CHAPTER III

Taylor and Laurent Expansions. Singular Points and Residues

1. Cauchy's Inequalities; Liouville's Theorem

EGRAL FORMULA FOR THE TAYLOR COEFFICIENTS

e seen (chapter II, § 2, no. 6. theorem 3) that, if $f(z)$ is holomorphic
ɔen disc D centred at the origin, then $f(z)$ is the sum of a power
$\sum a_n z^n$ which converges in D. The coefficients a_n of this power
:e given by the relation

$$a_n = \frac{1}{n!} f^{(n)}(0).$$

words, the a_n are the coefficients of the Taylor expansion of $f(z)$
rigin. This power series is called the Taylor series of $f(z)$. We
pose to express the coefficients a_n in terms of integrals involving
ction f.

$re^{i\theta}$ for $0 \leqslant r < \rho$, where ρ denotes the radius of the disc D. We

$$f(re^{i\theta}) = \sum_{n \geqslant 0} a_n r^n e^{in\theta}.$$

r, allowing θ to vary, $f(re^{i\theta})$ is a periodic function of θ, and the
lation gives the Fourier expansion of this function. We observe
y the $e^{in\theta}$ occur in this expansion for the various integers $n \geqslant 0$.
, we know that the coefficients in the Fourier expansion of a
us function of period 2π are expressible as integrals involving the

function. In the present context, the series (1.1) converges normally when θ varies, r remaining fixed; we can then integrate term by term and obtain

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta = \sum_{p \geqslant 0} \frac{1}{2\pi}\int_0^{2\pi} a_p r^p e^{i(p-n)\theta}\, d\theta;$$

on the right hand side, all the integrals are zero except that which corresponds to $p = n$, and we obtain the fundamental formula

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta, \tag{1.2}$$

which we could also have deduced from relation (6.1) of chapter II, § 2. This integral formula gives an upper bound for the coefficient a_n: let $M(r)$ be the upper bound of $|f(re^{i\theta})|$ as θ varies, that is the upper bound of the values of f on the circumference of radius r. The absolute value of the right hand side of (1.2) is then bounded above by $M(r)$, and relation (1.2) thus gives the fundamental inequalities

$$|a_n| \leqslant \frac{M(r)}{r^n}, \qquad n \text{ an integer} \geqslant 0. \tag{1.3}$$

These inequalities are known as the *Cauchy inequalities.*

2. Liouville's theorem

Theorem. *A bounded, holomorphic function $f(z)$ in the whole plane is constant.*

Proof. We apply inequality (1.3) for any integer $n \geqslant 1$. The quantity $M(r)$ is, by hypothesis, less than some number M independent of r. Hence

$$|a_n| \leqslant \frac{M}{r^n}$$

no matter how big r is. Since the right hand side of this inequality tends to 0 as r tends to infinity (n being $\geqslant 1$), we see that $a_n = 0$ for $n \geqslant 1$, thus $f(z) = a_0$ is constant.

Application : d'Alembert's theorem. We shall show that any polynomial with complex coefficients which is not constant has at least one complex root. Let $P(z)$ be such a polynomial, we shall use *reductio ad absurdum* by supposing that $P(z) \neq 0$ for any complex number z. Then, the function $\frac{1}{P(z)}$ is holomorphic in the whole plane. It is bounded; for,

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 = z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right), \quad a_n \neq 0,$$

tends to infinity as $|z|$ tends to infinity, so there is a compact disc outside of which $\left|\frac{1}{P(z)}\right|$ is bounded; on the other hand, $\left|\frac{1}{P(z)}\right|$ is bounded in the compact disc because it is continuous function. Hence, $\frac{1}{P(z)}$ is bounded in the whole of the plane and so is constant by Liouville theorem. It follows that $P(z)$ is a constant, contrary to hypothesis.

2. Mean Value Property; Maximum Modulus Principle

1. Mean value property

We apply relation (1.2) of § 1 in the particular case when $n = 0$. Then,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, d\theta,$$

or

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, d\theta. \tag{1.2}$$

This equation says that the value of f at the point 0 is equal to the mean value of f on the circle of centre 0 and radius r. It follows, more generally, that, if S is a closed disc contained in an open set D in which f is holomorphic, the value of f at the centre of S is equal to the mean of the values of f on the frontier circle of S (this mean being calculated with respect to the arc of the circle). We shall say that a real- or complex-valued, continuous function f defined in an open set D has the *mean value property* if, for any compact disc S contained in D, the value of f at the centre of S is equal to the mean value of f on the frontier circle of S. We shall see later that the functions with the mean value property are precisely the *harmonic functions.* From now on, we can say that *any holomorphic function has the mean value property.* It is clear that, if a complex-valued function has the mean value property, then so have its real and imaginary parts. Thus, the real and imaginary parts of a holomorphic function have the mean value property.

2. Maximum modulus principle

This principle will apply to any (real- or complex-valued) function which has the mean value property (that is to say, as we shall see later, to any harmonic function).

THEOREM 1. (maximum modulus principle). *Let f be a continuous (complex-valued) function in an open set D of the plane* **C**. *If f has the mean value property and if $|f|$ has a relative maximum at a point $a \in D$* (i.e. *if $|f(z)| \leqslant |f(a)|$ for any z sufficiently near to a*), *then f is constant in a neighbourhood of a.*

Proof. If $f(a) = 0$, the theorem is obvious; suppose then that $f(a) \neq 0$; by multiplying f by a complex constant if necessary, we can reduce the theorem to the case when $f(a)$ is real and > 0, which we shall assume from now on. For sufficiently small $r \geqslant 0$, let

$$M(r) = \sup_{\theta} |f(a + re^{i\theta})|.$$

For sufficiently small $r \geqslant 0$, we have $M(r) \leqslant f(a)$ by hypothesis. Moreover, the mean value property gives

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\, d\theta, \tag{2.1}$$

whence $f(a) \leqslant M(r)$ and consequently $f(a) = M(r)$. It follows that the function

$$g(z) = \mathrm{Re}\,(f(a) - f(z))$$

is $\geqslant 0$ for sufficiently small $|z - a| = r$, and that $g(z) = 0$ if and only if $f(z) = f(a)$. By (2.1), the mean value of $g(z)$ on the circle

$$|z - a| = r$$

is zero; since g is continuous and $\geqslant 0$, this requires that g is identically zero on this circle, and, consequently, $f(z) = f(a)$ when $|z - a| = r$ is sufficiently small. This completes the proof.

COROLLARY. *Let D be a bounded, connected, open set of the plane* **C**; *let f be a (complex-valued) continuous function defined in the closure $\overline{D}$ and having the mean value property in D; and, let M be the upper bound of $|f(z)|$ when z describes the frontier of D. Then,*

(i) $|f(z)| \leqslant M$ *for* $z \in D$;

(ii) *if* $|f(a)| = M$ *at a point* $a \in D$, *f is constant.*

Proof. Let M' be the upper bound of $|f(z)|$ for $z \in \overline{D}$, a bound which is attained at at least one point a of the compact set $\overline{D}$ (since $|f(z)|$ is continuous). If $a \in D$, f is constant in some neighbourhood of a by theorem 1; theorem 1 also shows that the subset of D where f takes the value $f(a)$ is open, and, as it is obviously closed and non-empty, this subset must be the whole of D (because D is connected); since f is continuous in $\overline{D}$, we

also have $f(z) = f(a)$ for $z \in \overline{D}$ which shows that $M = M'$ and establishes statements (i) and (ii). The other case to be proved is when $|f(a)| \neq M'$ for any point $a \in D$; but, then $M = M'$ (which proves (i)), and (ii) is trivially true because we do not have $|f(a)| = M$ for any point a of D.

Note. The maximum modulus principle is applied especially to the following case : if a continuous function f in a closed disc is holomorphic in the interior of the disc, the upper bound of $|f|$ on the boundary of the disc bounds $|f|$ above in the interior of the disc. In particular, in the Cauchy inequalities (1.3), $M(r)$ is not only the upper bound of $|f(z)|$ for $|z| = r$ but also for $|z| \leqslant r$.

3. Schwarz' Lemma

THEOREM (Schwarz' Lemma). *Let $f(z)$ be a holomorphic function in the disc $|z| < 1$ and suppose that*

$$f(0) = 0, \qquad |f(z)| < 1 \qquad \textit{for } |z| < 1.$$

Then :

1° *we have $|f(z)| \leqslant |z|$ for $|z| < 1$;*

2° *if, for a $z_0 \neq 0$, the equality $|f(z_0)| = |z_0|$ holds, then*

$$f(z) = \lambda z \qquad \textit{identically and} \qquad |\lambda| = 1.$$

Proof. In the Taylor expansion $f(z) = \sum_{n \geqslant 0} a_n z^n$, the coefficient a_0 is zero because $f(0) = 0$. It follows that $f(z)/z$ is holomorphic for $|z| < 1$. Since $|f(z)| < 1$ by hypothesis, we have

$$\left|\frac{f(z)}{z}\right| < \frac{1}{r} \qquad \text{for} \qquad |z| = r.$$

This inequality holds also for $|z| \leqslant r$ because of the maximum modulus principle. If we fix z in the disc $|z| < 1$, we have $|f(z)| < |z|/r$ for any $r \geqslant |z|$ and < 1. In the limit, we have then $|f(z)| \leqslant |z|$, which establishes assertion 1° of the theorem. If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, the holomorphic function $f(z)/z$ attains the upper bound of its modulus at a point interior to the disc $|z| < 1$; thus, by the maximum modulus principle, this function is contant and we have then the identity $f(z)/z = \lambda$, $|\lambda| = 1$. This completes the proof.

4. Laurent's Expansion

1. Laurent's series

Here we consider formal power series $\sum_n a_n X^n$, where the (formal) summation is taken over all integers n, positive, negative, or 0. To such a series, we associate two formal series (in the usual sense), $\sum_{n\geqslant 0} a_n X^n$ and $\sum_{n<0} a_n X^{-n}$. Let ρ_1 and $\frac{1}{\rho_2}$ be the radii of convergence of these two series. Consider the convergent series

$$(1.1) \qquad f_1(z) = \sum_{n\geqslant 0} a_n z^n \quad \text{for} \quad |z| < \rho_1,$$

$$(1.2) \qquad f_2(z) = \sum_{n<0} a_n z^n \quad \text{for} \quad |z| > \rho_2.$$

We shall show that $f_2(z)$ is a *holomorphic* function of z. Put $z = \frac{1}{u}$; the function

$$g(u) = \sum_{n>0} a_{-n} u^n$$

is holomorphic for $|u| < 1/\rho_2$ and its derivative is given by the formula

$$g'(u) = \sum_{n>0} n a_{-n} u^{n-1}.$$

The formula for the differentiation of a composite function shows that $f_2(z)$ has a derivative equal to

$$f_2'(z) = -\frac{1}{z^2} g'(1/z) = \sum_{n<0} n a_n z^{n-1}.$$

Hence, series (1.2) is differentiable term by term for $|z| > \rho_2$. Suppose from now on that $\rho_2 < \rho_1$. Then, the sum $f(z)$ of the series

$$(1.3) \qquad \sum_{-\infty<n<+\infty} a_n z^n$$

is holomorphic in the annulus $\rho_2 < |z| < \rho_1$ and its derivative $f'(z)$ is the sum of the series $\sum n a_n z^{n-1}$ obtained by differentiating term by term.

The series $\sum a_n z^n$ is called the *Laurent series* in the annulus $\rho_2 < |z| < \rho_1$.

