Elementary Theory of Analytic Functions of One or Several Complex Variables

Henri Cartan

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PREFACE

The present volume contains the substance, with some additions, of a course of lectures given at the Faculty of Science in Paris for the requirements of the *licence d'enseignement* during the academic sessions 1957-1958, 1958-1959 and 1959-1960. It is basically concerned with the theory of analytic functions of a complex variable. The case of analytic functions of several real or complex variables is, however, touched on in chapter IV if only to give an insight into the harmonic functions of two real variables as analytic functions and to permit the treatment in chapter VII of the existence theorem for the solutions of differential systems in cases where the data is analytic.

The subject matter of this book covers that part of the " Mathematics II " certificate syllabus given to analytic functions. This same subject matter was already included in the " Differential and integral calculus " certificate of the old *licence*.

As the syllabuses of certificates for the *licence* are not fixed in detail, the teacher usually enjoys a considerable degree of freedom in choosing the subject matter of his course. This freedom is mainly limited by tradition and, in the case of analytic functions of a complex variable, the tradition in France is fairly well established. It will therefore perhaps be useful to indicate here to what extent I have departed from this tradition. In the first place I decided to begin by offering not Cauchy's point of view (differentiable functions and Cauchy's integral) but the Weierstrass point of view, i.e. the theory of convergent power series (chapter I). This is itself preceded by a brief account of formal operations on power series, i.e. what is called nowadays the theory of formal series. I have also made something of an innovation by devoting two paragraphs of chapter VI to a systematic though very elementary exposition of the theory of abstract complex manifolds of one complex dimension. What is referred to here as a complex manifold is simply what used to be called a Riemann surface and is often still given that name; for our part, we decided to keep the term Riemann surface for the double datum of a complex manifold and a holomorphic mapping of this manifold into the complex plane (or, more generally, into another complex manifold). In this way a distinction is made between the two ideas with a clarity unattainable with orthodox terminology. With a subject as well established as the theory of analytic functions of a complex variable, which has been in the past the subject of so many treatises and still is in all countries, there could be no question of laying claim to originality. If the present treatise differs in any way from its forerunners in France, it does so perhaps because it conforms to a recent practice which is becoming increasingly prevalent: a mathematical text must contain precise statements of propositions or theorems — statements which are adequate in themselves and to which reference can be made at all times. With a very few exceptions which are clearly indicated, complete proofs are given of all the statements in the text. The somewhat ticklish problems of plane topology in relation to Cauchy's integral and the discussion of many-valued functions are approached quite openly in chapter II. Here again it was thought that a few precise statements were preferable to vague intuitions and hazy ideas. On these problems of plane topology, I drew my inspiration from the excellent book by L. Ahlfors (Complex Analysis), without however conforming completely with the points of view he develops. The basic concepts of general Topology are assumed to be familiar to the reader and are employed frequently in the present work; in fact this course is addressed to students of 'Mathematics II' who are expected to have already studied the 'Mathematics I' syllabus.

I express my hearty thanks to Monsieur Reiji Takahashi, who are from experience gained in directing the practical work of students, has consented to supplement the various chapters of this book with exersices and problems. It is hoped that the reader will thus be in a position to make sure that he has understood and assimilated the theoretical ideas set out in the text.

HENRI CARTAN

Die (Drôme), August 4th, 1960

CHAPTER I

Power Series in One Variable

1. Formal Power Series

1. Algebra of polynomials

Let K be a commutative field. We consider the formal polynomials in one symbol (or 'indeterminate') X with coefficients in K (for the moment we do not give a value to X). The laws of addition of two polynomials and of multiplication of a polynomial by a 'scalar' makes the set K[X] of polynomials into a *vector space* over K with the infinite base

$$1, X, \ldots, X^n, \ldots$$

Each polynomial is a finite linear combination of the X^n with coefficients in K and we write it $\sum_{n \geqslant 0} a_n X^n$, where it is understood that only a finite number of the coefficients a_n are non-zero in the infinite sequence of these coefficients. The multiplication table

$$X^p . X^q = X^{p+q}$$

defines a multiplication in K[X]; the product

$$\left(\sum_p a_p X^p\right) \cdot \left(\sum_q b_q X^q\right)$$

is $\sum_n c_n X^n$, where

$$c_n = \sum_{p+q=n} a_p b_q. \tag{1.1}$$

This multiplication is commutative and associative. It is bilinear in the sense that

$$\begin{cases} (P_1 + P_2) \cdot Q = P_1 Q + P_2 Q \\ (\lambda P) \cdot Q = \lambda \ (PQ) \end{cases} \tag{1.2}$$

for all polynomials P, P_1, P_2, Q and all scalars λ. It admits as unit element (denoted by 1) the polynomial $\sum_{n\geq 0} a_n X^n$ such that $a_0 = 1$ and $a_n = 0$ for $n > 0$. We express all these properties by saying that $K[X]$, provided with its vector space structure and its multiplication, is a *commutative algebra* with a unit element over the field K; it is, in particular, a commutative ring with a unit element.

