma, \xi\rangle + \langle c\,\sigma, \xi\rangle \Big)\,\mathrm{d}t = \mathrm{d}A(\sigma)\xi.$$

Therefore σ is in the null-space of $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ if and only if it is a critical point of A, that is if and only if it is a (smooth) solution of the Euler-Lagrange system (III.15). ■

Remark III.2. In case $\mathscr{L}$ is the autonomous Lagrangian of the geodesics, i.e.

$$\mathscr{L}(q,v) = |v|_q^2, \qquad \forall (q,v) \in \mathrm{T}M,$$

the null-space of $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ is given by the 1-periodic **Jacobi vector fields** along the closed geodesics $\gamma_{\boldsymbol{q}'}$, and the Euler-Lagrange system (III.15) is called the **Jacobi system**. This latter can also be intrinsically expressed as

$$\nabla_t^2\sigma + R(\sigma,\dot\gamma_{\boldsymbol{q}'})\dot\gamma_{\boldsymbol{q}'} = 0,$$

where ∇_t and R are respectively the covariant derivative and the Riemann tensor of the Riemannian manifold $(M, \langle\cdot,\cdot\rangle_\cdot)$. ■

As a consequence of lemmas III.8 and III.9, the null-space of $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ is contained in $\mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$, and therefore it is contained in the null-space of the Hessian of the restricted action $\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})$. This inclusion is actually an equality, as shown by the following.

Lemma III.10. $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ *and* $\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})$ *have the same null-space, and in particular* $\nu(\mathscr{A},\gamma_{\boldsymbol{q}'}) = \nu(\mathscr{A}|_{\Lambda_k},\gamma_{\boldsymbol{q}'})$.

Proof. We only need to show that any loop $\sigma \in \mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ that is not everywhere smooth cannot be in the null-space of $\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})$. In fact, since σ is always smooth outside the points j/k (for $j \in \mathbb{Z}_k$), for each $\xi \in \mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ we have

$$\begin{aligned}
\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})[\sigma,\xi] &= \sum_{j=0}^{k-1} \int_{j/k}^{(j+1)/k} \Big(\langle a\,\dot\sigma, \dot\xi\rangle + \langle b\,\sigma, \dot\xi\rangle + \langle b^T\,\dot\sigma, \xi\rangle + \langle c\,\sigma, \xi\rangle \Big)\,\mathrm{d}t \\
&= \sum_{j=0}^{k-1} \int_{j/k}^{(j+1)/k} \langle \underbrace{-a\,\ddot\sigma - b\,\dot\sigma - \dot a\,\dot\sigma - \dot b\,\sigma + b^T\,\dot\sigma + c\,\sigma}_{=0}, \xi\rangle\,\mathrm{d}t \\
&\quad + \sum_{j=0}^{k-1} \langle a\,\dot\sigma + b\,\sigma, \xi\rangle \Big|_{(j/k)^+}^{((j+1)/k)^-} \\
&= \sum_{j=0}^{k-1} \langle a(\tfrac{j}{k})[\dot\sigma(\tfrac{j^-}{k}) - \dot\sigma(\tfrac{j^+}{k})], \xi(\tfrac{j}{k})\rangle
\end{aligned} \tag{III.17}$$

By assumption we have $\dot\sigma(\frac{j^+}{k}) \neq \dot\sigma(\frac{j^-}{k})$ for some $j \in \mathbb{Z}_k$ and therefore $a(\frac{j}{k})[\dot\sigma(\frac{j^+}{k}) - \dot\sigma(\frac{j^-}{k})]$ is a nonzero vector (here we are using the fact that, by assumption **(Q1)**, $a(\frac{j}{k})$ is invertible, see section II.3). Now, considering $\xi \in \mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ such that

$$\xi(\tfrac{h}{k}) = \begin{cases} a(\tfrac{j}{k})[\dot\sigma(\tfrac{j^+}{k}) - \dot\sigma(\tfrac{j^-}{k})], & h = j, \\ \mathbf{0}, & h \neq j, \end{cases}$$

we conclude that σ is not in the null-space of $\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})$, since by (III.17) we have

$$\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})[\sigma,\xi] = \Big| a(\tfrac{j}{k})[\dot\sigma(\tfrac{j^+}{k}) - \dot\sigma(\tfrac{j^-}{k})] \Big|^2 \neq 0.$$ ■

Corollary III.11. *The discrete action functional $\mathscr{A}_k$ and the full action functional $\mathscr{A}$ have the same nullity at the critical points $\boldsymbol{q}'$ and $\gamma_{\boldsymbol{q}'}$ respectively, i.e.*

$$\nu(\mathscr{A}_k, \boldsymbol{q}') = \nu(\mathscr{A}, \gamma_{\boldsymbol{q}'}).$$

Proof. First of all, notice that $\mathrm{d}\lambda_k(\boldsymbol{q}') : \mathbb{R}^m \to \mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ is an isomorphism, and we have

$$\mathrm{Hess}\mathscr{A}_k(\boldsymbol{q}')[\boldsymbol{v},\boldsymbol{w}] = \mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})[\mathrm{d}\lambda_k(\boldsymbol{q}')\boldsymbol{v}, \mathrm{d}\lambda_k(\boldsymbol{q}')\boldsymbol{w}], \qquad \forall \boldsymbol{v},\boldsymbol{w} \in \mathbb{R}^m.$$

This implies that $\mathscr{A}_k$ and $\mathscr{A}|_{\Lambda_k}$ have the same nullity at $\boldsymbol{q}'$ and $\gamma_{\boldsymbol{q}'}$ respectively, i.e. $\nu(\mathscr{A}_k, \boldsymbol{q}') = \nu(\mathscr{A}|_{\Lambda_k}, \gamma_{\boldsymbol{q}'})$, and by lemma III.10 we obtain the assertion. ■

So far we have proved that $\mathscr{A}_k$ and $\mathscr{A}$ have the same Morse nullity at the corresponding critical points $\boldsymbol{q}'$ and $\gamma_{\boldsymbol{q}'}$. Now we want to prove that they also have the same Morse index provided that $k \in \mathbb{N}$ is sufficiently big. First of all, we need some preliminaries.

Let $\boldsymbol{E}$ be a real Hilbert space and $\mathscr{B} : \boldsymbol{E} \otimes \boldsymbol{E} \to \mathbb{R}$ a bounded symmetric bilinear form. We recall that the **Morse index** $\iota(\mathscr{B})$ of this form is the supremum of the dimension of the vector subspaces of $\boldsymbol{E}$ on which $\mathscr{B}$ is negative definite. Now, let us fix an infinite sequence $\{\boldsymbol{E}_n \,|\, n \in \mathbb{N}\}$ of Hilbert subspaces of $\boldsymbol{E}$ such that

$$\boldsymbol{E}_1 \subset \boldsymbol{E}_2 \subset \boldsymbol{E}_3 \subset ... \subset \boldsymbol{E},$$

and their union is dense in $\boldsymbol{E}$, i.e.

$$\overline{\bigcup_{n\in\mathbb{N}} \boldsymbol{E}_n} = \boldsymbol{E}. \tag{III.18}$$

The following holds.

Lemma III.12. *If the Morse index of $\mathscr{B}$ is finite then, for all the sufficiently big $n \in \mathbb{N}$, it coincides with the Morse index of $\mathscr{B}$ restricted to $\boldsymbol{E}_n$, i.e.*

$$\iota(\mathscr{B}) = \iota(\mathscr{B}|_{\boldsymbol{E}_n\otimes\boldsymbol{E}_n}).$$

Proof. The inequality $\iota(\mathscr{B}) \geq \iota(\mathscr{B}|_{\boldsymbol{E}_n\otimes\boldsymbol{E}_n})$ is trivial, hence we only have to prove that $\iota(\mathscr{B}) \leq \iota(\mathscr{B}|_{\boldsymbol{E}_n\otimes\boldsymbol{E}_n})$. For each $n \in \mathbb{N}$, we denote by $P_n : \boldsymbol{E} \to \boldsymbol{E}_n$ the orthogonal projector onto $\boldsymbol{E}_n$. Let $\boldsymbol{V}$ be a vector subspace of $\boldsymbol{E}$ of dimension $\iota = \iota(\mathscr{B}) \in \mathbb{N}$, such that $\mathscr{B}$ is negative definite on $\boldsymbol{V}$. We denote by $S(\boldsymbol{V})$ the $(\iota - 1)$-dimensional sphere in $\boldsymbol{V}$, i.e.

$$S(\boldsymbol{V}) = \{\boldsymbol{e} \in \boldsymbol{V} \,|\, \|\boldsymbol{e}\|_{\boldsymbol{E}} = 1\}.$$

By (III.18) and since $\mathscr{B}$ is continuous, for each $\boldsymbol{e} \in S(\boldsymbol{V})$ there exists a positive integer $n_{\boldsymbol{e}} \in \mathbb{N}$ and a neighborhood $U_{\boldsymbol{e}} \subseteq S(\boldsymbol{V})$ of $\boldsymbol{e}$ such that

$$\mathscr{B}(P_n\boldsymbol{f}, P_n\boldsymbol{f}) < 0, \qquad \forall n \geq n_{\boldsymbol{e}},\ \boldsymbol{f} \in U_{\boldsymbol{e}}.$$

By compactness, $S(\boldsymbol{V})$ admits a finite cover $U_{\boldsymbol{e}_1}, ..., U_{\boldsymbol{e}_s}$. For every integer $n \geq \max\{n_{\boldsymbol{e}_1}, ..., n_{\boldsymbol{e}_s}\}$ and for every nonzero $\boldsymbol{v} = P_n\boldsymbol{w} \in P_n\boldsymbol{V}$ we have

$$\mathscr{B}(\boldsymbol{v},\boldsymbol{v}) = \|\boldsymbol{w}\|^2_{\boldsymbol{E}} \underbrace{\mathscr{B}\left(P_n\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_{\boldsymbol{E}}}, P_n\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|_{\boldsymbol{E}}}\right)}_{<0} < 0,$$

and therefore $\mathscr{B}$ is negative definite on $P_n\boldsymbol{V}$. In order to conclude the proof we just need to show that $P_n\boldsymbol{V}$ has still dimension $\iota = \iota(\mathscr{B})$ provided n is sufficiently big. This is easily seen as follows. Let $\boldsymbol{v}_1, ..., \boldsymbol{v}_\iota$ be a basis for $\boldsymbol{V}$. By (III.18), for each $j = 1, ..., \iota$, we have that $P_n\boldsymbol{v}_i \to \boldsymbol{v}_i$ as $n \to \infty$. This implies that, for n sufficiently big, the vectors $P_n\boldsymbol{v}_1, ..., P_n\boldsymbol{v}_\iota$ are still linearly independent, and therefore $P_n\boldsymbol{V}$ has dimension ι. ∎

In order to apply this abstract lemma to our situation, we first need the following remark about the density of the spaces of broken affine loops in $W^{1,2}(\mathbb{T};\mathbb{R}^m)$. For each $k\in\mathbb{N}$, we define the **k-broken affine loop space** of $\mathbb{R}^m$ as

$$\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m) := \left\{\sigma:\mathbb{T}\to\mathbb{R}^m \;\middle|\; \sigma(\tfrac{j+s}{k}) = (1-s)\,\sigma(\tfrac{j}{k}) + s\,\sigma(\tfrac{j+1}{k}) \quad \forall s\in[0,1],\ j\in\mathbb{Z}_k\right\}.$$

This space is isomorphic to the k-fold product $\mathbb{R}^m\times\ldots\times\mathbb{R}^m$ by the linear map $\alpha_k:\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)\to\mathbb{R}^m\times\ldots\times\mathbb{R}^m$ given by

$$\alpha_k(\sigma) = \big(\sigma(0),\sigma(\tfrac{1}{k}),\ldots,\sigma(\tfrac{k-1}{k})\big), \qquad \forall\sigma\in\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m).$$

In particular, being $\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)$ a finite dimensional vector space, it is a Hilbert subspace of $W^{1,2}(\mathbb{T};\mathbb{R}^m)$.

Lemma III.13. *The union of the spaces* $\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)$*, for all* $k\in\mathbb{N}$*, is dense in* $W^{1,2}(\mathbb{T};\mathbb{R}^m)$*, i.e.*

$$\overline{\bigcup_{k\in\mathbb{N}}\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)} = W^{1,2}(\mathbb{T};\mathbb{R}^m).$$

Proof. It is well known that $C^\infty(\mathbb{T};\mathbb{R}^m)$ is dense in $W^{1,2}(\mathbb{T};\mathbb{R}^m)$, see for instance [Ad, page 54]. Hence, all we have to do in order to prove the lemma is to show that, for an arbitrary $\gamma\in C^\infty(\mathbb{T};\mathbb{R}^m)$, there exists a sequence $\{\gamma_k\,|\,k\in\mathbb{N}\}$ such that $\gamma_k\in\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)$ and $\gamma_k\to\gamma$ in $W^{1,2}(\mathbb{T};\mathbb{R}^m)$. A candidate for this sequence is built by defining γ_k to be the map in $\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)$ such that $\gamma_k(\frac{j}{k})=\gamma(\frac{j}{k})$ for each $j\in\mathbb{Z}_k$, see figure III.1.

Since γ is smooth and 1-periodic, if we fix an arbitrary $\varepsilon>0$ there exists a positive $\delta>0$ such that, for each $t\in\mathbb{T}$ and for each $\delta_0,\delta_1>0$ with $0<\delta_0+\delta_1\leq\delta$, we have

$$\left|\dot\gamma(t) - \frac{\gamma(t+\delta_1)-\gamma(t-\delta_0)}{\delta_1+\delta_0}\right| \leq \varepsilon. \tag{III.19}$$

Now, notice that, for each $k\in\mathbb{N}$, the periodic curve $\gamma_k:\mathbb{T}\to\mathbb{R}$ is differentiable outside the points j/k (for each $j\in\mathbb{Z}_k$) and we have

$$\dot\gamma_k(t) = k\left[\gamma\left(\tfrac{\lfloor kt\rfloor+1}{k}\right) - \gamma\left(\tfrac{\lfloor kt\rfloor}{k}\right)\right], \qquad t\in\mathbb{T}\setminus\left\{\tfrac{j}{k}\,|\,j\in\mathbb{Z}_k\right\},$$

where $\lfloor\cdot\rfloor$ gives the integer part of its argument. In particular, for each irrational number $t\in\mathbb{R}\setminus\mathbb{Q}$, or more precisely for each $t\in(\mathbb{R}\setminus\mathbb{Q})/\mathbb{Z}$, we have

$$\lim_{k\to\infty}\dot\gamma_k(t) = \dot\gamma(t).$$

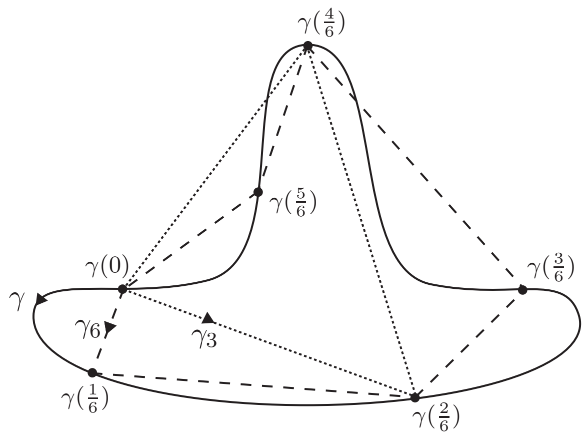


Figure III.1. Example of a smooth loop γ (solid line) and the corresponding piecewise affine loops γ_3 (dotted line) and γ_6 (dashed line) of the sequence $\{\gamma_k \,|\, k \in \mathbb{N}\}$.

Since the rational numbers have Lebesgue measure 0, the above equation implies that $\dot{\gamma}_k$ converges to $\dot{\gamma}$ almost everywhere as $k \to \infty$. By (III.19), for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $k \geq \delta^{-1}$ and for almost every $t \in \mathbb{T}$, we have

$$|\dot{\gamma}(t) - \dot{\gamma}_k(t)| = \left| \dot{\gamma}(t) - \frac{\gamma\left(\frac{\lfloor kt \rfloor + 1}{k}\right) - \gamma\left(\frac{\lfloor kt \rfloor}{k}\right)}{\frac{1}{k}} \right| \leq \varepsilon.$$

We can then apply the dominated convergence theorem to conclude that $\dot{\gamma}_k \to \dot{\gamma}$ in $L^2(\mathbb{T}; \mathbb{R}^m)$ as $k \to \infty$, and therefore $\gamma_k \to \gamma$ in $W^{1,2}(\mathbb{T}; \mathbb{R}^m)$ as $k \to \infty$. ∎

Once this preliminaries are established, we can prove the announced result about the Morse index of the discrete action functionals.

Lemma III.14. *For all $k \in \mathbb{N}$ sufficiently big, the Morse index of $\mathscr{A}_k$ at $\boldsymbol{q}'$ coincides with the Morse index of $\mathscr{A}$ at $\gamma_{\boldsymbol{q}'}$, i.e.*

$$\iota(\mathscr{A}_k, \boldsymbol{q}') = \iota(\mathscr{A}, \gamma_{\boldsymbol{q}'}).$$

Proof. Since the functionals $\mathscr{A}_k$ and $\mathscr{A}|_{\Lambda_k}$ have the same Morse index and nullity pair (see the proof of corollary III.11), we just need to prove that $\iota(\mathscr{A}|_{\Lambda_k}, \gamma_{\boldsymbol{q}'}) = \iota(\mathscr{A}, \gamma_{\boldsymbol{q}'})$ for all the sufficiently big $k \in \mathbb{N}$.

By definition of Morse index, there exists a $\iota(\mathscr{A}, \gamma_{\boldsymbol{q}'})$-dimensional vector subspace $\boldsymbol{V} \subseteq W^{1,2}(\mathbb{T}; \mathbb{R}^m)$ over which $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ is negative definite, i.e.

$$\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})[\sigma, \sigma] < 0, \quad \forall \sigma \in \boldsymbol{V} \setminus \{\boldsymbol{0}\}. \tag{III.20}$$

By lemmas III.13 and III.12, we can choose $\boldsymbol{V}$ to be a subspace of $\mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)$. Then, let us define a linear map $K : \boldsymbol{V} \to \mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ as $K(\sigma) = \tilde{\sigma}$, where $\tilde{\sigma}$ is the unique element in $\mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$ such that

$$\sigma(\tfrac{j}{k}) = \tilde{\sigma}(\tfrac{j}{k}), \qquad \forall j \in \mathbb{Z}_k.$$

Notice that K is injective. In fact, if $K(\sigma) = \mathbf{0}$, we have $\sigma(\frac{j}{k}) = \mathbf{0}$ for each $j \in \mathbb{Z}_k$, and since $\sigma \in \mathrm{Aff}_k(\mathbb{T};\mathbb{R}^m)$ we conclude that $\sigma = \mathbf{0}$. Hence, $\tilde{\boldsymbol{V}} = K\boldsymbol{V}$ is a $\iota(\mathscr{A},\gamma_{\boldsymbol{q}'})$-dimensional vector subspace of $\mathrm{T}_{\gamma_{\boldsymbol{q}'}}\Lambda_k$.

In order to conclude we just have to show that $\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})$ is negative definite over the vector space $\tilde{\boldsymbol{V}}$. First of all, notice that, for each $\sigma \in W^{1,2}(\mathbb{T};\mathbb{R}^m)$, we have

$$\begin{aligned}\mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})[\sigma,\sigma] &= \int_0^1 \big(\langle a\,\dot\sigma,\dot\sigma\rangle + \langle b\,\sigma,\dot\sigma\rangle + \langle b^T\,\dot\sigma,\sigma\rangle + \langle c\,\sigma,\sigma\rangle\big)\,\mathrm{d}t \\ &= 2\int_0^1 L(t,\sigma(t),\dot\sigma(t))\,\mathrm{d}t = 2\,A(\sigma).\end{aligned} \tag{III.21}$$

Now, consider an arbitrary $\tilde{\sigma} \in \tilde{\boldsymbol{V}} \setminus \{\mathbf{0}\}$ and put $\sigma = K^{-1}(\tilde{\sigma}) \in \boldsymbol{V} \setminus \{\mathbf{0}\}$. For each $j \in \mathbb{Z}_k$ the curve $\tilde{\sigma}|_{[j/k,(j+1)/k]}$ is an action minimizer with respect to L, and therefore $A(\tilde{\sigma}) \leq A(\sigma)$. By (III.20) and (III.21) we conclude

$$\mathrm{Hess}\mathscr{A}|_{\Lambda_k}(\gamma_{\boldsymbol{q}'})[\tilde{\sigma},\tilde{\sigma}] = 2\,A(\tilde{\sigma}) \leq 2\,A(\sigma) = \mathrm{Hess}\mathscr{A}(\gamma_{\boldsymbol{q}'})[\sigma,\sigma] < 0. \qquad \blacksquare$$

III.5 Homotopic approximation of the action sublevels

The discretization technique introduced in this chapter can also be used to build finite dimensional homotopic approximations of the sublevels of the action functional. Let $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ be a 1-periodic convex quadratic-growth Lagrangian with associated action $\mathscr{A} : W^{1,2}(\mathbb{T};M) \to \mathbb{R}$, as in the previous sections, and consider the constants $\varepsilon_0 = \varepsilon_0(\mathscr{L}) > 0$ and $\rho_0 = \rho_0(\mathscr{L}) > 0$ given by theorem III.2. We will need the following statement.

Proposition III.15. *For each $c \in \mathbb{R}$ there exists a real $\bar\varepsilon = \bar\varepsilon(\mathscr{L}, c) > 0$ such that, for each $\zeta \in W^{1,2}(\mathbb{T};M)$ with $\mathscr{A}(\zeta) < c$ and for each interval $[t_0,t_1] \subset \mathbb{R}$ with $0 < t_1 - t_0 \leq \bar\varepsilon$ we have* $\mathrm{dist}\,(\zeta(t_0),\zeta(t_1)) < \rho_0$.

Proof. Up to summing a positive constant to the convex quadratic-growth Lagrangian $\mathscr{L}$, we can always assume that there exists a constant $\underline{\ell} > 0$ such that

$$\mathscr{L}(t,q,v) \geq \underline{\ell}\,|v|_q^2, \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M.$$

Then, let us consider an arbitrary loop $\zeta \in W^{1,2}(\mathbb{T};M)$ such that $\mathscr{A}(\zeta) < c$. For each interval $[t_0, t_1] \subset \mathbb{R}$ with $0 < t_1 - t_0 \leq 1$ we have

$$\begin{aligned}\operatorname{dist}(\zeta(t_0), \zeta(t_1))^2 &\leq \left(\int_{t_0}^{t_1} |\dot\zeta(t)|_{\zeta(t)} \mathrm{d}t\right)^2 \leq (t_1 - t_0) \int_{t_0}^{t_1} |\dot\zeta(t)|^2_{\zeta(t)} \, \mathrm{d}t \\ &\leq (t_1 - t_0) \int_{t_0}^{t_1} \underline{\ell}^{-1} \mathscr{L}(t, \zeta(t), \dot\zeta(t)) \, \mathrm{d}t \leq (t_1 - t_0) \underline{\ell}^{-1} \mathscr{A}(\zeta) \\ &< (t_1 - t_0) \underline{\ell}^{-1} c.\end{aligned}$$

Hence, for $\bar\varepsilon = \bar\varepsilon(\mathscr{L}, c) := \rho_0^2 \, \underline{\ell} \, c^{-1}$, we obtain the claim. ■

From now on, we will briefly denote the open sublevels of the action and of the discrete action by

$$(\mathscr{A})_c := \mathscr{A}^{-1}(-\infty, c), \qquad (\mathscr{A}_k)_c := \mathscr{A}_k^{-1}(-\infty, c), \qquad \forall c \in \mathbb{R}.$$

Let us consider the integer

$$\bar k(\mathscr{L}, c) := \left\lceil \max\left\{ \frac{1}{\varepsilon_0(\mathscr{L})}, \frac{1}{\bar\varepsilon(\mathscr{L}, c)} \right\} \right\rceil \in \mathbb{N}.$$

We want to show that, for each $c \in \mathbb{R}$ and for each integer $k \geq \bar k(\mathscr{L}, c)$, the map $\lambda_k : \Delta_k \hookrightarrow W^{1,2}(\mathbb{T}; M)$ (see the definition after (III.11)) restricts to a homotopy equivalence

$$\lambda_k : (\mathscr{A}_k)_c \overset{\simeq}{\hookrightarrow} (\mathscr{A})_c.$$

Equivalently, since $\mathscr{A}_k = \mathscr{A} \circ \lambda_k$ and since λ_k maps Δ_k diffeomorphically onto Λ_k, we can show that the inclusion of the open sublevel $(\mathscr{A}|_{\Lambda_k})_c := (\mathscr{A})_c \cap \Lambda_k$ into $(\mathscr{A})_c$ is a homotopy equivalence.

In order to prove this claim, let us introduce the retraction

$$r_k : (\mathscr{A})_c \to (\mathscr{A}|_{\Lambda_k})_c$$

that maps a loop $\zeta \in (\mathscr{A})_c$ to the loop $r_k(\zeta)$ such that $r_k(\zeta)|_{[j/k,(j+1)/k]}$ is the unique action minimizer (with respect to $\mathscr{L}$) with endpoints $\zeta(\frac{j}{k})$ and $\zeta(\frac{j+1}{k})$, for each $j \in \mathbb{Z}_k$. Then, we build a homotopy

$$R_k : [0,1] \times (\mathscr{A})_c \to (\mathscr{A})_c$$

as follows: for each $j \in \mathbb{Z}_k$, $s \in [\frac{j}{k}, \frac{j+1}{k}]$ and $\zeta \in (\mathscr{A})_c$, the loop $R_k(s, \zeta)$ is defined as $R_k(s,\zeta)|_{[0,j/k]} = r_k(\zeta)|_{[0,j/k]}$, $R_k(s,\zeta)|_{[s,1]} = \zeta|_{[s,1]}$ and $R_k(s,\zeta)|_{[j/k,s]}$ is the unique action minimizer with endpoints $\zeta(\frac{j}{k})$ and $\zeta(s)$, see figure III.2. By proposition III.15, the homotopy R_k and the map r_k are well defined and we have

$$\mathscr{A}(R_k(s, \zeta)) \leq \mathscr{A}(\zeta) \qquad \forall \zeta \in (\mathscr{A})_c, \ s \in [0,1].$$

Moreover R_k is a strong deformation retraction. In fact, for each $\zeta \in (\mathscr{A})_c$ we have $R_k(0, \zeta) = \zeta$, $R_k(1, \zeta) = r_k(\zeta)$ and, if ζ also belongs to Λ_k, we further have

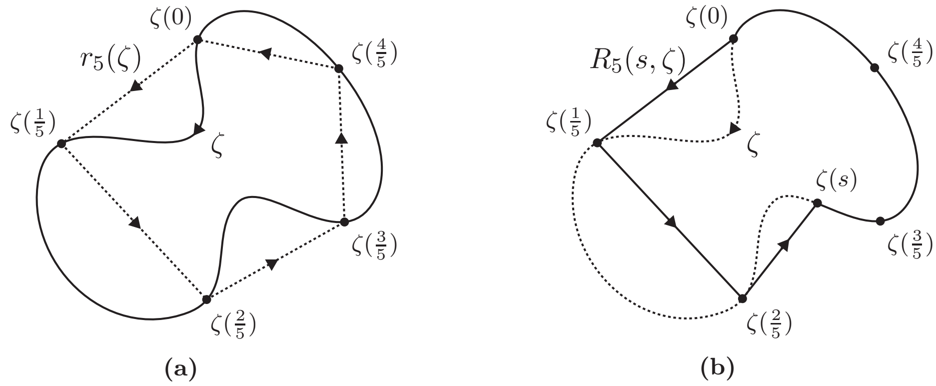


Figure III.2. **(a)** Example of a loop ζ (solid line) and the corresponding $r_5(\zeta)$ (dotted line), for the case of the autonomous Lagrangians of the geodesics on the flat $\mathbb{R}^2$, i.e. $\mathscr{L}(t,q,v) = \mathscr{L}(v) = v_1^2 + v_2^2$. **(b)** Homotopied curve $R_5(s,\zeta)$ (solid line).

$R_k(s,\zeta) = \zeta$ for all $s \in [0,1]$. This proves that the retraction r_k is the homotopic inverse of the inclusion $(\mathscr{A}|_{\Lambda_k})_c \hookrightarrow (\mathscr{A})_c$.

If $c_1 < c_2 \leq c$, the same deformation retraction R_k can be used to show that the pair $((\mathscr{A})_{c_2}, (\mathscr{A})_{c_1})$ deformation retracts strongly onto $((\mathscr{A}|_{\Lambda_k})_{c_2}, (\mathscr{A}|_{\Lambda_k})_{c_1})$. Furthermore, if $\gamma \in W^{1,2}(\mathbb{T}; M)$ is a critical point of $\mathscr{A}$ with $\mathscr{A}(\gamma) = c$, up to increasing k we have that $\gamma \in \Lambda_k$ and R_k gives a deformation retraction of the pair $((\mathscr{A})_c \cup \{\gamma\}, (\mathscr{A})_c)$ onto $((\mathscr{A}|_{\Lambda_k})_c \cup \{\gamma\}, (\mathscr{A}|_{\Lambda_k})_c)$.

Summing up, we have obtained the following.

Lemma III.16.

(i) *Let $c_1 < c_2 < \infty$. Then, there exists a positive integer $\bar{k} = \bar{k}(\mathscr{L}, c_2)$ such that, for every integer $k \geq \bar{k}$, the embedding λ_k restricts to a homotopy equivalence of topological pairs*

$$\lambda_k : ((\mathscr{A}_k)_{c_2}, (\mathscr{A}_k)_{c_1}) \overset{\simeq}{\hookrightarrow} ((\mathscr{A})_{c_2}, (\mathscr{A})_{c_1}).$$

(ii) *Let $\boldsymbol{q} \in \Delta_k$ be a critical point of $\mathscr{A}_k$ such that $\mathscr{A}_k(\boldsymbol{q}) = c$. Then, there exists a positive integer $\bar{k} = \bar{k}(\mathscr{L}, c)$ such that, for every integer $k \geq \bar{k}$, the embedding λ_k restricts to a homotopy equivalence of topological pairs*

$$\lambda_k : ((\mathscr{A}_k)_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_c) \overset{\simeq}{\hookrightarrow} ((\mathscr{A})_c \cup \{\gamma_{\boldsymbol{q}}\}, (\mathscr{A})_c),$$

where $\gamma_{\boldsymbol{q}} = \lambda_k(\boldsymbol{q})$.

In the forthcoming chapters, we will mainly apply the above lemma to show that λ_k induces isomorphisms between invariant groups that are fundamental in Morse theory: the **local homology groups**. We recall that the local homology groups of $\mathscr{A}_k$ at a critical point $\boldsymbol{q}$ are defined as

$$\mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) := \mathrm{H}_*\left((\mathscr{A}_k)_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_c\right),$$

where H_* in the right hand side denotes the singular homology functor (with arbitrary coefficient group). The local homology groups of the full action functional $\mathscr{A}$ at the corresponding critical point $\gamma_{\boldsymbol{q}} = \lambda_k(\boldsymbol{q})$ are defined analogously as

$$\mathrm{H}_*(\mathscr{A}, \gamma_{\boldsymbol{q}}) = \mathrm{H}_*\left((\mathscr{A})_c \cup \{\gamma_{\boldsymbol{q}}\}, (\mathscr{A})_c\right).$$

Hence, point (ii) of the above lemma III.16 has the following immediate consequence.

Corollary III.17. *For each integer $k > \bar{k}(\mathscr{L}, c)$ the embedding λ_k induces the homology isomorphism*

$$\mathrm{H}_*(\lambda_k) : \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}, \gamma_{\boldsymbol{q}}).$$

It is well known that the local homology groups of a C^2 functional at a critical point are trivial in dimension that is smaller than the Morse index or bigger than the sum of the Morse index and the nullity (cf. corollary A.10 in appendix A). Here, we can recover this result for the C^1 action functional $\mathscr{A} : W^{1,2}(\mathbb{T}; M) \to \mathbb{R}$, at least for contractible critical points.