In the above, we do not exclude the case where $\rho_2 = 0$, nor the case where $\rho_1 = +\infty$. The convergence of series (1.3) is normal in any annulus $r_2 \leqslant |z| \leqslant r_1$, with

$$\rho_2 < r_2 < r_1 < \rho_1.$$

2. Laurent series expansion of a function holomorphic in an annulus

Definition. A function $f(z)$ defined in an annulus

$$\rho_2 < |z| < \rho_1$$

is said to have a *Laurent expansion* in this annulus if there is a Laurent series $\sum a_n z^n$ which converges in this annulus and whose sum is equal to $f(z)$ at any point of the annulus.

By the results of no. 1, $f(z)$ is then holomorphic in the annulus and the convergence is normal in any closed annulus $r_2 \leqslant |z| \leqslant r_1$ such that

$$\rho_2 < r_2 < r_1 < \rho_1;$$

moreover, we shall show that the Laurent series, if it exists, is unique. For, put $z = re^{i\theta}$ $(\rho_2 < r < \rho_1)$; by integrating the normally convergent expansion

$$f(re^{i\theta}) = \sum_{-\infty<n<+\infty} a_n r^n e^{in\theta}$$

term by term with respect to θ, we obtain, exactly as in § 1 (no. 1), the integral formula

$$(2.1) \qquad a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta, \qquad \text{for } n \text{ an integer} \geqslant 0 \text{ or } < 0.$$

We see that, if the function f is given, the coefficients a_n of a Laurent expansion of f when it exists, are determined uniquely by relation (2.1). It is called the Laurent expansion of f.

Theorem. *Any holomorphic function $f(z)$ in an annulus $\rho_2 < |z| < \rho_1$ has a Laurent expansion in this annulus.*

Proof. Choose two numbers r_1 and r_2 such that

$$\rho_2 < r_2 < r_1 < \rho_1.$$

We shall show that there exists a Laurent series which converges normally in the annulus $r_2 \leqslant |z| \leqslant r_1$ and whose sum is equal to $f(z)$ in this annulus. By the uniqueness of the Laurent expansion, which follows from the integral formula (2.1), the Laurent series thus obtained will not depend on the choice of r_1 and r_2. Thus, this Laurent series will converge to $f(z)$ in the whole of the annulus $\rho_2 < |z| < \rho_1$, which will prove the theorem.

Having chosen the numbers r_1 and r_2, we choose two numbers r_1' and r_2' such that $\rho_2 < r_2' < r_2 < r_1 < r_1' < \rho_1$. Consider the compact annulus

$$r_2' \leqslant |z| \leqslant r_1'$$

whose oriented boundary is the difference of the circle γ_1 of centre o and radius r_1' described in the positive sense, and the circle γ_2 of centre o and radius r_2' described in the positive sense. By Cauchy's integral formula (chapter II, § 2, theorem 5), we have, for $r_2 \leqslant |z| \leqslant r_1$,

$$(2.2) \qquad f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)\,dt}{t-z} - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(t)\,dt}{t-z}.$$

Consider the first integral; we have $|t| = r_1'$ and $|z| \leqslant r_1 < r_1'$; we can then write the series expansion

$$\frac{1}{t-z} = \sum_{n\geqslant 0} \frac{z^n}{t^{n+1}},$$

which converges normally when t describes the circle of centre o and radius r_1'. We replace $\frac{1}{t-z}$ in the first integral by this series; we can integrate it term by term because of the normal convergence, whence

$$(2.3) \qquad \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)\,dt}{t-z} = \sum_{n\geqslant 0} a_n z^n,$$

where

$$(2.4) \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(t)\,dt}{t^{n+1}}, \qquad n \geqslant 0.$$

Consider now the second integral; we have

$$|t| = r_2' \quad \text{and} \quad |z| \geqslant r_2 > r_2',$$

so

$$\frac{1}{t-z} = -\frac{1}{z}\,\frac{1}{1-t/z} = -\sum_{n<0} \frac{z^n}{t^{n+1}}.$$

Replace $\frac{1}{t-z}$ in the second integral by this series; since this series converges normally, we can integrate it term by term to obtain

$$(2.5) \qquad -\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(t)\,dt}{t-z} = \sum_{n<0} a_n z^n,$$

where

$$(2.6) \qquad a_n = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(t)\,dt}{t^{n+1}}, \quad n < 0.$$

Finally, relation (2.2) shows that

$$f(z) = \sum_{-\infty<n<+\infty} a_n z^n \quad \text{for} \quad r_2 \leqslant |z| \leqslant r_1,$$

the convergence being normal. The theorem is thus proved.

3. Decomposition of a holomorphic function in an annulus

Proposition 3.1. *Given a holomorphic function $f(z)$ in an annulus $\rho_2 < |z| < \rho_1$, there exists a holomorphic function $f_1(z)$ in the disc $|z| < \rho_1$ and a holomorphic function $f_2(z)$ for $|z| > \rho_2$ such that*

$$(4.1) \qquad f(z) = f_1(z) + f_2(z).$$

This decomposition is unique if we stipulate that the function f_2 tends to o as $|z|$ tends to ∞.

For, let $f(z) = \sum_{-\infty<n<+\infty} a_n z^n$ be the Laurent expansion of f. Put

$$(4.2) \qquad f_1(z) = \sum_{n\geqslant 0} a_n z^n, \qquad f_2(z) = \sum_{n<0} a_n z^n.$$

Relation (4.1) is obviously satisfied, and $|f_2(z)|$ tends to o as $|z|$ tends to ∞. Suppose that

$$f(z) = g_1(z) + g_2(z)$$

is another such decomposition; let us show that $f_1 = g_1$ and $f_2 = g_2$. Let h be the holomorphic function which is equal to $f_1 - g_1$ for $|z| < \rho_1$ and equal to $g_2 - f_2$ for $|z| > \rho_2$; this function h is holomorphic in the whole plane and tends to o as $|z|$ tends to ∞. By the maximum modulus principle (§ 2, no. 2), or by Liouville's theorem (§ 1, no. 2), the function h is identically zero. This completes the proof.

4. Cauchy's inequalities; application to the study of isolated singularities

Consider the integral formula (2.1). If $M(r)$ denotes the upper bound of $|f(z)|$ for $|z| = r$, the right hand side of (2.1) has its modulus bounded above by $M(r)$, whence the Cauchy inequality

$$(4.1) \qquad |a_n| \leqslant \frac{M(r)}{r^n}, \quad \text{with } n \text{ an integer} \geqslant 0 \text{ or } < 0.$$

We shall consider a holomorphic function $f(z)$ in the punctured disc $0 < |z| < \rho$. We ask if this function can be extended to a holomorphic function in the complete disc $|z| < \rho$, centre included. This extension is obviously unique if it exists (by the principle of analytic continuation, or, in this case, simply because of continuity).

Proposition 4.1. *A necessary and sufficient condition for this extension to be possible is that the function $f(z)$ is bounded in some neighbourhood of o.*

The condition is obviously necessary. We shall show that it is sufficient.

In the punctured disc $0 < |z| < \rho$, the function f has a Laurent expansion $\sum_{-\infty<n<+\infty} a_n z^n$. By hypothesis, there exists a number $M > 0$ which bounds $|f(z)|$ above for $|z| = r$ with any sufficiently small r. By Cauchy's inequality (4. 1), we have

$$|a_n| \leqslant \frac{M}{r^n}$$

for all small r, and for $n < 0$ this implies that $a_n = 0$. Thus the Laurent expansion of f reduces to a Taylor series and this defines the required extension of $f(z)$.

Definition. Let $f(z)$ be a holomorphic function in the punctured disc $0 < |z| < \rho$. The origin 0 is said to be an *isolated singularity* of f if the function f cannot be extended to a holomorphic function on the entire disc $|z| < \rho$.

A necessary and sufficient condition for 0 to be an isolated singularity is that the coefficients a_n in the Laurent expansion are not all zero for $n < 0$. We see that there are two possible cases :

1 *st. case* : there are only a finite number of integers $n < 0$ for which $a_n \neq 0$. In this case, there is a positive integer n such that $z^n f(z)$ is a holomorphic function $g(z)$ in some neighbourhood of the origin. Thus $f(z) = g(z)/z^n$ is *meromorphic* in some neighbourhood of the origin.

2 *nd. case* : there is an infinity of integers $n < 0$ such that $a_n \neq 0$. In this case the function $f(z)$ is not a meromorphic function in a neighbourhood of the origin.

Definition. In the first case, we say that the point 0 is a *pole* of the function f; in the second case, we say that 0 is an *essential singularity* of the function f.

Theorem (Weierstrass). *If 0 is an isolated essential singularity of a holomorphic function $f(z)$ in the punctured disc $0 < |z| < \rho$, then, for any $\varepsilon > 0$, the image of the punctured disc $0 < |z| < \varepsilon$ under f is everywhere dense in the plane* $\mathbf{C}$.

Proof. We use *reductio ad absurdum* by supposing that there exists a disc centred at a of radius $r > 0$ which is *outside the image* of the punctured disc $0 < |z| < \varepsilon$ under f. We have then

$$(4.2) \qquad |f(z) - a| \geqslant r \quad \text{for} \quad 0 < |z| < \varepsilon.$$

The function $g(z) = \dfrac{1}{f(z) - a}$ will then be holomorphic and bounded in the punctured disc $0 < |z| < \varepsilon$. By proposition 4. 1, this function can be extended to a holomorphic function in the disc $|z| < \varepsilon$, again

denoted by $g(z)$. Thus, $\dfrac{1}{g(z)}$ will be meromorphic in the disc $|z| < \varepsilon$ and $f(z) = a + \dfrac{1}{g(z)}$ will also be meromorphic, which contradicts the hypothesis that 0 is an essential singularity of $f(z)$.

Note. The case when z_0 is an essential singularity is obviously reduced to the case when $z_0 = 0$ by replacing z by $z - z_0$.

The following theorem, which we shall not prove, is much more precise than the Weierstrass theorem :

Picard's theorem. *If 0 is an isolated essential singularity of the holomorphic function $f(z)$, then the image by f of any punctured disc $0 < |z| < \varepsilon$ is either the whole plane* $\mathbf{C}$, *or the plane* $\mathbf{C}$ *with one point missing.*

Example. The function $e^{1/z} = \sum_{n\geqslant 0} \frac{1}{n!}\frac{1}{z^n}$ is holomorphic in the punctured plane $z \neq 0$ and has an isolated essential singularity at the origin since the coefficient of $\frac{1}{z^n}$ is $\neq 0$ for all $n \geqslant 0$. This function never takes the value 0; a worthwhile exercise is to show that it takes any value $\neq 0$ in any punctured disc $0 < |z| < \varepsilon$.

5. Introduction of the Point at Infinity. Residue Theorem

1. Riemann sphere

In the space $\mathbf{R}^3$, let x, y, u be the coordinates of a point and let us consider the unit sphere $\mathbf{S}_2$,

$$x^2 + y^2 + u^2 = 1.$$

The sphere $\mathbf{S}_2$, with the topology induced by that of the space $\mathbf{R}^3$, is a compact space since $\mathbf{S}_2$ is a bounded closed subset of $\mathbf{R}^3$. Let P and P′ be two points of $\mathbf{S}_2$ whose coordinates are respectively (0, 0, 1) and (0, 0, −1). We shall consider stereographic projection from the pole P. It associates with any point M of $\mathbf{S}_2$ other than P the point of the plane $u = 0$ collinear with P and M. The complex coordinate z of this point is given by the formula

$$(1.1) \qquad z = \frac{x + iy}{1 - u},$$

where x, y, u are the coordinates of the point M. Similary, we consider stereographic projection from the pole P′ but we take the point of the plane $u = 0$ which is the complex conjugate of the point corresponding to $M(x, y, u)$ under this stereographic projection. Its complex coordinate z' is given by the formula

$$z' = \frac{x - iy}{1 + u}. \qquad (1.2)$$

Note that, for any point $M(x, y, u)$ other than P or P′, the corresponding complex numbers z and z' are related by

$$zz' = 1. \qquad (1.3)$$

The mapping $(x, y, u) \to z$ is a homeomorphism of $\mathbf{S}_2 - P$ onto $\mathbf{C}$; we say that we have a chart of $\mathbf{S}_2 - P$ on the complex plane $\mathbf{C}$. Similarly, the mapping $(x, y, u) \to z'$ is a chart of $\mathbf{S}_2 - P'$ onto the complex plane $\mathbf{C}$. Provided with these two charts, $\mathbf{S}_2$ is called the *Riemann sphere*.

Let D be an open set of $\mathbf{S}_2$. We say that a function f defined in D is *holomorphic* in D if, in some neighbourhood of any point $M \in D$ distinct from P, it can be expressed as a holomorphic function of z, and if, in some neighbourhood of any point $M \in D$ distinct from P′, it can be expressed as a holomorphic function of z'. We note that, in a neighbourhood of a point distinct from both P and P′, any holomorphic function of z is a holomorphic function of z', and conversely, because of relation (1.3). By means of relation (1.1), we shall always identify the complex plane $\mathbf{C}$ with the sphere $\mathbf{S}_2$ with the point P excluded. We see that $\mathbf{S}_2$ is obtained by adding 'a point at infinity', to $\mathbf{C}$. To study a function in a neighbourhood of the point at infinity P we use the complex variable $z' = 1/z$, which is zero at the point P. The open sets $|z| > r$ in $\mathbf{C}$ form a fundamental system of neighbourhoods of the point at infinity. A function $f(z)$ defined in such an open set is 'holomorphic at infinity' if, by the change of variable $z = 1/z'$, it is expressed as a holomorphic function of z' for $|z'| < 1/r$.

Similarly, a function $f(z)$ is *meromorphic at infinity* if, when expressed as a function of z', it is meromorphic in a neighbourhood of $z' = 0$. Finally, a holomorphic function $f(z)$ for $|z| > r$ has an isolated essential singularity at the point at infinity if the function $f(1/z')$ has an isolated essential singularity at the origin $z' = 0$.

If

$$f(z) = \sum_n a_n z^n$$

is the Laurent expansion of $f(z)$ for $|z| > r$, a necessary and sufficient condition for the point at infinity to be a pole of f is that $a_n = 0$ for all the integers $n \geqslant 0$ except for a finite number of them; the condition for an essential singularity at the point at infinity is that there exist an infinity of integers $n \geqslant 0$ such that $a_n \neq 0$.

The concepts of *differentiable path, closed path*, and *oriented boundary of a compact set* can be defined for the sphere $\mathbf{S}_2$.

2. Residue theorem.

First, let us consider a holomorphic function $f(z)$ in an annulus $\rho_2 < |z| < \rho_1$ centred at the origin.

Proposition 2. 1. *If γ is a closed path contained in the annulus, then*

$$\frac{1}{2\pi i}\int_\gamma f(z)\, dz = I(\gamma, 0)\, a_{-1}, \qquad (2.1)$$

where $I(\gamma, 0)$ is the index of the path γ with respect to the origin 0 and a_{-1} is the coefficient of $1/z$ in the Laurent expansion of f.

Proof. We have

$$f(z) = a_{-1}/z + g(z),$$

where

$$g(z) = \sum_{n \neq -1} a_n z^n$$

is holomorphic on the annulus and has a primitive in it equal to

$$\sum_{n \neq -1} \frac{a_n}{n+1} z^{n+1} \qquad \text{(cf. § 4, n° 1).}$$

Thus, we have the relation

$$\int_\gamma f(z)\, dz = a_{-1}\int_\gamma dz/z + \int_\gamma g(z)\, dz. \qquad (2.2)$$

But, $\int_\gamma g(z)\, dz = 0$ since g has a primitive, and

$\int_\gamma dz/z = 2\pi i\, I(\gamma, 0)$ by the definition of the index.

These two relations, along with (2.2), give (2.1).

Formula (2.1) is applied particularly in the case when the function f has an isolated singularity at the origin 0 (either a pole, or an essential singularity). In this case, γ is a closed path in some neighbourhood of 0 which does not pass through 0. The coefficient a_{-1} of the Laurent expansion is then called the *residue* of the function f at the singular point 0. In particular, if γ is a circle centred at 0 with small radius described in the positive sense then

$$\int_\gamma f(z)\, dz = 2\pi i a_{-1}. \qquad (2.3)$$

The residue at any isolated singularity situated at any point of the plane $\mathbf{C}$ is defined in a similar way.

A *residue at the point at infinity* needs a special definition : let $f(z)$ be a holomorphic function for $|z| > r$ and put $z = \frac{1}{z'}$; then,

$$f(z)\,dz = -\frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'.$$

By definition, the residue of f at the point at infinity is equal to the residue of the function $-\frac{1}{z'^2} f\left(\frac{1}{z'}\right)$ at the point $z' = 0$. Thus, if $\sum_n a_n z^n$ is the Laurent expansion of $f(z)$ in a neighbourhood of the point at infinity, the residue of f at infinity is $-a_{-1}$.

RESIDUE THEOREM. *Let* D *be an open set of the Riemann sphere* $\mathbf{S}_2$ *and let* f *be a holomorphic function in* D *except perhaps at isolated points which are singularities of* f. *Let* Γ *be the oriented boundary of a compact subset* A *of* D *and suppose that* Γ *does not pass through any singularities of* f, *or the point at infinity. Then, only a finite number of singularities* z_k *are contained in* A, *and*

$$\int_\Gamma f(z)\,dz = 2\pi i\left(\sum_k \operatorname{Res}(f, z_k)\right), \tag{2.4}$$

where $\operatorname{Res}(f, z_k)$ *denotes the residue of the function* f *at the point* z_k; *the summation extends over all the singularities* $z_k \in$ A *including the point at infinity if it qualifies.*

Proof. We distinguish between the two cases where the point at infinity belongs or does not belong to A.

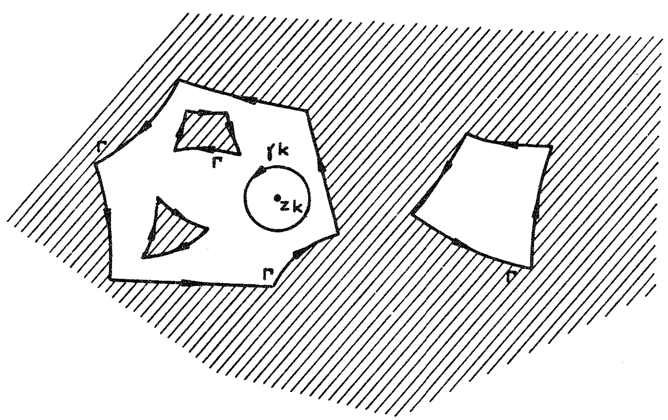

Fig. 4.
N. B. The shaded parts represent the complement of the compact set A.

1st. case. The point at infinity does not belong to A; A is then a (bounded) compact set of the plane $\mathbf{C}$ (cf. fig. 4); each singular point z_k is the centre

of a closed disc S_k in the interior of A and we can choose the radii of these discs small enough for the discs to be disjoint. Let γ_k be the boundary of the disc δ_k described in the positive sense. Let A′ be the compact set obtained by removing the interiors of the above discs from A; the oriented boundary of A′ is the difference between Γ (the oriented boundary of A) and the circles γ_k. Since f is holomorphic in some neighbourhood of A′, we have (cf. chapter II, § 2, no. 8, theorem 5)

$$\int_\Gamma f(z)\,dz = \sum_k \int_{\gamma_k} f(z)\,dz. \tag{2.5}$$

On the other hand, by (2. 3)

$$\int_{\gamma_k} f(z)\,dz = 2\pi i \operatorname{Res}(f, z_k),$$

and substituting this in (2. 5) gives the required relation (2. 4).

2nd. case. The point at infinity belongs to A. Let $|z| \geqslant r$ be a neighbourhood of the point at infinity which does not intersect Γ and such that $f(z)$ is holomorphic in this neighbourhood (the point at infinity being excluded). Let A″ be the compact set obtained by removing the open set $|z| > r$ from A (cf. fig. 5)

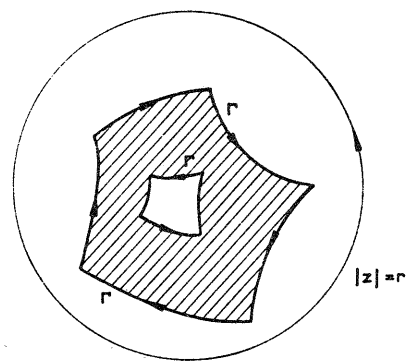

Fig. 5.
N. B. The shaded portion represents the complement of A.

The oriented boundary of A″ is the sum of the oriented boundary Γ of A and of the circle $|z| = r$ described in the positive sense. By applying the results we have proved in the first case to A″, we obtain

$$\int_\Gamma f(z)\,dz + \int_{|z|=r} f(z)\,dz = 2\pi i \sum_k \operatorname{Res}(f, z_k), \tag{2.6}$$

where the sum on the right hand side extends over all singularities z_k

contained in A *other than the point at infinity*. Moreover, by the definition of the residue at infinity, we have

$$\int_{|z|=r} f(z)\,dz = -2\pi i \operatorname{Res}(f, \infty),$$

and this substituted in (2. 6) gives

$$\int_{\Gamma} f(z)\,dz = 2\pi i \Big(\operatorname{Res}(f, \infty) + \sum_k \operatorname{Res}(f, z_k)\Big)$$

which is none other than the required relation (2. 4) when the point at infinity is one of the singularities z_k.

Note. Consider in particular the case where the compact set is the whole sphere S_2. In this case, the boundary is empty, and relation (2. 4) becomes:

$$\sum_k \operatorname{Res}(f, z_k) = 0. \tag{2.7}$$

For example, *the sum of the residues of a rational function (including the residue at infinity) is zero.*

3. Practical calculation of residues

The case of a simple pole which is not at infinity. Let z_0 be a simple pole of f; then

$$f(z) = \frac{1}{z - z_0} g(z),$$

where g is holomorphic in some neighbourhood of z_0 with $g(z_0) \neq 0$. Let

$$g(z) = \sum_{n \geqslant 0} a_n (z - z_0)^n$$

be the Taylor expansion of $g(z)$ in a neighbourhood of z_0; we see that, in the Laurent expansion of $f(z)$, the coefficient of $\frac{1}{z - z_0}$ is equal to $g(z_0)$. Thus,

$$\operatorname{Res}(f, z_0) = \lim_{\substack{z \to z_0 \\ z \neq z_0}} (z - z_0) f(z). \tag{3.1}$$

If f is given in the form of a quotient P/Q, where P and Q are holomorphic in a neighbourhood of z_0 and where z_0 is a simple zero of Q with $P(z_0) \neq 0$, then

$$\operatorname{Res}(f, z_0) = \frac{P(z_0)}{Q'(z_0)}, \tag{3.2}$$

Q′ denoting the derivative of Q.