Corollary III.18. *Let $\gamma : \mathbb{T} \to M$ be a contractible loop that is critical point of the action functional $\mathscr{A}$. Then, the local homology groups $\mathrm{H}_*(\mathscr{A}, \gamma)$ are trivial if $*$ is less than $\iota(\mathscr{A}, \gamma)$ or greater than $\iota(\mathscr{A}, \gamma) + \nu(\mathscr{A}, \gamma)$.*

Proof. For each sufficiently big $k \in \mathbb{N}$, there exists $\boldsymbol{q} \in \Delta_k$ such that

$$\gamma = \gamma_{\boldsymbol{q}} = \lambda_k(\boldsymbol{q})$$

and $\boldsymbol{q}$ is a critical point of the discrete action $\mathscr{A}_k$ (see section III.4). By corollary III.11 and lemma III.14, up to increasing k we have that

$$(\iota(\mathscr{A}_k, \boldsymbol{q}), \nu(\mathscr{A}_k, \boldsymbol{q})) = (\iota(\mathscr{A}, \gamma_{\boldsymbol{q}}), \nu(\mathscr{A}, \gamma_{\boldsymbol{q}})).$$

By the above corollary III.17, up to further increasing k we have

$$\mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) \simeq \mathrm{H}_*(\mathscr{A}, \gamma_{\boldsymbol{q}}).$$

Therefore our claim follows from corollary A.10 applied to the local homology group $\mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q})$. ■

III.6 Arbitrary period

So far we have only dealt with 1-periodic loop spaces, but our arguments extend to every period $n \in \mathbb{N}$ as follows. Consider the smooth 1-periodic convex quadratic-growth Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ of the previous sections. For each $k \in \mathbb{N}$ we define $C_k^\infty(\mathbb{T}^{[n]}; M)$ as the space of continuous n-periodic curves $\zeta : \mathbb{T}^{[n]} \to M$ such that, for each $j \in \mathbb{Z}_{nk}$, the restriction $\zeta|_{[j/k,(j+1)/k]}$ is smooth. We endow this space with the broken smooth topology that turns it into a Fréchet manifold, analogously at what we did for $C^\infty_k(\mathbb{T}; M)$. If $k \geq 1/\varepsilon_0(\mathscr{L})$, we define a smooth embedding

$$\lambda_k^{[n]} = \lambda_{k,\mathscr{L}}^{[n]} : \Delta_{nk} \hookrightarrow C^\infty_k(\mathbb{T}^{[n]}; M)$$

in the following way: for each $\boldsymbol{q} = (q_0, ..., q_{nk-1}, q_{nk}) \in \Delta_{nk}$ we put $\lambda_k^{[n]}(\boldsymbol{q}) := \gamma_{\boldsymbol{q}}$, where $\gamma_{\boldsymbol{q}}$ is the only loop whose restrictions $\gamma_{\boldsymbol{q}}|_{[j/k,(j+1)/k]}$ are the unique action minimizers (with respect to $\mathscr{L}$) with endpoints q_j and q_{j+1}, for each $j \in \mathbb{Z}_{nk}$. We define the **k-broken n-periodic Euler-Lagrange loop space** (with respect to $\mathscr{L}$) as the image

$$\Lambda_k^{[n]} = \Lambda_{k,\mathscr{L}}^{[n]} := \lambda_k^{[n]}(\Delta_{nk})$$

of the embedding $\lambda_k^{[n]}$. As for the 1-periodic case, $\Lambda_k^{[n]}$ is a smooth submanifold of $C^\infty_k(\mathbb{T}^{[n]}; M)$ (and of $W^{1,2}(\mathbb{T}^{[n]}; M)$) with finite dimension nkm, where m is the dimension of M. Finally, we define the **discrete mean action functional**

$$\mathscr{A}_k^{[n]} : \Delta_{nk} \to \mathbb{R}$$

as the composition of the mean action functional $\mathscr{A}^{[n]}$ of $\mathscr{L}$ (cf. section II.3) with $\lambda_k^{[n]}$, namely

$$\mathscr{A}_k^{[n]}(\boldsymbol{q}) = \frac{1}{n}\int_0^n \mathscr{L}(t, \gamma_{\boldsymbol{q}}(t), \dot\gamma_{\boldsymbol{q}}(t))\,\mathrm{d}t, \qquad \forall \boldsymbol{q} \in \Delta_{nk}.$$

All the results of sections III.3, III.4 and III.5 still hold if we replace $\mathscr{A}$ and $\mathscr{A}_k$ with the mean versions $\mathscr{A}^{[n]}$ and $\mathscr{A}_k^{[n]}$, for each $n \in \mathbb{N}$.

The n^{th}-iteration map $\psi^{[n]} : W^{1,2}(\mathbb{T}; M) \hookrightarrow W^{1,2}(\mathbb{T}^{[n]}; M)$ restrict to a continuous embedding of k-broken Euler-Lagrange loop spaces

$$\psi^{[n]} : \Lambda_k \hookrightarrow \Lambda_k^{[n]}.$$

Moreover, by means of the diffeomorphisms λ_k and $\lambda_k^{[n]}$ we can define the **discrete n^{th}-iteration map**

$$\psi_k^{[n]} : \Delta_k \hookrightarrow \Delta_{nk},$$

in such a way that the following diagram commutes.

$$\begin{array}{ccc} \Delta_k & \overset{\psi_k^{[n]}}{\hookrightarrow} & \Delta_{nk} \\ \lambda_k \downarrow \simeq & & \simeq \downarrow \lambda_k^{[n]} \\ \Lambda_k & \overset{\psi^{[n]}}{\hookrightarrow} & \Lambda_k^{[n]} \end{array}$$

Namely, for each $\boldsymbol{q} \in \Delta_k$, we have $\psi_k^{[n]}(\boldsymbol{q}) = \boldsymbol{q}^{[n]}$, where

$$\boldsymbol{q}^{[n]} := (\underbrace{\boldsymbol{q}, ..., \boldsymbol{q}}_{n \text{ times}}).$$

Chapter IV

Local homology and Hilbert subspaces

The aim of this chapter is to prove an abstract Morse-theoretic result, that will be applied to the Lagrangian action functional of convex quadratic-growth Lagrangians. Roughly speaking, the theorem states that the local homology of a functional defined on a Hilbert space does not change if we restrict the functional to a Hilbert subspace containing the involved critical point, provided that this Hilbert subspace is invariant for the gradient flow of the functional. The proof of this result is technical, and in particular requires to reconsider the proof of two classical Morse-theoretic results that are often used as black boxes: the generalized Morse lemma (cf. lemma A.1) and the characterization of the local homology groups as homology of Gromoll-Meyer pairs (cf. theorem A.7).

As an application of this theorem, we obtain that the n^{th}-iteration map induces an isomorphism between the local homology of the action functional $\mathscr{A}$ at a critical point γ and the local homology of the mean action functional $\mathscr{A}^{[n]}$ at $\gamma^{[n]}$, provided that the Morse index and nullity pairs of $\mathscr{A}$ and $\mathscr{A}^{[n]}$ at γ and $\gamma^{[n]}$ respectively are the same. For the special case of Lagrangian systems on the m-torus, with a fiberwise quadratic Lagrangian, this result is due to Long (cf. [Lo, theorem 3.7]). Our statement may be considered an extension of Long's one.

In section IV.1 we introduce the above mentioned result. The proof will be carried out in section IV.4, after establishing several preliminaries in sections IV.2 and IV.3. Finally, in section IV.5 we discuss the applications to the Lagrangian action functional.

IV.1 The abstract result

Let us consider an open set $\mathscr{U}$ of a Hilbert space $\boldsymbol{E}$ and a C^2 functional $\mathscr{F}:\mathscr{U}\to\mathbb{R}$ that satisfies the Palais-Smale condition. Let $\boldsymbol{E}_\bullet$ be a Hilbert subspace of $\boldsymbol{E}$ such that $\mathscr{U}_\bullet := \mathscr{U} \cap \boldsymbol{E}_\bullet \neq \varnothing$ and $\mathrm{Grad}\mathscr{F}(\boldsymbol{y}) \in \boldsymbol{E}_\bullet$ for all $\boldsymbol{y} \in \mathscr{U}_\bullet$. This latter condition is equivalently expressed via the isometric inclusion $J : \boldsymbol{E}_\bullet \hookrightarrow \boldsymbol{E}$ as

$$(\mathrm{Grad}\mathscr{F}) \circ J = J \circ \mathrm{Grad}(\mathscr{F} \circ J). \tag{IV.1}$$

Let $\boldsymbol{x} \in \mathscr{U}$ be an isolated critical point of $\mathscr{F}$ that sits in the subspace $\boldsymbol{E}_\bullet$, i.e. $\boldsymbol{x} \in \mathrm{Crit}\mathscr{F} \cap \boldsymbol{E}_\bullet$, and let us further assume that the Morse index and nullity of $\mathscr{F}$ at $\boldsymbol{x}$ are finite, i.e. $\iota(\mathscr{F}, \boldsymbol{x}), \nu(\mathscr{F}, \boldsymbol{x}) < \infty$. We denote by $H = H(\boldsymbol{x})$ the bounded self-adjoint linear operator on $\boldsymbol{E}$ associated to the Hessian of $\mathscr{F}$ at $\boldsymbol{x}$, i.e.

$$\mathrm{Hess}\mathscr{F}(\boldsymbol{x})[\boldsymbol{v}, \boldsymbol{w}] = \langle H\boldsymbol{v}, \boldsymbol{w}\rangle_{\boldsymbol{E}}, \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{E}. \tag{IV.2}$$

We require that H is a Fredholm operator, so that the functional $\mathscr{F}$ satisfies the hypotheses of the generalized Morse lemma (lemma A.1).

Throughout this chapter, for simplicity, all the homology groups are assumed to have coefficients in a field $\mathbb{F}$ (in this way we avoid the torsion terms that appear in the Künneth formula). We recall that the **local homology groups** of the functional $\mathscr{F}$ at $\boldsymbol{x}$ are defined as

$$\mathrm{H}_*(\mathscr{F}, \boldsymbol{x}) = \mathrm{H}_*\left((\mathscr{F})_c \cup \{\boldsymbol{x}\}, (\mathscr{F})_c\right),$$

where $c = \mathscr{F}(\boldsymbol{x})$ and $(\mathscr{F})_c := \mathscr{F}^{-1}(-\infty, c)$. If we denote by $\mathscr{F}_\bullet : \mathscr{U}_\bullet \to \mathbb{R}$ the restricted functional $\mathscr{F}|_{\mathscr{U}_\bullet}$, then $\boldsymbol{x}$ is a critical point of $\mathscr{F}_\bullet$ as well and the local homology groups $\mathrm{H}_*(\mathscr{F}_\bullet, \boldsymbol{x})$ are defined analogously as $\mathrm{H}_*((\mathscr{F}_\bullet)_c \cup \{\boldsymbol{x}\}, (\mathscr{F}_\bullet)_c)$. The inclusion J restricts to a continuous map of pairs

$$J : ((\mathscr{F}_\bullet)_c \cup \{\boldsymbol{x}\}, (\mathscr{F}_\bullet)_c) \hookrightarrow ((\mathscr{F})_c \cup \{\boldsymbol{x}\}, (\mathscr{F})_c).$$

In this way, it induces the homology homomorphism

$$\mathrm{H}_*(J) : \mathrm{H}_*(\mathscr{F}_\bullet, \boldsymbol{x}) \to \mathrm{H}_*(\mathscr{F}, \boldsymbol{x}).$$

In this chapter we want to show that, if the Morse index and nullity pair at $\boldsymbol{x}$ does not change under restriction to the Hilbert subspace $\boldsymbol{E}_\bullet$, then the above homomorphism $\mathrm{H}_*(J)$ is an isomorphism. Namely, our main result is the following.

Theorem IV.1. *If the Morse index and nullity pair of $\mathscr{F}$ and $\mathscr{F}_\bullet$ at $\boldsymbol{x}$ coincide, i.e.*

$$(\iota(\mathscr{F}, \boldsymbol{x}), \nu(\mathscr{F}, \boldsymbol{x})) = (\iota(\mathscr{F}_\bullet, \boldsymbol{x}), \nu(\mathscr{F}_\bullet, \boldsymbol{x})),$$

then $\mathrm{H}_*(J)$ is an isomorphism of local homology groups.

The proof of this theorem will be carried out in section IV.4, after several preliminaries.

Remark IV.1. One might ask if theorem IV.1 still holds without the assumption (IV.1). This is true in case $\boldsymbol{x}$ is a non-degenerate critical point: briefly, a relative cycle that represents a generator of $\mathrm{H}_{\iota(\mathscr{F},\boldsymbol{x})}(\mathscr{F}_\bullet,\boldsymbol{x})$ also represents a generator of $\mathrm{H}_{\iota(\mathscr{F},\boldsymbol{x})}(\mathscr{F},\boldsymbol{x})$, and all the other local homology groups $\mathrm{H}_*(\mathscr{F}_\bullet,\boldsymbol{x})$ and $\mathrm{H}_*(\mathscr{F},\boldsymbol{x})$, with $* \neq \iota(\mathscr{F},\boldsymbol{x}) = \iota(\mathscr{F}_\bullet,\boldsymbol{x})$, are trivial (see theorem A.3). However, in the general case, assumption (IV.1) is necessary, as it is shown by the following simple example. Consider the function F of example A.4 in appendix A, i.e. $F : \mathbb{R}^2 \to \mathbb{R}$ given by

$$F(x,y) = (y - x^2)(y - 2x^2), \qquad \forall (x,y) \in \mathbb{R}^2.$$

The origin $\mathbf{0}$ is clearly an isolated critical point of F, and the corresponding Hessian is given in matrix form by

$$\text{(IV.3)} \qquad \mathrm{Hess}F(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now, let us consider the inclusion $J : \mathbb{R} \hookrightarrow \mathbb{R}^2$ given by $J(x) = (x,0)$, namely the inclusion of the x-axis in $\mathbb{R}^2$. The Morse index of F at the origin is 0 and coincides with the Morse index of the restricted function $F \circ J$. Analogously, the nullity of F and $F \circ J$ at the origin are both equal to 1. However the gradient of F on the x-axis is given by

$$\mathrm{Grad}F(x,0) = (8x^3, -3x^2), \qquad \forall x \in \mathbb{R},$$

hence condition (IV.1) is not satisfied for the inclusion J, i.e.

$$(\mathrm{Grad}F) \circ J \neq \mathrm{Grad}(F \circ J).$$

The local homology groups of F and $F \circ J$ at the origin are not isomorphic (and consequently $\mathrm{H}_*(J)$ is not an isomorphism). In fact, examining the sublevel $(F)_0$ (see figure A.6 in appendix A) it is clear that the origin is a saddle for F and a minimum for $F \circ J$. Therefore we have

$$\mathrm{H}_*(F,\mathbf{0}) = \begin{cases} \mathbb{F} & * = 1, \\ 0 & * \neq 1, \end{cases} \qquad \mathrm{H}_*(F \circ J, 0) = \begin{cases} \mathbb{F} & * = 0, \\ 0 & * \neq 0, \end{cases} \qquad \blacksquare$$

Remark IV.2. Also the hypothesis of C^2 regularity of the involved functional is essential in order to obtain the assertion of theorem IV.1. In fact, let us modify the functional F of the previous remark in the following way

$$F(x,y) = (y - x^2)(y - 2x^2) + 3x^6 \arctan\left(\frac{y}{x^4}\right), \qquad \forall (x,y) \in \mathbb{R}^2.$$

This function is C^1 and twice Gateaux differentiable, but it is not C^2 at the origin, which is again a critical point of F. Equation (IV.3) still holds, but the gradient of F on the x-axis is now given by

$$\mathrm{Grad}F(x,0) = (8x^3, 0), \qquad \forall x \in \mathbb{R},$$

so that condition (IV.1) is satisfied, i.e.

$$(\mathrm{Grad}F) \circ J = \mathrm{Grad}(F \circ J),$$

where $J : \mathbb{R} \hookrightarrow \mathbb{R}^2$ is given by $J(x) = (x, 0)$. As in the previous remark, the Morse index and nullity pair of F at the origin is $(0, 1)$ and coincides with the Morse index and nullity pair of $F \circ J$ at 0. Moreover, 0 is a local minimum of $F \circ J$, which implies

$$\mathrm{H}_*(F \circ J, 0) = \begin{cases} \mathbb{F} & * = 0, \\ 0 & * \neq 0, \end{cases}$$

However, the origin $\mathbf{0} \in \mathbb{R}^2$ is not a local minimum of the functional F. In fact, a straightforward computation shows that $\mathbf{0}$ is a local maximum of the functional F restricted to the parabola $y = \frac{3}{2}x^2$, namely $0 \in \mathbb{R}$ is a local maximum of the function

$$x \mapsto F\left(x, \tfrac{3}{2}x^2\right) = -\frac{1}{4}x^4 + 3x^6 \arctan\left(\frac{3}{2x^2}\right).$$

This readily implies that $\mathrm{H}_0(F, \mathbf{0}) = 0 \neq \mathrm{H}_0(F \circ J, 0)$, which contradicts the assertion of theorem IV.1. ∎

IV.2 The generalized Morse lemma revisited

In order to prove theorem IV.1, we need to give a more precise statement of the generalized Morse lemma (lemma A.1). Everything that we will claim already follows from the classical proof (see for instance [Ch, page 44]), but for the reader's convenience we include a full treatment.

In order to simplify the notation, from now on we will assume, without loss of generality, that $\boldsymbol{x} = \mathbf{0} \in \boldsymbol{E}$ and hence $\mathscr{U} \subset \boldsymbol{E}$ is an open neighborhood of $\mathbf{0}$. According to the operator H associated to the Hessian of $\mathscr{F}$ at the critical point $\mathbf{0}$, we have an orthogonal splitting $\boldsymbol{E} = \boldsymbol{E}^+ \oplus \boldsymbol{E}^- \oplus \boldsymbol{E}^0$, where $\boldsymbol{E}^+$ [resp. $\boldsymbol{E}^-$] is a closed subspace in which H is positive definite [resp. negative definite], while $\boldsymbol{E}^0$ is the finite-dimensional kernel of H (cf. section A.1). We denote by $P^\pm : \boldsymbol{E} \to \boldsymbol{E}^\pm$ the linear projector onto $\boldsymbol{E}^\pm := \boldsymbol{E}^+ \oplus \boldsymbol{E}^-$. On $\boldsymbol{E}^\pm \setminus \{\mathbf{0}\}$ we introduce the local flow Θ_H defined by $\Theta_H(s, \sigma(0)) = \sigma(s)$, where $\sigma : (s_0, s_1) \to \boldsymbol{E}^\pm \setminus \{\mathbf{0}\}$ (with $s_0 < 0 < s_1$) is a curve that satisfies

$$\dot\sigma(s) = -\frac{H\sigma(s)}{\|H\sigma(s)\|_{\boldsymbol{E}}}, \qquad \forall s \in (s_0, s_1). \tag{IV.4}$$

We also set $\Theta_H(0, \mathbf{0}) := \mathbf{0}$.

Then, the generalized Morse lemma may be restated as follows.

Lemma IV.2 (Generalized Morse Lemma revisited). *With the above assumptions on $\mathscr{F}$, there exists an open neighborhood $\mathscr{V} \subseteq \mathscr{U}$ of $\mathbf{0}$, a homeomorphism onto its image*

$$\phi : (\mathscr{V}, \mathbf{0}) \to (\mathscr{U}, \mathbf{0})$$

and a C^1 map

$$\psi : (\mathscr{V} \cap \boldsymbol{E}^0, \mathbf{0}) \to (\boldsymbol{E}^\pm, \mathbf{0}),$$

such that the following assertions hold.

(i) *For each* $\boldsymbol{v} \in \mathscr{V}$, *if we write* $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^\pm$ *according to the splitting* $\boldsymbol{E} = \boldsymbol{E}^0 \oplus \boldsymbol{E}^\pm$, *we have*

$$\mathscr{F} \circ \phi(\boldsymbol{v}) = \underbrace{\mathscr{F}\left(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)\right)}_{=:\mathscr{F}^0(\boldsymbol{v}^0)} + \underbrace{\tfrac{1}{2}\left\langle H\boldsymbol{v}^\pm, \boldsymbol{v}^\pm \right\rangle_{\boldsymbol{E}}}_{=:\mathscr{F}^\pm(\boldsymbol{v}^\pm)}.$$

(ii) *The origin* $\mathbf{0}$ *is a critical point of both* $\mathscr{F}^0$ *and* $\mathscr{F}^\pm$.

(iii) *The map* ψ *is implicitly defined by*

$$P^\pm\left(\mathrm{Grad}\mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0))\right) = \mathbf{0}, \qquad \forall \boldsymbol{v}^0 \in \mathscr{V} \cap \boldsymbol{E}^0,$$
$$\psi(\mathbf{0}) = \mathbf{0}.$$

(iv) *The homeomorphism* ϕ *is given by*

$$\phi^{-1}(\boldsymbol{v}) = \boldsymbol{v}^0 + \Theta_H(\tau(\boldsymbol{v} - \psi(\boldsymbol{v}^0)), \boldsymbol{v}^\pm - \psi(\boldsymbol{v}^0)), \qquad \forall \boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^\pm \in \phi(\mathscr{V}),$$

where τ *is a continuous function defined in the following way: for each* $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^\pm$ *that belongs to its domain, if* $\boldsymbol{v}^\pm \neq \mathbf{0}$ *then* $\tau(\boldsymbol{v})$ *is the only real number satisfying*

$$|\tau(\boldsymbol{v})| < \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}},$$
$$\mathscr{F}\left(\boldsymbol{v} + \psi(\boldsymbol{v}^0)\right) - \mathscr{F}\left(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)\right) = \mathscr{F}^\pm(\Theta_H(\tau(\boldsymbol{v}), \boldsymbol{v}^\pm)),$$

otherwise, if $\boldsymbol{v}^\pm = \mathbf{0}$, *we have* $\tau(\boldsymbol{v}) = \tau(\boldsymbol{v}^0) = 0$.

Proof. Since the Hessian of $\mathscr{F}$ at $\mathbf{0}$ is non-degenerate when restricted to $\boldsymbol{E}^\pm \otimes \boldsymbol{E}^\pm$, by the implicit function theorem we can find an open neighborhood $\mathscr{V}^0 \subset \mathscr{U} \cap \boldsymbol{E}^0$ of $\mathbf{0}$ and a C^1 map $\psi : (\mathscr{V}^0, \mathbf{0}) \to (\boldsymbol{E}^\pm, \mathbf{0})$ that is implicitly defined as in point (iii), for

$$\left\langle P^\pm\left(\mathrm{Grad}\mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0))\right), \,\cdot\, \right\rangle_{\boldsymbol{E}} = \mathrm{d}\mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0))|_{\boldsymbol{E}^\pm}.$$

Since $\mathbf{0} = \psi(\mathbf{0})$ is a critical point of $\mathscr{F}$, we immediately obtain that it is also a critical point of the functional $\mathscr{F}^0$ defined in the statement. Analogously, $\mathbf{0}$ is clearly a critical point of the quadratic functional $\mathscr{F}^\pm : \boldsymbol{E}^\pm \to \mathbb{R}$ of the statement.

All we have to do in order to conclude is to show that, in some neighborhood $\mathscr{V}' \subset \mathscr{U}$ of $\mathbf{0}$, there is a continuous function $\tau : \mathscr{V}' \to \mathbb{R}$ that is defined as in point (iv). Then, we can consider a continuous map $\phi^{-1} : \mathscr{V}' \to \boldsymbol{E}$, defined as in the first equation of point (iv), that is a homeomorphism onto its image since we can build its inverse in an analogous way by the flow Θ_H. Assertion (i) follows as a direct consequence.

Now, notice that the flow Θ_H is well defined on

$$\left\{(s, \boldsymbol{v}^\pm) \in \mathbb{R} \times \boldsymbol{E}^\pm \,|\, \boldsymbol{v} \neq 0,\ |s| < \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}}\right\}.$$

In fact, for an arbitrary $\boldsymbol{v}^\pm$, consider the maximal flow line $\sigma : (s_0, s_1) \to \boldsymbol{E}^\pm \setminus \{\mathbf{0}\}$ of Θ_H (cf. equation (IV.4)) with $s_0 < 0 < s_1$ and $\sigma(0) = \boldsymbol{v}^\pm$, and assume by contradiction that $s_1 < \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}}$. Since $\|\dot\sigma(s)\|_{\boldsymbol{E}} = 1$ for each $s \in (s_0, s_1)$, we have $\|\sigma(s) - \boldsymbol{v}^\pm\|_{\boldsymbol{E}} \leq |s|$ and therefore $0 < \|\sigma(s_1)\|_{\boldsymbol{E}} < 2\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}}$. This shows that $\dot\sigma(s_1) = -H\sigma(s_1)/\|H\sigma(s_1)\|_{\boldsymbol{E}}$ is well defined, and consequently σ can be further extended over s_1, which contradicts the maximality of s_1. The same argument shows that $s_0 \leq -\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}}$.

For each $r > 0$, let us denote by $\boldsymbol{B}^\pm_r$ and $\boldsymbol{B}^0_r$ the open balls of radius r in $\boldsymbol{E}^\pm$ and $\boldsymbol{E}^0$ respectively, i.e.

$$\boldsymbol{B}^\pm_r = \left\{\boldsymbol{v}^\pm \in \boldsymbol{E}^\pm \mid \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} < r\right\}, \qquad \boldsymbol{B}^0_r = \left\{\boldsymbol{v}^0 \in \boldsymbol{E}^0 \mid \|\boldsymbol{v}^0\|_{\boldsymbol{E}} < r\right\}.$$

Since $\mathscr{F}$ is C^2, for each $\varepsilon > 0$ there exists $\delta > 0$ such that, for each $\boldsymbol{v} = \boldsymbol{v}^\pm + \boldsymbol{v}^0 \in \boldsymbol{B}^\pm_\delta \oplus \boldsymbol{B}^0_\delta$, we have

$$\begin{aligned} &|\mathscr{F}(\boldsymbol{v} + \psi(\boldsymbol{v}^0)) - \mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)) - \mathscr{F}^\pm(\boldsymbol{v}^\pm)| \\ &\quad = \left| \int_0^1\!\!\int_0^1 s\left(\mathrm{Hess}\mathscr{F}(rs\boldsymbol{v}^\pm + \boldsymbol{v}^0)[\boldsymbol{v}^\pm, \boldsymbol{v}^\pm] - \mathrm{Hess}\mathscr{F}(\boldsymbol{0})[\boldsymbol{v}^\pm, \boldsymbol{v}^\pm]\right) \mathrm{d}r\, \mathrm{d}s \right| \\ &\quad \leq \varepsilon \|\boldsymbol{v}^\pm\|^2_{\boldsymbol{E}}. \end{aligned} \tag{IV.5}$$

Then, notice that the flow lines of Θ_H are actually the anti-gradient flow lines (up to a reparametrization) of the Morse functional $\mathscr{F}^\pm$, and therefore, for each $\boldsymbol{v}^\pm \in \boldsymbol{E}^\pm \setminus \{\boldsymbol{0}\}$, the function $s \mapsto \mathscr{F}^\pm \circ \Theta_H(s, \boldsymbol{v}^\pm)$ is strictly decreasing. Moreover,

$$\begin{aligned} |\mathscr{F}^\pm(\Theta_H(s, \boldsymbol{v}^\pm)) - \mathscr{F}^\pm(\boldsymbol{v}^\pm)| &= \int_0^{|s|} \|H\,\Theta_H(r, \boldsymbol{v}^\pm)\|_{\boldsymbol{E}}\, \mathrm{d}r \\ &\geq c_H \int_0^{|s|} \|\Theta_H(r, \boldsymbol{v}^\pm)\|_{\boldsymbol{E}}\, \mathrm{d}r \\ &\geq c_H \int_0^{|s|} \left(\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} - r\right) \mathrm{d}r \\ &\geq c_H \left(\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}}\, |s| - \frac{s^2}{2}\right), \end{aligned} \tag{IV.6}$$

where $c_H > 0$ is a constant determined by the spectrum of H. By (IV.5) and (IV.6), for each $\boldsymbol{v} = \boldsymbol{v}^\pm + \boldsymbol{v}^0 \in \boldsymbol{B}^\pm_\delta \oplus \boldsymbol{B}^0_\delta$ with $\boldsymbol{v}^\pm \neq \boldsymbol{0}$ and for each $s \in \mathbb{R}$ with $|s| < \|\boldsymbol{v}^\pm\|$, we have

$$\begin{aligned} &|\mathscr{F}(\boldsymbol{v} + \psi(\boldsymbol{v}^0)) - \mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)) - \mathscr{F}^\pm(\Theta_H(s, \boldsymbol{v}^\pm))| \\ &\quad \geq |\mathscr{F}^\pm(\boldsymbol{v}^\pm) - \mathscr{F}^\pm(\Theta_H(s, \boldsymbol{v}^\pm))| - |\mathscr{F}(\boldsymbol{v} + \psi(\boldsymbol{v}^0)) - \mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)) - \mathscr{F}^\pm(\boldsymbol{v}^\pm)| \\ &\quad \geq \underbrace{c_H \left(\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}}\, |s| - \frac{s^2}{2}\right) - \varepsilon \|\boldsymbol{v}^\pm\|^2_{\boldsymbol{E}}}_{(*)}. \end{aligned}$$

The quantity $(*)$ is nonnegative for

$$\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} \left(1 - \sqrt{1 - \frac{2\varepsilon}{c_H}}\right) \leq |s| \leq \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} \left(1 + \sqrt{1 - \frac{2\varepsilon}{c_H}}\right),$$

and since the function $s \mapsto \mathscr{F}^\pm \circ \Theta_H(s, \boldsymbol{v}^\pm)$ is strictly decreasing we obtain

$$\mathscr{F}^\pm(\Theta_H(s, \boldsymbol{v}^\pm)) \leq \mathscr{F}(\boldsymbol{v} + \psi(\boldsymbol{v}^0)) - \mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)) \leq \mathscr{F}^\pm(\Theta_H(-s, \boldsymbol{v}^\pm))$$
$$\forall s \in \left[\|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} \left(1 - \sqrt{1 - \frac{2\varepsilon}{c_H}}\right), \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} \right).$$

This implies that there exists a unique $\tau(\boldsymbol{v}) \in \mathbb{R}$, with

$$|\tau(\boldsymbol{v})| < \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}} \left(1 - \sqrt{1 - \frac{2\varepsilon}{c_H}}\right) < \|\boldsymbol{v}^\pm\|_{\boldsymbol{E}},$$

such that

$$\mathscr{F}^\pm(\Theta_H(\tau(\boldsymbol{v}), \boldsymbol{v}^\pm)) = \mathscr{F}(\boldsymbol{v} + \psi(\boldsymbol{v}^0)) - \mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)).$$

By the implicit function theorem, the obtained function τ is continuous, and can be continuously extended to each $\boldsymbol{v} = \boldsymbol{v}^\pm + \boldsymbol{v}^0 \in \boldsymbol{B}^\pm_\delta \oplus \boldsymbol{B}^0_\delta$ with $\boldsymbol{v}^\pm = \boldsymbol{0}$ by $\tau(\boldsymbol{v}) = \tau(\boldsymbol{v}^0) = 0$. ■

IV.3 Naturality of the Morse lemma

Let $H_\bullet$ be the bounded self-adjoint linear operator on $\boldsymbol{E}_\bullet \subset \boldsymbol{E}$ associated to the Hessian of the restricted functional $\mathscr{F}_\bullet = \mathscr{F}|_{\boldsymbol{E}_\bullet}$ at $\boldsymbol{0}$. Then $H \circ J = J \circ H_\bullet$. This follows immediately from condition (IV.1), since

$$H \circ J = \mathrm{d}(\mathrm{Grad}\mathscr{F})(\boldsymbol{0}) \circ J = \mathrm{d}(\underbrace{(\mathrm{Grad}\mathscr{F}) \circ J}_{J \circ \mathrm{Grad}(\mathscr{F}_\bullet)})(\boldsymbol{0}) = J \circ \mathrm{d}(\mathrm{Grad}(\mathscr{F}_\bullet))(\boldsymbol{0}) = J \circ H_\bullet.$$

Namely, the restriction of H to $\boldsymbol{E}_\bullet$ coincides with $H_\bullet$, i.e.

$$H|_{\boldsymbol{E}_\bullet} = H_\bullet, \tag{IV.7}$$

and in particular it is a self-adjoint Fredholm operator on $\boldsymbol{E}_\bullet$. If we denote by $\boldsymbol{E}_\bullet = \boldsymbol{E}^0_\bullet \oplus \boldsymbol{E}^+_\bullet \oplus \boldsymbol{E}^-_\bullet$ the orthogonal splitting defined by the operator $H_\bullet$, equation (IV.7) readily implies that

$$\boldsymbol{E}^0_\bullet \subseteq \boldsymbol{E}^0, \quad \boldsymbol{E}^+_\bullet \subseteq \boldsymbol{E}^+, \quad \boldsymbol{E}^-_\bullet \subseteq \boldsymbol{E}^-, \tag{IV.8}$$

and moreover, if we denote by $P^\pm_\bullet : \boldsymbol{E}_\bullet \to \boldsymbol{E}^\pm_\bullet$ the orthogonal projector onto $\boldsymbol{E}^\pm_\bullet = \boldsymbol{E}^+_\bullet \oplus \boldsymbol{E}^-_\bullet$, this latter turns out to be the restriction of the projector $P^\pm : \boldsymbol{E} \to \boldsymbol{E}^\pm$ to $\boldsymbol{E}_\bullet$, i.e.

$$P^\pm|_{\boldsymbol{E}_\bullet} = P^\pm_\bullet. \tag{IV.9}$$

The hypotheses of the generalized Morse Lemma are fulfilled by both the functional $\mathscr{F}$ and its restriction $\mathscr{F}_\bullet$. The following is the long list of the symbols involved in the statement of lemma IV.2, and we write in the subsequent line the corresponding list of symbols involved in the statement referred to the restricted functional $\mathscr{F}_\bullet$:

$$\begin{array}{cccccccccc} \boldsymbol{E}^\pm, & \boldsymbol{E}^0, & P^\pm, & \mathscr{V}, & \Theta_H, & \phi, & \psi, & \tau, & \mathscr{F}^0, & \mathscr{F}^\pm, \\ \boldsymbol{E}_\bullet^\pm, & \boldsymbol{E}_\bullet^0, & P_\bullet^\pm, & \mathscr{V}_\bullet, & \Theta_{H_\bullet}, & \phi_\bullet, & \psi_\bullet, & \tau_\bullet, & \mathscr{F}_\bullet^0, & \mathscr{F}_\bullet^\pm. \end{array}$$

We want to show that, under the hypotheses of theorem IV.1, the decomposition $\mathscr{F}^\pm + \mathscr{F}^0$ of $\mathscr{F}$, given by the generalized Morse lemma, restricts to the corresponding decomposition $\mathscr{F}_\bullet^\pm + \mathscr{F}_\bullet^0$ of $\mathscr{F}_\bullet$.

Lemma IV.3.
(i) *If* $\iota(\mathscr{F}, \mathbf{0}) = \iota(\mathscr{F}_\bullet, \mathbf{0})$, *then* $\boldsymbol{E}^- = \boldsymbol{E}_\bullet^-$.

(ii) *If* $\nu(\mathscr{F}, \mathbf{0}) = \nu(\mathscr{F}_\bullet, \mathbf{0})$, *then* $\boldsymbol{E}^0 = \boldsymbol{E}_\bullet^0$.

Proof. The claims follow at once from (IV.8), since

$$\dim \boldsymbol{E}_\bullet^- = \iota(\mathscr{F}_\bullet, \mathbf{0}) = \iota(\mathscr{F}, \mathbf{0}) = \dim \boldsymbol{E}^-,$$
$$\dim \boldsymbol{E}_\bullet^0 = \nu(\mathscr{F}_\bullet, \mathbf{0}) = \nu(\mathscr{F}, \mathbf{0}) = \dim \boldsymbol{E}^0. \qquad \blacksquare$$

Proposition IV.4. *If* $\nu(\mathscr{F}, \mathbf{0}) = \nu(\mathscr{F}_\bullet, \mathbf{0})$, *the following equalities hold (on some neighborhood of the critical point* $\mathbf{0}$ *where the involved maps are defined):*

(i) $\psi = \psi_\bullet$,

(ii) $\phi|_{\boldsymbol{E}_\bullet} = \phi_\bullet$.

Proof. By lemma IV.3(ii), the domains of the maps ψ and $\psi_\bullet$ are open neighborhoods of $\mathbf{0}$ in $\boldsymbol{E}^0 = \boldsymbol{E}_\bullet^0$. Up to shrinking these neighborhoods, we can assume that both ψ and $\psi_\bullet$ have common domain $\mathscr{V}^0 \subset \boldsymbol{E}^0$. By lemma IV.2(iii) we have $\psi(\mathbf{0}) = \psi_\bullet(\mathbf{0}) = \mathbf{0}$, and all we have to do in order to conclude the proof of (i) is to show that, for each $\boldsymbol{v}^0 \in \mathscr{V}^0 \setminus \{\mathbf{0}\}$, the maps $\psi_\bullet(\boldsymbol{v}^0)$ and $\psi(\boldsymbol{v}^0)$ are implicitly defined by the same equation, that is

$$P^\pm\left(\mathrm{Grad}\mathscr{F}(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0))\right) = \mathbf{0} = P^\pm\left(\mathrm{Grad}\mathscr{F}(\boldsymbol{v}^0 + \psi_\bullet(\boldsymbol{v}^0))\right).$$

This is easily verified since, by (IV.1) and (IV.9), we have

$$P^\pm(\mathrm{Grad}\mathscr{F}(\boldsymbol{v}^0 + \psi_\bullet(\boldsymbol{v}^0))) = P_\bullet^\pm(\mathrm{Grad}\mathscr{F}_\bullet(\boldsymbol{v}^0 + \psi_\bullet(\boldsymbol{v}^0))) = \mathbf{0}.$$

For (ii), up to shrinking the domains of ϕ and $\phi_\bullet$, we can assume that they are maps of the form $\phi : \mathscr{V} \to \mathscr{U}$ and $\phi_\bullet : \mathscr{V}_\bullet \to \mathscr{U}_\bullet$, where $\mathscr{V}_\bullet = \mathscr{V} \cap \boldsymbol{E}_\bullet$. Being ϕ and $\phi_\bullet$ homeomorphisms onto their images, we can equivalently prove that $\phi^{-1} = \phi_\bullet^{-1}$ on the open set $\phi_\bullet(\mathscr{V}_\bullet) \subset \mathscr{U}_\bullet$. To begin with, notice that (IV.7) readily implies that the flow $\Theta_{H_\bullet}$ is the restriction of the flow Θ_H to $\boldsymbol{E}_\bullet^\pm \setminus \{\mathbf{0}\}$, i.e.

$$\Theta_H(\cdot, \boldsymbol{v}^\pm) = \Theta_{H_\bullet}(\cdot, \boldsymbol{v}^\pm), \qquad \forall \boldsymbol{v}^\pm \in \boldsymbol{E}_\bullet^\pm \setminus \{\mathbf{0}\}.$$

By lemma IV.2(iv) and since $\psi = \psi_\bullet$, for each $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^\pm \in \phi_\bullet(\mathscr{V}_\bullet)$ we have

$$\phi^{-1}(\boldsymbol{v}) = \boldsymbol{v}^0 + \Theta_H(\tau(\boldsymbol{v} - \psi(\boldsymbol{v}^0)), \boldsymbol{v}^\pm - \psi(\boldsymbol{v}^0)),$$
$$\phi_\bullet^{-1}(\boldsymbol{v}) = \boldsymbol{v}^0 + \Theta_H(\tau_\bullet(\boldsymbol{v} - \psi(\boldsymbol{v}^0)), \boldsymbol{v}^\pm - \psi(\boldsymbol{v}^0)).$$

Hence, in order to conclude the proof of (ii) we just need to show that, for each $\boldsymbol{v}$ in the domain of $\tau_\bullet$, we have $\tau(\boldsymbol{v}) = \tau_\bullet(\boldsymbol{v})$. This is easily verified since, by lemma IV.2(iv), $\tau(\boldsymbol{v})$ and $\tau_\bullet(\boldsymbol{v})$ are implicitly defined by the same equation

$$\begin{aligned}\frac{1}{2}&\left\langle H\,\Theta_H(\tau(\boldsymbol{v}), \boldsymbol{v}^\pm), \Theta_H(\tau(\boldsymbol{v}), \boldsymbol{v}^\pm)\right\rangle_{\boldsymbol{E}} \\ &= \mathscr{F}\left(\boldsymbol{v} + \psi(\boldsymbol{v}^0)\right) - \mathscr{F}\left(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)\right) \\ &= \frac{1}{2}\left\langle H\,\Theta_H(\tau_\bullet(\boldsymbol{v}), \boldsymbol{v}^\pm), \Theta_H(\tau_\bullet(\boldsymbol{v}), \boldsymbol{v}^\pm)\right\rangle_{\boldsymbol{E}}.\end{aligned}$$ ■

Corollary IV.5. *If* $\nu(\mathscr{F}, \mathbf{0}) = \nu(\mathscr{F}_\bullet, \mathbf{0})$ *then* $\mathscr{F}^\pm|_{\boldsymbol{E}_\bullet} = \mathscr{F}_\bullet^\pm$ *and* $\mathscr{F}^0 = \mathscr{F}_\bullet^0$.

IV.4 Local Homology

In this section we will carry out the proof of theorem IV.1. Before going to this proof, we need to establish another naturality property, this time for the isomorphism between the local homology of $\mathscr{F}$ at $\mathbf{0}$ and the homology of corresponding Gromoll-Meyer pairs (see section A.5 in appendix A).

Lemma IV.6. *Let* $(\mathscr{W}, \mathscr{W}_-)$ *be a Gromoll-Meyer pair for* $\mathscr{F}$ *at* $\mathbf{0}$*. Then, the following holds.*

(i) *The pair* $(\mathscr{W}_\bullet, \mathscr{W}_{\bullet-}) := (\mathscr{W} \cap \boldsymbol{E}_\bullet, \mathscr{W}_- \cap \boldsymbol{E}_\bullet) = (J^{-1}(\mathscr{W}), J^{-1}(\mathscr{W}_-))$ *is a Gromoll-Meyer pair for* $\mathscr{F}_\bullet = \mathscr{F}|_{\boldsymbol{E}_\bullet}$ *at* $\mathbf{0}$.

(ii) *Consider the restrictions of* $J : \boldsymbol{E}_\bullet \hookrightarrow \boldsymbol{E}$ *given by*

$$J : ((\mathscr{F}_\bullet)_c \cup \{\boldsymbol{x}\}, (\mathscr{F}_\bullet)_c) \hookrightarrow ((\mathscr{F})_c \cup \{\boldsymbol{x}\}, (\mathscr{F})_c),$$
$$J : (\mathscr{W}_\bullet, \mathscr{W}_{\bullet-}) \hookrightarrow (\mathscr{W}, \mathscr{W}_-).$$

These restrictions induce the homology homomorphisms

$$\mathrm{H}_*(J) : \mathrm{H}_*(\mathscr{F}_\bullet, \mathbf{0}) \to \mathrm{H}_*(\mathscr{F}, \mathbf{0}),$$
$$\mathrm{H}_*(J) : \mathrm{H}_*(\mathscr{W}_\bullet, \mathscr{W}_{\bullet-}) \to \mathrm{H}_*(\mathscr{W}, \mathscr{W}_-).$$

Then, there exist homology isomorphisms $\iota_{(\mathscr{W}_\bullet, \mathscr{W}_{\bullet-})}$ *and* $\iota_{(\mathscr{W}, \mathscr{W}_-)}$ *such that the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{H}_*(\mathscr{F}_\bullet, \mathbf{0}) & \xrightarrow{\mathrm{H}_*(J)} & \mathrm{H}_*(\mathscr{F}, \mathbf{0}) \\ {\scriptstyle \iota_{(\mathscr{W}_\bullet, \mathscr{W}_{\bullet-})}}\downarrow \simeq & & \simeq \downarrow {\scriptstyle \iota_{(\mathscr{W}, \mathscr{W}_-)}} \\ \mathrm{H}_*(\mathscr{W}_\bullet, \mathscr{W}_{\bullet-}) & \xrightarrow{\mathrm{H}_*(J)} & \mathrm{H}_*(\mathscr{W}, \mathscr{W}_-) \end{array}$$

Proof. Part (i) just requires the straightforward verification that the pair $(\mathscr{W}_\bullet, \mathscr{W}_{\bullet -})$ satisfies conditions **(GM1)**,...,**(GM4)** (cf. section A.5) in the definition of Gromoll-Meyer pair. Part (ii) requires to examine the proof of theorem A.7 in appendix A, and we refer the reader to [Ch, page 48] for more details on what we claim. The point, here, is to show that the isomorphism between the local homology and the homology of a Gromoll-Meyer pair is given by the composition of homology isomorphisms induced by maps, so that the assertion follows from the functoriality of singular homology.

We denote by $\Phi_{\mathrm{Grad}\mathscr{F}}$ the anti-gradient flow of $\mathscr{F}$, as in section A.3 of appendix A. Notice that, by condition (IV.1), $\Phi_{\mathrm{Grad}\mathscr{F}}$ restricts on $\boldsymbol{E}_\bullet$ to the anti-gradient flow $\Phi_{\mathrm{Grad}\mathscr{F}_\bullet}$ of the restricted functional $\mathscr{F}_\bullet$. We introduce the sets $\mathscr{Y}$ and $\mathscr{Y}_\bullet$ given by

$$\begin{aligned} \mathscr{Y} &:= \Phi_{\mathrm{Grad}\mathscr{F}}\left([0,\infty) \times \mathscr{W}\right), \\ \mathscr{Y}_\bullet &:= \Phi_{\mathrm{Grad}\mathscr{F}_\bullet}\left([0,\infty) \times \mathscr{W}_\bullet\right) = \Phi_{\mathrm{Grad}\mathscr{F}}\left([0,\infty) \times \mathscr{W}_\bullet\right) = \mathscr{Y} \cap \boldsymbol{E}_\bullet, \end{aligned}$$

and we consider the following diagram.

$$\begin{array}{ccc}
\mathrm{H}_*(\mathscr{F}_\bullet, \mathbf{0}) & \xrightarrow{\mathrm{H}_*(J)} & \mathrm{H}_*(\mathscr{F}, \mathbf{0}) \\
\uparrow \simeq & & \uparrow \simeq \\
\mathrm{H}_*\left(\mathscr{Y}_\bullet \cap (\mathscr{F}_\bullet)_c \cup \{\mathbf{0}\}, \mathscr{Y}_\bullet \cap (\mathscr{F}_\bullet)_c\right) & \longrightarrow & \mathrm{H}_*\left(\mathscr{Y} \cap (\mathscr{F})_c \cup \{\mathbf{0}\}, \mathscr{Y} \cap (\mathscr{F})_c\right) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathrm{H}_*\left(\mathscr{Y}_\bullet, \mathscr{Y}_\bullet \cap (\mathscr{F}_\bullet)_c\right) & \longrightarrow & \mathrm{H}_*\left(\mathscr{Y}, \mathscr{Y} \cap (\mathscr{F})_c\right) \\
\uparrow \simeq & & \uparrow \simeq \\
\mathrm{H}_*(\mathscr{W}_\bullet, \mathscr{W}_{\bullet -}) & \xrightarrow{\mathrm{H}_*(J)} & \mathrm{H}_*(\mathscr{W}, \mathscr{W}_-)
\end{array}$$

In this diagram, all the arrows are homology homomorphisms induced by inclusions. Moreover, all the vertical arrows are isomorphisms (this fact is proved by anti-gradient flow deformations and excisions), and we define the isomorphisms $\iota_{(\mathscr{W}_\bullet, \mathscr{W}_{\bullet -})}$ and $\iota_{(\mathscr{W}, \mathscr{W}_-)}$ as the composition of the whole left vertical line and right vertical line respectively. By the functoriality of singular homology, this diagram is commutative, and the claim of part (ii) follows. ■

After these preliminaries, let us go back to the proof of theorem IV.1. First of all, if we assume $\nu(\mathscr{F}, \mathbf{0}) = \nu(\mathscr{F}_\bullet, \mathbf{0})$, corollary IV.5 implies that $\mathscr{F}^0 = \mathscr{F}^0_\bullet$. Hence the inclusion J restrict to the identity map on the pair

$$((\mathscr{F}^0_\bullet)_c \cup \{\mathbf{0}\}, (\mathscr{F}^0_\bullet)_c) = ((\mathscr{F}^0)_c \cup \{\mathbf{0}\}, (\mathscr{F}^0)_c),$$

and therefore

$$\mathrm{H}_*(\mathscr{F}^0_\bullet, \mathbf{0}) = \mathrm{H}_*(\mathscr{F}^0, \mathbf{0}). \tag{IV.10}$$

For the Morse functionals $\mathscr{F}^\pm_\bullet$ and $\mathscr{F}^\pm$ we have the following result.

Lemma IV.7. *If* $(\iota(\mathscr{F}, \mathbf{0}), \nu(\mathscr{F}, \mathbf{0})) = (\iota(\mathscr{F}_\bullet, \mathbf{0}), \nu(\mathscr{F}_\bullet, \mathbf{0}))$ *then the inclusion* J*, restricted as a map*

$$J : ((\mathscr{F}^\pm_\bullet)_c \cup \{\mathbf{0}\}, (\mathscr{F}^\pm_\bullet)_c) \hookrightarrow ((\mathscr{F}^\pm)_c \cup \{\mathbf{0}\}, (\mathscr{F}^\pm)_c), \tag{IV.11}$$

induces the homology isomorphism

$$\mathrm{H}_*(J) : \mathrm{H}_*(\mathscr{F}^\pm_\bullet, \mathbf{0}) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{F}^\pm, \mathbf{0}).$$

Proof. The fact that J restricts to a map of the form (IV.11) is guaranteed by corollary IV.5. Moreover, lemma IV.3(i) guarantees that $\boldsymbol{E}^-_\bullet = \boldsymbol{E}^-$, hence J further restricts to a homeomorphism

$$\tilde{J} : (\boldsymbol{E}^- \cap (\mathscr{F}^\pm_\bullet)_c \cup \{\mathbf{0}\}, \boldsymbol{E}^- \cap (\mathscr{F}^\pm_\bullet)_c) \hookrightarrow (\boldsymbol{E}^- \cap (\mathscr{F}^\pm)_c \cup \{\mathbf{0}\}, \boldsymbol{E}^- \cap (\mathscr{F}^\pm)_c),$$

and we obtain the following commutative diagram of inclusions.

$$\begin{array}{ccc}
((\mathscr{F}^\pm_\bullet)_c \cup \{\mathbf{0}\}, (\mathscr{F}^\pm_\bullet)_c) & \xhookrightarrow{\quad J \quad} & ((\mathscr{F}^\pm)_c \cup \{\mathbf{0}\}, (\mathscr{F}^\pm)_c) \\
k_\bullet \Big\uparrow \sim & & k \Big\uparrow \sim \\
(\boldsymbol{E}^- \cap (\mathscr{F}^\pm_\bullet)_c \cup \{\mathbf{0}\}, \boldsymbol{E}^- \cap (\mathscr{F}^\pm_\bullet)_c) & \xrightarrow[\simeq]{\quad \tilde{J} \quad} & (\boldsymbol{E}^- \cap (\mathscr{F}^\pm)_c \cup \{\mathbf{0}\}, \boldsymbol{E}^- \cap (\mathscr{F}^\pm)_c)
\end{array}$$

It is well known that $k_\bullet$ and k are homotopy equivalences, and in particular $\mathrm{H}_*(k_\bullet)$ and $\mathrm{H}_*(k)$ are isomorphisms. Therefore $\mathrm{H}_*(J) = \mathrm{H}_*(k) \circ \mathrm{H}_*(\tilde{J}) \circ \mathrm{H}_*(k_\bullet)^{-1}$ is an isomorphism as well. ∎

Proof of theorem IV.1. The homeomorphisms ϕ and $\phi_\bullet$ obtained by the Morse lemma induce local homology isomorphisms $\mathrm{H}_*(\phi)$ and $\mathrm{H}_*(\phi_\bullet)$ such that the following diagram commutes.

$$\begin{array}{ccc}
\mathrm{H}_*(\mathscr{F}_\bullet, \mathbf{0}) & \xrightarrow{\quad \mathrm{H}_*(J) \quad} & \mathrm{H}_*(\mathscr{F}, \mathbf{0}) \\
\mathrm{H}_*(\phi_\bullet) \Big\uparrow \simeq & & \simeq \Big\uparrow \mathrm{H}_*(\phi) \\
\mathrm{H}_*(\mathscr{F}^0_\bullet + \mathscr{F}^\pm_\bullet, \mathbf{0}) & \xrightarrow{\quad \mathrm{H}_*(J) \quad} & \mathrm{H}_*(\mathscr{F}^0 + \mathscr{F}^\pm, \mathbf{0})
\end{array}$$

Hence, we only need to prove that the lower horizontal homomorphism $\mathrm{H}_*(J)$ is an isomorphism. We consider Gromoll-Meyer pairs $(\mathscr{W}^{\pm}, \mathscr{W}^{\pm}_{-})$ and $(\mathscr{W}^{0}, \mathscr{W}^{0}_{-})$ for $\mathscr{F}^{\pm}$ and $\mathscr{F}^{0}$ respectively at $\mathbf{0}$, so that the cross product of these pairs, that is

$$(\mathscr{W}, \mathscr{W}_{-}) := (\mathscr{W}^{\pm} \times \mathscr{W}^{0}, (\mathscr{W}^{\pm}_{-} \times \mathscr{W}^{0}) \cup (\mathscr{W}^{\pm} \times \mathscr{W}^{0}_{-})),$$

is a Gromoll-Meyer pair for $\mathscr{F}^0 + \mathscr{F}^{\pm}$ at $\mathbf{0}$. Then, by lemma IV.6, we obtain Gromoll-Meyer pairs for the functionals $\mathscr{F}^{\pm}_{\bullet}$, $\mathscr{F}^{0}_{\bullet}$ and $\mathscr{F}^{0}_{\bullet} + \mathscr{F}^{\pm}_{\bullet}$ at $\mathbf{0}$ respectively as

$$\begin{aligned}
(\mathscr{W}^{\pm}_{\bullet}, \mathscr{W}^{\pm}_{\bullet -}) &:= (\mathscr{W}^{\pm} \cap \boldsymbol{E}_{\bullet}, \mathscr{W}^{\pm}_{-} \cap \boldsymbol{E}_{\bullet}) = (J^{-1}(\mathscr{W}^{\pm}), J^{-1}(\mathscr{W}^{\pm}_{-})),\\
(\mathscr{W}^{0}_{\bullet}, \mathscr{W}^{0}_{\bullet -}) &:= (\mathscr{W}^{0} \cap \boldsymbol{E}_{\bullet}, \mathscr{W}^{0}_{-} \cap \boldsymbol{E}_{\bullet}) = (J^{-1}(\mathscr{W}^{0}), J^{-1}(\mathscr{W}^{0}_{-})),\\
(\mathscr{W}_{\bullet}, \mathscr{W}_{\bullet -}) &:= (\mathscr{W} \cap \boldsymbol{E}_{\bullet}, \mathscr{W}_{-} \cap \boldsymbol{E}_{\bullet}) = (J^{-1}(\mathscr{W}), J^{-1}(\mathscr{W}_{-}))
\end{aligned}$$

and, together with the Künneth formula, we obtain the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{H}_*(\mathscr{F}^{0}_{\bullet} + \mathscr{F}^{\pm}_{\bullet}, \mathbf{0}) & \xrightarrow{\ \mathrm{H}_*(J)\ } & \mathrm{H}_*(\mathscr{F}^{0} + \mathscr{F}^{\pm}, \mathbf{0}) \\
{\scriptstyle \iota_{(\mathscr{W}_{\bullet}, \mathscr{W}_{\bullet -})}} \downarrow \simeq & & \simeq \downarrow {\scriptstyle \iota_{(\mathscr{W}, \mathscr{W}_{-})}} \\
\mathrm{H}_*(\mathscr{W}_{\bullet}, \mathscr{W}_{\bullet -}) & \xrightarrow{\ \mathrm{H}_*(J)\ } & \mathrm{H}_*(\mathscr{W}, \mathscr{W}_{-}) \\
{\scriptstyle \text{Künneth}} \downarrow \simeq & & \simeq \downarrow {\scriptstyle \text{Künneth}} \\
\mathrm{H}_*(\mathscr{W}^{\pm}_{\bullet}, \mathscr{W}^{\pm}_{\bullet -}) \otimes \mathrm{H}_*(\mathscr{W}^{0}_{\bullet}, \mathscr{W}^{0}_{\bullet -}) & \xrightarrow{\ \mathrm{H}_*(J) \otimes \mathrm{H}_*(J)\ } & \mathrm{H}_*(\mathscr{W}^{\pm}, \mathscr{W}^{\pm}_{-}) \otimes \mathrm{H}_*(\mathscr{W}^{0}, \mathscr{W}^{0}_{-}) \\
{\scriptstyle \iota_{(\mathscr{W}^{\pm}_{\bullet}, \mathscr{W}^{\pm}_{\bullet -})} \otimes \iota_{(\mathscr{W}^{0}_{\bullet}, \mathscr{W}^{0}_{\bullet -})}} \uparrow \simeq & & \simeq \uparrow {\scriptstyle \iota_{(\mathscr{W}^{\pm}, \mathscr{W}^{\pm}_{-})} \otimes \iota_{(\mathscr{W}^{0}, \mathscr{W}^{0}_{-})}} \\
\mathrm{H}_*(\mathscr{F}^{\pm}_{\bullet}, \mathbf{0}) \otimes \mathrm{H}_*(\mathscr{F}^{0}_{\bullet}, \mathbf{0}) & \underset{\simeq}{\xrightarrow{\ \mathrm{H}_*(J) \otimes \mathrm{H}_*(J)\ }} & \mathrm{H}_*(\mathscr{F}^{\pm}, \mathbf{0}) \otimes \mathrm{H}_*(\mathscr{F}^{0}, \mathbf{0})
\end{array}$$

The commutativity of the upper and lower squares follows from lemma IV.6, while the commutativity of the central square follows from the naturality of the Künneth formula (see for instance [Hat, page 275]). By (IV.10) and lemma IV.7, the lower horizontal homomorphism $\mathrm{H}_*(J) \otimes \mathrm{H}_*(J)$ is an isomorphism, and so must be all the others horizontal homomorphisms. ■

IV.5 Application to the action functional

Let M be an m-dimensional smooth closed manifold and $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ a smooth 1-periodic convex quadratic-growth Lagrangian, with associated action functional $\mathscr{A} : W^{1,2}(\mathbb{T}; M) \to \mathbb{R}$. We also consider, for each $n \in \mathbb{N}$, the mean action functional $\mathscr{A}^{[n]} : W^{1,2}(\mathbb{T}^{[n]}; M) \to \mathbb{R}$ (cf. section II.3) and, for each $k \in \mathbb{N}$ big enough, the discrete action functional $\mathscr{A}_k : \Delta_k \to \mathbb{R}$ (cf. section III.3) and the discrete mean action functional $\mathscr{A}_k^{[n]} : \Delta_{nk} \to \mathbb{R}$ (cf. section III.6).

We show that the abstract theorem IV.1 applies when J is the discrete iteration map $\psi_k^{[n]} : \Delta_k \hookrightarrow \Delta_{nk}$ and $\mathscr{F}$ is the discrete mean action functional $\mathscr{A}_k^{[n]}$.

Theorem IV.8. *Let $\boldsymbol{q} \in \Delta_k$ be a critical point of the discrete action functional $\mathscr{A}_k$ such that $\mathscr{A}_k(\boldsymbol{q}) = c$ and, for some $n \in \mathbb{N}$, we have*

$$(\iota(\mathscr{A}_k, \boldsymbol{q}), \nu(\mathscr{A}_k, \boldsymbol{q})) = (\iota(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]}), \nu(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]})).$$

Then, the discrete iteration map $\psi_k^{[n]}$, restricted to a map

$$\psi_k^{[n]} : ((\mathscr{A}_k)_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_c) \hookrightarrow ((\mathscr{A}_k^{[n]})_c \cup \{\boldsymbol{q}^{[n]}\}, (\mathscr{A}_k^{[n]})_c),$$

induces the homology isomorphism

$$\mathrm{H}_*(\psi_k^{[n]}) : \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]}).$$

Proof. Applying the localization argument of remark III.1 we can assume that our Lagrangian function has the form $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times U \times \mathbb{R}^m \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^m$, and the corresponding action and mean action have the form $\mathscr{A} : W^{1,2}(\mathbb{T}; U) \to \mathbb{R}$ and $\mathscr{A}^{[n]} : W^{1,2}(\mathbb{T}^{[n]}; U) \to \mathbb{R}$. In this way, we are considering $\boldsymbol{q}$ as lying in the open set $W_k = \lambda_k^{-1}(W^{1,2}(\mathbb{T}; U))$ of the k-fold product $\mathbb{R}^m \times ... \times \mathbb{R}^m$, and correspondingly

$$\boldsymbol{q}^{[n]} \in W_{nk} = (\lambda_k^{[n]})^{-1}(W^{1,2}(\mathbb{T}^{[n]}; U)) \subset \underbrace{\mathbb{R}^m \times ... \times \mathbb{R}^m}_{nk \text{ times}}.$$

The iteration map $\psi^{[n]} : W^{1,2}(\mathbb{T}; U) \hookrightarrow W^{1,2}(\mathbb{T}^{[n]}; U)$ is now the restriction of a bounded linear embedding $\psi^{[n]} : W^{1,2}(\mathbb{T}; \mathbb{R}^m) \hookrightarrow W^{1,2}(\mathbb{T}^{[n]}; \mathbb{R}^m)$, namely it is the restriction of the n^{th}-iteration map on the Hilbert space $W^{1,2}(\mathbb{T}; \mathbb{R}^m)$. Analogously, the discrete iteration map $\psi_k^{[n]}$ is the restriction of the linear embedding $\psi_k^{[n]} : \mathbb{R}^{km} \hookrightarrow \mathbb{R}^{nkm}$ given by

$$\psi_k^{[n]}(\boldsymbol{w}) = \boldsymbol{w}^{[n]} = (\underbrace{\boldsymbol{w}, ..., \boldsymbol{w}}_{n \text{ times}}), \qquad \forall \boldsymbol{w} \in \mathbb{R}^{km}.$$

This latter embedding is an isometry with respect to the standard inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\mathbb{R}^{km}$ and the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle^{[n]}$ on $\mathbb{R}^{nkm}$ obtained multiplying by n^{-1} the standard one, i.e.

$$\langle\!\langle \boldsymbol{w}, \boldsymbol{z}\rangle\!\rangle = \sum_{j=0}^{k-1} \langle w_j, z_j\rangle, \qquad \forall \boldsymbol{w} = (w_0, ..., w_{k-1}), \boldsymbol{z} = (z_0, ..., z_{k-1}) \in \mathbb{R}^{km},$$

$$\langle\!\langle \boldsymbol{w}', \boldsymbol{z}'\rangle\!\rangle^{[n]} = \frac{1}{n}\sum_{j=0}^{nk-1} \langle w'_j, z'_j\rangle, \qquad \forall \boldsymbol{w}' = (w'_0, ..., w'_{nk-1}), \boldsymbol{z}' = (z'_0, ..., z'_{nk-1}) \in \mathbb{R}^{nkm},$$

where we denote by $\langle\cdot,\cdot\rangle$ the standard inner product of $\mathbb{R}^m$.

Now, the functionals $\mathscr{A}_k$ and $\mathscr{A}_k^{[n]}$ are smooth (cf. proposition III.4), and therefore our claim follows from the abstract theorem IV.1, provided that we verify that condition (IV.1) holds in our setting. Namely, we must verify that

$$\mathrm{Grad}\mathscr{A}_k^{[n]}(\boldsymbol{w}^{[n]}) = \psi^{[n]} \circ \mathrm{Grad}\mathscr{A}_k(\boldsymbol{w}), \qquad \forall \boldsymbol{w} \in W_k.$$

This condition follows easily by proposition III.6. In fact, if we fix an arbitrary $\boldsymbol{w} \in W_k$ and we define $\gamma_{\boldsymbol{w}} := \lambda_k(\boldsymbol{w})$ and $\boldsymbol{g} = (g_0, ..., g_{k-1})$ with

$$g_j = \partial_v \mathscr{L}(\tfrac{j}{k}, \gamma_{\boldsymbol{w}}(\tfrac{j}{k}), \dot\gamma_{\boldsymbol{w}}(\tfrac{j^-}{k})) - \partial_v \mathscr{L}(\tfrac{j}{k}, \gamma_{\boldsymbol{w}}(\tfrac{j}{k}), \dot\gamma_{\boldsymbol{w}}(\tfrac{j^+}{k})) \in \mathbb{R}^m, \qquad \forall j \in \mathbb{Z}_k,$$

we have

$$\mathrm{Grad}\mathscr{A}_k(\boldsymbol{w}) = \boldsymbol{g}, \qquad \mathrm{Grad}\mathscr{A}_k^{[n]}(\boldsymbol{w}^{[n]}) = \boldsymbol{g}^{[n]}. \qquad \blacksquare$$

Theorem IV.1 does not immediately apply to the mean action functional $\mathscr{A}^{[n]}$, since this latter is not C^2 in general. However, despite this lack of regularity, we can still obtain the assertion of the theorem as a consequence of the results of chapter III and the above theorem IV.8.

Corollary IV.9. *Let $\gamma \in W^{1,2}(\mathbb{T}; M)$ be a critical point of the action functional $\mathscr{A}$ such that $\mathscr{A}(\gamma) = c$ and, for some $n \in \mathbb{N}$, we have*

$$(\iota(\mathscr{A}, \gamma), \nu(\mathscr{A}, \gamma)) = (\iota(\mathscr{A}^{[n]}, \gamma^{[n]}), \nu(\mathscr{A}^{[n]}, \gamma^{[n]})).$$

Then, the iteration map $\psi^{[n]}$, restricted to a map

$$\psi^{[n]} : ((\mathscr{A})_c \cup \{\gamma\}, (\mathscr{A})_c) \hookrightarrow ((\mathscr{A}^{[n]})_c \cup \{\gamma^{[n]}\}, (\mathscr{A}^{[n]})_c),$$

induces the homology isomorphism

$$\mathrm{H}_*(\psi^{[n]}) : \mathrm{H}_*(\mathscr{A}, \gamma) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}^{[n]}, \gamma^{[n]}).$$

Proof. As we have already remarked at the beginning of section III.4, for each sufficiently big $k \in \mathbb{N}$ there exists $\boldsymbol{q} \in \Delta_k$ such that

$$\gamma = \gamma_{\boldsymbol{q}} = \lambda_k(\boldsymbol{q}),$$

and $\boldsymbol{q}$ is a critical point of the discrete action $\mathscr{A}_k$. By corollary III.11 and lemma III.14, up to increasing k we have that

$$\begin{gathered}(\iota(\mathscr{A}_k,\boldsymbol{q}),\nu(\mathscr{A}_k,\boldsymbol{q})) = (\iota(\mathscr{A},\gamma_{\boldsymbol{q}}),\nu(\mathscr{A},\gamma_{\boldsymbol{q}})),\\ (\iota(\mathscr{A}_k^{[n]},\boldsymbol{q}^{[n]}),\nu(\mathscr{A}_k^{[n]},\boldsymbol{q}^{[n]})) = (\iota(\mathscr{A}^{[n]},\gamma_{\boldsymbol{q}}^{[n]}),\nu(\mathscr{A}^{[n]},\gamma_{\boldsymbol{q}}^{[n]})),\end{gathered}$$

and by our assumptions these four Morse index and nullity pairs coincide. By corollary III.17, up to further increasing k, the embeddings λ_k and $\lambda_k^{[n]}$ induce homology isomorphisms such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{H}_*(\mathscr{A},\gamma_{\boldsymbol{q}}) & \xrightarrow{\mathrm{H}_*(\psi^{[n]})} & \mathrm{H}_*(\mathscr{A}^{[n]},\gamma_{\boldsymbol{q}}^{[n]}) \\ {\scriptstyle \mathrm{H}_*(\lambda_k)}\uparrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\uparrow{\scriptstyle \mathrm{H}_*(\lambda_k^{[n]})} \\ \mathrm{H}_*(\mathscr{A}_k,\boldsymbol{q}) & \xrightarrow[\simeq]{\mathrm{H}_*(\psi_k^{[n]})} & \mathrm{H}_*(\mathscr{A}_k^{[n]},\boldsymbol{q}^{[n]}) \end{array}$$

Here, by the above theorem IV.8, the homomorphism $\mathrm{H}_*(\psi_k^{[n]})$ is an isomorphism, and so must be $\mathrm{H}_*(\psi^{[n]})$. ∎

Chapter V

The Conley conjecture for Tonelli systems

This final chapter is devoted to the proof of the Conley conjecture for Tonelli Hamiltonian systems on the cotangent bundle of a closed manifold, that, in the Lagrangian formulation, states the following:

> *Let M be a closed manifold and $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ a Tonelli Lagrangian with global Euler-Lagrange flow. Then, the Euler-Lagrange system of $\mathscr{L}$ admits infinitely many integer periodic solutions.*

Actually, the result that we will prove (cf. theorem V.9) is even stronger: in particular it implies the existence of infinitely many contractible periodic orbits with bounded mean action and, if only finitely many of them are 1-periodic, it also implies the existence of contractible periodic orbits of arbitrarily high period.