Example. Let $f(z) = \frac{e^{iz}}{z^2 + 1}$; the function has two simple poles $z = \pm i$; we have $P/Q' = \frac{1}{2z} e^{iz}$, and, consequently, the residue of f at the pole i is equal to $-\frac{i}{2e}$.

The case of a multiple pole. Let $f(z) = \frac{1}{(z - z_0)^k} g(z)$, where $g(z)$ is holomorphic in a neighbourhood of the point z_0 with $g(z_0) \neq 0$. The residue of $f(z)$ is equal to the coefficient of $(z - z_0)^{k-1}$ in the Taylor expansion of $g(z)$ at the point z_0. The problem is reduced, then, to finding a limited expansion of $g(z)$. To this end, it is often convenient to take a new variable $t = z - z_0$.

Example. Let $f(z) = \frac{e^{iz}}{z(z^2 + 1)^2}$ and let us calculate residue of $f(z)$ at the double pole $z = i$. In this case,

$$g(z) = \frac{e^{iz}}{z(z + i)^2}.$$

Put $z = i + t$, so we must find coefficient of t in the Taylor expansion of

$$h(t) = \frac{e^{i(i+t)}}{(i + t)(2i + t)^2}.$$

It is sufficient to write down the limited expansion of degree 1 of each of the terms

$$e^{i(i+t)} = e^{-1}(1 + it + \cdots),$$
$$(i + t)^{-1} = -i(1 - it)^{-1} = -i(1 + it + \cdots),$$
$$(2i + t)^{-2} = -\frac{1}{4}\left(1 - \frac{i}{2}t\right)^{-2} = -\frac{1}{4}(1 + it + \cdots),$$

whence

$$h(t) = \frac{i}{4e}(1 + 3it + \cdots),$$

and the required residue is $-\frac{3}{4e}$.

Application. Residue of a logarithmic derivative. Let $f(z)$ be a meromorphic function in a neighbourhood of z_0. We propose to find the residue of the logarithmic derivative f'/f at the point z_0. We have

$$f(z) = (z - z_0)^k g(z)$$

where g is holomorphic at the point z_0 and $g(z_0) \neq 0$; the integer k is $\geqslant 0$

if f is holomorphic at z_0, and $k < 0$ if z_0 is a pole of f; taking the logarithmic derivative of the two sides gives

$$f'/f = \frac{k}{z - z_0} + g'/g;$$

thus f'/f has z_0 as a *simple pole* and *the residue of this pole is equal to the integer* k, the order of multiplicity of the zero or pole z_0 (counted positively for a zero and negatively for a pole).

4. Application to finding the number of poles and zeros of a meromorphic function.

Proposition 4. 1. *Let $f(z)$ be a meromorphic function which is not constant in an open set* D *and let* Γ *be the oriented boundary of a compact set* K *contained in* D. *Suppose that the function f has no poles on Γ and does not take the value a on Γ. Then,*

$$(4.\ 1) \qquad \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)\,dz}{f(z) - a} = \mathrm{Z} - \mathrm{P},$$

where Z *denotes the sum of the orders of multiplicity of the roots of the equation*

$$f(z) - a = 0$$

contained in K, *and* P *denotes the sum of the orders of multiplicity of the poles of f contained in* K.

This is an immediate consequence of the residue theorem and of the explicit calculation of the residues of the function $\dfrac{f'(z)}{f(z) - a}$.

In particular, when f is holomorphic, the integral on the left hand side of (4. 1) is equal to the number of zeros of $f(z) - a$ contained in K, it being understood that each zero is counted as many times as its order of multiplicity.

You will notice that the value of the integral on the left hand side of (4. 1) is equal to the quotient by 2π of *the variation of the argument of* $f(z) - a$ when z describes the closed path Γ (cf. chapter II, § 1, no. 5).

Proposition 4. 2. *Let z_0 be a root of order k of the equation $f(z) = a$, f being a non-constant, holomorphic function in some neighbourhood of z_0. For any sufficiently small neighbourhood* V *of z_0, and for any b sufficiently near to a and $\neq a$, the equation $f(z) = b$ has exactly k simple solutions in* V.

For, let γ be a circle centred at z_0 with sufficiently small radius to ensure that z_0 is the only solution to the equation $f(z) = a$ contained in the closed disc bounded by γ. Suppose also that the radius of γ is sufficiently small

to ensure that $f'(z)$ is $\neq 0$ at any point of the disc except the centre z_0. We consider the integral

$$(4.\ 2) \qquad \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - b}.$$

We know that (4. 2) remains constant when b varies in a connected component of the complement of the image of γ under f (cf. chapter II, § 1, no. 8). Thus, for any b sufficiently near to a, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - b} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)\,dz}{f(z) - a} = k,$$

and, consequently, the equation $f(z) = b$ has exactly k roots in the interior of γ, if each root is counted with its order of multiplicity. But, for b sufficiently near to a but $\neq a$, the roots of the equation $f(z) = b$ are all simple because the derivative $f'(z)$ is $\neq 0$ at any point of z sufficiently near to z_0 and $\neq z_0$. Hence, proposition 4. 2 is proved.

5. Application to doubly periodic functions

Let e_1 and e_2 be two complex numbers, which are linearly independent over the real field **R**, that is to say, such that $e_1 \neq 0$ and that their quotient e_2/e_1 is not real. The totality of vectors of the form $n_1e_1 + n_2e_2$, where n_1 and n_2 are arbitrary integers, forms a discrete subgroup Ω of the additive group of the field **C**. We say that a function $f(z)$ defined on the plane has the group Ω as group of periods if

$$(5.\ 1) \qquad f(z + n_1e_1 + n_2e_2) = f(z)$$

for all z and for all integers n_1 and n_2. A necessary and sufficient condition for this is that

$$(5.\ 2) \qquad f(z + e_1) = f(z) \qquad f(z + e_2) = f(z).$$

Let z_0 be any complex number. We consider the (closed) parallelogram with vertices z_0, $z_0 + e_1$, $z_0 + e_2$, $z_0 + e_1 + e_2$. It consists of all points of the form $z_0 + t_1e_1 + t_2e_2$, where $0 \leqslant t_1 \leqslant 1$ and $0 \leqslant t_2 \leqslant 1$. Such a parallelogramm is called a *parallelogram of periods* with first vertex z_0. Let $f(z)$ now be a meromorphic function in the whole plane which has Ω as its group of periods, and choose z_0 in such a way that $f(z)$ has no poles on the boundary γ of parallelogram of periods with z_0 as first vertex. We can consider the integral $\int_\gamma f(z)\,dz$, whose value is zero because of the periodicity; for

$$\int_\gamma f(z)\,dz = \int_0^1 [f(z_0 + te_1) - f(z_0 + e_2 + te_1)]\,dt + \int_0^1 [f(z_0 + e_1 + te_2) - f(z_0 + te_2)]\,dt.$$

By appling this result to the logarithmic derivative f'/f and using proposition 4. 1, we obtain :

PROPOSITION 5. 1. *If $f(z)$ is a non-constant meromorphic function in the whole plane which has Ω as group of periods, the number of zeros of this function contained in a parallelogramm of periods is equal to the number of poles contained in the same parallelogram, if no zeros or poles of the function f occur on the boundary of the parallelogram.*

COROLLARY. *A holomorphic function in $\mathbf{C}$ having Ω as group of periods is constant.*

Otherwise, the number of zeros of $f(z) - a$ would be equal to the number of poles, that is zero; but, this is true for all a, which is absurd.
Moreover, consider the function $zf'(z)/(f(z) - a)$. It is not periodic, so we can no longer say that its integral round the boundary γ of some parallelogram of periods is zero. We shall show that the value of the integral

$$(5.3) \qquad \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)\,dz}{f(z) - a}$$

belongs to the group Ω of periods. For, it is equal to

$$-\frac{e_2}{2\pi i}\int_{\gamma_1}\frac{f'(z)\,dz}{f(z)-a} + \frac{e_1}{2\pi i}\int_{\gamma_2}\frac{f'(z)\,dz}{f(z)-a},$$

where γ_1 denotes the side of the parallelogram starting at z_0 and ending at $z_0 + e_1$, and γ_2 denotes the side of the parallelogram starting at z_0 and ending at $z_0 + e_2$. However, the values of the integrals $\frac{1}{2\pi i}\int_{\gamma_1}\frac{f'(z)\,dz}{f(z)-a}$ and $\frac{1}{2\pi i}\int_{\gamma_2}\frac{f'(z)\,dz}{f(z)-a}$ are *integers*.

On the other hand, the integral (5. 3) is equal to the sum of the residues of the function $zf'(z)/(f(z) - a)$. Let us calculate these residues. The poles are at most the poles of $f(z)$ and the zeros of $f(z) - a$. If β_i is a pole and k is its order of multiplicity, then the residue for this pole is equal to $-k\beta_i$. Similarly, the residue of a zero α_i of multiplicity k of $f(z) - a$ is equal to $k\alpha_i$.

This is summed up by the following :

PROPOSITION 5. 2. *Let $f(z)$ be a non-constant, meromorphic function in the whole plane having Ω as group of periods. For any complex number a, we have*

$$\sum_i \alpha_i \equiv \sum_i \beta_i \qquad \text{mod. } \Omega,$$

where the α_i denote the roots of the equation $f(z) = a$ (each occurring as often as its multiplicity) and the β_i denote the poles (occurring as often as their multiplicity) contained in a parallelogram of periods.

In particular, the sum $\sum_i \alpha_i$ taken modulo Ω is independent of a.

6. Evaluation of Integrals by the Method of Residues

We propose to evaluate definite integrals without finding a primitive of the integrand, but by interpreting the value of the integral as the sum of the residues at the singular points of a suitably chosen holomorphic function. There is no general method of dealing with this problem. We shall limit ourselves to some classical types and indicate, for each of them, the procedure by which the problem can be transformed into a residue calculation.

1 st type. Consider an integral of the form

$$I = \int_0^{2\pi} R(\sin t, \cos t)\,dt,$$

where $R(x, y)$ is a rational function without a pole on the circle $x^2 + y^2 = 1$. Put $z = e^{it}$, so that z describes the unit circle as t increases form 0 to 2π. Thus, I is equal to $2\pi i$ times the sum of the residues of the function

$$\frac{1}{iz}R\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)$$

at the poles contained in the unit disc.
We then have

$$I = 2\pi \sum \operatorname{Res}\left\{\frac{1}{z}R\left(\frac{1}{2i}\left(z - \frac{1}{z}\right), \frac{1}{2}\left(z + \frac{1}{z}\right)\right)\right\},$$

the sum extending over poles contained in the unit disc.

Example. Let $\int_0^{2\pi}\frac{dt}{a + \sin t}$, where a is a real number > 1. Then,

$$I = 2\pi \sum \operatorname{Res}\frac{2i}{z^2 + 2iaz - 1}.$$

The only pole contained in the unit disc is $z_0 = -ia + i\sqrt{a^2 - 1}$; its residue is $\frac{i}{z_0 + ia} = \frac{1}{\sqrt{a^2 - 1}}$, so $I = \frac{2\pi}{\sqrt{a^2 - 1}}$.