Our arguments are inspired by a work of Long [Lo], who proved the above Conley conjecture in the special case of fiberwise quadratic Lagrangians on the m-torus $\mathbb{T}^m$. More precisely, he proved it for Lagrangians $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}\mathbb{T}^m \to \mathbb{R}$ of the form

$$\mathscr{L}(t, q, v) = \langle A(q)v, v\rangle + V(t, q), \qquad \forall (t, q, v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}\mathbb{T}^m,$$

where $A : \mathbb{T}^m \to \mathrm{GL}(m)$ takes values in the space of positive definite symmetric matrices, $\langle \cdot, \cdot \rangle$ is the standard flat Riemannian metric on $\mathbb{T}^m$ and $V : \mathbb{R}/\mathbb{Z} \times \mathbb{T}^m \to \mathbb{R}$ is a smooth function. Recently, Lu [Lu] extended Long's proof to the case of a convex quadratic-growth Lagrangian[1] on a closed configuration space. He also established the existence of infinitely many integer periodic orbits which are homotopic to some

[1]However, Lu's proof is valid only for Lagrangians that are fiberwise quadratic. In fact, his arguments need the C^2 regularity of the Lagrangian action functional over the $W^{1,2}$ free loop space, which he assumes citing the erroneous [AF, proposition 4.1]. See proposition II.4 and remark II.3 for the correct statements about the regularity of the action functional.

iteration of a given free loop, provided the connected component of the loop in the free loop space have nontrivial homology in some positive degree.

Our proof, as well as Long's and Lu's ones, is Morse-theoretic in nature, and can be roughly described as follows: assuming by contradiction that the considered Euler-Lagrange system admits only finitely many integer periodic solutions, then it is possible to find a periodic orbit whose local homology persists under iteration, contradicting an established homological vanishing property. Under the Tonelli assumptions, we need to deal with several problems while carrying out this proof scheme. These problems are mainly due to the fact that we do not know a functional setting in which the Tonelli action functional is both regular and satisfies the Palais-Smale condition, the minimum requirements to perform Morse theory. In order to overcome these difficulties we apply the machinery of convex quadratic modifications, recently developed by Abbondandolo and Figalli [AF]. Their idea consists in modifying the Tonelli Lagrangian $\mathscr{L}$ outside a sufficiently big neighborhood of the zero section of $\mathrm{T}M$, making it fiberwise quadratic there, and then performing the Morse-theoretic analysis to the action functional of the modified Lagrangian. In a prescribed period $n \in \mathbb{N}$, a suitable a priori estimate on the n-periodic Euler-Lagrange orbits of the modified Lagrangian that belong to a given action sublevel shows that these orbits must lie in the region where the Lagrangian is not modified. Combining the machinery of convex quadratic modifications with our discretization technique, developed in chapter III, we will recover suitable local homology groups for the Tonelli action (or, more precisely, for a discretized version of it).

In section V.1 we recall the definition and the basic properties of convex quadratic modifications, following [AF, section 5]. In section V.2 we introduce the discrete Tonelli action and we prove the crucial properties of its local homology. In section V.3 we establish the above mentioned homological vanishing under iteration. Finally, in section V.4 we state and prove our main result about periodic orbits of Tonelli Lagrangian systems.

V.1 Convex quadratic modifications

Let us fix, once for all, a smooth closed manifold M of dimension m with a fixed Riemannian metric $\langle\cdot,\cdot\rangle_\cdot$, and a smooth 1-periodic Tonelli Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$. We denote by $\mathscr{A}(\zeta) \in \mathbb{R} \cup \{+\infty\}$ the action of an absolutely continuous 1-periodic curve $\zeta : \mathbb{R}/\mathbb{Z} \to M$, i.e.

$$\mathscr{A}(\zeta) = \int_0^1 \mathscr{L}(t, \zeta(t), \dot{\zeta}(t))\, \mathrm{d}t,$$

and in general, for each $n \in \mathbb{N}$, we denote by $\mathscr{A}^{[n]}(\zeta) \in \mathbb{R} \cup \{+\infty\}$ the mean action of an absolutely continuous n-periodic curve $\zeta : \mathbb{R}/n\mathbb{Z} \to M$, i.e.

$$\mathscr{A}^{[n]}(\zeta) = \frac{1}{n} \int_0^n \mathscr{L}(t, \zeta(t), \dot{\zeta}(t))\, \mathrm{d}t.$$

By the uniform fiberwise superlinearity of $\mathscr{L}$ (cf. remark I.1) there exists a real constant $C(\mathscr{L}) > 0$ such that

$$\mathscr{L}(t, q, v) \geq |v|_q - C(\mathscr{L}), \qquad \forall (t, q, v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M.$$

For each real $R > 0$, we say that a smooth convex quadratic-growth Lagrangian $\mathscr{L}_R : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ is a **convex quadratic R-modification** (or simply an **R-modification**) of a Tonelli Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ when:

(M1) $\mathscr{L}_R(t, q, v) = \mathscr{L}(t, q, v)$ for each $(t, q, v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M$ with $|v|_q \leq R$,

(M2) $\mathscr{L}_R(t, q, v) \geq |v|_q - C(\mathscr{L})$ for each $(t, q, v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M$.

Proposition V.1 (Existence of modifications). *For every real $R > 0$, there exists a convex quadratic modification $\mathscr{L}_R$ of a Tonelli Lagrangian $\mathscr{L}$.*

Proof. We denote by ∂^2_{vv} the fiberwise Hessian operator on $\mathrm{T}M$. For instance, $\partial^2_{vv}\mathscr{L}(t, q, v) : \mathrm{T}_qM \otimes \mathrm{T}_qM \to \mathbb{R}$ is the symmetric bilinear form given in local coordinates by

$$\partial^2_{vv}\mathscr{L}(t, q, v)[w, z] = \sum_{j,h=1}^{m} \frac{\partial^2 \mathscr{L}}{\partial v^j \, \partial v^h}(t, q, v) w^j z^h, \qquad \forall w, z \in \mathrm{T}_qM.$$

From now on, we will write inequalities between symmetric bilinear forms meaning that the inequalities hold for the associated quadratic forms. For instance, if $\ell \in \mathbb{R}$ and $\mathscr{Q} : \mathrm{T}M \to \mathbb{R}$, we will write $\partial^2_{vv}\mathscr{Q}(q, v) \geq \ell$ meaning

$$\partial^2_{vv}\mathscr{Q}(q, v)[w, w] \geq \ell \, |w|^2_q, \qquad \forall w \in \mathrm{T}_qM.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\begin{cases} \phi(r) = r, & \forall r \leq 1, \\ \phi(r) = \text{const}, & \forall r \geq 2. \end{cases}$$

For each real parameter c_1, the affine function $r \mapsto c_1 r - 2c_1 R^2$ is negative on $(-\infty, R^2]$ and positive on $[4R^2, \infty)$, therefore we can find a smooth convex function such that

$$\begin{cases} \psi(r) = 0, & \forall r \leq R^2, \\ \psi(r) = c_1 r - 2c_1 R^2, & \forall r \geq 4R^2. \end{cases}$$

We introduce the smooth Lagrangians $\mathscr{B} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ and $\mathscr{Q} : \mathrm{T}M \to \mathbb{R}$ given by

$$\mathscr{B}(t, q, v) := c_2 \, \phi\left(\frac{\mathscr{L}(t, q, v)}{c_2}\right), \qquad \mathscr{Q}(q, v) := \psi(|v|^2_q),$$
$$\forall (t, q, v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M,$$

where c_2 is again a positive real parameter.

The Lagrangian $\mathscr{B}$ is uniformly bounded, being constant outside a compact set of $\mathbb{R}/\mathbb{Z} \times \mathrm{T}M$. This, in turn, follows by the coercivity of $\mathscr{L}$, namely it follows from the fact that $\mathscr{L}(t,q,v) \to \infty$ as $|v|_q \to \infty$ uniformly in $(t,q) \in \mathbb{R}/\mathbb{Z} \times M$. By the definition of ϕ, we have that $\mathscr{B}(t,q,v) = \mathscr{L}(t,q,v)$ whenever $|v|_q \leq 2R$, provided the constant c_2 is such that

$$c_2 \geq \max \{\mathscr{L}(t,q,v) \,|\, (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M,\ |v|_q \leq 2R\}\,. \tag{V.1}$$

The autonomous Lagrangian $\mathscr{Q}$ is fiberwise convex and fiberwise quadratic outside a compact neighborhood of the zero section of $\mathrm{T}M$. Its fiberwise Hessian satisfies

$$\partial^2_{vv}\mathscr{Q}(q,v) = 2c_1, \qquad \forall (q,v) \in \mathrm{T}M \text{ with } |v|_q \geq 2R.$$

Up to fixing the above constants $c_1, c_2 > 0$, we define the smooth Lagrangian $\mathscr{L}_R : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ by

$$\mathscr{L}_R(t,q,v) := \mathscr{B}(t,q,v) + \mathscr{Q}(q,v), \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M.$$

Then, we want to show that, for c_1 and c_2 sufficiently big, $\mathscr{L}_R$ is convex quadratic-growth (in fact, it is even fiberwise quadratic outside a compact subset of $\mathbb{R} \times \mathrm{T}M$), and it is an R-modification of $\mathscr{L}$.

Choosing the constant c_2 as in (V.1), we have that

$$\mathscr{L}_R(t,q,v) = \mathscr{L}(t,q,v) + \mathscr{Q}(q,v), \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \text{ with } |v|_q \leq 2R,$$

hence

$$\partial^2_{vv}\mathscr{L}_R(t,q,v) \geq \partial^2_{vv}\mathscr{L}(t,q,v), \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \text{ with } |v|_q \leq 2R,$$

and, up to choosing a sufficiently big constant c_1, we can assume that

$$\partial^2_{vv}\mathscr{B}(t,q,v) \geq -c_1, \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M.$$

This implies

$$\partial^2_{vv}\mathscr{L}_R(t,q,v) \geq \partial^2_{vv}\mathscr{B}(t,q,v) + \partial^2_{vv}\mathscr{Q}(q,v) \geq c_1,$$
$$\forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \text{ with } |v|_q \geq 2R.$$

and therefore $\mathscr{L}_R$ satisfies **(Q1)**. For each $(t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M$ with $|v|_q$ large enough, we have

$$\mathscr{L}_R(t,q,v) = \mathrm{const} + \mathscr{Q}(q,v) = \mathrm{const}' + c_1|v|_q^2,$$

hence $\mathscr{L}_R$ also satisfies **(Q2)**, namely it is a convex quadratic-growth Lagrangian.

By the definition of $\mathscr{B}$ and $\mathscr{Q}$, it readily follows that $\mathscr{L}_R$ satisfies **(M1)**. It only remains to verify condition **(M2)**. To begin with, we have

$$\mathscr{L}_R(t,q,v) \geq \mathscr{L}(t,q,v) \geq |v|_q - C(\mathscr{L}), \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \text{ with } |v|_q \leq 2R.$$

On the other hand, for c_1 big enough, we have

$$\mathscr{L}_R(t,q,v) \geq \min\{\mathscr{B}\} + c_1|v|_q^2 - 2c_1R^2 \geq |v|_q - C(\mathscr{L}), \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \text{ with } |v|_q \geq 2R. \quad \blacksquare$$

From now on, we will always denote by $\mathscr{L}_R$ an arbitrary R-modification of the Tonelli Lagrangian $\mathscr{L}$, and by $\mathscr{A}_R : W^{1,2}(\mathbb{T};M) \to \mathbb{R}$ and $\mathscr{A}_R^{[n]} : W^{1,2}(\mathbb{T}^{[n]};M) \to \mathbb{R}$ (for each $n \in \mathbb{N}$) the associated action and mean action, i.e.

$$\mathscr{A}_R(\zeta) = \int_0^1 \mathscr{L}_R(t,\zeta(t),\dot\zeta(t))\,\mathrm{d}t, \qquad \forall \zeta \in W^{1,2}(\mathbb{T};M),$$
$$\mathscr{A}_R^{[n]}(\zeta) = \frac{1}{n}\int_0^n \mathscr{L}_R(t,\zeta(t),\dot\zeta(t))\,\mathrm{d}t, \qquad \forall \zeta \in W^{1,2}(\mathbb{T}^{[n]};M).$$

Notice that, if a curve $\gamma \in W^{1,2}(\mathbb{T};M)$ is such that $|\dot\gamma(t)|_{\gamma(t)} \leq R$ for almost every $t \in \mathbb{T}$, then $\mathscr{A}(\gamma) = \mathscr{A}_R(\gamma)$ (and the same holds in any period n for the mean action). Moreover, if we further assume that γ is a solution of the Euler-Lagrange system of $\mathscr{L}$, then it is a critical point of $\mathscr{A}_R$ and the Gateaux Hessian $\mathrm{Hess}\mathscr{A}_R(\gamma)$ coincides with the bilinear form $\mathscr{B}_\gamma$ (cf. equation (I.12)) associated to the Lagrangian $\mathscr{L}$. In particular, $\mathrm{Hess}\mathscr{A}_R(\gamma)$ does not depend on the chosen $R \geq \max\{|\dot\gamma(t)|_{\gamma(t)} \,|\, t \in \mathbb{R}/\mathbb{Z}\}$ and on the chosen R-modification $\mathscr{L}_R$. Motivated by this fact, from now on we will informally speak about the Morse index and nullity pair of $\mathscr{A}$ at γ as the pair of integers

$$(\iota(\mathscr{A},\gamma),\nu(\mathscr{A},\gamma)) := (\iota(\mathscr{B}_\gamma),\nu(\mathscr{B}_\gamma)) = (\iota(\mathscr{A}_R,\gamma),\nu(\mathscr{A}_R,\gamma)).$$

Analogously, we will speak about the Morse index an nullity pair of $\mathscr{A}^{[n]}$ at $\gamma^{[n]}$ as

$$(\iota(\mathscr{A}^{[n]},\gamma^{[n]}),\nu(\mathscr{A}^{[n]},\gamma^{[n]})) := (\iota(\mathscr{A}_R^{[n]},\gamma^{[n]}),\nu(\mathscr{A}_R^{[n]},\gamma^{[n]})),$$

and about the mean Morse index of $\mathscr{A}$ at γ as

$$\widehat{\iota}(\mathscr{A},\gamma) := \widehat{\iota}(\mathscr{A}_R,\gamma).$$

One of the important features of convex quadratic modifications in the study of Tonelli Lagrangian systems is given by the following a priori estimate.

Lemma V.2. *Assume that the Euler-Lagrange flow* $\Phi_{\mathscr{L}}$ *of* $\mathscr{L}$ *is global (cf. section I.1). Then, for each* $\tilde{a} > 0$ *and* $\tilde{n} \in \mathbb{N}$*, there exists* $\tilde{R} = \tilde{R}(\tilde{a}, \tilde{n}) > 0$ *such that, for any R-modification* $\mathscr{L}_R$ *of* $\mathscr{L}$ *with* $R > \tilde{R}$ *and for any* $n \in \{1, ..., \tilde{n}\}$*, the following holds: if* γ *is a critical point of* $\mathscr{A}_R^{[n]}$ *such that* $\mathscr{A}_R^{[n]}(\gamma) \leq \tilde{a}$*, then*

$$|\dot{\gamma}(t)|_{\gamma(t)} \leq \tilde{R}, \qquad \forall t \in \mathbb{R}/n\mathbb{Z}.$$

In particular, γ *is an extremal curve of* $\mathscr{A}^{[n]}$ *and* $\mathscr{A}^{[n]}(\gamma) = \mathscr{A}_R^{[n]}(\gamma)$*.*

Proof. We introduce the compact subset of $\mathrm{T}M$ given by

$$K = K(\tilde{a}, \tilde{n}) = \left\{ \Phi_{\mathscr{L}}^{t_1,t_0}(q, v) \,|\, t_0, t_1 \in [-\tilde{n}, \tilde{n}],\ (q, v) \in \mathrm{T}M,\ |v|_q \leq \tilde{a} + C(\mathscr{L}) \right\},$$

and we define

$$\tilde{R} = \tilde{R}(\tilde{a}, \tilde{n}) = \max \{ |v|_q \,|\, (q, v) \in K \}.$$

Now, we consider $R > \tilde{R}$, $n \in \{1, ..., \tilde{n}\}$ and a critical point γ of $\mathscr{A}_R^{[n]}$ such that $\mathscr{A}_R^{[n]}(\gamma) \leq \tilde{a}$, as in the statement. There exists $t_0 \in \mathbb{R}/n\mathbb{Z}$ such that

$$\mathscr{L}_R(t_0, \gamma(t_0), \dot{\gamma}(t_0)) \leq \mathscr{A}_R^{[n]}(\gamma),$$

hence, by **(M2)**, we have

$$|\dot{\gamma}(t_0)|_{\gamma(t_0)} \leq \mathscr{L}_R(t_0, \gamma(t_0), \dot{\gamma}(t_0)) + C(\mathscr{L}) \leq \mathscr{A}_R^{[n]}(\gamma) + C(\mathscr{L}) \leq \tilde{a} + C(\mathscr{L}),$$

and, in particular, $(\gamma(t_0), \dot{\gamma}(t_0)) \in K$. Let I be the closed subset of $\mathbb{R}/n\mathbb{Z}$ given by

$$I = \{ t \in \mathbb{R}/n\mathbb{Z} \,|\, (\gamma(t), \dot{\gamma}(t)) \in K \}.$$

If $t \in I$ we have $|\dot{\gamma}(t)|_{\gamma(t)} \leq \tilde{R} < R$, hence the Lagrangian functions $\mathscr{L}$ and $\mathscr{L}_R$ coincide on a neighborhood of $(t, \gamma(t), \dot{\gamma}(t))$. This implies that there exists $\varepsilon > 0$ such that

$$(\gamma(s), \dot{\gamma}(s)) = \Phi_{\mathscr{L}_R}^{s,t}(\gamma(t), \dot{\gamma}(t)) = \Phi_{\mathscr{L}}^{s,t}(\gamma(t), \dot{\gamma}(t)), \qquad \forall s \in (t - \varepsilon, t + \varepsilon),$$

and, in particular, $(t - \varepsilon, t + \varepsilon) \subset I$. This shows that I is also open in $\mathbb{R}/n\mathbb{Z}$, hence $I = \mathbb{R}/n\mathbb{Z}$ and $|\dot{\gamma}(t)|_{\gamma(t)} \leq \tilde{R}$ for each $t \in \mathbb{R}/n\mathbb{Z}$. ∎

V.2 Discrete Tonelli action

Let γ be an integer periodic solution of the Euler-Lagrange system associated to the Tonelli Lagrangian $\mathscr{L}$. In order to simplify the notation, we can assume that the period of γ is 1, so that it is a map of the form $\gamma : \mathbb{R}/\mathbb{Z} \to M$. We choose a constant $U \in \mathbb{R}$ such that

$$U > \max \left\{ |\dot{\gamma}(t)|_{\gamma(t)} \,|\, t \in \mathbb{R}/\mathbb{Z} \right\}, \tag{V.2}$$

and we consider a convex quadratic-growth Lagrangian $\mathscr{L}_U : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ that is a U-modification of $\mathscr{L}$. By our choice of the constant U, γ is a critical point of the action $\mathscr{A}_U$, and $\mathscr{A}_U(\gamma) = \mathscr{A}(\gamma)$.

If we apply the discretization technique of chapter III to $\mathscr{L}_U$, for each $k \in \mathbb{N}$ greater than the constant $1/\varepsilon_0(\mathscr{L}_U)$ given by theorem III.2, we obtain a map

$$\lambda_{k,\mathscr{L}_U} : \Delta_k \hookrightarrow C_k^\infty(\mathbb{T}; M) \subset W^{1,2}(\mathbb{T}; M),$$

as explained in section III.2. Then, for each $k \in \mathbb{N}$, we define the open set of $C_k^\infty(\mathbb{T}; M)$ given by

$$\mathscr{U}_k := \left\{ \zeta \in C_k^\infty(\mathbb{T}; M) \,\middle|\, \operatorname*{ess\,sup}_{t \in \mathbb{R}/\mathbb{Z}} \left\{ |\dot{\zeta}(t)|_{\zeta(t)} \right\} < U \right\}. \tag{V.3}$$

Since $\lambda_{k,\mathscr{L}_U}$ is smooth, we get an open set

$$U_k := \lambda_{k,\mathscr{L}_U}^{-1}(\mathscr{U}_k) \subseteq \Delta_k. \tag{V.4}$$

For $k \in \mathbb{N}$ big enough, there exists a $\boldsymbol{q} \in U_k$ such that

$$\gamma = \lambda_{k,\mathscr{L}_U}(\boldsymbol{q}).$$

Now, notice that the action $\mathscr{A}_U$ coincides with the Tonelli action $\mathscr{A}$ on the open set $\mathscr{U}_k$. This allows us to define the **discrete Tonelli action** $\mathscr{A}_k : U_k \to \mathbb{R}$ as

$$\mathscr{A}_k := \mathscr{A} \circ \lambda_{k,\mathscr{L}_U}|_{U_k} = \mathscr{A}_U \circ \lambda_{k,\mathscr{L}_U}|_{U_k}.$$

This functional is smooth (cf. proposition III.4) and $\boldsymbol{q}$ is a critical point of it.

From now on we will just write $\lambda_k : U_k \hookrightarrow C_k^\infty(\mathbb{T}; M)$ for $\lambda_{k,\mathscr{L}_U}|_{U_k}$. Even if this embedding does depend on the chosen U-modification $\mathscr{L}_U$, its germ at $\boldsymbol{q}$ does not, as stated by the following.

Lemma V.3. *For every real number $R > U$, if $\mathscr{L}_R$ is an R-modification of $\mathscr{L}$, there exists an open neighborhood $V_k \subset U_k$ of $\boldsymbol{q}$ such that*

$$\lambda_k|_{V_k} = \lambda_{k,\mathscr{L}_U}|_{V_k} = \lambda_{k,\mathscr{L}_R}|_{V_k}.$$

Proof. The Lagrangian functions $\mathscr{L}_U$ and $\mathscr{L}_R$ coincide on a neighborhood of the support of the lifted curve $(\gamma, \dot{\gamma})$ in $\mathrm{T}M$. Therefore the k-broken Euler-Lagrange curves of $\mathscr{L}_U$ and $\mathscr{L}_R$ that are close to γ are the same, and the claim follows. ■

Let $c = \mathscr{A}_k(\boldsymbol{q}) = \mathscr{A}(\gamma)$. By corollary III.17 and the excision property, if $k \geq \bar{k}(\mathscr{L}_U, c)$, the embedding λ_k restricts to a map

$$\lambda_k : ((\mathscr{A}_k)_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_c) \hookrightarrow ((\mathscr{A}_U)_c \cup \{\gamma\}, (\mathscr{A}_U)_c)$$

that induces the homology isomorphism

$$\text{(V.5)} \qquad \mathrm{H}_*(\lambda_k) : \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}_U, \gamma).$$

For each R-modification $\mathscr{L}_R$ of $\mathscr{L}$, with $R > U$, the action $\mathscr{A}_R$ coincides with $\mathscr{A}$ on the image of λ_k. Hence, the embedding λ_k also restricts to a map

$$\text{(V.6)} \qquad \lambda_k : ((\mathscr{A}_k)_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_c) \hookrightarrow ((\mathscr{A}_R)_c \cup \{\gamma\}, (\mathscr{A}_R)_c).$$

A priori, this map might induce a homology homomorphism that is not an isomorphism. However, the next lemma guarantees that this cannot happen.

Lemma V.4. *For each $R > U$, the map λ_k, restricted as in* (V.6), *induces a homology isomorphism*

$$\mathrm{H}_*(\lambda_k) : \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}_R, \gamma).$$

Proof. For each $h \in \mathbb{N}$ we define the embedding

$$j_h = \lambda_{hk}^{-1} \circ \lambda_k : U_k \hookrightarrow U_{hk}.$$

Since $hk \geq k \geq \bar{k}(\mathscr{L}_U, c)$, by corollary III.17 and the excision property we obtain that the map

$$\lambda_{hk} : ((\mathscr{A}_{hk})_c \cup \{j_h(\boldsymbol{q})\}, (\mathscr{A}_{jk})_c) \hookrightarrow ((\mathscr{A}_U)_c \cup \{\gamma\}, (\mathscr{A}_U)_c)$$

induces the homology isomorphism $\mathrm{H}_*(\lambda_{hk}) : \mathrm{H}_*(\mathscr{A}_{hk}, j_h(\boldsymbol{q})) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}_U, \gamma)$, analogously to λ_k in (V.5). Hence, the map j_h, restricted as in the following commutative diagram

$$\begin{array}{ccc} ((\mathscr{A}_k)_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_c) & \xhookrightarrow{\quad\lambda_k\quad} & ((\mathscr{A}_U)_c \cup \{\gamma\}, (\mathscr{A}_U)_c) \\ \big\downarrow j_h & \nearrow \lambda_{hk} & \\ ((\mathscr{A}_{hk})_c \cup \{j_h(\boldsymbol{q})\}, (\mathscr{A}_{hk})_c) & & \end{array}$$

induces the homology isomorphism

$$\mathrm{H}_*(j_h) = \mathrm{H}_*(\lambda_{hk})^{-1} \circ \mathrm{H}_*(\lambda_k) : \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) \xrightarrow{\simeq} \mathrm{H}_*(\mathscr{A}_{hk}, j_h(\boldsymbol{q})).$$

Now, let us consider an R-modification $\mathscr{L}_R$ of $\mathscr{L}$ as in the statement, with associated maps

$$\lambda_{hk,\mathscr{L}_R} : \Delta_{hk} \hookrightarrow C_k^\infty(\mathbb{T}; M), \qquad \forall h \in \mathbb{N}.$$

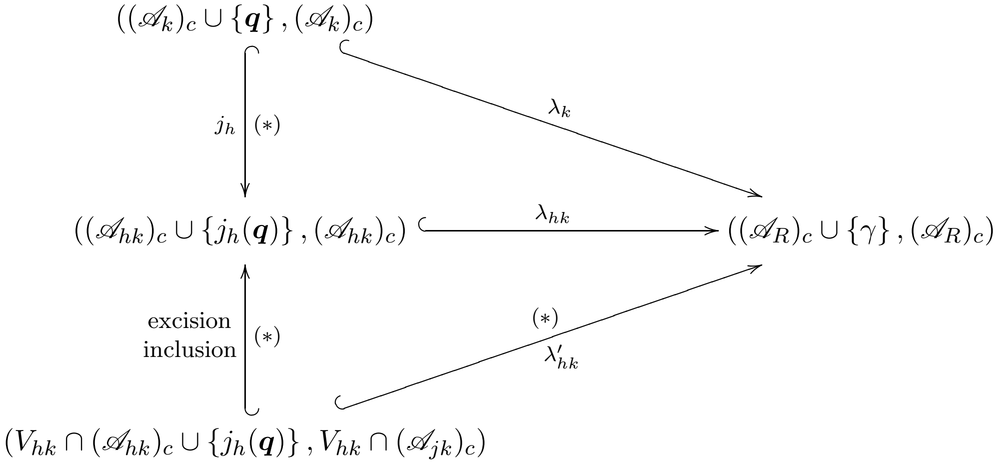
By lemma V.3, for each $h \in \mathbb{N}$, we know that there exists a neighborhood $V_{hk} \subset U_{hk}$ of $j_h(\boldsymbol{q})$ such that

$$\lambda_{hk}|_{V_{hk}} = \lambda_{hk,\mathscr{L}_R}|_{V_{hk}}.$$

If h is big enough so that $hk \geq \bar{k}(\mathscr{L}_R, c)$, by corollary III.17 and the excision property we obtain that both λ_{hk} and $\lambda_{hk,\mathscr{L}_R}$ restrict to give the same map

$$\lambda'_{hk} : (V_{hk} \cap (\mathscr{A}_{hk})_c \cup \{j_h(\boldsymbol{q})\}, V_{hk} \cap (\mathscr{A}_{jk})_c) \hookrightarrow ((\mathscr{A}_R)_c \cup \{\gamma\}, (\mathscr{A}_R)_c)$$

that induces an isomorphism in homology. Finally, consider the following commutative diagram of maps.



We already know that the maps marked with $(*)$ induce isomorphisms in homology. Therefore, the maps λ_{hk} and λ_k also induce isomorphisms in homology. ∎

Remark V.1. As a consequence of the above lemma, we immediately obtain that the local homology groups $\mathrm{H}_*(\mathscr{A}_R, \gamma)$ do not depend (up to isomorphism) on the chosen real constant $R \geq U$ and on the chosen R-modification $\mathscr{L}_R$. ∎

V.3 Homological vanishing by iteration

All the arguments of the last section can be carried out word by word if we work in an arbitrary period $n \in \mathbb{N}$. Briefly, we introduce the open sets

$$\mathscr{U}_k^{[n]} := \left\{ \zeta \in C_k^\infty(\mathbb{T}^{[n]}; M) \,\middle|\, \operatorname*{ess\,sup}_{t \in \mathbb{R}/n\mathbb{Z}} \left\{ |\dot{\zeta}(t)|_{\zeta(t)} \right\} < U \right\} \subset C_k^\infty(\mathbb{T}^{[n]}; M),$$

$$U_k^{[n]} := (\lambda_{k,\mathscr{L}_U}^{[n]})^{-1}(\mathscr{U}_k^{[n]}) \subset \Delta_{nk},$$

and we define the **discrete mean Tonelli action**

$$\mathscr{A}_k^{[n]} := \mathscr{A}^{[n]} \circ \lambda_k^{[n]} : U_k^{[n]} \to \mathbb{R},$$

where $\lambda_k^{[n]} := \lambda_{k,\mathscr{L}_U}^{[n]}$. Then lemmas V.3 and V.4 go through.

We recall from section III.6 that we have a discrete iteration map $\psi_k^{[n]} : U_k \hookrightarrow U_k^{[n]}$ such that the following diagram commutes

$$\begin{array}{ccc} U_k & \overset{\psi_k^{[n]}}{\hookrightarrow} & U_{nk} \\ {\scriptstyle \lambda_k} \big\downarrow & & \big\downarrow {\scriptstyle \lambda_k^{[n]}} \\ W^{1,2}(\mathbb{T}; M) & \overset{\psi^{[n]}}{\hookrightarrow} & W^{1,2}(\mathbb{T}^{[n]}; M) \end{array}$$

Now, consider $\infty \geq c_2 > c_1 = c = \mathscr{A}(\gamma)$. For each $R \geq U$, the embeddings λ_k, $\lambda_k^{[n]}$ and $\psi^{[n]}$ restrict to give maps such that the following diagram commutes.

$$\begin{array}{ccc} ((\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_{c_1}) & \overset{\psi_k^{[n]}}{\hookrightarrow} & ((\mathscr{A}_k^{[n]})_{c_1} \cup \{\boldsymbol{q}^{[n]}\}, (\mathscr{A}_k^{[n]})_{c_1}) \\ {\scriptstyle \lambda_k} \big\downarrow & & \big\downarrow {\scriptstyle \lambda_k^{[n]}} \\ ((\mathscr{A}_R)_{c_2}, (\mathscr{A}_R)_{c_1}) & \overset{\psi^{[n]}}{\hookrightarrow} & ((\mathscr{A}_R^{[n]})_{c_2}, (\mathscr{A}_R^{[n]})_{c_1}) \end{array}$$

This latter, in turn, induces the following commutative diagram of homology groups.

$$\begin{array}{ccc} \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) & \xrightarrow{\mathrm{H}_*(\psi_k^{[n]})} & \mathrm{H}_*(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]}) \\ {\scriptstyle \mathrm{H}_*(\lambda_k)} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{H}_*(\lambda_k^{[n]})} \\ \mathrm{H}_*((\mathscr{A}_R)_{c_2}, (\mathscr{A}_R)_{c_1}) & \xrightarrow{\mathrm{H}_*(\psi^{[n]})} & \mathrm{H}_*((\mathscr{A}_R^{[n]})_{c_2}, (\mathscr{A}_R^{[n]})_{c_1}) \end{array}$$

The main result of this section is the following homological vanishing theorem, that may be considered an extension of a celebrated result by Bangert and Klingenberg (cf. [BK, theorem 2]) and Long (cf. [Lo, section 5]).

Theorem V.5 (Homological vanishing). *Let $[\eta]$ be an element in the local homology group $\mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q})$, where the critical point $\boldsymbol{q}$ of $\mathscr{A}_k$ is not a local minimum. Then, for each $j \in \mathbb{N}$, there exist $\bar{R} = \bar{R}([\eta], j) \geq U$ and $\bar{n} = \bar{n}([\eta], j) \in \mathbb{N}$ that is a power of j such that, for each real $R \geq \bar{R}$, we have*

$$\mathrm{H}_*(\psi^{[\bar{n}]}) \circ \mathrm{H}_*(\lambda_k)[\eta] = 0 \qquad \text{in } \mathrm{H}_*((\mathscr{A}_R^{[\bar{n}]})_{c_2}, (\mathscr{A}_R^{[\bar{n}]})_{c_1}).$$

Equivalently, we have that

$$\mathrm{H}_*(\psi_k^{[\bar n]})[\eta] \in \ker \left[\mathrm{H}_*(\lambda_k^{[\bar n]}) : \mathrm{H}_*(\mathscr{A}_k^{[\bar n]}, \boldsymbol{q}^{[\bar n]}) \to \mathrm{H}_*((\mathscr{A}_R^{[\bar n]})_{c_2}, (\mathscr{A}_R^{[\bar n]})_{c_1}) \right].$$

The proof of this theorem, which will take the remaining of this section, is based on a homotopic technique that is essentially due to Bangert (cf. [Ba, section 3]).

Lemma V.6 (Bangert homotopy). *Let $\sigma : \Delta^p \to W^{1,2}(\mathbb{T}; M)$ be a continuous p-singular simplex such that*

$$\begin{aligned}
&\max_{\boldsymbol{z}\in\Delta^p} \{\mathscr{A}(\sigma(\boldsymbol{z}))\} < c_2,\\
&\max_{\boldsymbol{z}\in\partial\Delta^p} \{\mathscr{A}(\sigma(\boldsymbol{z}))\} < c_1,\\
&\sup_{\boldsymbol{z}\in\Delta^p} \operatorname*{ess\,sup}_{t\in\mathbb{R}/\mathbb{Z}} \left\{ \left| \frac{\mathrm{d}}{\mathrm{d}t}\sigma(\boldsymbol{z})(t) \right|_{\sigma(\boldsymbol{z})(t)} \right\} \leq \bar r(\sigma),
\end{aligned}$$

where $\bar r(\sigma)$ is a real constant. Then, there exist a positive integer $\bar n(\sigma)$, a positive real $\bar R(\sigma) \geq \bar r(\sigma)$ and, for each $n \in \mathbb{N}$, a homotopy[2]

$$\mathrm{Ban}_\sigma^{[n]} : [0,1] \times \Delta^p \to W^{1,2}(\mathbb{T}; M) \qquad \textit{relative } \partial\Delta^p$$

satisfying the following properties:

(i) $\mathrm{Ban}_\sigma^{[n]}(0,\cdot) = \psi^{[n]} \circ \sigma$, *for each $n \in \mathbb{N}$;*

(ii) *for each integer $n \geq \bar n(\sigma)$ we have*

$$\begin{aligned}
&\max_{(s,\boldsymbol{z})\in[0,1]\times\Delta^p} \left\{ \mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z})) \right\} < c_2,\\
&\max_{(s,\boldsymbol{z})\in[0,1]\times\partial\Delta^p} \left\{ \mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z})) \right\} < c_1;\\
&\max_{\boldsymbol{z}\in\Delta^p} \left\{ \mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(1,\boldsymbol{z})) \right\} < c_1;
\end{aligned}$$

(iii) *for each $n \in \mathbb{N}$ we have*

$$\sup_{(s,\boldsymbol{z})\in[0,1]\times\Delta^p} \operatorname*{ess\,sup}_{t\in\mathbb{R}/n\mathbb{Z}} \left\{ \left| \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z})(t) \right|_{\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z})(t)} \right\} \leq \bar R(\sigma).$$

Hereafter, the homotopies $\mathrm{Ban}_\sigma^{[n]}$ will be called **Bangert homotopies**. Before going to the proof of the above lemma, we want to show that it readily gives the following homotopical vanishing result.

[2] See section A.3 in appendix A for a review of the classical terminology in homotopy theory.

Theorem V.7 (Homotopical vanishing). *Consider a continuous p-singular simplex $\sigma : (\Delta^p, \partial\Delta^p) \to ((\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_{c_1})$. Then, there exist a positive integer $\bar{n}(\sigma)$ and a real number $\bar{R}(\sigma)$ such that, for each real $R \geq \bar{R}(\sigma)$ and for each integer $n \geq \bar{n}(\sigma)$, we have*[3]

$$\pi_p(\psi^{[n]}) \circ \pi_p(\lambda_k)[\sigma] = 0 \qquad \text{in } \pi_p((\mathscr{A}_R^{[n]})_{c_2}, (\mathscr{A}_R^{[n]})_{c_1}).$$

Equivalently, we have that

$$\pi_p(\psi_k^{[n]})[\eta] \in \ker \left[\pi_p(\lambda_k^{[n]}) : \pi_p((\mathscr{A}_k^{[n]})_{c_1} \cup \{\boldsymbol{q}^{[n]}\}, (\mathscr{A}_k^{[n]})_{c_1}) \to \pi_p((\mathscr{A}_R^{[n]})_{c_2}, (\mathscr{A}_R^{[n]})_{c_1})\right].$$

Proof. Let $U > 0$ be the constant chosen in (V.2). The singular simplex $\tilde{\sigma} := \lambda_k \circ \sigma : \Delta^p \to W^{1,2}(\mathbb{T}; M)$ satisfies

$$\begin{aligned} \mathscr{A}(\tilde{\sigma}(\boldsymbol{z})) &= \mathscr{A}_k(\sigma(\boldsymbol{z})) \leq c_1 < c_2, && \forall \boldsymbol{z} \in \Delta^p, \\ \mathscr{A}(\tilde{\sigma}(\boldsymbol{z})) &= \mathscr{A}_k(\sigma(\boldsymbol{z})) < c_1, && \forall \boldsymbol{z} \in \partial\Delta^p. \end{aligned}$$

Moreover, since $\tilde{\sigma}(\Delta^p) \subset \lambda_k(U_k) \subset \mathscr{U}_k$ (cf. definitions in (V.3) and (V.4)), we further have

$$\sup_{\boldsymbol{z} \in \Delta^p} \ \operatorname*{ess\,sup}_{t \in \mathbb{R}/\mathbb{Z}} \left\{ \left| \frac{\mathrm{d}}{\mathrm{d}t} \tilde{\sigma}(\boldsymbol{z})(t) \right|_{\tilde{\sigma}(\boldsymbol{z})(t)} \right\} \leq U =: \bar{r}(\tilde{\sigma}).$$

By lemma V.6, we obtain $\bar{n}(\tilde{\sigma}) \in \mathbb{N}$, $\bar{R}(\tilde{\sigma}) \geq \bar{r}(\tilde{\sigma}) > 0$ and Bangert homotopies $\mathrm{Ban}_{\tilde{\sigma}}^{[n]}$ for each $n \in \mathbb{N}$. For each $R \geq \bar{R}(\tilde{\sigma})$ and $n \geq \bar{n}(\tilde{\sigma})$, $\mathrm{Ban}_{\tilde{\sigma}}^{[n]}$ is a homotopy of maps of pairs of the form

$$\mathrm{Ban}_{\tilde{\sigma}}^{[n]} : [0,1] \times (\Delta^p, \partial\Delta^p) \to ((\mathscr{A}_R^{[n]})_{c_2}, (\mathscr{A}_R^{[n]})_{c_1}),$$

with $\mathrm{Ban}_{\tilde{\sigma}}^{[n]}(0, \cdot) = \psi^{[n]} \circ \tilde{\sigma}$ and $\mathrm{Ban}_{\tilde{\sigma}}^{[n]}(\{1\} \times \Delta^p) \subset (\mathscr{A}_R^{[n]})_{c_1}$. Hence

$$0 = [\psi^{[n]} \circ \tilde{\sigma}] = \pi_p(\psi^{[n]})[\tilde{\sigma}] = \pi_p(\psi^{[n]}) \circ \pi_p(\lambda_k)[\sigma] \qquad \text{in } \pi_p((\mathscr{A}_R^{[n]})_{c_2}, (\mathscr{A}_R^{[n]})_{c_1}). \blacksquare$$

Proof of lemma V.6. First of all, let us introduce some notation. For a path $\alpha : [x_0, x_1] \to M$, we denote by $\overline{\alpha} : [x_0, x_1] \to M$ the inverse path

$$\overline{\alpha}(x) = \alpha(x_0 + x_1 - x), \qquad \forall x \in [x_0, x_1].$$

If we consider another path $\beta : [x_0', x_1'] \to M$ with $\alpha(x_1) = \beta(x_0')$, we denote by $\alpha \bullet \beta : [x_0, x_1 + x_1' - x_0'] \to M$ the concatenation of the paths α and β, namely

$$\alpha \bullet \beta(x) = \begin{cases} \alpha(x) & x \in [x_0, x_1], \\ \beta(x - x_1 + x_0) & x \in [x_1, x_1 + x_1' - x_0']. \end{cases}$$

[3]Notice that $(\Delta^p, \partial\Delta^p)$ is homeomorphic to the pair (D^p, S^{p-1}), so that we can consider the singular simplex σ of the statement as an element of the homotopy group $\pi_p((\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}, (\mathscr{A}_k)_{c_1})$.

Consider a continuous map $\vartheta : [x_0, x_1] \to W^{1,2}(\mathbb{T}; M)$ where $[x_0, x_1] \subset \mathbb{R}$. For each $n \in \mathbb{N}$, by composition with the iteration map we obtain a map $\vartheta^{[n]} := \psi^{[n]} \circ \vartheta : [x_0, x_1] \to W^{1,2}(\mathbb{T}^{[n]}; M)$. Now, we want to build a continuous map $\vartheta^{\langle n \rangle} : [x_0, x_1] \to W^{1,2}(\mathbb{T}^{[n]}; M)$ as explained in the following.

To begin with, let us denote by $\mathrm{ev} : W^{1,2}(\mathbb{T}; M) \to M$ the **evaluation map**, given by

$$\mathrm{ev}(\zeta) = \zeta(0), \qquad \forall \zeta \in W^{1,2}(\mathbb{T}; M).$$

This map is smooth (see the proof of proposition II.2), hence the initial point curve $\mathrm{ev} \circ \vartheta : [x_0, x_1] \to M$ is a uniformly continuous path and, in particular, there exists a constant $\rho(\vartheta) > 0$ such that, for each $x, x' \in [x_0, x_1]$ with $|x - x'| \leq \rho(\vartheta)$, we have

$$\mathrm{dist}(\mathrm{ev} \circ \vartheta(x), \mathrm{ev} \circ \vartheta(x')) < \mathrm{injrad}(M).$$

In this inequality we have denoted by dist the Riemannian distance on M, and by $\mathrm{injrad}(M)$ the injectivity radius of M. Now, for each $x, x' \in [x_0, x_1]$ with $0 \leq x' - x \leq \rho(\vartheta)$, we define the **horizontal geodesic** $\vartheta_x^{x'} : [x, x'] \to M$ as the shortest geodesic[4] that connects the points $\mathrm{ev} \circ \vartheta(x)$ and $\mathrm{ev} \circ \vartheta(x')$. Then, let $J \in \mathbb{N}$ be such that $x_0 + J\rho \leq x_1 \leq x_0 + (J+1)\rho$. For each $x \in [x_0, x_1]$ we further choose $j \in \mathbb{N}$ such that $x_0 + j\rho \leq x \leq x_0 + (j+1)\rho$, and we define the **horizontal broken geodesics** $\vartheta_{x_0}^{x} : [x_0, x] \to M$ and $\vartheta_x^{x_1} : [x, x_1] \to M$ by

$$\begin{aligned}
\vartheta_{x_0}^{x} &:= \vartheta_{x_0}^{x_0+\rho} \bullet \vartheta_{x_0+\rho}^{x_0+2\rho} \bullet \dots \bullet \vartheta_{x_0+j\rho}^{x}, \\
\vartheta_{x}^{x_1} &:= \vartheta_{x}^{x_0+(j+1)\rho} \bullet \vartheta_{x_0+(j+1)\rho}^{x_0+(j+2)\rho} \bullet \dots \bullet \vartheta_{x_0+J\rho}^{x_1}.
\end{aligned}$$

We define a preliminary map $\tilde{\vartheta}^{\langle n \rangle} : [x_0, x_1] \to W^{1,2}(\mathbb{T}^{[n]}; M)$ in the following way. For each $j \in \{1, ..., n-2\}$ and $y \in [0, \frac{x_1 - x_0}{n}]$ we put

$$\begin{aligned}
\tilde{\vartheta}^{\langle n \rangle}(x_0 + y) &= \vartheta^{[n-1]}(x_0) \bullet \vartheta_{x_0}^{x_0+ny} \bullet \vartheta(x_0 + ny) \bullet \overline{\vartheta_{x_0}^{x_0+ny}}, \\
\tilde{\vartheta}^{\langle n \rangle}\big(x_0 + \tfrac{j}{n}(x_1 - x_0) + y\big) &= \vartheta^{[n-j-1]}(x_0) \bullet \vartheta_{x_0}^{x_0+ny} \bullet \vartheta(x_0 + ny) \bullet \vartheta_{x_0+ny}^{x_1} \\
&\quad \bullet \vartheta^{[j]}(x_1) \bullet \overline{\vartheta_{x_0}^{x_1}}, \\
\tilde{\vartheta}^{\langle n \rangle}\big(x_0 + \tfrac{n-1}{n}(x_1 - x_0) + y\big) &= \vartheta(x_0 + ny) \bullet \vartheta_{x_0+ny}^{x_1} \bullet \vartheta^{[n-1]}(x_1) \bullet \overline{\vartheta_{x_0+ny}^{x_1}}.
\end{aligned}$$

For each $x \in [x_0, x_1]$, we reparametrize the loop $\tilde{\vartheta}^{\langle n \rangle}(x)$ as follows: in the above formulas, each fixed part $\vartheta(x_0)$ and $\vartheta(x_1)$ spends the original time 1, while the moving parts $\vartheta(x_0 + ny)$ and the pieces of horizontal broken geodesics share the remaining time 1 proportionally to their original parametrizations. We define $\vartheta^{\langle n \rangle} : [x_0, x_1] \to W^{1,2}(\mathbb{T}^{[n]}; M)$ as the obtained continuous path in the loop space (see the example in figure V.1(a)).

[4]In the following, we will implicitely use the fact that the short geodesics lying in $C^\infty([t_0, t_1]; M)$ depend smoothly on their endpoints, cf. theorem III.3.

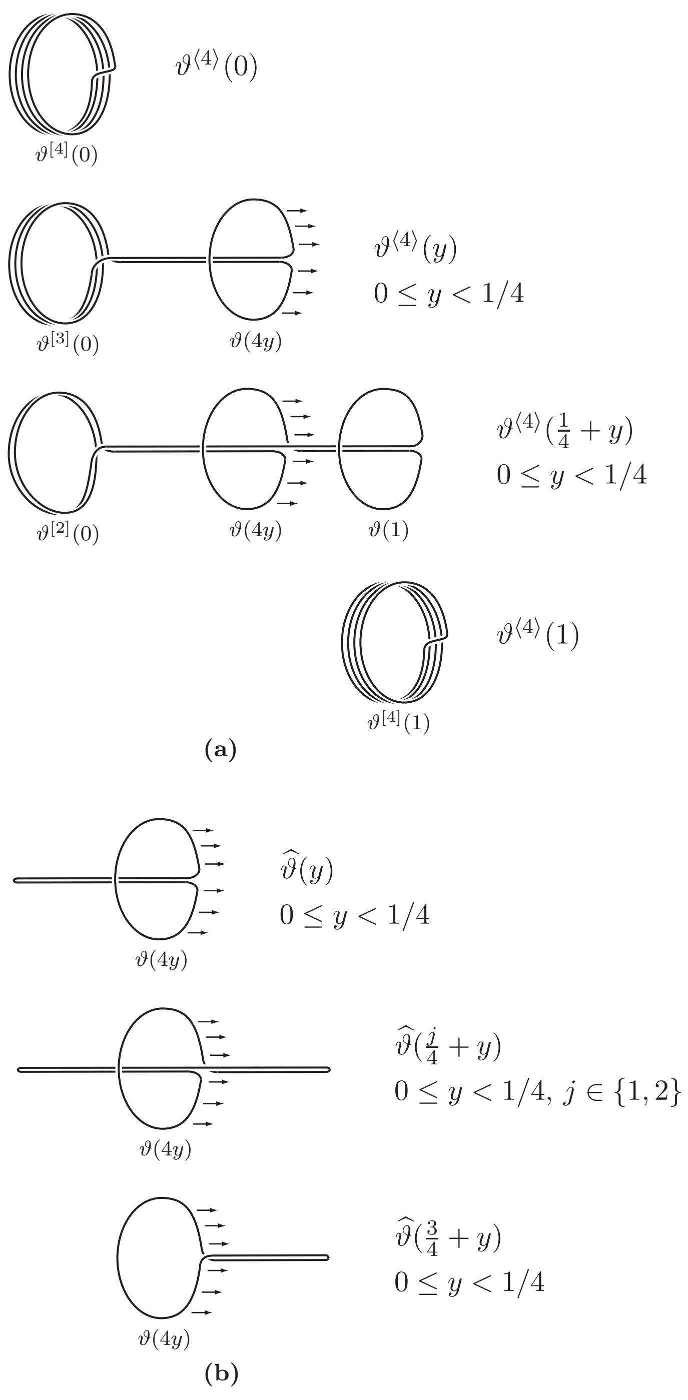


Figure V.1. **(a)** Description of $\vartheta^{\langle 4\rangle} : [0,1] \to W^{1,2}(\mathbb{T}^{[4]};M)$, obtained from a continuous map $\vartheta : [0,1] \to W^{1,2}(\mathbb{T};M)$. Here, for simplicity, we are assuming that the diameter of $\vartheta([x_0,x_1])$ is less than the injectivity radius of M, so that the horizontal geodesics are not broken. The arrows show the direction in which the loop $\vartheta(4y)$ is pulled as y grows. **(b)** Description of the map of pulling loops $\widehat{\vartheta} : [0,1] \to W^{1,2}(\mathbb{T};M)$.

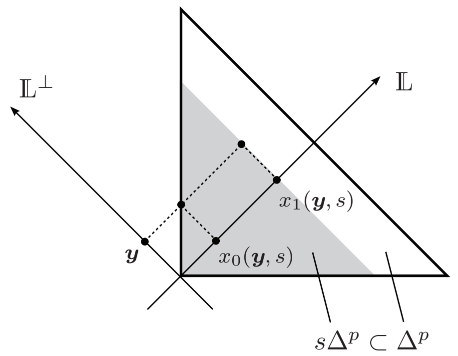


Figure V.2.

For each $x \in [x_0, x_1]$, we define the **pulling loop** $\widehat{\vartheta}(x) : \mathbb{T} \to M$ as the loop obtained erasing from the formula of $\widetilde{\vartheta}^{\langle n \rangle}(x)$ the fixed parts $\vartheta(x_0)$ and $\vartheta(x_1)$ and reparametrizing on $[0, 1]$ (see the example in figure V.1(b)). Notice that $\widehat{\vartheta}$ is independent of the integer $n \in \mathbb{N}$ and, for each $x \in \mathbb{N}$, the action $\mathscr{A}(\widehat{\vartheta}(x))$ is finite and depends continuously on x. In particular we obtain a finite constant

$$C(\vartheta) := \max_{x \in [x_0, x_1]} \left\{ \mathscr{A}(\widehat{\vartheta}(x)) \right\} = \max_{x \in [x_0, x_1]} \left\{ \int_0^1 \mathscr{L}\left(t, \widehat{\vartheta}(x)(t), \frac{\mathrm{d}}{\mathrm{d}t} \widehat{\vartheta}(x)(t)\right) \mathrm{d}t \right\} < \infty,$$

and, for each $n \in \mathbb{N}$, the estimate

$$\text{(V.7)} \qquad \begin{aligned} \mathscr{A}^{[n]}(\vartheta^{\langle n \rangle}(x)) &\leq \frac{1}{n} \Big[(n-1) \max \left\{ \mathscr{A}(\vartheta(x_0)), \mathscr{A}(\vartheta(x_1)) \right\} + \mathscr{A}(\widehat{\vartheta}(x)) \Big] \\ &\leq \max \left\{ \mathscr{A}(\vartheta(x_0)), \mathscr{A}(\vartheta(x_1)) \right\} + \frac{C(\vartheta)}{n}. \end{aligned}$$

Now, let $\mathbb{L} \subseteq \mathbb{R}^p$ be the straight line passing through the origin and the barycenter of the standard p-simplex $\Delta^p \subset \mathbb{R}^p$. We have an orthogonal decomposition of $\mathbb{R}^p$ as $\mathbb{L}^\perp \oplus \mathbb{L}$, and according to this decomposition we can write every $\boldsymbol{z} \in \mathbb{R}^p$ as

$$\boldsymbol{z} = (\boldsymbol{y}, x) \in \mathbb{L}^\perp \oplus \mathbb{L}.$$

For each $s \in [0, 1]$ we denote by $s\Delta^p$ the rescaled p-simplex

$$s\Delta^p = \left\{ s\boldsymbol{z} \,|\, \boldsymbol{z} \in \Delta^p \right\}.$$

Varying s from 1 to 0 we obtain a deformation retraction of Δ^p onto the origin of $\mathbb{R}^p$. For each $(\boldsymbol{y}, x) \in s\Delta^p$, we denote by $[x_0(\boldsymbol{y}, s), x_1(\boldsymbol{y}, s)] \subseteq \mathbb{L}$ the maximum interval such that $(\boldsymbol{y}, x')$ belongs to $s\Delta^p$ for all $x' \in [x_0(\boldsymbol{y}, s), x_1(\boldsymbol{y}, s)]$ (see figure V.2).

Consider the p-singular simplex σ of the statement. For each $n \in \mathbb{N}$, we define the **Bangert homotopy**

$$\mathrm{Ban}_\sigma^{[n]} : [0, 1] \times \Delta^p \to W^{1,2}(\mathbb{T}^{[n]}; M)$$

by

$$\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z}) := \begin{cases} (\sigma|_{[x_0(\boldsymbol{y},s),x_1(\boldsymbol{y},s)]})^{\langle n\rangle}(x) & \boldsymbol{z}=(\boldsymbol{y},x)\in s\Delta^p,\\ \psi^{[n]}\circ\sigma(\boldsymbol{z}) & \boldsymbol{z}\notin s\Delta^p, \end{cases}$$

for each $(s,\boldsymbol{z})\in[0,1]\times\Delta^p$. This homotopy $\mathrm{Ban}_\sigma^{[n]}$ is relative $\partial\Delta^p$, for

$$\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z}) = \psi^{[n]}\circ\sigma(\boldsymbol{z}), \qquad \forall (s,\boldsymbol{z})\in[0,1]\times\partial\Delta^p,$$

and clearly $\mathrm{Ban}_\sigma^{[n]}(0,\cdot)=\psi^{[n]}\circ\sigma$, proving assertion (i) of the lemma. By the assumptions on σ, there exists $\varepsilon>0$ such that

$$\max_{\boldsymbol{z}\in\Delta^p}\mathscr{A}(\sigma(\boldsymbol{z}))\leq c_2-\varepsilon, \qquad \max_{\boldsymbol{z}\in\partial\Delta^p}\mathscr{A}(\sigma(\boldsymbol{z}))\leq c_1-\varepsilon.$$

For each $s\in[0,1]$, $n\in\mathbb{N}$ and $\boldsymbol{z}=(\boldsymbol{y},x)\in s\Delta^p$, by the estimate in (V.7) we have

$$\mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z}))\leq\max\{\mathscr{A}(\sigma(x_0(\boldsymbol{y},s))),\mathscr{A}(\sigma(x_1(\boldsymbol{y},s)))\}+\frac{C(\sigma|_{[x_0(\boldsymbol{y},s),x_1(\boldsymbol{y},s)]})}{n},$$

while, for each $\boldsymbol{z}\in\Delta^p\setminus s\Delta^p$, we have

$$\mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z}))=\mathscr{A}(\sigma(\boldsymbol{z})).$$

In particular, there exists a finite constant

$$C(\sigma):=\max\big\{C(\sigma|_{[x_0(\boldsymbol{y},s),x_1(\boldsymbol{y},s)]})\,|\,s\in[0,1],\ (\boldsymbol{y},x)\in s\Delta^p\big\}$$

such that, for each $n\in\mathbb{N}$ and $(s,\boldsymbol{z})\in[0,1]\times\Delta^p$, we have

$$\mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(s,\boldsymbol{z}))\leq\max_{\boldsymbol{w}\in\Delta^p}\{\mathscr{A}(\sigma(\boldsymbol{w})\}+\frac{C(\sigma)}{n}\leq c_2-\varepsilon+\frac{C(\sigma)}{n},$$
$$\mathscr{A}^{[n]}(\mathrm{Ban}_\sigma^{[n]}(1,\boldsymbol{z}))\leq\max_{\boldsymbol{w}\in\partial\Delta^p}\{\mathscr{A}(\sigma(\boldsymbol{w})\}+\frac{C(\sigma)}{n}\leq c_1-\varepsilon+\frac{C(\sigma)}{n},$$

proving assertion (ii). Finally, for all $s\in[0,1]$ and $\boldsymbol{z}=(\boldsymbol{y},x)\in s\Delta^p$, there is a uniform bound $\bar{r}'(\sigma)>0$ for the derivative of the pulling loops associated to the paths $\sigma|_{[x_0(\boldsymbol{y},s),x_1(\boldsymbol{y},s)]}$. Therefore, for $\bar{R}(\sigma):=\max\{\bar{r}(\sigma),\bar{r}'(\sigma)\}$, assertion (iii) follows. ■

The idea of the proof of the homological vanishing theorem (theorem V.5) is to apply the Bangert homotopy lemma successively to all the faces of the singular simplices that compose the relative cicle η. In order to do this, we will make use of the following easy result from algebraic topology, that was stated without proof in [BK, lemma 1]. For the sake of completeness, we include a detailed proof here (basically, the argument is a variation of the one that proves that singular homology is a homotopic invariant, see for instance [Hat, page 112]).

Lemma V.8. *Let (X, Y) be a pair of topological spaces, μ a relative p-cycle*[5] *in (X, Y) and $\Sigma(\mu)$ the set of singular simplices that compose μ together with their faces. Suppose that, for each singular simplex $\sigma : \Delta^q \to X$ that belongs to $\Sigma(\mu)$, where $0 \leq q \leq p$, there exists a homotopy*

$$P_\sigma : \Delta^q \times [0, 1] \to X$$

such that

(i) $P_\sigma(\cdot, 0) = \sigma$;

(ii) $P_\sigma(\cdot, s) = \sigma$ *for each* $s \in [0, 1]$, *if* $\sigma(\Delta^q) \subset Y$;

(iii) $P_\sigma(\Delta^q \times \{1\}) \subset Y$;

(iv) $P_\sigma(F_j(\cdot), \cdot) = P_{\sigma \circ F_j}(\cdot, \cdot)$ *for each* $j = 0, ..., p$, *where* $F_j : \Delta^{q-1} \to \Delta^q$ *is the standard affine map onto the* j^{th}*-face of* Δ^q.

Then $[\mu] = 0$ *in* $\mathrm{H}_p(X, Y)$.

Proof. If $\boldsymbol{v}_0, ..., \boldsymbol{v}_h$ are points in $\mathbb{R}^q$, we will denote by $\langle \boldsymbol{v}_0, ..., \boldsymbol{v}_h \rangle$ their convex hull, that is the minimal convex close subset of $\mathbb{R}^q$ containing these points. We will denote by ∂ the usual boundary operator from algebraic topology, i.e. $\partial \langle \boldsymbol{v}_0, ..., \boldsymbol{v}_h \rangle$ is the following formal sum of $(h-1)$-simplices

$$\partial \langle \boldsymbol{v}_0, ..., \boldsymbol{v}_h \rangle := \sum_{j=0}^{h} (-1)^j \langle \boldsymbol{v}_0, ..., \widehat{\boldsymbol{v}}_j, ..., \boldsymbol{v}_h \rangle ,$$

where, as usual, we add a hat over $\boldsymbol{v}_j$ to denote that it is missing in the corresponding term, namely

$$\langle \boldsymbol{v}_0, ..., \widehat{\boldsymbol{v}}_j, ..., \boldsymbol{v}_h \rangle := \langle \boldsymbol{v}_0, ..., \boldsymbol{v}_{j-1}, \boldsymbol{v}_{j+1}, ..., \boldsymbol{v}_h \rangle .$$

If $\boldsymbol{e}_1, ..., \boldsymbol{e}_q$ is the standard basis of $\mathbb{R}^q$ and $\boldsymbol{e}_0$ is the origin, then $\langle \boldsymbol{e}_0, ..., \boldsymbol{e}_q \rangle$ is the standard q-simplex Δ^q. We identify $\mathbb{R}^q$ with $\mathbb{R}^q \times \{0\} \subset \mathbb{R}^{q+1}$ and we define $\boldsymbol{f}_j = (\boldsymbol{e}_j, 1)$ for each $j \in \{0, ..., q\}$. The product $\Delta^q \times [0, 1]$ can be decomposed as the union of $(q+1)$-simplices as follows:

$$\Delta^q \times [0, 1] = \bigcup_{j=0}^{q} \langle \boldsymbol{e}_0, ..., \boldsymbol{e}_j, \boldsymbol{f}_j, ..., \boldsymbol{f}_q \rangle .$$

For each $j \in \{0, ..., q-1\}$, $\langle \boldsymbol{e}_0, ..., \boldsymbol{e}_j, \boldsymbol{f}_j, ..., \boldsymbol{f}_q \rangle$ intersect $\langle \boldsymbol{e}_0, ..., \boldsymbol{e}_{j+1}, \boldsymbol{f}_{j+1}, ..., \boldsymbol{f}_q \rangle$ in the q-simplex face $\langle \boldsymbol{e}_0, ..., \boldsymbol{e}_j, \boldsymbol{f}_{j+1}, ..., \boldsymbol{f}_q \rangle$ (see the example in figure V.3).

After these preliminaries, consider the relative cycle μ of the lemma. For each q-simplex $\sigma : \Delta^q \to X$ that belongs to $\Sigma(\mu)$, we define an associated $q+1$-chain p_σ by

$$p_\sigma = \sum_{j=0}^{q} (-1)^j P_\sigma|_{\langle \boldsymbol{e}_0, ..., \boldsymbol{e}_j, \boldsymbol{f}_j, ..., \boldsymbol{f}_q \rangle}.$$

[5] Namely $[\mu] \in \mathrm{H}_p(X, Y)$.

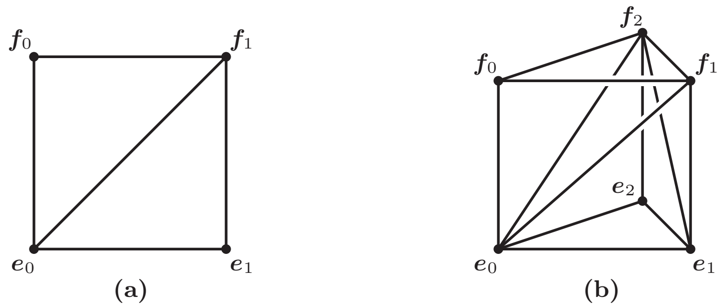


Figure V.3. **(a)** Decomposition of $\Delta^1 \times [0,1] = [0,1] \times [0,1]$ into 2-simplices. **(b)** Decomposition of $\Delta^2 \times [0,1]$ into 3-simplices.