2 nd type. Consider an integral of the form

$$I = \int_{-\infty}^{+\infty} R(x)\, dx,$$

where R is a rational function without a real pole. We also need to assume that the integral is convergent, and a necessary and sufficient condition for this is that the principal part of $R(x)$ at infinity is of the form $\frac{1}{x^n}$ with the integer $n \geqslant 2$. An equivalent condition is that

(2. 1) $$\lim_{|x|\to\infty} xR(x) = 0.$$

To calculate this integral I, we shall integrate the function $R(z)$ of the complex variable z along the boundary γ of a half-disc of centre o and radius r situated in the half-plane $y \geqslant 0$ (fig. 6). For sufficiently large r,

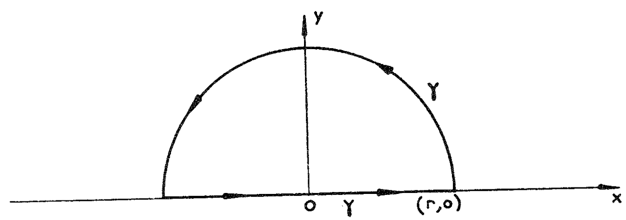

Fig. 6.

the function $R(z)$ is holomorphic on the boundary γ and the integral $\int_\gamma R(z)\, dz$ is equal to the sum of the residues of the poles of R which are inside γ. Then

(2. 2) $$\int_{-r}^{+r} R(x)\, dx + \int_{\delta(r)} R(z)\, dz = 2\pi i \sum \text{Res}\,(R(z)),$$

where $\delta(r)$ denotes the half-circle centred at o of radius r described in the positive sense, and where the summation extends over the residues of poles situated in the half-plane $y > 0$. As r tends to $+\infty$, the first integral on the left hand side of (2. 2) tends to I; we shall show that the second integral on the left hand side of (2. 2) tends to o. This will give

(2. 3) $$\int_{-\infty}^{+\infty} R(x)\, dx = 2\pi i \sum \text{Res}\,(R(z)),$$

the sum extending over all the poles of R in the upper half-plane $y > 0$. Similarly,

$$\int_{-\infty}^{+\infty} R(x)\, dx = -2\pi i \sum \text{Res}\,(R(z)),$$

the sum this time being taken over all the poles in the lower half-plane $y < 0$.
It remains to be proved that $\int_{\delta(r)} R(z)\, dz$ tends to o as r tends to $+\infty$, which will be an immediate consequence of the following lemma:

LEMMA 1. *Let $f(z)$ be a continuous function defined in the sector*

$$\theta_1 \leqslant \theta \leqslant \theta_2,$$

r and θ denoting the modulus and argument of z. If

$$\lim_{|z|\to\infty} zf(z) = 0 \quad (\theta_1 \leqslant \arg z \leqslant \theta_2),$$

then the integral $\int f(z)\, dz$ extended over the arc of the circle $|z| = r$ contained in the sector tends to o as r tends to $+\infty$.

For, let $M(r)$ be the upper bound of $|f(z)|$ on the arc of the circle $|z| = r$. Then

$$\left|\int f(z)\, dz\right| \leqslant M(r)\, r(\theta_2 - \theta_1),$$

and the lemma follows immediately from this.

We could prove the following lemma similarly:

LEMMA 2. *Let $f(z)$ be a continuous function defined in a sector*

$$\theta_1 \leqslant \theta \leqslant \theta_2,$$

r and θ being the modulus and argument of z. If

$$\lim_{z\to 0} zf(z) = 0 \quad (\theta_1 \leqslant \arg z \leqslant \theta_2),$$

then the integral $\int f(z)\, dz$ over the arc of the circle $|z| = r$ contained in the sector tends to o as r tends to o.

Example. To evaluate the integral

$$I = \int_0^{+\infty} \frac{dx}{1 + x^6}.$$

The function $\frac{1}{1+z^6}$ has six poles, all on the unit circle; the three poles which are in the upper half-plane are

$$e^{i\frac{\pi}{6}}, \quad e^{i\frac{\pi}{2}}, \quad e^{5i\frac{\pi}{6}}.$$

The residue of such a pole is equal to $\frac{1}{6z^5} = -\frac{z}{6}$. Hence,

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{1+x^6} = -\frac{\pi i}{6}\left(e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{2}} + e^{5i\frac{\pi}{6}}\right)$$
$$= \frac{\pi}{6}\left(2\sin\frac{\pi}{6} + 1\right) = \frac{\pi}{3}.$$

3 rd type. We shall study integrals of the the form

$$I = \int_{-\infty}^{+\infty} f(x)\, e^{ix}\, dx,$$

where $f(z)$ is a holomorphic function in a neighbourhood of each point of the closed half-plane $y \geqslant 0$, except perhaps at a *finite* number of points. We shall first consider the case when the singularities are not on the real axis. Then, the integral

$$\int_{-r}^{+r} f(x)\, e^{ix}\, dx$$

has a meaning, and, as r tends to $+\infty$, its value tends to

$$\int_{-\infty}^{+\infty} f(x)\, e^{ix}\, dx$$

if the latter integral is convergent.

We shall prove the following result :

Proposition 3. 1. *If* $\lim\limits_{|z|\to\infty} f(z) = 0$ *for* $y \geqslant 0$, *then*

(3. 1)
$$\lim_{r\to+\infty} \int_{-r}^{+r} f(x)\, e^{ix}\, dx = 2\pi i \sum \text{Res}\,(f(z)\, e^{iz}),$$

the summation extending over the singularities of $f(z)$ contained in the upper half-plane $y > 0$.

First, we note that, if the integral $\int_{-\infty}^{+\infty} |f(x)|\, dx$ is convergent, the proposed integral $\int_{-\infty}^{+\infty} f(x) e^{ix}\, dx$ is absolutely convergent; relation (3. 1) then gives

(3. 2)
$$\int_{-\infty}^{+\infty} f(x) e^{ix}\, dx = 2\pi i \sum \text{Res}\,(f(z)\, e^{iz}).$$

The integral $\int_{-\infty}^{+\infty} f(x)\, e^{ix}\, dx$ can also be convergent without being absolutely convergent; for example it is well-known that, if the function $f(x)$ is real and monotonic for $x > 0$ and tends to 0 as x tends to $+\infty$, then the

integral $\int_0^{+\infty} f(x)\, e^{ix}\, dx$ is convergent (by applying the second mean value theorem); in such a case, relation (3. 2) is again true.

Before starting the proof of proposition 3. 1, we note that $|e^{iz}| \leqslant 1$ in the half-plane $y \geqslant 0$. This leads us to integrate on the half-plane $y \geqslant 0$ along the contour already used above for the second type of integral. With the same notations as in (2. 2), we shall show that *the integral* $\int_{\delta(r)} f(z) e^{iz}\, dz$ *tends to 0 as r tends to $+\infty$.* Proposition 3. 1 will follow obviously from this.

If we knew that $\lim\limits_{|z|\to\infty} z f(z) = 0$, it would be sufficient to apply lemma 1. Relation (3. 1) is thus proved in this case. For example, consider the integral

$$\int_0^{+\infty} \frac{\cos x}{x^2+1}\, dx = \frac{1}{2}\,\text{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{ix}}{x^2+1}\, dx\right);$$

its value is equal to $\pi i \sum \text{Res}\left(\frac{e^{iz}}{z^2+1}\right)$, the summation extending over poles situated in the upper half-plane. There is only one pole $z = i$, it is simple, and its residue is

$$\frac{e^{-1}}{2i}, \qquad \text{whence} \qquad \int_0^{+\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{2e}.$$

To prove that $\int_{\delta(r)} f(z)\, e^{iz}\, dz$ tends to zero *with only the hypothesis of proposition* 3. 1, we use the following lemma :

Lemma 3. *Let $f(z)$ be a function defined in a sector of the half-plane* $y \geqslant 0$. *If* $\lim\limits_{|z|\to\infty} f(z) = 0$, *the integral* $\int f(z)\, e^{iz}\, dz$ *extended over the arc of the circle* $|z| = r$ *contained in the sector tends to 0 as r tends to* $+\infty$.

For, let us put $z = re^{i\theta}$ and let $M(r)$ be the upper bound of $|f(re^{i\theta})|$ as θ varies, the point $e^{i\theta}$ remaining in the sector. Then

(3. 3)
$$\left|\int f(z)\, e^{iz}\, dz\right| \leqslant M(r) \int_0^{\pi} e^{-r\sin\theta}\, r\, d\theta$$

We shall show that $\int_0^{\pi} e^{-r\sin\theta} r\, d\theta$ is bounded above by a fixed number independent of r, which will complete the proof of lemma 3. In fact,

(3. 4)
$$\int_0^{\pi} e^{-r\sin\theta} r\, d\theta = 2\int_0^{\frac{\pi}{2}} e^{-r\sin\theta} r\, d\theta \leqslant \pi.$$

Proof of (3. 4) : we have

$$\frac{2}{\pi} \leqslant \frac{\sin \theta}{\theta} \leqslant 1 \quad \text{for} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2},$$

whence

$$\int_0^{\frac{\pi}{2}} e^{-r \sin \theta} r \, d\theta \leqslant \int_0^{\frac{\pi}{2}} e^{-\frac{2}{\pi} r\theta} r \, d\theta \leqslant \int_0^{+\infty} e^{-\frac{2}{\pi} r\theta} r \, d\theta = \frac{\pi}{2}.$$

Hence the proposition 3. 1 is completely proved.

We now examine the case when $f(z)$ can have singularities on the real axis. We shall limit ourselves to one example, the case when $f(z)$ has a simple pole at the origin. In this case, it is appropriate to modify the path of integration to make it by-pass the origin along a semicircle $\gamma(\varepsilon)$ of small radius $\varepsilon > 0$ centred *at the origin and situated in the upper half-plane* (fig. 7). We use the following lemma :

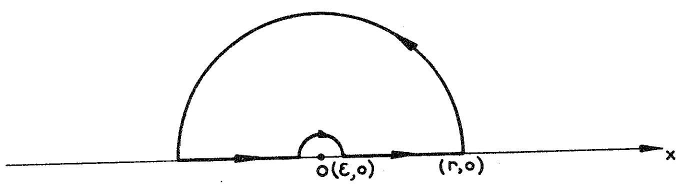

Fig. 7.

Lemma 4. *If $z = 0$ is a simple pole of $g(z)$, then*

(3. 4) $$\lim_{\varepsilon \to 0} \int_{\gamma(\varepsilon)} g(z) \, dz = \pi i \operatorname{Res}(g, 0),$$

$\gamma(\varepsilon)$ being described in the direction of increasing argument.

For, we have $g(z) = \frac{a}{z} + h(z)$, where h is a holomorphic function at the origin. The integral $\int_{\gamma(\varepsilon)} h(z) \, dz$ tends to 0 as ε tends to 0, and the integral $\int_{\gamma(\varepsilon)} \frac{a}{z} \, dz$ is equal to $\pi i a$. This gives relation (3. 4).

This lemma will be applied to the function $g(z) = f(z) e^{iz}$.

Example. To evaluate the integral

$$I = \int_0^{+\infty} \frac{\sin x}{x} \, dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{\sin x}{x} \, dx = \frac{1}{2i} \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x} \, dx + \int_{+\varepsilon}^{+\infty} \frac{e^{ix}}{x} \, dx \right].$$

By figure 7, this is equal to

$$\frac{1}{2i} \lim_{\varepsilon \to 0} \int_{\gamma(\varepsilon)} \frac{e^{iz}}{z} \, dz = \frac{\pi}{2} \operatorname{Res}\left(\frac{e^{iz}}{z}, 0\right) = \frac{\pi}{2}.$$

Important note. If, instead of $\int_{-\infty}^{+\infty} f(x) e^{ix} \, dx$, we had to calculate the integral $\int_{-\infty}^{+\infty} f(x) \, e^{-ix} \, dx$, then it would be necessary to integrate *in the lower half-plane* instead of the upper half-plane; for, the function $|e^{-iz}|$ is bounded in the lower half-plane $y \leqslant 0$ and it is in this half-plane that lemma 3 is applicable (*mutatis mutandis*). More generally, an integral of the form $\int_{-\infty}^{+\infty} f(x) e^{ax} \, dx$ (where a is complex constant) can be evaluated by integrating in the half-plane where $|e^{az}| \leqslant 1$.