Then, we extend this definition for chains η that are composed by simplices that belong to $\Sigma(\mu)$, in the following way: if we write η as formal sum of q-simplices

$$\eta = \sum_{v=1}^{V} \eta_v,$$

then we put

$$p_\eta := \sum_{v=1}^{V} p_{\eta_v}.$$

Notice that p_η is well defined: if η_v and $\eta_{v'}$ have a common face, say $\eta_v \circ F_j = \eta_{v'} \circ F_{j'}$, then assumption (iv) guarantees

$$p_{\eta_v} \circ F_j = p_{\eta_v \circ F_j} = p_{\eta_{v'} \circ F_{j'}} = p_{\eta_{v'}} \circ F_{j'}.$$

We define the chain $\tilde{\mu}$ that is "homotopic" to the relative cycle μ (by the homotopies given in the lemma) in the following way: if we write μ as formal sum of q-simplices

$$\mu = \sum_{w=1}^{W} \mu_w,$$

then we put

$$\tilde{\mu} := \sum_{w=1}^{W} \tilde{\mu}_w,$$

where $\tilde{\mu}_w = P_{\mu_w}(\cdot, 1) : \Delta^p \to Y$ for each $w \in \{1, ..., W\}$. Here, the fact that $\tilde{\mu}_w(\Delta^p) \subset Y$ for each $w \in \{1, ..., W\}$ is guaranteed by assumption (iii), and it implies that $\tilde{\mu}$ is a relative cycle whose homology class in $\mathrm{H}_p(X, Y)$ is zero, i.e.

$$[\tilde{\mu}] = 0 \qquad \text{in } \mathrm{H}_p(X, Y). \tag{V.8}$$

Now,

$$\partial p_\mu = \sum_{w=1}^{W} \partial p_{\mu_w} = \sum_{w=1}^{W} \Bigg[\sum_{j\leq i} (-1)^{i+j} P_{\mu_w}|_{\langle \boldsymbol{e}_0,...,\widehat{\boldsymbol{e}}_j,...,\boldsymbol{e}_i,\boldsymbol{f}_i,...,\boldsymbol{f}_q\rangle} + \sum_{j\geq i} (-1)^{i+j+1} P_{\mu_w}|_{\langle \boldsymbol{e}_0,...,\boldsymbol{e}_i,\boldsymbol{f}_i,...,\widehat{\boldsymbol{f}}_j,...,\boldsymbol{f}_q\rangle} \Bigg].$$

The terms with $i = j$ in the two inner sums cancel, except for $P_{\mu_w}|_{\langle \widehat{\boldsymbol{e}}_0,\boldsymbol{f}_0,...,\boldsymbol{f}_q\rangle} = \tilde{\mu}_w$ and $-P_{\mu_w}|_{\langle \boldsymbol{e}_0,...,\boldsymbol{e}_q,\widehat{\boldsymbol{f}}_q\rangle} = -\mu_w$. The terms with $i \neq j$ are precisely $p_{\partial\mu_w}$, for

$$p_{\partial\mu_w} = \sum_{i<j} (-1)^{i+j} P_{\mu_w}|_{\langle \boldsymbol{e}_0,...,\boldsymbol{e}_i,\boldsymbol{f}_i,...,\widehat{\boldsymbol{f}}_j,...,\boldsymbol{f}_q\rangle} + \sum_{i>j} (-1)^{i-1+j} P_{\mu_w}|_{\langle \boldsymbol{e}_0,...,\widehat{\boldsymbol{e}}_j,...,\boldsymbol{e}_i,\boldsymbol{f}_i,...,\boldsymbol{f}_q\rangle}.$$

Therefore, we have obtained

$$\tilde{\mu} - \mu = \partial p_\mu - p_{\partial\mu}. \tag{V.9}$$

The above equality must be understood in the p^{th}-relative chain group of (X, Y). By assumption (ii) and since μ is a relative cycle, the singular simplexes that compose $p_{\partial\mu}$ have image inside Y, hence $p_{\partial\mu}$ is a relative cycle whose homology class in $\mathrm{H}_p(X, Y)$ is zero, i.e.

$$[p_{\partial\mu}] = 0 \qquad \text{in } \mathrm{H}_p(X, Y). \tag{V.10}$$

By (V.8), (V.9) and (V.10), we conclude

$$[\mu] = [\tilde{\mu}] - [\partial p_\mu] + [p_{\partial\mu}] = 0 \qquad \text{in } \mathrm{H}_p(X, Y).$$ ■

Proof of theorem V.5. Let $U > 0$ be the constant chosen in (V.2). We denote by $\Sigma(\eta)$ the set of singular simplices in η together with all their faces, and by $\mathbb{K} \subset \mathbb{N}$ the set of nonnegative integer powers of j, i.e. $\mathbb{K} = \{j^n \mid n \in \mathbb{N} \cup \{0\}\}$. For each singular simplex $\sigma : \Delta^p \to (\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}$ that belongs to $\Sigma(\eta)$ we will find $\bar{n} = \bar{n}(\sigma, j) \in \mathbb{K}$, a positive real $\bar{R} = \bar{R}(\sigma, j) \geq U$ and a homotopy

$$P_\sigma^{[\bar{n}]} : [0, 1] \times \Delta^p \to W^{1,2}(\mathbb{T}^{[\bar{n}]}; M)$$

such that

(i) $P_\sigma^{[\bar{n}]}(0, \cdot) = \psi^{[\bar{n}]} \circ \lambda_k \circ \sigma$;

(ii) if $\sigma(\Delta^p) \subset (\mathscr{A}_k)_{c_1}$, then $P_\sigma^{[\bar{n}]}(s, \cdot) = \psi^{[\bar{n}]} \circ \lambda_k \circ \sigma$ for each $s \in [0, 1]$;

(iii) $\mathscr{A}^{[\bar{n}]}(P_\sigma^{[\bar{n}]}(s, \boldsymbol{z})) < c_2$ and $\mathscr{A}^{[\bar{n}]}(P_\sigma^{[\bar{n}]}(1, \boldsymbol{z})) < c_1$ for each $(s, \boldsymbol{z}) \in [0, 1] \times \Delta^p$;

(iv) $P_\sigma^{[\bar{n}]}(\cdot, F_i(\cdot)) = P_{\sigma\circ F_i}^{[\bar{n}]}(\cdot, \cdot)$ for each $i = 0, ..., p$, where $F_i : \Delta^{p-1} \to \Delta^p$ is the standard affine map onto the i^{th}-face of Δ^p.

(v) $$\sup_{(s,\boldsymbol{z})\in[0,1]\times\Delta^p} \operatorname*{ess\,sup}_{t\in\mathbb{T}^{[\bar n]}} \left\{ \left| \frac{\mathrm{d}}{\mathrm{d}t} P_\sigma^{[\bar n]}(s,\boldsymbol{z})(t) \right|_{P_\sigma^{[\bar n]}(s,\boldsymbol{z})(t)} \right\} < \bar R.$$

For each $n \in \mathbb{K}$ greater than $\bar n$, we define a homotopy

$$P_\sigma^{[n]} : [0,1] \times \Delta^q \to W^{1,2}(\mathbb{T}^{[n]}; M)$$

by $P_\sigma^{[n]} := \psi^{[n/\bar n]} \circ P_\sigma^{[\bar n]}$. This homotopy satisfies the analogous properties (i),...,(v) in period n. Notice that property (iv) implicitly requires that $\bar n(\sigma, j) \geq \bar n(\sigma \circ F_i, j)$ for each $i = 0, ..., p$.

Now, assume that such homotopies exist and put

$$\bar R = \bar R([\eta], j) := \max\{\bar R(\sigma, j) \mid \sigma \in \Sigma(\eta)\},$$
$$\bar n = \bar n([\eta], j) := \max\{\bar n(\sigma, j) \mid \sigma \in \Sigma(\eta)\}.$$

Then, for each $R \geq \bar R$, the set of homotopies $\{P_\sigma^{[\bar n]} \mid \sigma \in \Sigma(\eta)\}$ satisfies the hypotheses of lemma V.8 with respect to the relative cicle $\psi^{[\bar n]} \circ \lambda_k \circ \eta$ in $((\mathscr{A}_R^{[\bar n]})_{c_2}, (\mathscr{A}_R^{[\bar n]})_{c_1})$, and we conclude

$$\mathrm{H}_*(\psi^{[\bar n]}) \circ \mathrm{H}_*(\lambda_k)[\eta] = [\psi^{[\bar n]} \circ \lambda_k \circ \eta] = 0 \qquad \text{in } \mathrm{H}_*((\mathscr{A}_R^{[\bar n]})_{c_2}, (\mathscr{A}_R^{[\bar n]})_{c_1}).$$

In order to conclude the proof, we only need to build the above homotopies. We do it inductively on the dimension of the relative cycle η. If η is a 0-relative cycle, $\Sigma(\eta)$ is a finite set of points $\{\boldsymbol{w}_1, ..., \boldsymbol{w}_h\}$ that is contained in $(\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}$. Since we are assuming that $\boldsymbol{q}$ is not a minimum, the sublevel $(\mathscr{A}_k)_{c_1}$ is not empty. Hence, for each $\boldsymbol{w} \in \Sigma(\eta)$, we can find a path $\Gamma_{\boldsymbol{w}} : [0,1] \to (\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}$ such that $\Gamma_{\boldsymbol{w}}(0) = \boldsymbol{w}$ and $\Gamma_{\boldsymbol{w}}(s) \in (\mathscr{A}_k)_{c_1}$ for each $s \in (0,1]$ (if $\boldsymbol{w} \in (\mathscr{A}_k)_{c_1}$, we simply choose $\Gamma_{\boldsymbol{w}}(s) := \boldsymbol{w}$ for each $s \in [0,1]$). Then, we set $\bar R(\boldsymbol{w}, j) := U$, $\bar n(\boldsymbol{w}, j) := 1$ and

$$P_{\boldsymbol{w}}^{[1]} := \lambda_k \circ \Gamma_{\boldsymbol{w}} : [0,1] \times \{\boldsymbol{0}\} \to W^{1,2}(\mathbb{T}; M).$$

If η is a p-relative cycle, with $p \geq 1$, we can apply the inductive hypothesis: for each nonnegative integer $i < p$ and for each i-singular simplex $\nu \in \Sigma(\eta)$, we obtain $\bar n(\nu, j) \in \mathbb{K}$, $\bar R(\nu, j) \geq U$ and, for each $n \in \mathbb{K}$ greater or equal than $\bar n(\nu, j)$, a homotopy $P_\nu^{[n]}$ satisfying the above properties (i),...,(v). Now, consider a p-singular simplex $\sigma : \Delta^p \to (\mathscr{A}_k)_{c_1} \cup \{\boldsymbol{q}\}$ that belongs to $\Sigma(\eta)$. If $\sigma(\Delta^p) \subset (\mathscr{A}_k)_{c_1}$ we simply set $\bar R(\sigma, j) := U$, $\bar n(\sigma, j) := 1$ and $P_\sigma^{[1]}(s, \cdot) := \lambda_k \circ \sigma$ for each $s \in [0,1]$. In the other case, $\sigma(\Delta^p) \not\subset (\mathscr{A}_k)_{c_1}$, we proceed as follows. We denote by $\bar R' = \bar R'(\sigma, j)$ and $\bar n' = \bar n'(\sigma, j)$ respectively the maximum of the $\bar R(\nu, j)$'s and $\bar n(\nu, j)$'s for all the proper faces ν of σ. Thus, for each $n \in \mathbb{K}$ greater or equal than $\bar n'(\sigma, j)$, every proper face ν of σ has an associated homotopy

$$P_\nu^{[n]} : [0,1] \times \Delta^{p-1} \to W^{1,2}(\mathbb{T}^{[n]}; M).$$

For technical reasons we assume that $P_\nu^{[n]}(s, \cdot) = P_\nu^{[n]}(\frac{1}{2}, \cdot)$ for $s \in [\frac{1}{2}, 1]$. Patching together the homotopies of the proper faces of σ, we obtain

$$P_\sigma^{[n]} : ([0, \tfrac{1}{2}] \times \partial\Delta^p) \cup (0 \times \Delta^p) \to W^{1,2}(\mathbb{T}^{[n]}; M), \qquad \forall n \in \mathbb{K},\ n \geq \bar n',$$

such that $P_\sigma^{[n]}(0,\cdot) = \psi^{[n]} \circ \lambda_k \circ \sigma$ and $P_\sigma^{[n]}(\cdot, F_i(\cdot)) = P_{\sigma\circ F_i}^{[n]}$ for all $i = 1, ..., p$. By retracting $([0,\frac{1}{2}] \times \Delta^p)$ onto $([0,\frac{1}{2}] \times \partial\Delta^p) \cup (0 \times \Delta^p)$, we can extend the homotopies $P_\sigma^{[n]}$ to the whole $([0,\frac{1}{2}] \times \Delta^p)$, obtaining

$$P_\sigma^{[n]} : [0,\tfrac{1}{2}] \times \Delta^p \to W^{1,2}(\mathbb{T}^{[n]}; M), \qquad \forall n \in \mathbb{K},\ n \geq \bar{n}'. \tag{V.11}$$

Let us briefly denote the singular simplex $P_\sigma^{[\bar{n}']}(\frac{1}{2},\cdot) : \Delta^p \to W^{1,2}(\mathbb{T}^{[\bar{n}']}; M)$ by $\tilde{\sigma}$. Notice that

$$\begin{aligned}
&\max_{\boldsymbol{z}\in\Delta^p} \left\{ \mathscr{A}^{[\bar{n}']}(\tilde{\sigma}(\boldsymbol{z})) \right\} < c_2,\\
&\max_{\boldsymbol{z}\in\partial\Delta^p} \left\{ \mathscr{A}^{[\bar{n}']}(\tilde{\sigma}(\boldsymbol{z})) \right\} < c_1,\\
&\sup_{\boldsymbol{z}\in\Delta^p} \ \operatorname*{ess\,sup}_{t\in\mathbb{R}/\bar{n}'\mathbb{Z}} \left\{ \left| \frac{\mathrm{d}}{\mathrm{d}t} \tilde{\sigma}(\boldsymbol{z})(t) \right|_{\tilde{\sigma}(\boldsymbol{z})(t)} \right\} \leq \bar{R}'.
\end{aligned}$$

Hence we can apply lemma V.6, obtaining an integer $\bar{n}(\tilde{\sigma})$, a positive real $\bar{R}(\tilde{\sigma}) \geq \bar{R}'(\sigma, j)$ and, if we choose the smallest $\bar{n}'' \in \mathbb{K}$ greater or equal than $\bar{n}(\tilde{\sigma})$, a Bangert homotopy

$$\mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']} : [0,1] \times \Delta^p \to W^{1,2}(\mathbb{T}^{[\bar{n}'\bar{n}'']}; M) \qquad \text{relative } \partial\Delta^p$$

such that $\mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(0,\cdot) = \psi^{[\bar{n}'']} \circ \tilde{\sigma}$ and

$$\begin{aligned}
&\max_{(s,\boldsymbol{z})\in[0,1]\times\Delta^p} \left\{ \mathscr{A}^{[\bar{n}'\bar{n}'']}(\mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(s,\boldsymbol{z})) \right\} < c_2,\\
&\max_{(s,\boldsymbol{z})\in[0,1]\times\partial\Delta^p} \left\{ \mathscr{A}^{[\bar{n}'\bar{n}'']}(\mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(s,\boldsymbol{z})) \right\} < c_1;\\
&\max_{\boldsymbol{z}\in\Delta^p} \left\{ \mathscr{A}^{[\bar{n}'\bar{n}'']}(\mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(1,\boldsymbol{z})) \right\} < c_1;\\
&\sup_{(s,\boldsymbol{z})\in[0,1]\times\Delta^p} \ \operatorname*{ess\,sup}_{t\in\mathbb{R}/\bar{n}'\bar{n}''\mathbb{Z}} \left\{ \left| \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(s,\boldsymbol{z})(t) \right|_{\mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(s,\boldsymbol{z})(t)} \right\} \leq \bar{R}(\sigma).
\end{aligned}$$

Finally, we set $\bar{n} = \bar{n}(\sigma, j) := \bar{n}'\bar{n}''$, $\bar{R}(\sigma, j) := \bar{R}(\tilde{\sigma})$ and we build the homotopy $P_\sigma^{[\bar{n}]} : [0,1] \times \Delta^p \to W^{1,2}(\mathbb{T}^{[\bar{n}]}; M)$ extending the one in (V.11) by

$$P_\sigma^{[\bar{n}]}(s,\cdot) := \mathrm{Ban}_{\tilde{\sigma}}^{[\bar{n}'']}(2s-1,\cdot), \qquad \forall s \in [\tfrac{1}{2}, 1].$$

■

V.4 The main result

We are now ready to state and prove the main result of this chapter, that confirms the Conley conjecture for Tonelli Lagrangian systems.

Theorem V.9. *Let M be a smooth closed manifold, $\mathscr{L} : \mathbb{R}/\mathbb{Z}\times \mathrm{T}M \to \mathbb{R}$ a smooth 1-periodic Tonelli Lagrangian with global Euler-Lagrange flow and $a \in \mathbb{R}$ a constant greater than*

$$\max_{q\in M} \left\{ \int_0^1 \mathscr{L}(t, q, 0)\, \mathrm{d}t \right\}. \tag{V.12}$$

Assume that only finitely many contractible 1-periodic solutions of the Euler-Lagrange system of $\mathscr{L}$ have action less than a. Then, for each prime $p \in \mathbb{N}$, the Euler-Lagrange system of $\mathscr{L}$ admits infinitely many contractible periodic solutions with period that is a power of p and mean action less than a.

Here it is worthwhile to point out that the infinitely many periodic orbits that we find are geometrically distinct in the phase-space of our system. If we consider the Tonelli Hamiltonian $\mathscr{H}$ that is Legendre-dual to the Tonelli Lagrangian $\mathscr{L}$ (cf. sections I.1 and I.2), theorem V.9 can be rephrased in the Hamiltonian formulation. The two statements are completely equivalent.

Theorem V.9 (Hamiltonian formulation). *Let M be a smooth closed manifold, $\mathscr{H} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}^*M \to \mathbb{R}$ a smooth 1-periodic Tonelli Hamiltonian with global Hamiltonian flow and $a \in \mathbb{R}$ a constant greater than*

$$-\min_{q\in M} \left\{ \int_0^1 \min_{p\in \mathrm{T}^*_q M} \{\mathscr{H}(t, q, p)\}\, \mathrm{d}t \right\}.$$

Assume that only finitely many contractible 1-periodic solutions of the Hamilton system of $\mathscr{H}$ have action less than a. Then, for each prime $p \in \mathbb{N}$, the Hamilton system of $\mathscr{H}$ admits infinitely many contractible periodic solutions with period that is a power of p and mean action less than a.

In this statement, the mean action of a periodic orbit $\Gamma : \mathbb{R}/n\mathbb{Z} \to \mathrm{T}^*M$ is meant to be its **Hamiltonian mean action**, defined by

$$\frac{1}{n} \int_0^n \Big(\Gamma^*\lambda - \mathscr{H}(t, \Gamma(t))\Big)\, \mathrm{d}t,$$

where λ is the Liouville form on T^*M (cf. section I.1). This quantity coincides with the usual Lagrangian mean action of the associated Lagrangian periodic orbit, i.e. if $\mathscr{L}$ is Legendre-dual to $\mathscr{H}$ and we write Γ as (γ, ρ), where $\gamma : \mathbb{R}/n\mathbb{Z} \to M$ is a periodic solution of the Euler-Lagrange system of $\mathscr{L}$, we have

$$\begin{aligned} \frac{1}{n} \int_0^n \Big(\Gamma^*\lambda - \mathscr{H}(t, \Gamma(t))\Big)\, \mathrm{d}t &= \frac{1}{n} \int_0^n \Big(\rho(t)[\dot\gamma(t)] - \mathscr{H}(t, \gamma(t), \rho(t))\Big)\, \mathrm{d}t \\ &= \frac{1}{n} \int_0^n \mathscr{L}(t, \gamma(t), \dot\gamma(t))\, \mathrm{d}t. \end{aligned}$$

Proof of theorem V.9 Throughout the proof, all the homology groups that appear are assumed to have coefficient in the finite field $\mathbb{Z}_2$. Moreover, for each $n \in \mathbb{N}$ and $R > 0$, we implicitely restrict the mean action functional $\mathscr{A}_R^{[n]}$ of any R-modification $\mathscr{L}_R$ of $\mathscr{L}$ to the connected component of $W^{1,2}(\mathbb{T}^{[n]}; M)$ given by the contractible loops. In particular, for any $c \in \mathbb{R}$, the action sublevel $(\mathscr{A}_R^{[n]})_c$ is understood to be contained in this connected component.

Let us fix a prime $p \in \mathbb{N}$. We will denote by $\mathbb{K} \subset \mathbb{N}$ the set of nonnegative integer powers of p, i.e.

$$\mathbb{K} = \{p^n \,|\, n \in \mathbb{N} \cup \{0\}\}\,.$$

We will proceed by contradiction, assuming that the only contractible periodic solutions of the Euler-Lagrange system of $\mathscr{L}$ with period in $\mathbb{K}$ and mean action less than a are

$$\gamma_1, ..., \gamma_r.$$

Without loss of generality, we can assume that all these orbits have period $1 = p^0$. This can be easily seen in the following way. First of all, if p^n is the maximum of their periods, then they are all p^n-periodic, i.e.

$$\gamma_v(t + p^n) = \gamma_v(t), \qquad \forall t \in \mathbb{R},\ v \in \{1, ..., r\}\,.$$

Then, we can build a Tonelli Lagrangian $\tilde{\mathscr{L}} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ by time-rescaling of $\mathscr{L}$ as

$$\tilde{\mathscr{L}}(t, q, v) := \mathscr{L}(p^n t, q, p^{-n} v), \qquad \forall (t, q, v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M.$$

For each $j \in \mathbb{N}$, a curve $\tilde{\gamma} : \mathbb{R} \to M$ is a j-periodic solution of the Euler-Lagrange system of $\tilde{\mathscr{L}}$ if and only if the rescaled curve $\gamma : \mathbb{R} \to M$, given by $\gamma(t) := \tilde{\gamma}(p^{-n} t)$ for each $t \in \mathbb{R}$, is a $p^n j$-periodic solution of the Euler-Lagrange system of $\mathscr{L}$. Moreover, $\tilde{\gamma}$ and γ have the same mean action (with respect to the Lagrangians $\tilde{\mathscr{L}}$ and $\mathscr{L}$ respectively).

Let m be the dimension of the manifold M, and consider the constant $a \in \mathbb{R}$ given in the statement of the theorem. For each $R > 0$ and $n \in \mathbb{N}$, the homology of the sublevel $(\mathscr{A}_R^{[n]})_a$ is non-trivial in dimension m, i.e.

$$\mathrm{H}_m((\mathscr{A}_R^{[n]})_a) \neq 0, \qquad \forall R > 0, n \in \mathbb{N}. \tag{V.13}$$

This is a straightforward extension of proposition II.2, and can be proved in the following way. To begin with, notice that the quantity in (V.12) is finite (due to the compactness of M) and may be interpreted in the following way. For each integer $n \in \mathbb{N}$, if we denote by $\iota^{[n]} : M \hookrightarrow W^{1,2}(\mathbb{T}^{[n]}; M)$ the embedding that maps a point to the constant loop at that point, the quantity in (V.12) is equal to the maximum of the function $\mathscr{A}^{[n]} \circ \iota^{[n]} : M \to \mathbb{R}$. By our choice of the constant a we have

$$\mathscr{A}_R^{[n]} \circ \iota^{[n]}(q) = \mathscr{A}^{[n]} \circ \iota^{[n]}(q) < a, \qquad \forall q \in M,$$

therefore $\iota^{[n]}$ can be seen as a map of the form $\iota^{[n]} : M \hookrightarrow (\mathscr{A}_R^{[n]})_a$. Since M is an m-dimensional closed manifold and we consider homology groups with $\mathbb{Z}_2$ coefficients, we have that $\mathrm{H}_m(M)$ is non trivial. Then, the following commutative diagram readily implies that $\mathrm{H}_m(\iota^{[n]})$ is a monomorphism, and the claim follows.

$$\begin{array}{ccccc} & & \mathrm{H}_m((\mathscr{A}_R^{[n]})_a) & & \\ & \overset{\mathrm{H}_m(\iota^{[n]})}{\nearrow} & & \overset{\mathrm{H}_m(\mathrm{ev})}{\searrow} & \\ 0 \neq \mathrm{H}_m(M) & & \xrightarrow[\simeq]{\mathrm{H}_m(\mathrm{id}_M)} & & \mathrm{H}_m(M) \end{array}$$

Now, we want to show that there exists $\gamma \in \{\gamma_1, ..., \gamma_r\}$ having mean Morse index[6] $\widehat{\iota}(\mathscr{A}, \gamma)$ equal to zero. In fact, assume by contradiction that $\widehat{\iota}(\mathscr{A}, \gamma_v) > 0$ for each $v \in \{1, ..., r\}$. By the iteration inequality for the Morse index and nullity pair (cf. proposition II.6) there exists $n \in \mathbb{K}$ such that

$$\iota(\mathscr{A}^{[n]}, \gamma_v^{[n]}) > m, \qquad \forall v \in \{1, ..., r\}\,. \tag{V.14}$$

By lemma V.2, if we choose a real constant $R > \tilde{R}(a, n)$, we know that the only critical points of $\mathscr{A}_R^{[n]}$ in the open sublevel $(\mathscr{A}_R^{[n]})_a$ are $\gamma_1^{[n]}, ..., \gamma_r^{[n]}$ and, by (V.14), we have

$$\iota(\mathscr{A}_R^{[n]}, \gamma_v^{[n]}) = \iota(\mathscr{A}^{[n]}, \gamma_v^{[n]}) > m, \qquad \forall v \in \{1, ..., r\}\,.$$

In particular, for each $v \in \mathbb{N}$, the local homology of $\mathscr{A}_R^{[n]}$ at $\gamma_v^{[n]}$ vanishes in dimension m (cf. corollary III.18), i.e.

$$\mathrm{H}_m(\mathscr{A}_R^{[n]}, \gamma_v^{[n]}) = 0, \qquad \forall v \in \{1, ..., r\}\,.$$

By the Morse inequality (cf. corollary A.6) of the action functional $\mathscr{A}_R^{[n]}$ in the a-sublevel, i.e.

$$\dim \mathrm{H}_m((\mathscr{A}_R^{[n]})_a) \leq \sum_{v=1}^{r} \dim \mathrm{H}_m(\mathscr{A}_R^{[n]}, \gamma_v^{[n]}),$$

we readily obtain that $\mathrm{H}_m((\mathscr{A}_R^{[n]})_a) = 0$, contradicting (V.13).

Hence we can assume that $\gamma_1, ..., \gamma_s$, with $1 \leq s \leq r$, are periodic solutions with mean Morse index equal to zero, while $\gamma_{s+1}, ..., \gamma_r$ (if $s < r$) are the ones with strictly positive mean Morse index. By the iteration inequality for the Morse index and nullity pair we have

$$\iota(\mathscr{A}^{[n]}, \gamma_v^{[n]}) + \nu(\mathscr{A}^{[n]}, \gamma_v^{[n]}) \leq m, \qquad \forall n \in \mathbb{K},\ v \in \{1, ..., s\}\,.$$

[6]See the discussion in section V.1 about the Morse index and nullity pair and the mean Morse index of the Tonelli action.

In particular

$$\iota(\mathscr{A}^{[n]},\gamma_v^{[n]}),\nu(\mathscr{A}^{[n]},\gamma_v^{[n]})\in\{0,...,m\}\,,\qquad \forall n\in\mathbb{K},\ v\in\{1,...,s\}\,,$$

and therefore we can find an infinite subset $\mathbb{K}'\subseteq\mathbb{K}$ such that

$$\Big(\iota(\mathscr{A}^{[n]},\gamma_v^{[n]}),\nu(\mathscr{A}^{[n]},\gamma_v^{[n]})\Big)=\Big(\iota(\mathscr{A}^{[n']},\gamma_v^{[n']}),\nu(\mathscr{A}^{[n']},\gamma_v^{[n']})\Big)\,,\qquad \forall n,n'\in\mathbb{K}',\ v\in\{1,...,s\}\,.$$

For each $n,n'\in\mathbb{K}'$ with $n<n'$ and for each $R>0$ big enough (namely, $R>|\dot\gamma_v(t)|_{\gamma_v(t)}$ for each $t\in\mathbb{R}/\mathbb{Z}$ and $v\in\{1,...,s\}$), corollary IV.9 guarantees that the iteration map $\psi^{[n'/n]}$ induces the homology isomorphism

$$\text{(V.15)}\qquad \mathrm{H}_*(\psi^{[n'/n]}):\mathrm{H}_*(\mathscr{A}_R^{[n]},\gamma_v^{[n]})\xrightarrow{\simeq}\mathrm{H}_*(\mathscr{A}_R^{[n']},\gamma_v^{[n']}),\qquad \forall v\in\{1,...,s\}\,.$$

If $s<r$, by the iteration inequality for the Morse index and nullity there exists $n\in\mathbb{K}'$ big enough so that the periodic solutions $\gamma_{s+1},...,\gamma_r$ with strictly positive mean Morse index satisfy

$$\iota(\mathscr{A}^{[n]},\gamma_v^{[n]})\geq n\,\widehat{\iota}(\mathscr{A},\gamma_v)-m>m,\qquad \forall v\in\{s+1,...r\}\,.$$

This implies (cf. corollary III.18) that, for $R>0$ big enough (namely, $R>|\dot\gamma_v(t)|_{\gamma_v(t)}$ for each $t\in\mathbb{R}/\mathbb{Z}$ and $v\in\{s+1,...,r\}$), we have

$$\mathrm{H}_m(\mathscr{A}_R^{[n]},\gamma_v^{[n]})=0,\qquad \forall v\in\{s+1,...,r\}\,.$$

If $s=r$ we just set $n=1$. If $\tilde R(a,n)$ is the constant given by lemma V.2 and $R>\tilde R(a,n)$, the Morse inequality

$$0\neq\dim\,\mathrm{H}_m((\mathscr{A}_R^{[n]})_a)\leq\sum_{v=1}^{r}\dim\,\mathrm{H}_m(\mathscr{A}_R^{[n]},\gamma_v^{[n]})=\sum_{v=1}^{s}\dim\,\mathrm{H}_m(\mathscr{A}_R^{[n]},\gamma_v^{[n]})$$

implies that there is a $\gamma\in\{\gamma_1,...,\gamma_s\}$ such that

$$\mathrm{H}_m(\mathscr{A}_R^{[n]},\gamma^{[n]})\neq 0.$$

At this point, let us assume without loss of generality that $1\in\mathbb{K}'$ (this can be achieved by time-rescaling, as we discussed above at the beginning of the proof). Hence, (V.15) can be more easily expressed for γ as

$$\text{(V.16)}\qquad \mathrm{H}_*(\psi^{[n]}):\mathrm{H}_*(\mathscr{A}_R,\gamma)\xrightarrow{\simeq}\mathrm{H}_*(\mathscr{A}_R^{[n]},\gamma^{[n]})\neq 0,\qquad \forall n\in\mathbb{K}'.$$

Now, we apply the discretization technique of section V.2: choosing $U>0$ as in (V.2), we obtain an embedding $\lambda_k^{[n]}:U_k^{[n]}\hookrightarrow W^{1,2}(\mathbb{T}^{[n]};M)$ and the discrete mean Tonelli action functional $\mathscr{A}_k^{[n]}=\mathscr{A}^{[n]}\circ\lambda_k:U_k^{[n]}\to\mathbb{R}$, for $k\in\mathbb{N}$ sufficiently big and for each $n \in \mathbb{N}$. Let $\boldsymbol{q} := \lambda_k^{-1}(\gamma)$. For each $R \geq U$, the homology isomorphism induced by the iteration map in (V.16) fits into the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q}) & \xrightarrow{\mathrm{H}_*(\psi_k^{[n]})} & \mathrm{H}_*(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]}) \\ {\scriptstyle \mathrm{H}_*(\lambda_k)} \downarrow \simeq & & \simeq \downarrow {\scriptstyle \mathrm{H}_*(\lambda_k^{[n]})} \\ \mathrm{H}_*(\mathscr{A}_R, \gamma) & \xrightarrow[\simeq]{\mathrm{H}_*(\psi^{[n]})} & \mathrm{H}_*(\mathscr{A}_R^{[n]}, \gamma^{[n]}) \end{array}$$

showing that $\mathrm{H}_*(\psi_k^{[n]})$ is an isomorphism.

Let $c = \mathscr{A}(\gamma)$ and let $\varepsilon > 0$ be small enough so that $c + \varepsilon < a$ and there are no $\gamma_v \in \{\gamma_1, ..., \gamma_r\}$ with $\mathscr{A}(\gamma_v) \in (c, c+\varepsilon)$. By lemma V.2, for all $n \in \mathbb{K}'$ and $R > \tilde{R}(a,n)$ the action functional $\mathscr{A}_R^{[n]}$ does not have any critical point with critical value in $(c, c+\varepsilon)$. By theorem A.4(i), the inclusion

$$((\mathscr{A}_R^{[n]})_c \cup \{\gamma\}, (\mathscr{A}_R^{[n]})_c) \hookrightarrow ((\mathscr{A}_R^{[n]})_{c+\varepsilon}, (\mathscr{A}_R^{[n]})_c)$$

induces a monomorphism in homology. Hence, the embedding $\lambda_k^{[n]}$, seen as a map

$$\lambda_k^{[n]} : ((\mathscr{A}_k^{[n]})_c \cup \{\boldsymbol{q}\}, (\mathscr{A}_k^{[n]})_c) \hookrightarrow ((\mathscr{A}_R^{[n]})_{c+\varepsilon}, (\mathscr{A}_R^{[n]})_c),$$

induces a monomorphism in homology as well. Summing up, for each $R \geq \tilde{R}(a,n)$ and $n \in \mathbb{K}'$, we have obtained the following commutative diagram.

$$\begin{array}{ccc} 0 \neq \mathrm{H}_m(\mathscr{A}_k, \boldsymbol{q}) & \xrightarrow[\simeq]{\mathrm{H}_m(\psi_k^{[n]})} & \mathrm{H}_m(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]}) \\ {\scriptstyle \mathrm{H}_m(\lambda_k)} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{H}_m(\lambda_k^{[n]})} \\ \mathrm{H}_m((\mathscr{A}_R)_{c+\varepsilon}, (\mathscr{A}_R)_c) & \xrightarrow{\mathrm{H}_m(\psi^{[n]})} & \mathrm{H}_m((\mathscr{A}_R^{[n]})_{c+\varepsilon}, (\mathscr{A}_R^{[n]})_c) \end{array}$$

This diagram contradicts the homological vanishing (theorem V.5). In fact, since the local homology group $\mathrm{H}_m(\mathscr{A}_k, \boldsymbol{q})$ is nontrivial and $m > 0$, the point $\boldsymbol{q}$ is not a local minimum of $\mathscr{A}_k$. For each nonzero $[\eta] \in \mathrm{H}_m(\mathscr{A}_k, \boldsymbol{q})$, there exist $\bar{R} = \bar{R}([\eta], p) \geq U$ and $\bar{n} = \bar{n}([\eta], p) \in \mathbb{K}$ such that, for each real $R \geq \bar{R}$ and for each $n \in \mathbb{K}$ greater or equal than $\bar{n}$, we have

$$\mathrm{H}_m(\psi^{[n]}) \circ \mathrm{H}_m(\lambda_k)[\eta] = \mathrm{H}_m(\psi^{[n/\bar{n}]}) \circ \underbrace{\mathrm{H}_m(\psi^{[\bar{n}]}) \circ \mathrm{H}_m(\lambda_k)[\eta]}_{=0} = 0,$$

therefore

$$\mathrm{H}_m(\psi_k^{[n]})[\eta] \in \ker \left[\mathrm{H}_m(\lambda_k^{[n]}) : \mathrm{H}_m(\mathscr{A}_k^{[n]}, \boldsymbol{q}^{[n]}) \to \mathrm{H}_m((\mathscr{A}_R^{[n]})_{c+\varepsilon}, (\mathscr{A}_R^{[n]})_c) \right]. \quad ✿$$

Appendix A

An overview of Morse theory

Morse theory is a beautiful subject that sits between differential geometry, topology and calculus of variations. It was started by Marston Morse[1] in the middle 1920s and further developed, among many others, by Thom, Bott, Milnor, Palais, Smale, Gromoll, Meyer, Witten and Floer. The general philosophy of the theory is that the topology of a smooth manifold is related in a very particular way to the number and "type" of critical points that a smooth functional defined over it can have. In this brief appendix we would like to give an overview of the topic, from the classical[2] point of view of Morse, but in the more recent extensions of Palais, Gromoll and Meyer that allow the theory to deal with so called degenerate functionals on infinite dimensional manifolds. A full treatment of the subject can be found in the first chapter of the book of Chang [Ch].

We will try to keep our exposition as elementary as possible. To this aim, we will renounce to give the results in their maximal generality whenever this saves us from technicalities. Nevertheless, in view of the application of this machinery to study the critical point of the action functional of Lagrangian dynamics, we insist on the regularity that the functional under consideration must have, that will be mostly C^1, and occasionally C^2.

A.1 Preliminaries

Throughout this appendix we will denote by $\mathscr{M}$ a Hilbert manifold, i.e. a paracompact Hausdorff topological space that is locally homeomorphic to a real separable

[1] *Relations between the critical points of a real function on n independent variables*, [Mo].

[2] There is also another, more recent, approach to the theory, that is based on the so called Morse complex. It was pioneered by René Thom [Th] in 1949 and further developed by Steve Smale [Sm] to solve the Poincaré conjecture in dimensions greater than four (see the beautiful book of Milnor [Mi2] for an account of that stage of the theory). The definition of Morse complex appeared in 1981 on a paper by Edward Witten [Wi]. See the book of Schwarz [Sc], the one of Banyaga and Hurtubise [BH] or the survey of Abbondandolo and Majer [AM] for a modern treatment.

Hilbert space $\boldsymbol{E}$ with smooth change of charts. If $\boldsymbol{E}$ is finite dimensional, i.e. $\boldsymbol{E} = \mathbb{R}^m$ for some $m \in \mathbb{N}$, then $\mathscr{M} = M$ is just an ordinary m dimensional smooth manifold. In the context of Morse theory, the most relevant difference between the finite dimensional and the infinite dimensional cases is that in this latter the manifold $\mathscr{M}$ is not locally compact. We will come back to this point later on.

Remark A.1. From elementary functional analysis it is well known that every real separable Hilbert space with infinite dimension is isomorphic to the Hilbert space

$$\ell^2 = \left\{ \{a_n \,|\, n \in \mathbb{N}\} \,\middle|\, \sum_{n\in\mathbb{N}} |a_n|^2 < \infty \right\}$$

with interior product given by

$$\langle \{a_n\}, \{b_n\} \rangle_{\ell^2} = \sum_{n\in\mathbb{N}} a_n b_n.$$

■

From now on, we use calligraphic letters ($\mathscr{M}$, $\mathscr{N}$, $\mathscr{U}$ and so on) to denote possibly infinite dimensional manifolds, while we leave the roman ones (M, N, U and so on) when the object under consideration must have finite dimension for the current argument to be valid.

Throughout this appendix, $\mathscr{F} : \mathscr{M} \to \mathbb{R}$ will be a C^1 functional (we will write $F : M \to \mathbb{R}$ in the finite dimensional case), unless more regularity will be explicitly required. We recall that a point $p \in \mathscr{M}$ is called a **critical point** of $\mathscr{F}$ when the differential $\mathrm{d}\mathscr{F}(p)$ of $\mathscr{F}$ calculated at p vanishes, and the correspondent image $\mathscr{F}(p)$ is called a **critical value**. We denote by $\mathrm{Crit}\mathscr{F}$ the set of critical points of $\mathscr{F}$. In what follows, we will only deal with functionals $\mathscr{F}$ having isolated critical points, and we will investigate the relationship between this critical points and the topological properties (or, more precisely, the homological properties) of the underlying manifold $\mathscr{M}$.

Consider an open neighborhood $\mathscr{U}$ of the critical point p such that there exists a chart $\phi : \mathscr{U} \to \boldsymbol{E}$ of $\mathscr{M}$. We denote by $\mathscr{F}_\phi$ the functional $\mathscr{F} \circ \phi^{-1} : \boldsymbol{E} \to \mathbb{R}$. By the composition rule for the differential, we have

$$\mathrm{d}\mathscr{F}_\phi(\phi(p)) = \mathrm{d}\mathscr{F}(p) \circ \mathrm{d}\phi^{-1}(\phi(p)),$$

where d in the left hand side denotes the Fréchet derivative on the Hilbert space $\boldsymbol{E}$, i.e. $\mathrm{d}\mathscr{F}_\phi : \boldsymbol{E} \to \boldsymbol{E}^*$. Therefore $\phi(p)$ is a critical point of the functional $\mathscr{F}_\phi$. If $\mathscr{F}$ is C^2 or at least twice Gateaux differentiable, the same is true for $\mathscr{F}_\phi$. We recall that, in these cases, the second Gateaux derivative of $\mathscr{F}_\phi$ at $\boldsymbol{x}$ can be seen as a symmetric bounded bilinear form $\mathrm{Hess}\mathscr{F}_\phi(\boldsymbol{x}) : \boldsymbol{E} \otimes \boldsymbol{E} \to \mathbb{R}$ given by

$$\mathrm{Hess}\mathscr{F}_\phi(\boldsymbol{x})[\boldsymbol{v}, \boldsymbol{w}] = (\mathrm{d}(\mathrm{d}\mathscr{F}_\phi)(\boldsymbol{x})\boldsymbol{v})\, \boldsymbol{w}, \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{E}.$$

In the above formula, we have denoted by $\mathrm{d}(\mathrm{d}\mathscr{F}_\phi)$ the Gateaux derivative of $\mathrm{d}\mathscr{F}_\phi$, i.e.

$$\mathrm{d}(\mathrm{d}\mathscr{F}_\phi)(\boldsymbol{x})\boldsymbol{v} = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{d}\mathscr{F}_\phi(\boldsymbol{x} + \varepsilon\boldsymbol{v}) \qquad \forall \boldsymbol{x}, \boldsymbol{v} \in \boldsymbol{E}.$$

This latter coincides with the Fréchet derivative of $\mathrm{d}\mathscr{F}_\phi$ in case $\mathscr{F}$ is C^2. We define the **Hessian** of $\mathscr{F}$ at a critical point p as the symmetric bilinear form

$$\mathrm{Hess}\mathscr{F}(p) : \mathrm{T}_p\mathscr{M} \otimes \mathrm{T}_p\mathscr{M} \to \mathbb{R}$$

given by

$$\mathrm{Hess}\mathscr{F}(p)[v,w] = \mathrm{Hess}\mathscr{F}_\phi(\boldsymbol{x})[\boldsymbol{v}_\phi, \boldsymbol{w}_\phi], \qquad \forall v, w \in \mathrm{T}_p\mathscr{M} \simeq \boldsymbol{E}$$

where $\boldsymbol{x} = \phi(p)$ and $\boldsymbol{v}_\phi = \mathrm{d}\phi(p)v, \boldsymbol{w}_\phi = \mathrm{d}\phi(p)w \in \boldsymbol{E}$. It is easy to see that $\mathrm{Hess}\mathscr{F}(p)$ is intrinsically defined, i.e. its definition is independent of the chosen chart ϕ as long as p is a critical point of $\mathscr{F}$.

Remark A.2. If $\mathscr{M} = M$ is finite dimensional and $\boldsymbol{x} = (x^1, ..., x^m)$ is a system of local coordinates around p, the Hessian of $\mathscr{F} = F$ at the critical point p is given in local coordinates by

$$\mathrm{Hess}F(p) = \left(\frac{\partial^2 F}{\partial x^i \, \partial x^j} \, \mathrm{d}x^i \otimes \mathrm{d}x^j \right)(p),$$

where we implicitly assume summations[3] from 1 to m for the repeated indices i and j. Moreover, if ∇ is any linear connection on M, the Hessian of F at p is given by

$$\mathrm{Hess}F(p) = \nabla(\mathrm{d}F)(p).$$ ■

We denote by $\iota(\mathscr{F}, p)$ the supremum of the dimensions of subspaces of $\mathrm{T}_p\mathscr{M}$ in which the Hessian of $\mathscr{F}$ at p is negative definite. Analogously, we denote by $\nu(\mathscr{F}, p)$ the dimension of the nullspace of the Hessian of $\mathscr{F}$ at p, i.e. the Hilbert space consisting of all $v \in \mathrm{T}_p\mathscr{M}$ such that $\mathrm{Hess}\mathscr{F}(p)[v,w] = 0$ for all $w \in \mathrm{T}_p\mathscr{M}$. We call $\iota(\mathscr{F}, p)$ and $\nu(\mathscr{F}, p)$ respectively **Morse index** and **nullity** of the functional $\mathscr{F}$ at p. Notice that both may be infinite. Morse Theory was initially developed for so called **Morse functionals**, that are functionals whose critical points have nullity equal to zero and for this reason are called **non-degenerate**. Today we are able to deal with functional having possibly **degenerate** critical points.

Since the inner product $\langle \cdot, \cdot \rangle_{\boldsymbol{E}}$ of $\boldsymbol{E}$ is a nondegenerate bilinear form, there exists a self-adjoint linear continuous operator $H_\phi = H_\phi(p) : \boldsymbol{E} \to \boldsymbol{E}$ such that

$$\mathrm{Hess}\mathscr{F}(p)[v,w] = \langle H_\phi \boldsymbol{v}_\phi, \boldsymbol{w}_\phi \rangle_{\boldsymbol{E}}, \qquad \forall v, w \in \mathrm{T}_p\mathscr{M}$$

By the spectral theorem, this operator induces an orthogonal splitting

$$\boldsymbol{E} = \boldsymbol{E}^0_\phi \oplus \boldsymbol{E}^+_\phi \oplus \boldsymbol{E}^-_\phi,$$

where $\boldsymbol{E}^0_\phi$ is the kernel of H_ϕ, and $\boldsymbol{E}^-_\phi$ [resp. $\boldsymbol{E}^+_\phi$] is a subspace of $\boldsymbol{E}$ on which H_ϕ is negative definite [resp. positive definite], i.e.

$$\begin{aligned} &\boldsymbol{E}^0_\phi = \{\boldsymbol{v} \in \boldsymbol{E} \,|\, H_\phi \boldsymbol{v} = 0\}, \\ &\langle H_\phi \boldsymbol{v}, \boldsymbol{v} \rangle_{\boldsymbol{E}} < 0, \qquad && \forall \boldsymbol{v} \in \boldsymbol{E}^-_\phi \setminus \{\boldsymbol{0}\}, \\ &\langle H_\phi \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{E}} > 0, \qquad && \forall \boldsymbol{w} \in \boldsymbol{E}^+_\phi \setminus \{\boldsymbol{0}\}. \end{aligned}$$

Notice that $\iota(\mathscr{F}, p) = \dim \boldsymbol{E}^-_\phi$ and $\nu(\mathscr{F}, p) = \dim \boldsymbol{E}^0_\phi$.

[3]This is what differential geometers call "Einstein convention" on subscripts and superscripts.

A.2 The generalized Morse lemma

A starting point for Morse Theory might be the so called "Morse Lemma", that we will give here in its generalized form[4]. It is a local result, in the sense that it applies to functionals defined on an open set of a Hilbert space. Let $\mathbf{0}$ be an isolated critical point of a C^2 functional $\mathscr{F} : \mathscr{U} \to \mathbb{R}$, where $\mathscr{U}$ is an open set in a Hilbert space $\boldsymbol{E}$. We denote by $H : \boldsymbol{E} \to \boldsymbol{E}$ the self-adjoint linear operator associated to the second derivative of $\mathscr{F}$ at $\mathbf{0}$, i.e. $\mathrm{Hess}\mathscr{F}(\mathbf{0})[\boldsymbol{v}, \boldsymbol{w}] = \langle H\boldsymbol{v}, \boldsymbol{w}\rangle_{\boldsymbol{E}}$. As we remarked in the last section, this operator defines an orthogonal splitting $\boldsymbol{E}^0 \oplus \boldsymbol{E}^{\pm}$ of $\boldsymbol{E}$.

Lemma A.1 (Generalized Morse Lemma). *Assume that H is a Fredholm[5] operator (in particular $\nu(\mathscr{F}, \mathbf{0})$ is finite). Then, there exists an open neighborhood $\mathscr{V} \subseteq \mathscr{U}$ of $\mathbf{0}$, a homeomorphism onto its image*

$$\phi : (\mathscr{V}, \mathbf{0}) \to (\mathscr{U}, \mathbf{0})$$

and a C^1 map

$$\psi : (\mathscr{V} \cap \boldsymbol{E}^0, \mathbf{0}) \to (\boldsymbol{E}^{\pm}, \mathbf{0}),$$

such that, for each $\boldsymbol{v} \in \mathscr{V}$, we have

$$\mathscr{F} \circ \phi(\boldsymbol{v}) = \underbrace{\mathscr{F}\left(\boldsymbol{v}^0 + \psi(\boldsymbol{v}^0)\right)}_{=:\mathscr{F}^0(\boldsymbol{v}^0)} + \underbrace{\tfrac{1}{2}\left\langle H\boldsymbol{v}^{\pm}, \boldsymbol{v}^{\pm}\right\rangle_{\boldsymbol{E}}}_{=:\mathscr{F}^{\pm}(\boldsymbol{v}^{\pm})} \tag{A.1}$$

where we write $\boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^{\pm}$ according to the splitting $\boldsymbol{E} = \boldsymbol{E}^0 \oplus \boldsymbol{E}^{\pm}$. Moreover $\mathbf{0}$ is a critical point of both $\mathscr{F}^{\pm}$ and $\mathscr{F}^0$.

Notice that, in case the functional $\mathscr{F}$ happens to be Morse (i.e. $\nu(\mathscr{F}, \mathbf{0}) = 0$), equation (A.1) simplifies to

$$\mathscr{F} \circ \phi(\boldsymbol{v}) = \mathscr{F}(\mathbf{0}) + \mathscr{F}^{\pm}(\boldsymbol{v}) = \mathscr{F}(\mathbf{0}) + \frac{1}{2}\langle H\boldsymbol{v}, \boldsymbol{v}\rangle_{\boldsymbol{E}}.$$

This is a fundamental rigidity result for functionals around a non-degenerate critical point: it states that, up to a local "reparametrization" of the ambient space, every functional around a non-degenerate critical point is given by a quadratic form (plus a constant).

In the general case, equation (A.1) gives us a local representation of a functional $\mathscr{F}$ as a sum of a Morse function $\mathscr{F}^{\pm}$ and of a function $\mathscr{F}^0$. In this representation, the origin $\mathbf{0}$ is a non-degenerate critical point for $\mathscr{F}^{\pm}$ and a **fully degenerate** critical point for $\mathscr{F}^0$, i.e. the second derivative of $\mathscr{F}^0$ at $\mathbf{0}$ vanishes.

[4] The original Morse Lemma goes back to Marston Morse, and it was generalized to infinite dimensional manifolds by Palais [Pa]. The version that we give here, that admits possibly degenerate functionals, is basically due to Gromoll and Meyer [GM]. In the literature, the generalized Morse lemma is also known as **splitting lemma**.

[5] We recall that a continuous linear operator $H : \boldsymbol{E} \to \boldsymbol{E}$ is **Fredholm** when $H(\boldsymbol{E})$ is closed and H has finite dimensional kernel and cokernel.

A.3 Deformation of sublevels

At this point, let us recall some standard terminology from topology. Two continuous maps f_0 and f_1 from a topological space X to another topological space Y are **homotopic**, and we write it as $f_0 \sim f_1$, when there exists a continuous map $f : [0,1] \times X \to Y$, called **homotopy**, such that $f_0 = f(0,\cdot)$ and $f_1 = f(1,\cdot)$. If we consider the two maps as maps of topological pairs $f_1, f_2 : (X,A) \to (Y,B)$, i.e. $A \subseteq X$, $B \subseteq Y$ and $f(A) \subseteq B$, then they are homotopic (as maps of pairs) when there exists a homotopy f as before that further satisfies $f(t,A) \subseteq B$ for all $t \in [0,1]$, and we write it as $f : [0,1] \times (X,A) \to (Y,B)$ and still $f_0 \sim f_1$.

A homotopy $f : [0,1] \times X \to Y$ is said **relative** C when C is a subspace of X and $f(t,x) = f(0,x)$ for all $x \in X$ and for all $t \in [0,1]$. The same definition, *mutatis mutandis*, can be given for homotopies of maps of pairs.

A map $j : X \to Y$ is a **homotopy equivalence**, and we write it as

$$j : X \xrightarrow{\sim} Y$$

or $X \sim Y$ if j is implicit from the context, when there exists a map $l : Y \to X$, called **homotopy inverse**, such that $l \circ j \sim \mathrm{id}_X$ and $j \circ l \sim \mathrm{id}_Y$. If j is a map of topological pairs $(X,A) \to (Y,B)$ we say that it is a homotopy equivalence (of pairs) when the homotopy inverse is of the form $l : (Y,B) \to (X,A)$ and the homotopies with the identity are homotopies of maps of pairs.

Now, assume that we have an inclusion $\iota : X \hookrightarrow Y$ and a **retraction** $r : Y \to X$, i.e a surjective map r such that the restriction $r|_X$ is equal to the identity id_X. If there exists a homotopy $R : [0,1] \times (Y,X) \to (Y,X)$ such that $R(0,\cdot) = \mathrm{id}_Y$ and $R(1,\cdot) = r$, then we say that Y **deformation retracts** onto X and we call the homotopy R a **deformation retraction**. Notice that R is assumed to be a homotopy of maps of pairs, therefore $R(t,X) \subseteq X$ for all $t \in [0,1]$ and the inclusion ι turns out to be a homotopy equivalence. The deformation retraction R is called **strong** if we further assume that it is relative X, i.e. $R(t,x) = x$ for all $(t,x) \in [0,1] \times X$.

After this excursion, let us go back to Morse Theory. So far, we have just discussed local aspects, but Morse Theory allows us to say something about global properties of the ambient manifold $\mathscr{M}$. To start with, notice that a functional $\mathscr{F} : \mathscr{M} \to \mathbb{R}$ defines a "filtration" of the manifold $\mathscr{M}$. In fact, if $c \in \mathbb{R}$, let us denote by $(\mathscr{F})_c$ the open subspace $\mathscr{F}^{-1}(-\infty, c)$, called c-**sublevel** of $\mathscr{F}$. Then if $\{c_n \,|\, n \in \mathbb{N}\}$ is a monotone increasing sequence of real numbers tending to infinity, then we have the sequence of inclusions

$$(\mathscr{F})_{c_1} \subseteq (\mathscr{F})_{c_2} \subseteq (\mathscr{F})_{c_3} \subseteq \ldots \subseteq \mathscr{M}. \tag{A.2}$$

We may also define $(\mathscr{F})_{-\infty} = \varnothing$ and $(\mathscr{F})_{+\infty} = \mathscr{M}$. Having this filtration, we would like to investigate the relation between the homology of pairs of sublevels $((\mathscr{F})_b, (\mathscr{F})_a)$ and the critical points of $\mathscr{F}$ contained in the region $\mathscr{F}^{-1}(a,b)$, at least when the interval (a,b) contains a single critical value. Then, by standard algebraic topological manipulations, we would like to conclude something about the homology of pairs $((\mathscr{F})_b, (\mathscr{F})_a)$ for a and b arbitrarily distant.

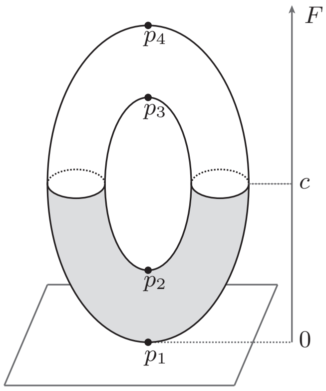


Figure A.1. Height functional on the 2-torus in $\mathbb{R}^3$. The shaded region is the sublevel $(F)_c$, for $F(p_2) < c < F(p_3)$.

Example A.1. From now on, it is useful to keep in mind a classical finite dimensional example: a torus $\mathscr{M} = M = \mathbb{T}^2$ in $\mathbb{R}^3$ sitting on a plane as shown in figure A.1. On this torus we consider as functional $\mathscr{F} = F$ the height, i.e. $F(p)$ is the height of $p \in \mathbb{T}^2$ above the plane. For simplicity, we will assume that F is a Morse functional, therefore its four critical points p_1, p_2, p_3 and p_4 have indices 0, 1, 1 and 2 respectively. This example will be useful to geometrically visualize some of the operations that we will perform, but of course our general setting is more complicated since $\mathscr{F}$ has possibly degenerate critical points and the ambient manifold $\mathscr{M}$ is not locally compact. ■

At this point, let us consider a **Hilbert-Riemannian metric** $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\cdot$ on $\mathscr{M}$, i.e. a smooth section of the bundle $\mathrm{T}^*\mathscr{M} \otimes \mathrm{T}^*\mathscr{M}$ that is symmetric and positive definite, meaning

$$\langle\!\langle v, w \rangle\!\rangle_q = \langle\!\langle w, v \rangle\!\rangle_q, \quad \langle\!\langle z, z \rangle\!\rangle_q > 0 \quad \forall q \in \mathscr{M},\ v, w, z \in \mathrm{T}_q\mathscr{M},\ z \neq 0.$$

In local coordinates given by a chart $\phi : \mathscr{U} \to \boldsymbol{E}$, this metric can be expressed in terms of the inner product of $\boldsymbol{E}$ as

$$\langle\!\langle v, w \rangle\!\rangle_q = \langle G_\phi(q) \boldsymbol{v}_\phi, \boldsymbol{w}_\phi \rangle_{\boldsymbol{E}}, \quad \forall q \in M,\ v, w \in \mathrm{T}_q\mathscr{M},$$

where G_ϕ is the unique map from $\mathscr{U}$ to the space of bounded self-adjoint operators on $\boldsymbol{E}$ that realizes this equality. If $\psi : \mathscr{V} \to \boldsymbol{E}$ is another chart of $\mathscr{M}$, then

$$G_\psi(q) = \big(\mathrm{d}(\phi \circ \psi^{-1})(\psi(q))\big)^* \circ G_\phi(q) \circ \mathrm{d}(\phi \circ \psi^{-1})(\psi(q)) \qquad \forall q \in \mathscr{U} \cap \mathscr{V}.$$

We denote by $\|\cdot\|_\cdot$ the **Hilbert-Finsler metric** obtained by $\|v\|_q = \langle\!\langle v, v \rangle\!\rangle_q$ for each $q \in \mathscr{M}$ and $v \in \mathrm{T}_q\mathscr{M}$. This metric induces, as usual, a corresponding Hilbert-Finsler metric (still denoted by $\|\cdot\|_\cdot$) on the cotangent bundle $\mathrm{T}^*\mathscr{M}$ as

$$\|\nu\|_q = \max\{\nu(v) \,|\, v \in \mathrm{T}_q\mathscr{M},\ \|v\|_q = 1\}, \qquad \forall q \in \mathscr{M},\ \nu \in \mathrm{T}^*_q\mathscr{M}.$$

For each C^1 curve $\sigma : [a,b] \to \mathscr{M}$, we define its **length** (with respect to the Hilbert-Finsler metric $\|\cdot\|_{\cdot}$) as

$$\int_a^b \|\dot\sigma(t)\|_{\sigma(t)}\,\mathrm{d}t.$$

If q and q' are two points that belongs to the same connected component, we can define their **distance** (again, with respect to the Hilbert-Finsler metric $\|\cdot\|_{\cdot}$) as the infimum of the lengths of all C^1 curves joining q and q'. If each component of $\mathscr{M}$ is a complete metric space with respect to this distance, we say that $\mathscr{M}$ is a **complete** Hilbert-Riemannian manifold.

A C^1 vector field $\mathscr{X}$ on $\mathscr{M} \setminus \mathrm{Crit}(\mathscr{F})$ is called a **pseudo-gradient** for $\mathscr{F}$ when, for each $q \in \mathscr{M}$, it satisfies

$$\|\mathscr{X}(q)\|_q \leq 2\|\mathrm{d}\mathscr{F}(q)\|_q, \qquad \mathrm{d}\mathscr{F}(q)\,\mathscr{X}(q) \geq \|\mathrm{d}\mathscr{F}(q)\|_q^2.$$

By means of a partition of unity, one can show that pseudo-gradient vector fields always exist on Hilbert manifolds. Integrating $\mathscr{X}$ we obtain its **(anti) pseudo-gradient flow**, that is a map

$$\Phi_{\mathscr{X}} : \mathscr{W} \to \mathscr{M},$$

where $\mathscr{W} \subseteq \mathbb{R} \times \mathscr{M} \setminus \mathrm{Crit}(\mathscr{F})$ is an open neighborhood of $\{0\} \times \mathscr{M}$, satisfying the following Cauchy problem:

$$\frac{\partial \Phi_{\mathscr{X}}}{\partial t}(t,q) = -\mathscr{X}(\Phi_{\mathscr{X}}(t,q)), \qquad \Phi_{\mathscr{X}}(0,\cdot) = \mathrm{id}_{\mathscr{M}}.$$

It is easy to verify that the functional $\mathscr{F}$ is decreasing along pseudo-gradient flow lines. In fact, for every $(t,q) \in \mathscr{W}$, we have

$$\begin{aligned}\mathscr{F}(\Phi_{\mathscr{X}}(t,q)) - \mathscr{F}(q) &= -\int_0^t \mathrm{d}\mathscr{F}(\Phi_{\mathscr{X}}(s,q))\,\mathscr{X}(\Phi_{\mathscr{X}}(s,q))\,\mathrm{d}s \\ &\leq -\int_0^t \underbrace{\|\mathrm{d}\mathscr{F}(\Phi_{\mathscr{X}}(s,q))\|^2_{\Phi_{\mathscr{X}}(s,q)}}_{>0}\,\mathrm{d}s \\ &< 0.\end{aligned}$$

Example A.2. If $\mathscr{F}$ is C^2, a pseudo-gradient is given by the **gradient** of $\mathscr{F}$, that is the vector field $\mathrm{Grad}\mathscr{F}$ defined by

$$\langle\!\langle \mathrm{Grad}\mathscr{F}(q), v \rangle\!\rangle_q = \mathrm{d}\mathscr{F}(q)v, \quad \forall q \in \mathscr{M},\ v \in \mathrm{T}_q\mathscr{M}.$$ ■

We would like to use the pseudo-gradient flow to deform a certain sublevel $(\mathscr{F})_{c_2}$ to a lower one, say $(\mathscr{F})_{c_1}$, for some $c_1 < c_2$ such that the interval $[c_1, c_2)$ does not contain critical values (see figure A.2). In case $\mathscr{M} = M$ is compact there are no obtacles in performing such an operation. However, if we just deal with a complete but non-compact manifold, something is needed to replace the lack of compactness.

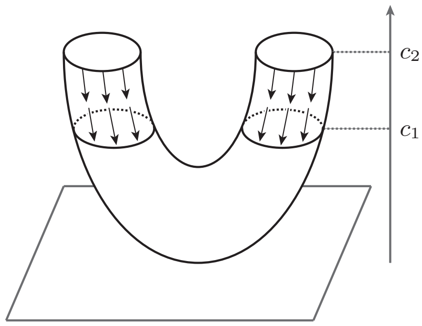


Figure A.2. Deformation of $(F)_{c_2}$ over $(F)_{c_1}$ along gradient flow lines in the torus example A.1.

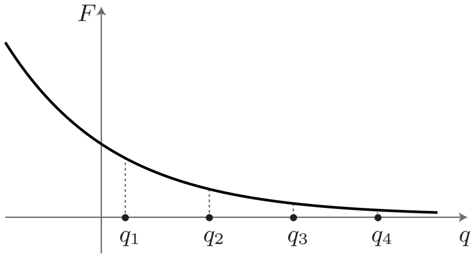


Figure A.3.

The right assumption on $\mathscr{F}$ was found in the 1960s by Palais and Smale[6], and it now carries their name: we say that $\mathscr{F}$ satisfyes the **Palais-Smale condition** at level c when, for each sequence $\{q_n \,|\, n \in \mathbb{N}\} \subseteq \mathscr{M}$ such that

$$\begin{aligned} &\lim_{n\to\infty} \mathscr{F}(q_n) = c, \\ &\lim_{n\to\infty} \|\mathrm{d}\mathscr{F}(q_n)\|_{q_n} = 0, \end{aligned} \tag{A.3}$$

there exists a subsequence convergent to a point $q \in \mathscr{M}$. By (A.3), the limit point q must be a critical point of $\mathscr{F}$. We just say that $\mathscr{F}$ satisfies the Palais-Smale condition if it satisfies it at every level $c \in \mathbb{R}$.

Example A.3. There are basically two slightly different examples of situations that the Palais-Smale condition wants to avoid.

- Consider the function $F(q) = \exp(-q)$ on $\mathscr{M} = M = \mathbb{R}$. For any divergent sequence $q_n \uparrow \infty$ we have that $F(q_n) \to 0$ and $F'(q_n) \to 0$, however $\{q_n\}$ does not admit any converging subsequence (see figure A.3).
- Consider a functional $F : \mathscr{M} \to \mathbb{R}$ such that, for a certain level $c \in \mathbb{R}$, the set $\mathrm{Crit}F \cap F^{-1}(c)$ is not compact (e.g. $F : \mathbb{R} \to \mathbb{R}$ with $F(q) = \sin(q)$ and

[6] *A generalized Morse theory*, [PS].

$c = 1$). Therefore we can find a sequence of critical points $\{q_n\}$ that does not admit any converging subsequence. ■

As promised, here is the important consequence of the Palais-Smale condition.

Lemma A.2 (Deformation Lemma). *Let $\mathscr{M}$ be a complete Hilbert-Riemannian manifold and assume that $\mathscr{F}$ satisfies the Palais-Smale condition at every level $c \in [a, b]$ and does not have any critical value in the interval $[a, b)$. Then the inclusion $(\mathscr{F})_a \hookrightarrow (\mathscr{F})_b$ is a homotopy equivalence.*

Notice that, by this lemma, the inclusion $(\mathscr{F})_a \hookrightarrow (\mathscr{F})_b$ induces the homology isomorphism[7] $\mathrm{H}_*((\mathscr{F})_a) \xrightarrow{\simeq} \mathrm{H}_*((\mathscr{F})_b)$ and $\mathrm{H}_*((\mathscr{F})_b, (\mathscr{F})_a) = 0$.

A.4 Passing a critical level

We now want to study the changes that occur, in term of homology, whenever we pass a critical level. To this aim, we have to introduce a fundamental invariant of an isolated critical point $p \in \mathscr{M}$ of our C^1 functional $\mathscr{F} : \mathscr{M} \to \mathbb{R}$. The **local homology** of $\mathscr{F}$ at p is defined as the homology group

$$\mathrm{H}_*(\mathscr{F}, p) := \mathrm{H}_*((\mathscr{F})_c \cup \{p\}, (\mathscr{F})_c),$$

where $c = \mathscr{F}(p)$. This is an invariant of the critical point p, in the sense that it depends only on the germ of $\mathscr{F}$ at p. In fact, if $\mathscr{U} \subseteq \mathscr{M}$ is an open neighborhood of p, by excision we obtain that the inclusion

$$(\mathscr{U} \cap (\mathscr{F})_c \cup \{p\}, \mathscr{U} \cap (\mathscr{F})_c) \hookrightarrow ((\mathscr{F})_c \cup \{p\}, (\mathscr{F})_c)$$

induces an isomorphism in homology, and therefore the local homology groups of $\mathscr{F}$ at p coincides with

$$\mathrm{H}_*(\mathscr{U} \cap (\mathscr{F})_c \cup \{p\}, \mathscr{U} \cap (\mathscr{F})_c).$$

Remark A.3. It is easy to see that the local homology groups of $\mathscr{F}$ at p, with $c = \mathscr{F}(p)$, can be equivalently defined as

$$\mathrm{H}_* \left(\overline{(\mathscr{F})}_c, \overline{(\mathscr{F})}_c \setminus \{p\} \right).$$

Moreover, if $\mathscr{U} \subseteq \mathscr{M}$ is an open neighborhood of p that does not contain other critical points of $\mathscr{F}$, the local homology groups of $\mathscr{F}$ at p further coincide with

$$\mathrm{H}_* \left(\mathscr{U} \cap \overline{(\mathscr{F})}_c, \mathscr{U} \cap (\mathscr{F})_c \right).$$ ■

[7]The same is true if we substitute the singular homology with any other homotopy invariant functor: equivariant singular homology (in an equivariant situation), K-theory, etc.

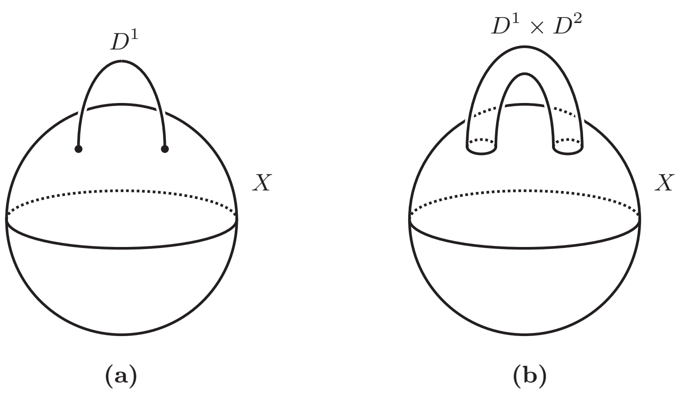


Figure A.4. **(a)** Attaching a 1-cell D^1 to $X = D^3$. **(b)** Attaching a thickened 1-cell $D^1 \times D^2$ to $X = D^3$. Notice that the interior of the obtained space is a smooth 3-manifold, but the boundary is not smooth anymore.

The role that local homology plays in Morse Theory is illustrated in the next two statements. We recall that a **λ-cell**, for $\lambda \in \mathbb{N}$, is simply a λ-dimensional closed disk D^λ. A topological space Y is obtained by the attachment of λ-cell to X when there exists a continuous map $f : \partial D^\lambda \to X$ such that

$$Y = X \cup_f D^\lambda := X \cup D^\lambda / \sim,$$

where $\sim$ is the identification given by $z \sim f(z)$ for each $z \in \partial D^\lambda$, see figure A.4(a). If X happens to be a smooth finite dimensional manifold with boundary and f maps ∂D^λ homeomorphically to a subset of ∂X, we can obtain another smooth manifold $\tilde{Y}$ (with non-smooth boundary) by attaching a so called **thickened λ-cell** $D^\lambda \times D^{\dim X - \lambda}$ as described in figure A.4(b). The spaces Y and $\tilde{Y}$ are clearly homotopy equivalent.