Remember that $\sin z$ and $\cos z$ *are not bounded in any half-plane.* To evaluate integrals of the forms

$$\int_{-\infty}^{+\infty} f(x) \sin^n x \, dx, \qquad \int_{-\infty}^{+\infty} f(x) \cos^n x \, dx,$$

one always expresses the trigonometric functions in terms of complex exponentials so that the preceding methods can be applied.

4 th type. Consider integrals of the form

$$I = \int_0^{+\infty} \frac{R(x)}{x^\alpha} \, dx,$$

where α denotes a real number such that $0 < \alpha < 1$, and $R(x)$ is a rational function with no pole on the positive real axis $x \geqslant 0$. It is clear that such an integral converges for the lower limit of integration 0. A necessary and sufficient condition for it to converge at the upper limit $+\infty$ is that the principal part of $R(x)$ at infinity is of the form $\frac{1}{x^n}$ with $n \geqslant 1$: in other words, it is necessary and sufficient that

(4. 1) $$\lim_{x \to +\infty} R(x) = 0.$$

To calculate such an integral, we consider the function $f(z) = \frac{R(z)}{z^\alpha}$ of the complex variable z, defined in the plane with the positive real axis $x \geqslant 0$ excluded. Let D be the open set thus defined. It is necessary to specify the branch of z^α chosen in D : we shall take the branch of the argument of z between 0 and 2π. With this convention, integrate $\frac{R(z)}{z^\alpha}$ along the closed path $\delta(r, \varepsilon)$ defined as follows: we describe, first, the real axis from $\varepsilon > 0$ to $r > 0$, then the circle $\gamma(r)$ of centre 0 and radius r in the positive sense, then the real axis from r to ε, and, finally, the circle $\gamma(\varepsilon)$ of centre 0 and radius ε in the negative sense (cf. figure 8). The integral

$$\int_{\delta(r, \varepsilon)} \frac{R(z)}{z^\alpha} \, dz$$

is equal to the sum of the residues of the poles of $\frac{R(z)}{z^\alpha}$ contained in D, if r has been chosen sufficiently large and ε sufficiently small. We have

$$\int_{\delta(r,\varepsilon)} \frac{R(z)}{z^\alpha}\,dz = \int_{\gamma(r)} \frac{R(z)}{z^\alpha}\,dz + \int_{\gamma(\varepsilon)} \frac{R(z)}{z^\alpha}\,dz + (1 - e^{-2\pi i\alpha})\int_\varepsilon^r \frac{R(x)}{x^\alpha}\,dx$$

because, when the argument of z is equal to 2π, we have $z^\alpha = e^{2\pi i\alpha}|z|^\alpha$. Since the argument of z remains bounded, $zf(z)$ tends to 0 when z tends

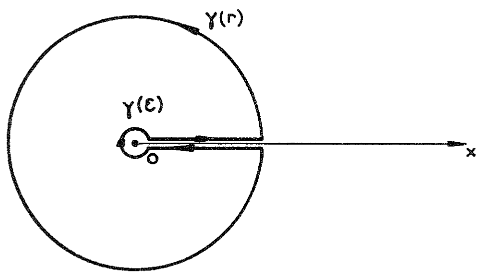

Fig. 8.

to 0 or when $|z|$ tends to infinity; thus the integrals along $\gamma(r)$ and $\gamma(\varepsilon)$ tend to 0 as r tends to ∞ and ε tends to 0 (lemmas 1 and 2). On the limit, we have

(4. 2) $$(1 - e^{-2i\pi\alpha})I = 2\pi i \sum \operatorname{Res}\left(\frac{R(z)}{z^\alpha}\right),$$

and this relation allows us to calculate I.

Example. To evaluate $I = \int_0^{+\infty} \frac{dx}{x^\alpha(1+x)}$, $(0 < \alpha < 1)$. Here we have $R(z) = \frac{1}{1+z}$; there is only one pole $z = -1$; because the branch of the argument of z is equal to π at this point, the residue of $\frac{R(z)}{z^\alpha}$ at this pole is equal to $\frac{1}{e^{\pi i\alpha}}$. Relation (4. 2) then gives

$$I = \frac{\pi}{\sin \pi\alpha}.$$

5 *th type.* Let us consider integrals of the form

$$\int_0^{+\infty} R(x) \log x\,dx,$$

where R is a rational function with no pole on the positive real axis $x \geqslant 0$, and such that $\lim_{x\to+\infty} xR(x) = 0$. This last condition ensures that the integral is convergent.

We consider the same open set D as for integrals of the 4th type and the same path of integration. Here again, we must specify the branch chosen for $\log z$; we shall choose the argument of z between 0 and 2π. For a reason which will soon be apparent, we shall integrate, not the function $R(z)\log z$, but the function $R(z)(\log z)^2$. Here again the integrals along the circles $\gamma(r)$ and $\gamma(\varepsilon)$ tend to 0 as r tends to ∞, and ε tends to 0 because of lemmas 1 and 2. When the argument of z is equal to 2π, we have

$$\log z = \log x + 2\pi i,$$

x denoting the modulus of z. Thus we have the relation

$$\int_0^{+\infty} R(x)(\log x)^2 dx - \int_0^{+\infty} R(x)(\log x + 2\pi i)^2 dx = 2\pi i \sum \operatorname{Res}\{R(z)(\log z)^2\};$$

hence,

(5. 1) $$-2\int_0^{+\infty} R(x)\log x\,dx - 2\pi i\int_0^{+\infty} R(x)\,dx = \sum \operatorname{Res}\{R(z)(\log z)^2\}.$$

Basically this only gives one relation between the two integrals $\int_0^{+\infty} R(x)\,dx$ and $\int_0^{+\infty} R(x)\log x\,dx$. Let us suppose, however, that the rational function R is *real* (that is, it takes real values for x real); by separating real and imaginary parts of the relation (5. 1), we obtain the two relations

(5. 2) $$\int_0^{+\infty} R(x)\log x\,dx = -\frac{1}{2}\operatorname{Re}\left(\sum \operatorname{Res}\{R(z)(\log z)^2\}\right),$$

(5. 3) $$\int_0^{+\infty} R(x)\,dx = -\frac{1}{2\pi}\operatorname{Im}\left(\sum \operatorname{Res}\{R(z)(\log z)^2\}\right).$$

The summation extends over all the poles of the rational function $R(z)$ contained in D.

Example. To evaluate the integral

$$I = \int_0^{+\infty} \frac{\log x}{(1+x)^3}\,dx.$$

The residue of $\frac{(\log z)^2}{(1+z)^3}$ at the pole $z = -1$ is equal to the coefficient of t^2 in the limited expansion of $(i\pi + \log(1-t))^2$; it is therefore $1 - i\pi$, and we find $I = -\frac{1}{2}$.

Note. By integrating the function $R(z)\log z$ in the same way, we obtain the formula

(5. 4) $$\int_0^{+\infty} R(x)\,dx = -\sum \operatorname{Res}\{R(z)\log z\}.$$

The above method can also be applied in some cases when the rational function R has a *simple pole* at $x = 1$; in this case, the integral $\int_0^{+\infty} R(x) \log x \, dx$ still has a meaning because the principal branch of $\log z$ has a simple zero at the point 1. It is then necessary to modify the contour of integration which we used before; when we integrate along the positive real axis with the argument of z equal to 2π, we must by-pass

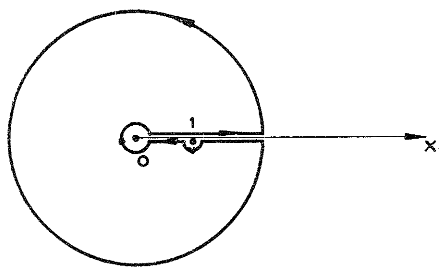

Fig. 9.

the point $z = 1$ along a semi-circle of centre 1 and small radius (fig. 9). The reader should prove that, when the function R is real, it satisfies

$$(5.5) \quad \int_0^{+\infty} R(x) \log x \, dx = \pi^2 \operatorname{Re} (\operatorname{Res} (R, 1)) - \frac{1}{2} \operatorname{Re} \left(\sum \operatorname{Res} (f)\right),$$

where f denotes the function $R(z) (\log z)^2$ and where the summation extends over all the poles of f other than $z = 1$. For example, it can be verified that

$$\int_0^{+\infty} \frac{\log x}{x^2 - 1} dx = \frac{\pi^2}{4}.$$

Exercises

1. Let $f(z)$ be holomorphic in $|z| < R$, $R > 1$. Evaluate the integrals

$$\int_{|z|=1} \left(2 \pm \left(z + \frac{1}{z}\right)\right) \frac{f(z)}{z} dz$$

taken over the unit circle in the positive sense in two different ways and deduce the following relations:

$$\begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^{2\pi} f(e^{i\theta}) \cos^2 \frac{\theta}{2} d\theta = 2f(0) + f'(0), \\ \dfrac{2}{\pi} \displaystyle\int_0^{2\pi} f(e^{i\theta}) \sin^2 \frac{\theta}{2} d\theta = 2f(0) - f'(0). \end{cases}$$

2. Let $f(z)$ be a holomorphic function in an open set containing the disc $|z| \leqslant R$ and let γ be the image of the circle $|z| = R$ under the mapping $z \to f(z)$; suppose that f is simple, i.e. $f(z) \neq f(z')$ if $z \neq z'$. Show that the length L of γ is equal to $R \int_0^{2\pi} |f'(Re^{i\theta})| \, d\theta$; deduce that

$$L \geqslant 2\pi R |f'(0)|.$$

Show that, under the same conditions, the area A of the image of the closed disc $|z| \leqslant R$ under the same mapping is given by

$$A = \iint_{|z| \leqslant R} |f'(x + iy)|^2 \, dx \, dy;$$

deduce the inequality

$$A \geqslant \pi R^2 |f'(0)|^2.$$

(Change to polar coordinates and note that, for $0 \leqslant r \leqslant R$,

$$|f'(0)|^2 = \frac{1}{4\pi^2} \left| \int_0^{2\pi} f'(re^{i\theta}) \, d\theta \right|^2 \leqslant \frac{1}{4\pi^2} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \int_0^{2\pi} d\theta = \frac{1}{2\pi} \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta,$$

because of the Cauchy-Schwarz inequality for integrals.)

3. Show that, if $f(z)$ is holomorphic in an open set containing the closed disc $|z| \leqslant 1$, then

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{\overline{f(z)}}{z - a} dz = \begin{cases} \overline{f(0)} & \text{if } |a| < 1, \\ \overline{f(0)} - \overline{f(1/\bar{a})} & \text{if } |a| > 1, \end{cases}$$

where the integral is taken in the positive sense. (Use exercise 1.b) of chapter II and Cauchy's integral formula.)

4. Let $f(z)$ be a holomorphic function in the whole plane, and suppose that there is an integer n and two positive real numbers R and M such that

$$|f(z)| \leqslant M . |z|^n \quad \text{for} \quad |z| \geqslant R.$$

Show then that $f(z)$ is a polynomial of degree at most n.

5. Let f be a non-constant, holomorphic function in a connected open set D, and let D′ be a connected open set whose closure $\overline{D'}$ is a compact subset of D. Show that, if $|f(z)|$ is constant on the frontier of D′, there is at least one zero of $f(z)$ in D′. (Use *reductio ad absurdum* by considering $1/f(z)$.)

6. Let D be a bounded, connected, open set and consider n points P_1, $P_2, \ldots, P_n$ in the plane $\mathbf{R}^2$. Show that the product $\overline{PP_1}.\overline{PP_2}\ldots\overline{PP_n}$ of the distances from a point P, which varies in the closure $\overline{D}$, to the points $P_1, P_2, \ldots, P_n$, attains its maximum at a frontier point of D.

7. Let $f(z)$ be a holomorphic function in the disc $|z| < R_1$ and put $M(r) = \sup_{|z|=r} |f(z)|$, for $0 \leqslant r < R$. Show that

a) $M(r)$ is a continuous, monotonic increasing (in the broad sense), function of r in $0 \leqslant r < R$,

b) if $f(z)$ is not constant, $M(r)$ is strictly increasing.

8. Hadamard's three circles theorem : let $f(z)$ be a holomorphic function in an open set containing the closed annulus

$$r_1 \leqslant |z| \leqslant r_2 \qquad (0 < r_1 < r_2),$$

and put $M(r) = \sup_{|z|=r} |f(z)|$ for $r_1 < r < r_2$. Show that the following inequality holds

$$(1) \qquad M(r) \leqslant M(r_1)^{\frac{\log r_2 - \log r}{\log r_2 - \log r_1}} . M(r_2)^{\frac{\log r - \log r_1}{\log r_2 - \log r_1}},$$

for $r_1 \leqslant r \leqslant r_2$. (Apply the maximum modulus principle to the function $z^p(f(z))^q$ with p, q integers and $q > 0$; choose α real such that $r_1^\alpha M(r_1) = r_2^\alpha M(r_2)$, and a sequence of pairs of integers (p_n, q_n) such that $\lim_{n\to\infty} p_n/q_n = \alpha$.) Verify that inequality (1) expresses that $\log M(r)$ is a convex function of $\log r$ for $r_1 \leqslant r \leqslant r_2$.

9. Let $f(z)$ be holomorphic in $|z| < R$ and put

$$I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta, \qquad \text{for} \qquad 0 \leqslant r < R.$$

Show that, if a_n denotes the n-th Talylor coefficient of $f(z)$ at the point $z = 0$, then

$$I_2(r) = \sum_{n\geqslant 0} |a_n|^2 r^{2n};$$

deduce that, if $0 \leqslant r < R$,

(i) $I_2(r)$ is a continuous, monotonic increasing (in the broad sense), function of r;

(ii) $|f(0)|^2 \leqslant I_2(r) \leqslant (M(r))^2$, ($M(r)$ has the same meaning as in 7.);

(iii) $\log I_2(r)$ is a convex function of $\log r$ in the case when f is not identically zero (show that, if

$$s = \log r, \quad J(s) = I_2(e^s) = \sum_{n\geqslant 0} |a_n|^2 e^{2ns}, \quad \text{then} \quad (\log J)'' = \frac{J''J - (J')^2}{J^2};$$

to show that $JJ'' - (J')^2 \geqslant 0$, use the Cauchy-Schwarz inequality for absolutely convergent series :

$$\Big|\sum_{n\geqslant 0} \alpha_n \bar{\beta}_n\Big|^2 \leqslant \Big(\sum_{n\geqslant 0} |\alpha_n|^2\Big)\Big(\sum_{n\geqslant 0} |\beta_n|^2\Big)\Big).$$

10. Let f be a holomorphic function in the disc $|z| < 1$, such that $|f(z)| < 1$ in this disc; if there exist two distinct points a and b in the disc such that $f(a) = a$ and $f(b) = b$, show that $f(z) = z$ in the disc. (Consider the function $g(z) = \dfrac{h(z) - a}{1 - \bar{a}h(z)}$, with $h(z) = f\Big(\dfrac{z + a}{1 + \bar{a}z}\Big)$, for which $g(0) = 0$, $g\Big(\dfrac{b - a}{1 - \bar{a}b}\Big) = \dfrac{b - a}{1 - \bar{a}b}$, and $|g(z)| < 1$ in the disc.)

11. Let f be a holomorphic function in an open set containing the disc $|z| \leqslant R$. For $0 \leqslant r \leqslant R$, put

$$A(r) = \sup_{0\leqslant\theta\leqslant 2\pi} \operatorname{Re}(f(re^{i\theta})).$$

(i) Show that $A(r)$ is a continuous, monotonic increasing (in the broad sense), function of r (note that $e^{\operatorname{Re} f(z)} = |e^{f(z)}|$).

(ii) Show that, if $f(0) = 0$ also, then, for $0 \leqslant r < R$,

$$M(r) \leqslant \frac{2r}{R - r} A(R).$$

(Consider the function $g(z) = f(z)/(2A(R) - f(z))$.)

(iii) Show that, for $0 \leqslant r < R_1$

$$M(r) \leqslant \frac{2r}{R - r} A(R) + \frac{R + r}{R - r} |f(0)|.$$

12. Let x be a complex parameter.

(i) Show that the Laurent expansion of the function

$$\exp\Big(x\Big(z + \frac{1}{z}\Big)\Big/2\Big)$$

at the origin $z = 0$, is of the following form :

$$\exp\Big(x\Big(z + \frac{1}{z}\Big)\Big/2\Big) = a_0 + \sum_{n\geqslant 1} a_n\Big(z^n + \frac{1}{z^n}\Big) \qquad \text{for} \qquad 0 < |z| < +\infty,$$

with

$$a_n = \frac{1}{\pi}\int_0^\pi e^{x\cos t} \cos nt \, dt, \qquad \text{for} \qquad n \geqslant 0.$$

Show similarly that the function $\exp\left(x\left(z-\frac{1}{z}\right)/2\right)$ has the expansion

$$\exp\left(x\left(z-\frac{1}{z}\right)/2\right)=b_0+\sum_{n\geqslant 1} b_n\left(z^n+\frac{(-1)^n}{z^n}\right) \quad \text{for} \quad 0<|z|<+\infty,$$

with

$$b_n=\frac{1}{\pi}\int_0^{\pi}\cos(nt-x\sin t)\,dt, \quad \text{for} \quad n\geqslant 0.$$

(Note that, if $z'=-1/z$, then

$$\exp(x(z'-1/z')/2)=\exp(x(z-1/z)/2) \quad \text{for} \quad 0<|z|<+\infty.)$$

(ii) Let m, n be two integers $\geqslant 0$. Show that

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{(z^2\pm 1)^m\,dz}{z^{m+n+1}}=\begin{cases}\dfrac{(\pm 1)^p(n+2p)!}{p!\,(n+p)!}, & \text{if } m=n+2p, \text{ with } p \text{ an integer} \geqslant 0,\\ 0 & \text{otherwise,}\end{cases}$$

and deduce the power series expansions of a_n, b_n as functions of the parameter x (b_n, as a function of x, is called Bessel's function of the first kind).

13. Let $f(z)$ be a meromorphic function in a neighbourhood of the origin $z=0$ with a simple pole at the origin. Let x be any complex number. Show that the Laurent expansion of the function of z

$$\frac{f'(z)}{f(z)-x}$$

is of the form

$$-\frac{1}{z}+u_1+u_2z+\cdots+u_{n+1}z^n+\cdots,$$

where u_n is a polynomial in x of degree n. (An identification can be made by using the Taylor expansion of the function $zf(z)$.)

14. Let $f(z)$ be a holomorphic function in the upper half-plane P^+ defined by $\operatorname{Im}(z)>0$, and suppose that $f(z+1)=f(z)$ for all $z\in P^+$. Show that there is a holomorphic function $g(t)$ in the punctured disc $0<t<1$, such that

$$f(z)=g(e^{2\pi iz}), \quad \text{for} \quad z\in P^+.$$

Deduce that $f(z)$ has an expansion of the form

$$f(z)=\sum_{-\infty<n<+\infty} a_n e^{2\pi inz},$$

where

$$a_n=\int_0^1 f(x+iy)e^{-2\pi in(x+iy)}\,dx,$$

for any $y>0$. Show that this series is normally convergent in any compact subset of P^+. Show also that, if there exists a constant $M>0$ and an integer n_0 such that

$$|f(x+iy)|\leqslant Me^{2\pi n_0 y} \quad \text{for all sufficiently large } y,$$

and uniformly in x, then the expansion is of the form

$$f(z)=\sum_{n\geqslant -n_0} a_n e^{2\pi inz}.$$

15. (i) Show that the function $f(z)=1/(e^z-1)$ is meromorphic in the whole plane $\mathbf{C}$ and has simple poles at the points $z=2p\pi i$, p an integer. Calculate its Laurent expansion at the point $z=2p\pi i$. If a_n ($n\geqslant -1$) denote the coefficients of the expansion for $p=0$, show that $a_{2q}=0$ for $q=1,2,\ldots$, and if

$$B_n=(-1)^{n-1}(2n)!\,a_{2n-1}, \quad \text{for } n\geqslant 1,$$

show that the following recurrence relation holds :

$$\frac{1}{(2n+1)!}-\frac{1}{2(2n)!}+\sum_{1\leqslant\nu\leqslant n}\frac{(-1)^{\nu-1}B_\nu}{(2\nu)!\,(2n-2\nu+1)!}=0,$$

for $n\geqslant 1$ (by equating coefficients on the two sides of the relation

$$\left(a_{-1}/z+\sum_{n\geqslant 0}a_nz^n\right)\left(\sum_{m\geqslant 1}z_m/m!\right)=1\Big).$$

(ii) For $n\geqslant 1$, put

$$f_{2n}(z)=\frac{1}{z^{2n}(e^z-1)},$$

and let γ_m be the perimeter of the square whose vertices have complex coordinates $\pm(2m+1)\pi\pm(2m+1)\pi i$. Show that

$$|f_{2n}(z)|\leqslant 2/((2m+1)\pi)^{2n} \quad \text{if } z \text{ is on } \gamma_m,$$

and deduce, by integrating $f_{2n}(z)$ round the contour γ_m in the positive sense and letting $m\to\infty$, that

$$\sum_{p\geqslant 1}1/p^{2n}=\frac{(2\pi)^{2n}B_n}{2(2n)!}.$$

(N.B. The numbers B_n are called the Bernoulli numbers.)

16. Let c be an essential singularity of a holomorphic function $f(z)$ in the punctured disc D given by $0 < |z - c| < \rho$.

(i) For any $\gamma \in \mathbf{C}$ and $\varepsilon > 0$, show that there exists a $z' \in D$ and a real number $\varepsilon' > 0$ such that

$$\overline{\Delta}(f(z'), \varepsilon') \subset \Delta \cap \Delta(\gamma, \varepsilon),$$

where Δ denotes the image of D under the mapping $z \to f(z)$ and where $\Delta(b, r)$ (resp. $\overline{\Delta}(b, r)$) is the open (resp. closed) disc of radius r centred at b (note that proposition 4. 2 of § 5 implies that Δ is *open* (this also follows from the theorem in chapter VI, § 1, no. 3), and use Weierstrass' theorem, no. 4 of § 4).