Theorem A.3. *Assume that the functional $\mathscr{F} : \mathscr{M} \to \mathbb{R}$ is C^2 and has a non-degenerate critical point p with $\mathscr{F}(p) = c$. Consider a neighborhood $\mathscr{U} \subseteq \mathscr{M}$ of p that does not contain any other critical point of $\mathscr{F}$, and $c_1, c_2 \in \mathbb{R}$ such that $c_1 < c \leq c_2$. Then $\overline{(\mathscr{F})_{c_2}} \cap \mathscr{U}$ is homotopy equivalent to $\overline{(\mathscr{F})_{c_1}} \cap \mathscr{U}$ with a $\iota(\mathscr{F}, p)$-cell attached. In particular the local homology $\mathrm{H}_*(\mathscr{F}, p)$ is trivial in case[8] $\iota(\mathscr{F}, p) = \infty$, otherwise*

$$\mathrm{H}_*(\mathscr{F}, p) = \begin{cases} \mathbb{F} & * = \iota(\mathscr{F}, p), \\ 0 & * \neq \iota(\mathscr{F}, p), \end{cases}$$

where $\mathbb{F}$ is the coefficient group of the homology.

Remark A.4. The above statement can be strengthen, saying that $\overline{(\mathscr{F})_{c_2}} \cap \mathscr{U}$ is diffeomorphic to $\overline{(\mathscr{F})_{c_1}} \cap \mathscr{U}$ with a thickened $\iota(\mathscr{F}, p)$-cell attached (see figure A.5). However, this is not relevant as far as homotopy (or homology) is concerned. ■

[8]This follows from the fact that the unit-sphere of an infinite-dimensional Hilbert space is contractible.

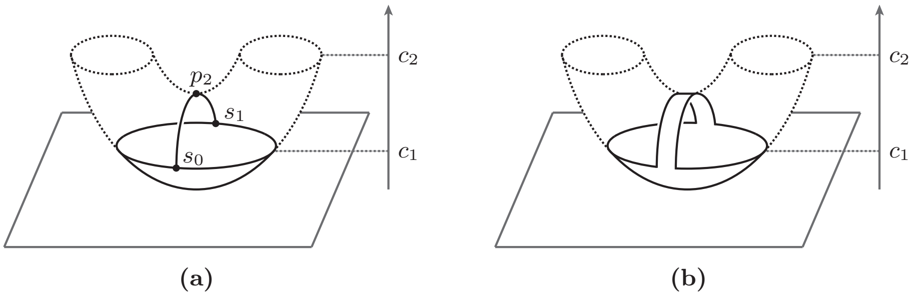


Figure A.5. **(a)** Attachment of a cell of dimension $1 = \iota(F, p_2)$ to the closed sublebel $\overline{(F)}_{c_1}$ of the height function in example A.1. The attaching region is the 0-dimensional **attaching sphere** $\{s_0, s_1\}$. Notice that the result of the attachment is homotopically equivalent to $\overline{(F)}_{c_2}$. **(b)** Attachment of a thickened 1-cell to the closed sublebel $\overline{(F)}_{c_1}$

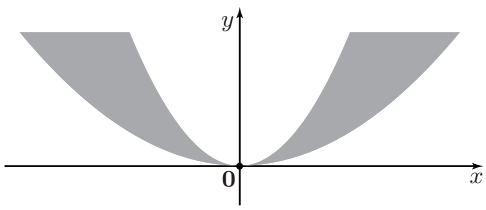


Figure A.6. Behaviour of $F(x, y) = (y - 2x^2)(y - x^2)$ around the critical point **0**. The shaded region corresponds to the sublevel $(F)_0 = F^{-1}(-\infty, 0)$.

Theorem A.3 tells us that, for a non-degenerate critical point of a C^2 functional, the knowledge of local homology at it coincides with the knowledge of its Morse index. This is no longer true in the degenerate case, where the knowledge of the Morse index and nullity of a critical point is not sufficient to determine its local homology. An easy example on $\mathscr{M} = M = \mathbb{R}^2$ is the following.

Example A.4. Both the functionals $F, G : \mathbb{R}^2 \to \mathbb{R}$ given by $F(x, y) = (y - 2x^2)(y - x^2)$ and $G(x, y) = x^4 + y^2$ have $\mathbf{0} = (0, 0)$ as critical point with $\iota(F, \mathbf{0}) = \iota(G, \mathbf{0}) = 0$ and $\nu(F, \mathbf{0}) = \nu(G, \mathbf{0}) = 1$; however $\mathbf{0}$ is a saddle point for F (see figure A.6) and a global minimum for G, therefore

$$\mathrm{H}_*(F, \mathbf{0}) = \begin{cases} \mathbb{F} & * = 1, \\ 0 & \text{otherwise}, \end{cases} \qquad \mathrm{H}_*(G, \mathbf{0}) = \begin{cases} \mathbb{F} & * = 0, \\ 0 & \text{otherwise}. \end{cases} \qquad \blacksquare$$

If the functional $\mathscr{F}$ is only C^1 with possibly degenerate critical points, but it satisfies the Palais-Smale condition, we can still describe the homological changes that occur passing a critical level in terms of local homology.

Theorem A.4. *Let $\mathscr{M}$ be a complete Hilbert-Riemannian manifold and $\mathscr{F} : \mathscr{M} \to \mathbb{R}$ a C^1 functional having only isolated critical points and satisfying the Palais-Smale condition. Consider $c \in \mathbb{R}$ and $\varepsilon > 0$ such that the interval $(c, c + \varepsilon) \subset \mathbb{R}$ does not contain any critical value of $\mathscr{F}$. The following claims hold:*

(i) *For each $p \in \mathrm{Crit}(\mathscr{F})$ with $\mathscr{F}(p) = c$, the inclusion*

$$j_p : ((\mathscr{F})_c \cup \{p\}, (\mathscr{F})_c) \hookrightarrow ((\mathscr{F})_{c+\varepsilon}, (\mathscr{F})_c)$$

induces the homology isomorphism

$$\mathrm{H}_*(j_p) : \mathrm{H}_*(\mathscr{F}, p) \xrightarrow{\simeq} \mathrm{H}_*((\mathscr{F})_{c+\varepsilon}, (\mathscr{F})_c).$$

(ii) *If $\{p_1, ..., p_s\} \subset \mathscr{M}$ is the set of critical point of $\mathscr{F}$ with critical value c, we have an isomorphism*

$$J_c : \mathrm{H}_*(\mathscr{F}, p_1) \oplus ... \oplus \mathrm{H}_*(\mathscr{F}, p_s) \xrightarrow{\simeq} \mathrm{H}_*((\mathscr{F})_{c+\varepsilon}, (\mathscr{F})_c),$$

where $J_c = \mathrm{H}_*(j_{p_1}) \oplus ... \oplus \mathrm{H}_*(j_{p_s})$.

Playing with the filtration (A.2) given by the sublevels of $\mathscr{F}$ and with associated long exact sequences in homology we can deduce the following statement.

Theorem A.5. *Assume that $\mathscr{F}$ satisfies the hypotheses of theorem A.4. We fix a bounded interval $[a, b] \subset \mathbb{R}$, so that $\{c_1, ..., c_t\} \subset (a, b)$ is the set of critical values of $\mathscr{F}$ inside (a, b), and we choose ε such that*

$$0 < \varepsilon < \min\{c_h - c_k \,|\, h, k = 1, ..., t,\ h \neq k\}).$$

Then, we have

$$\sum_{n=0}^{N} (-1)^{N-n} \operatorname{rank} \mathrm{H}_n((\mathscr{F})_b, (\mathscr{F})_a) \leq \sum_{n=0}^{N} (-1)^{N-n} \sum_{h=1}^{t} \operatorname{rank} \mathrm{H}_n((\mathscr{F})_{c_h+\varepsilon}, (\mathscr{F})_{c_h}), \qquad \forall N \in \mathbb{N}.$$

Moreover

$$\sum_{n=0}^{\infty} (-1)^{n} \operatorname{rank} \mathrm{H}_n((\mathscr{F})_b, (\mathscr{F})_a) = \sum_{n=0}^{\infty} (-1)^{n} \sum_{h=1}^{t} \operatorname{rank} \mathrm{H}_n((\mathscr{F})_{c_h+\varepsilon}, (\mathscr{F})_{c_h}),$$

provided the above series are finite.

As a consequence of this result and of theorem A.4, we obtain the celebrated Morse inequalities.

Corollary A.6 (Morse inequalities). *Assume that $\mathscr{F}$ satisfies the hypotheses of theorem A.4. For each bounded interval $[a, b] \subset \mathbb{R}$, if $\{p_1, ..., p_u\} \subset \mathscr{M}$ are the critical points of $\mathscr{F}$ with critical value inside (a, b), we have*

$$\operatorname{rank} \mathrm{H}_*((\mathscr{F})_b, (\mathscr{F})_a) \leq \sum_{h=1}^{u} \operatorname{rank} \mathrm{H}_*(\mathscr{F}, p_h). \tag{A.4}$$

A.5 Local homology and Gromoll-Meyer pairs

In the fundamental paper [GM], Gromoll and Meyer showed that the local homology groups of an isolated critical point can also be produced as the homology of an opportune closed neighborhood of the critical point, relative to a part of its boundary. The homotopy type of the pair

$$\text{(closed neighborhood, part of boundary)}$$

is what in the 1980s, after the seminal work of Conley [Co], would be called **Conley index**[9] of the critical point. Here, the critical point is considered as an isolated invariant set for the dynamical system defined by a pseudo-gradient flow.

For this section, let us assume that our functional $\mathscr{F} : \mathscr{M} \to \mathbb{R}$ is C^2, so that we can choose as a pseudo-gradient $\mathscr{X}$ the gradient of $\mathscr{F}$, see example A.2. Some, but not all, of the statements that we will give still hold with only the C^1 assumption (up to choose a C^1 pseudo-gradient).

Let $p \in \mathscr{M}$ be a critical point of $\mathscr{F}$ with $\mathscr{F}(p) = c$. A pair of topological spaces $(\mathscr{W}, \mathscr{W}_-)$ is called a **Gromoll-Meyer pair** for $\mathscr{F}$ at p when

(GM1) $\mathscr{W} \subset \mathscr{M}$ is a closed neighborhood of p that does not contain other critical points of $\mathscr{F}$,

(GM2) there exists $\varepsilon > 0$ such that $[c - \varepsilon, c)$ does not contain critical values of $\mathscr{F}$, and $\mathscr{W} \cap (\mathscr{F})_{c-\varepsilon} = \varnothing$,

(GM3) if $t_1 < t_2$ are such that $\Phi_{\mathscr{X}}(t_1, q), \Phi_{\mathscr{X}}(t_2, q) \in \mathscr{W}$ for some $q \in \mathscr{M}$, then $\Phi_{\mathscr{X}}(t, q) \in \mathscr{W}$ for all $t \in [t_1, t_2]$,

(GM4) $\mathscr{W}_-$ is defined as

$$\mathscr{W}_- = \{q \in \mathscr{W} \mid \Phi_{\mathscr{X}}((0, \infty) \times \{q\}) \subset \mathscr{M} \setminus \mathscr{W}\}$$

and it is a piecewise submanifold of $\mathscr{M}$ transversal to the flow $\Phi_{\mathscr{X}}$.

It is always possible to build a Gromoll-Meyer pair of an isolated critical point. Moreover, all the Gromoll-Meyer pairs of a critical point have the same homology type, and in fact they can serve as an alternative definition of local homology.

Theorem A.7. *With the above assumptions for the functional $\mathscr{F}$, if $(\mathscr{W}, \mathscr{W}_-)$ is a Gromoll-Meyer pair for $\mathscr{F}$ at the isolated critical point $p \in \mathscr{M}$, we have*

$$\mathrm{H}_*(\mathscr{W}, \mathscr{W}_-) \simeq \mathrm{H}_*(\mathscr{F}, p).$$

[9] Notice that this Conley index has nothing to do with the Conley-Zehnder index discussed in chapter I.

Now, applying the generalized Morse lemma (and the notation adopted therein), we may reduce to work in a coordinate neighborhood of the critical point p: without loss of generality, we can identify p with the origin in the Hilbert space $\boldsymbol{E}$, so that we can assume that $\mathscr{F}$ is defined on a neighborhood $\mathscr{V} \subset \boldsymbol{E}$ of $\mathbf{0}$ and has the form

$$\mathscr{F}(\boldsymbol{v}) = \mathscr{F}^0(\boldsymbol{v}^0) + \mathscr{F}^\pm(\boldsymbol{v}^\pm), \qquad \forall \boldsymbol{v} = \boldsymbol{v}^0 + \boldsymbol{v}^\pm \in \mathscr{V} \subset \boldsymbol{E} = \boldsymbol{E}^0 \oplus \boldsymbol{E}^\pm.$$

Here the origin is a non-degenerate critical point of $\mathscr{F}^\pm : \mathscr{V} \cap \boldsymbol{E}^\pm \to \mathbb{R}$ and a fully degenerate critical point of $\mathscr{F}^0 : \mathscr{V} \cap \boldsymbol{E}^0 \to \mathbb{R}$. If we consider Gromoll-Meyer pairs $(\mathscr{W}^\pm, \mathscr{W}^\pm_-)$ and $(\mathscr{W}^0, \mathscr{W}^0_-)$ for $\mathscr{F}^\pm$ and $\mathscr{F}^0$ respectively at $\mathbf{0}$, it is easy to verify that the cross product of this pairs, that is

$$(\mathscr{W}^\pm, \mathscr{W}^\pm_-) \times (\mathscr{W}^0, \mathscr{W}^0_-) = (\mathscr{W}^\pm \times \mathscr{W}^0, (\mathscr{W}^\pm_- \times \mathscr{W}^0) \cup (\mathscr{W}^\pm \times \mathscr{W}^0_-)),$$

is a Gromoll-Meyer pair for $\mathscr{F}$ at $\mathbf{0}$. Now, assume that the coefficient group $\mathbb{F}$ of the homology is a field. Then, by the Künneth formula we get an isomorphism of graded vector spaces

$$\mathrm{H}_*((\mathscr{W}^\pm, \mathscr{W}^\pm_-) \times (\mathscr{W}^0, \mathscr{W}^0_-)) \simeq \mathrm{H}_*(\mathscr{W}^\pm, \mathscr{W}^\pm_-) \otimes \mathrm{H}_*(\mathscr{W}^0, \mathscr{W}^0_-),$$

and by the above theorem A.7 we obtain the following.

Theorem A.8. $\mathrm{H}_*(\mathscr{F}, \mathbf{0}) \simeq \mathrm{H}_*(\mathscr{F}^\pm, \mathbf{0}) \otimes \mathrm{H}_*(\mathscr{F}^0, \mathbf{0})$.

By theorem A.3, the local homology of the non-degenerate functional $\mathscr{F}^\pm$ at $\mathbf{0}$ is nontrivial only in dimension $\iota(\mathscr{F}^\pm, \mathbf{0})$, in which coincides with the coefficient vector space $\mathbb{F}$. Notice that the Morse index of $\mathscr{F}^\pm$ at $\mathbf{0}$ is precisely the Morse index of $\mathscr{F}$ at $\mathbf{0}$, and therefore theorem A.8 readily gives the following fundamental result.

Theorem A.9 (Shifting). $\mathrm{H}_*(\mathscr{F}, \mathbf{0}) \simeq \mathrm{H}_{*-\iota(\mathscr{F},\mathbf{0})}(\mathscr{F}^0, \mathbf{0})$.

Corollary A.10. *If $\nu(\mathscr{F}, \mathbf{0}) < \infty$, the local homology groups $\mathrm{H}_*(\mathscr{F}, \mathbf{0})$ are trivial if $*$ is less than $\iota(\mathscr{F}, \mathbf{0})$ or greater than $\iota(\mathscr{F}, \mathbf{0}) + \nu(\mathscr{F}, \mathbf{0})$.*

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List of symbols

Chapter I

M	Smooth finite dimensional closed manifold	2
$\mathrm{T}M$	Tangent bundle of M	2
$\mathscr{L}$	Lagrangian function	2
$\mathscr{A}^{t_0,t_1}$	Action functional for curves parametrized on $[t_0, t_1]$	2
$\mathfrak{U}$	Atlas for M	2
(U_α, ϕ_α)	Chart in $\mathfrak{U}$	2
$\mathrm{T}\mathfrak{U}$	Atlas for $\mathrm{T}M$ associated to $\mathfrak{U}$	2
$(\mathrm{T}U_\alpha, \mathrm{T}\phi_\alpha)$	Chart in $\mathrm{T}\mathfrak{U}$	2
$\mathrm{T}^*\mathfrak{U}$	Atlas for T^*M associated to $\mathfrak{U}$	2
$(\mathrm{T}^*U_\alpha, \mathrm{T}^*\phi_\alpha)$	Chart in $\mathrm{T}^*\mathfrak{U}$	2
$\partial_v\mathscr{L}$	Fiberwise derivative of $\mathscr{L}$	2
$\gamma^*\mathrm{T}M$	Pullback of $\mathrm{T}M$ by γ	3
$X_\mathscr{L}$	Euler-Lagrange vector field of $\mathscr{L}$	4
$\Phi_\mathscr{L}$	Euler-Lagrange flow of $\mathscr{L}$	4
T^*M	Cotangent bundle of M	5
$\mathscr{H}$	Hamiltonian function	5
λ	Liouville form on T^*M	5
ω	Canonical symplectic form on T^*M	5
$X_\mathscr{H}$	Hamiltonian vector field of $\mathscr{H}$	5
$\lrcorner$	Interior product	6
$\Phi_\mathscr{H}$	Hamiltonian flow of $\mathscr{H}$	6
$\mathrm{Leg}_\mathscr{L}$	Legendre transform given by $\mathscr{L}$	6
L	Convex function on $\mathbb{R}^m$	8
H	Legendre dual of L	9
$\langle\cdot,\cdot\rangle.$	Riemannian metric	7
Lie	Lie derivative	11
$\Gamma^*\mathrm{T}\mathrm{T}^*M$	Pullback of $\mathrm{T}\mathrm{T}^*M$ by Γ	11
ω_0	Standard symplectic form of $\mathbb{R}^{2m}$	11
J_0	Standard complex structure of $\mathbb{R}^{2m}$	12
$\mathrm{Sp}(2m)$	Symplectic group of $\mathbb{R}^{2m}$	12
$\mathrm{GL}(2m)$	General linear group of $\mathbb{R}^{2m}$	12
$\mathrm{Sp}^0(2m)$	Automorphisms in $\mathrm{Sp}(2m)$ with eigenvalue 1	12
$\mathrm{Sp}^*(2m)$	Complementary of $\mathrm{Sp}^0(2m)$ in $\mathrm{Sp}(2m)$	12
$\mathrm{Sp}^\pm(2m)$	Connected components of $\mathrm{Sp}^*(2m)$	12
$\mathrm{O}(2m)$	Orthogonal group of $\mathbb{R}^{2m}$	12
r	Retraction of $\mathrm{Sp}(2m)$ onto $\mathrm{Sp}(2m) \cap \mathrm{O}(2m)$	12
$\mathrm{GL}(m,\mathbb{C})$	General linear group of $\mathbb{C}^m$	12

$\mathrm{U}(m)$	Unitary group of $\mathbb{C}^m$	12
$\det_{\mathbb{C}}$	Complex determinant	12
$\mathscr{P}$	Continuous paths in $\mathrm{Sp}(2m)$ from I	12
$\mathscr{P}^*$	Paths of $\mathscr{P}$ with endpoint in $\mathrm{Sp}^*(2m)$	12
$\mathscr{P}^0$	Paths of $\mathscr{P}$ with endpoint in $\mathrm{Sp}^0(2m)$	12
$\Psi \bullet \tilde{\Psi}$	Concatenation of the paths Ψ and $\tilde{\Psi}$	13
$\iota(\Psi)$	Conley-Zehnder index of Ψ	13
deg	Brouwer degree	13
$\nu(\Psi)$	Multiplicity of the eigenvalue 1 of $\Psi(1)$	14
$\Psi^{[n]}$	n^{th} iteration of the path Ψ	14
$\widehat{\iota}(\Psi)$	Mean Conley-Zehnder index of Ψ	14
$\mathbb{V}^m$	Vertical Lagrangian subspace of $\mathbb{R}^{2m}$	15
Γ_ϕ	Path in $\mathrm{Sp}(2m)$ defined by Γ and ϕ	16
$\mathrm{Sp}(2m, \mathbb{V}^m)$	Symplectic automorphisms preserving $\mathbb{V}^m$	16
π_1	Fundamental group functor	16
$(\iota(\mathscr{H}, \Gamma), \nu(\mathscr{H}, \Gamma))$	Conley-Zehnder-Long index pair of the orbit Γ	17
$\Gamma^{[n]}$	n^{th} iteration of the loop Γ	17
$\widehat{\iota}(\mathscr{H}, \Gamma)$	Mean Conley-Zehnder index of the orbit Γ	18
$\mathscr{A}$	Action functional for 1-periodic curves	18
$\mathscr{B}_\gamma$	Second variation of $\mathscr{A}$ at γ	19
$(\iota(\mathscr{B}_\gamma), \nu(\mathscr{B}_\gamma))$	Morse index and nullity pair of $\mathscr{B}_\gamma$	20

Chapter II

$\mathbb{T}$	Circle $\mathbb{R}/\mathbb{Z}$	21
$C^0(\mathbb{T}; M)$	Free loop space of M with C-regularity	21
$C^\infty(\mathbb{T}; M)$	Free loop space of M with C^∞-regularity	22
$W^{1,2}(\mathbb{T}; M)$	Free loop space of M with $W^{1,2}$-regularity	22
$W^{1,2}(\mathbb{T}; \mathbb{R}^k)$	Sobolev space	22
$W^{1,2}(\gamma^*\mathrm{T}M)$	$W^{1,2}$ sections of $\gamma^*\mathrm{T}M$	22
∇_t	Covariant derivative	22
$\langle\!\langle \cdot, \cdot \rangle\!\rangle_\gamma$	Inner product on $W^{1,2}(\gamma^*\mathrm{T}M)$	22
exp	Exponential map of $(M, \langle \cdot, \cdot \rangle_\cdot)$	23
$(\mathscr{U}_\gamma, \exp_\gamma^{-1})$	Chart of $W^{1,2}(\mathbb{T}; M)$	23
$\mathbb{I}$	Interval	23
$W^{1,2}(\mathbb{I}; M)$	Path space of M with $W^{1,2}$-regularity	23
$C^\infty(\mathbb{I}; M)$	Path space of M with C^∞-regularity	23
$(\mathscr{V}_\lambda, \Phi_\lambda)$	Chart of $W^{1,2}(\mathbb{I}; M)$	23
$\mathbb{V}_\gamma$	Vector subspace of $\mathbb{R}^m \times \mathbb{R}^m$	24
$W^{1,2}_{\mathbb{V}_\gamma}(\mathbb{I}; \mathbb{R}^m)$	Hilbert subspace of $W^{1,2}(\mathbb{I}; \mathbb{R}^m)$	24
$(\mathscr{U}_\gamma, \Theta_\gamma)$	Chart of $W^{1,2}(\mathbb{T}; M)$	24
$\mathbb{T}^{[n]}$	Circle $\mathbb{R}/n\mathbb{Z}$	24
$\psi^{[n]}$	n^{th} iteration map	25
$\{h_\varepsilon \mid \varepsilon > 0\}$	Approximate identity	26
$*$	Convolution operator	26
ev	Evaluation map	27
$(\iota(\mathscr{A}, \gamma), \nu(\mathscr{A}, \gamma))$	Morse index and nullity pair of $\mathscr{A}$ at γ	34
$\mathscr{A}^{[n]}$	Mean action functional on $W^{1,2}(\mathbb{T}^{[n]}; M)$	35
$\widehat{\iota}(\mathscr{A}, \gamma)$	Mean Morse index of $\mathscr{A}$ at γ	35
dist	Distance induced by the Riemannian metric on M	36

Chapter III

$\mathscr{A}^{t_0,t_1}$	Action functional for curves parametrized on $[t_0, t_1]$	40
γ_{q_0,q_1}	Unique action minimizer with endpoints q_0 and q_1	40
$\mathrm{Leb}(\mathfrak{U})$	Lebesgue number of the atlas $\mathfrak{U}$	41
$\overline{B(q,R)}$	Riemannian closed ball with center q and radius R	41
$\Delta(\rho_0)$	ρ_0-neighborhood of the diagonal submanifold of $M \times M$	44
$Q_{\mathscr{L}}$	Projection of the Euler-Lagrange flow onto M	45
Δ_k	Neighborhood of the diagonal submanifold of the k-fold product of M	46
$\mathbb{Z}_k$	Cyclic group of order k	46
$C_k^\infty(\mathbb{T}; M)$	Space of continuous and k-broken smooth loops	46
λ_k	Embedding of Δ_k into $C_k^\infty(\mathbb{T}; M)$	46
Λ_k	k-broken Euler-Lagrange loop space	46
$\gamma_{\boldsymbol{q}}$	Broken Euler-Lagrange loop between the k-uple of points $\boldsymbol{q}$	46
$\mathscr{A}_k$	Discrete action functional	47
L	Quadratic Lagrangian	52
A	Action functional associated to L	52
$\boldsymbol{E}$	Real Hilbert space	55
$\mathscr{B}$	Bounded symmetric bilinear form on $\boldsymbol{E}$	55
$\iota(\mathscr{B})$	Morse index of $\mathscr{B}$	55
$\{\boldsymbol{E}_n \mid n \in \mathbb{N}\}$	Family of Hilbert subspaces of $\boldsymbol{E}$	55
P_n	Orthogonal projector of $\boldsymbol{E}$ onto $\boldsymbol{E}_n$	55
$\boldsymbol{V}$	Finite dimensional subspace of $\boldsymbol{E}$	55
$S(\boldsymbol{V})$	Sphere of $\boldsymbol{V}$	55
$\mathrm{Aff}_k(\mathbb{T}; \mathbb{R}^m)$	k-broken affine loop space of $\mathbb{R}^m$	56
$\lfloor \cdot \rfloor$	Integer part	56
$(\mathscr{A})_c$	c-open sublevel of $\mathscr{A}$	59
$(\mathscr{A}_k)_c$	c-open sublevel of $\mathscr{A}_k$	59
r_k	Retraction of a sublevel of $\mathscr{A}$ onto a sublevel of $\mathscr{A}_k$	59
R_k	Deformation retraction from the identity to r_k	59
H_*	Singular homology functor	60
$\mathrm{H}_*(\mathscr{A}_k, \boldsymbol{q})$	Local homology of $\mathscr{A}_k$ at $\boldsymbol{q}$	60
$\mathrm{H}_*(\mathscr{A}, \gamma_{\boldsymbol{q}})$	Local homology of $\mathscr{A}$ at $\gamma_{\boldsymbol{q}}$	61
$\mathrm{H}_*(\lambda_k)$	Local homology induced homomorphism	61
$\lambda_k^{[n]}$	Embedding of Δ_{nk} into $C_k^\infty(\mathbb{T}^{[n]}; M)$	62
$\Lambda_k^{[n]}$	k-broken n-periodic Euler-Lagrange loop space	62
$\mathscr{A}_k^{[n]}$	Discrete mean action functional	62
$\psi_k^{[n]}$	Discrete n^{th}-iteration map	62
$\boldsymbol{q}^{[n]}$	n^{th}-iteration of $\boldsymbol{q}$	62

Chapter IV

$\boldsymbol{E}$	Hilbert space	63
$\mathscr{U}$	Open set in $\boldsymbol{E}$	63
$\mathscr{F}$	Functional over $\mathscr{U}$	63
$\boldsymbol{E}_\bullet$	Hilbert subspace of $\boldsymbol{E}$	63
$\mathrm{Grad}\mathscr{F}$	Gradient of $\mathscr{F}$	64
J	Inclusion map of $\boldsymbol{E}_\bullet$ in $\boldsymbol{E}$	64
H	Operator on $\boldsymbol{E}$ associated to the Hessian of $\mathscr{F}$ at $\boldsymbol{x}$	64
$\mathrm{H}_*(\mathscr{F}, \boldsymbol{x})$	Local homology of $\mathscr{F}$ at $\boldsymbol{x}$	64
$\mathscr{F}_\bullet$	Restriction of $\mathscr{F}$ to $\boldsymbol{E}_\bullet$	64
$\mathrm{H}_*(\mathscr{F}_\bullet, \boldsymbol{x})$	Local homology of $\mathscr{F}_\bullet$ at $\boldsymbol{x}$	64
$\mathrm{H}_*(J)$	Local homology homomorphism induced by J	64

$(\iota(\mathscr{F},\boldsymbol{x}),\nu(\mathscr{F},\boldsymbol{x}))$	Morse index and nullity pair of $\mathscr{F}$ at $\boldsymbol{x}$	64
$(\iota(\mathscr{F}_\bullet,\boldsymbol{x}),\nu(\mathscr{F}_\bullet,\boldsymbol{x}))$	Morse index and nullity pair of $\mathscr{F}_\bullet$ at $\boldsymbol{x}$	64
$\boldsymbol{E}^+\oplus\boldsymbol{E}^-\oplus\boldsymbol{E}^0$	Orthogonal splitting of $\boldsymbol{E}$	66
$\mathscr{F}^\pm+\mathscr{F}^0$	Decomposition of $\mathscr{F}$	67
$H_\bullet$	Operator on $\boldsymbol{E}_\bullet$ associated to the Hessian of $\mathscr{F}_\bullet$ at $\boldsymbol{x}$	69
$\boldsymbol{E}_\bullet^+\oplus\boldsymbol{E}_\bullet^-\oplus\boldsymbol{E}_\bullet^0$	Orthogonal splitting of $\boldsymbol{E}_\bullet$	69
$\mathscr{F}_\bullet^\pm+\mathscr{F}_\bullet^0$	Decomposition of $\mathscr{F}_\bullet$	70

Chapter V

$\mathscr{A}$	Tonelli Action functional	80
$\mathscr{A}^{[n]}$	Mean Tonelli Action functional	80
$\mathscr{L}_R$	R-modification of $\mathscr{L}$	81
∂^2_{vv}	Fiberwise Hessian operator	81
$\mathscr{A}_R$	Action functional of $\mathscr{L}_R$	83
$\mathscr{A}_R^{[n]}$	Mean Action functional of $\mathscr{L}_R$	83
$\mathscr{A}_k$	Discrete Tonelli action	85
$\mathscr{A}_k^{[n]}$	Discrete mean Tonelli action	87
Δ^p	Standard p-simplex in $\mathbb{R}^p$	90
$\overline{\alpha}$	Inversion of the path α	90
$\alpha\bullet\beta$	Associative composition of the paths α and β	90
ev	Evaluation map	91
$\vartheta^x_{x_0},\ \vartheta^{x_0}_x$	Horizontal broken geodesics	91
$\vartheta^{\langle n\rangle}$	Path obtained from ϑ by the Bangert construction	91
$\widehat{\vartheta}$	Path of pulling loops	93
$\mathrm{Ban}_\sigma^{[n]}$	Bangert homotopy	93
$\langle\boldsymbol{v}_1,...,\boldsymbol{v}_j\rangle$	Convex hull of $\boldsymbol{v}_1,...,\boldsymbol{v}_j$	95

Appendix A

$\boldsymbol{E}$	Hilbert space	106
$\mathscr{M}$	Hilbert manifold	105
M	Finite dimensional manifold	106
$\mathscr{U}$	Open set in a Hilbert manifold	106
U	Open set in a finite dimensional manifold	106
$\mathscr{F}$	Functional over a Hilbert manifold	106
F	Functional over a finite dimensional manifold	106
$\mathrm{Crit}\mathscr{F}$	Set of critical points of $\mathscr{F}$	106
$\mathscr{F}_\phi$	Pull-back of $\mathscr{F}$ via the chart ϕ	106
$\mathrm{Hess}\mathscr{F}_\phi$	Hessian of $\mathscr{F}_\phi$	107
$\mathrm{Hess}\mathscr{F}$	Hessian of $\mathscr{F}$	107
$\boldsymbol{v}_\phi$	Representation of $v\in\mathrm{T}_p\mathscr{M}$ via the chart ϕ	107
H_ϕ	Operator on $\boldsymbol{E}$ associated to the Hessian of $\mathscr{F}_\phi$ at $\phi(p)$	107
$\boldsymbol{E}_\phi^+\oplus\boldsymbol{E}_\phi^-\oplus\boldsymbol{E}_\phi^0$	Orthogonal splitting of $\boldsymbol{E}$	107
$\mathscr{F}^\pm+\mathscr{F}^0$	Decomposition of $\mathscr{F}$	108
$f_0\sim f_1$	Homotopic maps	109
$j:X\xrightarrow{\simeq}Y$	Homotopy equivalence	109
$X\sim Y$	Homotopy equivalent spaces	109
$(\mathscr{F})_c$	Sublevel $\mathscr{F}^{-1}(-\infty,c)$	109
$\mathscr{X}$	Pseudo-gradient vector field	111
$\Phi_{\mathscr{X}}$	Anti pseudo-gradient flow of $\mathscr{F}$	111

$\mathrm{Grad}\mathscr{F}$	Gradient of $\mathscr{F}$	111
H_*	Singular homology functor	113
$\mathrm{H}_*(\mathscr{F}, p)$	Local homology of $\mathscr{F}$ at p	113
D^λ	Closed disk of dimension λ	114
$\mathbb{F}$	Coefficient group for singular homology	114
$(\mathscr{W}, \mathscr{W}_-)$	Gromoll-Meyer pair	117

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