(ii) Let D be the punctured disc $0 < |z - c| < \rho/2^n$ and let Δ_n be its image under f, for $n \geqslant 0$. Given $\gamma_0 \in \mathbf{C}$, $\varepsilon_0 > 0$, show, by induction on n, the existence of a sequence $(\varepsilon_n)_{n \geqslant 1}$ of positive real numbers and a sequence $(z_n)_{n \geqslant 1}$ of points of D satisfying the following conditions :

$$z_n \in D_{n-1}, \quad \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \dots, \quad \overline{\Delta}(f(z_1), \varepsilon_1) \subset \Delta \cap \Delta(\gamma_0, \varepsilon_0)$$
$$\overline{\Delta}(f(z_{n+1}), \varepsilon_{n+1}) \subset \Delta_n \cap \Delta(f(z_n), \varepsilon_n) \quad \text{for} \quad n \geqslant 1,$$

deduce that there exists a sequence $(c_n)_{n \geqslant 0}$ of points in D such that

$$\lim c_n = c \quad \text{and} \quad f(c_n) = \gamma \quad \text{for all } n \text{, with} \quad |\gamma - \gamma_0| < \varepsilon_0,$$

and that $f(z)$ is not simple in any punctured disc $0 < |z - c| < r$ however small r is.

17. Let $\varphi : (x, y, u) \to z$ be the stereographic projection of $\mathbf{S}_2 - P$ onto $\mathbf{C}$.

(i) Express x, y and u as functions of z.

(ii) Show that, if C is a circle of $\mathbf{S}_2$, which does not pass through P, $\varphi(C)$ is a circle in the plane $\mathbf{C}$, and that, if C passes through P, $\varphi(C - P)$ is a line in $\mathbf{C}$.

(iii) Let $z_1, z_2 \in \mathbf{C}$; show that a necessary and sufficient condition for $\varphi^{-1}(z_1)$ and $\varphi^{-1}(z_2)$ to be antipodal is that $z_1\bar{z}_2 = -1$.

(iv) Show that the distance $\overline{P_1P_2}$ (in $\mathbf{R}^3$) between

$$P_1 = \varphi^{-1}(z_1) \quad \text{and} \quad P_2 = \varphi^{-1}(z_2)$$

is given by the formula

$$\overline{P_1P_2} = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}.$$

What happens to the formula when z_2 tends towards the point infinity?

18. Show that a meromorphic function on the Riemann sphere is necessarily rational. (Show first that such a function can only have a finite number of poles.)

19. Rouché's theorem. Let $f(z)$ and $g(z)$ be holomorphic functions in an open set D and let $\Gamma = (\Gamma_i)_{i \in I}$ be the oriented boundary of a compact subset K of D. If

$$|f(z)| > |g(z)| \quad \text{on} \quad \Gamma,$$

show that the number of zeros of $f(z) + g(z)$ in K is equal to the number of zeros of $f(z)$ in K. (Consider the closed paths $f \circ \Gamma_i$, $i \in I$ and apply proposition 4. 1 of § 5 and proposition 8.3 of chapter II, §1.)

Example. If $f(z)$ is holomorphic in an open set containing the closed disc $|z| \leqslant 1$ and if $|f(z)| < 1$ for $|z| = 1$, then the equation $f(z) = z^n$ has exactly n solutions in $|z| < 1$, for any integer $n \geqslant 0$.

20. Evaluate the following integrals by calculating residues :

(i) $\displaystyle\int_0^{+\infty} \frac{dx}{(a + bx^2)^n}$ $(a, b > 0)$, (ii) $\displaystyle\int_0^{+\infty} \frac{\cos 2ax - \cos 2bx}{x^2} dx$ $(a, b$ real$)$,

(iii) $\displaystyle\int_0^{+\infty} \frac{x^2 - a^2}{x^2 + a^2} \frac{\sin x}{x} dx$ $(a > 0)$, (iv) $\displaystyle\int_0^{\pi} \frac{\cos nt\, dt}{1 - 2a \cos t + a^2}$ $(|a| \neq 1)$

(integrate the function $z^n/(z - a)(z - 1/a)$ round the unit circle).

21. Integrate the function $f(z) = \dfrac{1}{(z^2 + a^2) \log z}$, where log denotes the branch such that $-\pi \leqslant \arg z \leqslant \pi$, along the closed path $\delta(r, \varepsilon)$ defined as follows : describe in turn the negative real axis from $-r$ to $-\varepsilon$, then the circle $\gamma(\varepsilon)$ of radius ε centred at 0 in the negative sense, then the negative real axis from $-\varepsilon$ to $-r$, and, finally, the circle $\gamma(r)$ of radius r centred at 0 anticlockwise $(0 < \varepsilon < a < r)$; deduce that

$$\int_0^{\infty} \frac{dx}{(x^2 + a^2)((\log x)^2 + \pi^2)} = \frac{\pi}{2a((\log a)^2 + \pi^2/4)} - \frac{1}{1 + a^2}.$$

22. Let a be > 0 and ν be real. Show that

$$\int_0^{\infty} \frac{\cos \nu x\, dx}{\cosh x + \cosh a} = \frac{\pi \sin \nu a}{\sinh \pi\nu \sinh a},$$

by integrating the function $e^{i\nu z}/(\cosh z + \cosh a)$ along the perimeter of the rectangle with vertices $\pm R$, $\pm R + 2\pi i$.

23. (i) Let n be an integer $\geqslant 2$. Show that

$$\int_0^{\infty} \frac{dx}{1 + x^n} = \frac{\pi/n}{\sin(\pi/n)},$$

by integrating the function $1/(1+z^n)$ along the contour formed by the segment $[0, R]$ of the positive real axis, the arc represented by Re^{it}, $0 \leqslant t \leqslant 2\pi/n$, and the segment represented by $re^{2\pi i/n}$, $0 \leqslant r \leqslant R$.

(ii) Let n be an integer $\geqslant 2$ and let α be a real number such that $n > 1 + \alpha > 0$. Evaluate, by the same method, the integral

$$\int_0^\infty \frac{x^\alpha\,dx}{1+x^n}.$$

24. Let p,q be two real numbers > 0 and let n be an integer $\geqslant 1$. By integrating the function $z^{n-1}e^{-z}$ along a contour analogous to the above (in exercise 23), but with a suitable choice of the angle at the origin, prove the following relations:

$$\int_0^\infty x^{n-1}e^{-px}\cos qx\,dx = \frac{(n-1)!\,\mathrm{Re}\,(p+iq)^n}{(p^2+q^2)^n},$$
$$\int_0^\infty x^{n-1}e^{-px}\sin qx\,dx = \frac{(n-1)!\,\mathrm{Im}\,(p+iq)^n}{(p^2+q^2)^n}.$$

$\left(\text{Recall that } \int_0^\infty x^{n-1}e^{-x}\,dx = (n-1)!.\right)$

25. (i) Show that the function $\pi\cot\pi z$ is meromorphic in the whole complex plane, that it has simple poles at the points $z = n$ for n an integer, and that its residue at the pole $z = n$ is equal to 1 for all n. Let

$$f(z) = P(z)/Q(z)$$

be a rational function such that $\deg Q > \deg P + 1$, and let $a_1, a_2, \ldots, a_m$ be its poles and let $b_1, b_2, \ldots, b_m$ be the corresponding residues. Suppose also that the a_q are not integers for $1 \leqslant q \leqslant m$. Let γ_n denote the perimeter of the square with vertices $\pm\left(n+\frac{1}{2}\right) \pm \left(n+\frac{1}{2}\right)i$, where n is a positive integer. Show that there exist two positive real numbers M_1, K independent of n such that

a) $\quad |\pi\cot\pi z| \leqslant M_1 \quad$ on $\quad \gamma_n$

b) $\quad |f(z)| \leqslant K/|z|^2 \quad$ for sufficiently large $|z|$.

Deduce that

$$\lim_{n\to\infty}\int_{\gamma_n} f(z)\pi\cot\pi z\,dz = 0.$$

and that

(1) $$\lim_{n\to\infty}\sum_{-n\leqslant p\leqslant n} f(p) = -\sum_{1\leqslant q\leqslant m} b_q\pi\cot\pi a_q.$$

(Note: b) implies that $\lim\limits_{n,n'\to\infty}\sum\limits_{-n\leqslant p\leqslant n'} f(p)$ exists, thus the left hand sidef of (1) can be replaced by $\sum\limits_{-\infty<p\leqslant\infty} f(p)$.)

Example. $\sum\limits_{n\geqslant 1} 1/(a+bn^2)$, $\sum\limits_{n\geqslant 1} n^2/(n^4+a^4)$ (a, b positive real numbers).

(ii) Show that the conclusion is valid even if we only have $\deg Q > \deg P$. (Show first that $f(z)$ can be written $g(z) + c/z$ with c a constant and $g(z)$ a rational function which satisfies the conditions of (i); show next that $\int_{\gamma_n}\frac{\cot\pi z}{z}\,dz = 0$ (the integrals along opposite sides cancel). Note: $\lim\limits_{n,n'\to\infty}\sum\limits_{-n\leqslant p\leqslant n'} f(p)$ does not exist in this case.)

Example. Calculate $\lim\limits_{n\to\infty}\sum\limits_{-n\leqslant p\leqslant n}\frac{1}{x-p}$, and deduce the value o $\sum\limits_{p\geqslant 1}\frac{1}{x^2-p^2}$ when x is not an integer.

(iii) Let α be a real number such that $-\pi < \alpha < \pi$. Show that:

c) there exists a positive real number M_2, which does not depend on n, such that

$$\left|\frac{e^{i\alpha z}}{\sin\pi z}\right| \leqslant M_2 \quad \text{on} \quad \gamma_n,$$

d) $$\lim_{n\to\infty}\int_{\gamma_n}\frac{e^{i\alpha z}}{z\sin\pi z}\,dz = 0.$$

(Note that

$$\int_{\gamma_n}\frac{e^{i\alpha z}}{z\sin\pi z}\,dz = 2i\int_{\gamma_n'}\frac{\sin\alpha z}{z\sin\pi z}\,dz + 2i\int_{\gamma_n''}\frac{\sin\alpha z}{z\sin\pi z}\,dz,$$

where γ_n' (resp. γ_n'') denotes the line segment represented by

$$z = n+\frac{1}{2}+iy, \quad |y| \leqslant n+\frac{1}{2}\left(\text{resp. } z = x+i\left(n+\frac{1}{2}\right), |x| \leqslant n+\frac{1}{2}\right),$$

and use exercise 14. of chapter 1.) Deduce finally that, if $f(z)$ is a rational function and satisfies the conditions of (ii), then

$$\lim_{n\to\infty}\sum_{-n\leqslant p\leqslant n}(-1)^p f(p)e^{i\alpha p} = -\pi\sum_{1\leqslant q\leqslant m} b_q\frac{e^{i\alpha a_q}}{\sin\pi a_q}.$$

Example. Take $f(z) = 1/(x-z)$ and show that, if $-\pi < \alpha < \pi$, then

$$\begin{cases}\sum\limits_{n\geqslant 1}(-1)^n\dfrac{\cos\alpha n}{x^2-n^2} = \dfrac{\pi\cos\alpha x}{2x\sin\pi x} - \dfrac{1}{2x^2},\\ \sum\limits_{n\geqslant 1}(-1)^n\dfrac{n\sin\alpha n}{x^2-n^2} = \dfrac{\pi}{2}\dfrac{\sin\alpha x}{\sin\pi x},\end{cases}$$

for $x \neq 0, \pm 1, \pm 2, \ldots$.