2. The algebra of formal series

A formal power series in X is a formal expression $\sum_{n\geq 0} a_n X^n$, where this time we no longer require that only a finite number of the coefficients a_n are non-zero. We define the sum of two formal series by

$$\Big(\sum_{n\geq 0} a_n X^n\Big) + \Big(\sum_{n\geq 0} b_n X^n\Big) = \sum_{n\geq 0} c_n X^n, \qquad \text{where} \qquad c_n = a_n + b_n,$$

and the product of a formal series with a scalar by

$$\lambda\Big(\sum_{n\geq 0} a_n X^n\Big) = \sum_{n\geq 0} (\lambda a_n) X^n.$$

The set $K[[X]]$ of formal series then forms a vector space over K. The neutral element of the addition is denoted by 0; it is the formal series with all its coefficients zero.

The product of two formal series is defined by the formula (1.1), which still has a meaning because the sum on the right hand side is over a finite number of terms. The multiplication is still commutative, associative and bilinear with respect to the vector structure. Thus $K[[X]]$ is an algebra over the field K with a unit element (denoted by 1), which is the series $\sum_{n\geq 0} a_n X^n$ such that $a_0 = 1$ and $a_n = 0$ for $n > 0$.

The algebra $K[X]$ is identified with a subalgebra of $K[[X]]$, the subalgebra of formal series whose coefficients are all zero except for a finite number of them.

3. The order of a formal series

Denote $\sum_{n\geq 0} a_n X^n$ by $S(X)$, or, more briefly, by S. The *order* $\omega(S)$ of this series is an integer which is only defined when $S \neq 0$; it is the smallest n such that $a_n \neq 0$. We say that a formal series S has order $\geq k$ if it is 0 or if $\omega(S) \geq k$. By *abus de langage*, we write $\omega(S) \geq k$ even when $S = 0$ although $\omega(S)$ is not defined in this case.

Note. We can make the convention that $\omega(0) = +\infty$. The S such that $\omega(S) \geq k$ (for a given integer k) are simply the series $\sum_{n\geq 0} a_n X^n$ such that $a_n = 0$ for $n < k$. They form a vector subspace of $K[[X]]$.

Definition. A family $(S_i(X))_{i\in I}$, where I denotes a set of indices, is said to be *summable* if, for any integer k, $\omega(S_i) \geq k$ for all but a finite number of the indices i. By definition, the *sum* of a summable family of formal series

$$S_i(X) = \sum_{n\geq 0} a_{n,i} X^n$$

is the series

$$S(X) = \sum_{n\geq 0} a_n X^n,$$

where, for each n, $a_n = \sum_i a_{n,i}$. This makes sense because, for fixed n, all but a finite number of the $a_{n,i}$ are zero by hypothesis. The operation of addition of formal series which form summable families generalizes the finite addition of the vector structure of $K[[X]]$. The generalized addition is commutative and associative in a sense which the reader should specify.

The formal notation $\sum_{n\geq 0} a_n X^n$ can then be justified by what follows. Let a *monomial* of degree p be a formal series $\sum_{n\geq 0} a_n X^n$ such that $a_n = 0$ for $n \neq p$ and let $a_p X^p$ denote such a monomial. The family of monomials $(a_n X^n)_{n\in N}$ (N being the set of integers ≥ 0) is obviously summable, and its sum is simply the formal series $\sum_{n\geq 0} a_n X^n$.

Note. The product of two formal series

$$\Big(\sum_p a_p X^p\Big)\cdot\Big(\sum_q b_q X^q\Big)$$

is merely the sum of the summable family formed by all the products

$$(a_p X^p)\cdot(b_q X^q) = (a_p b_q) X^{p+q}$$

of a monomial of the first series by one of the second.

PROPOSITION 3.1. *The ring* $K[[X]]$ *is an integral domain* (this means that $S \neq 0$ and $T \neq 0$ imply $ST \neq 0$).

Proof. Suppose that $S(X) = \sum_p a_p X^p$ and $T(X) = \sum_q b_q X^q$ are non-zero. Let $p = \omega(S)$ and $q = \omega(T)$, let

$$S(X)\cdot T(X) = \sum_n c_n X^n;$$

obviously $c_n = 0$ for $n < p + q$ and $c_{p+q} = a_p b_q$. Since K is a field and since $a_p \neq 0$, $b_q \neq 0$, we have that $c_{p+q} \neq 0$, so ST is not zero.

What is more, we have proved that

$$\omega(S \cdot T) = \omega(S) + \omega(T) \quad \text{for} \quad S \neq 0 \text{ and } T \neq 0. \tag{3.1}$$

Note. One can consider formal series with coefficients in a commutative ring A with a unit element which is not necessarily a field K; the above proof then establishes that, *if* A *is an integral domain, then so is* A[[X]].

4. Substitution of a formal series in another

Consider two formal series

$$S(X) = \sum_{n \geq 0} a_n X^n, \qquad T(Y) = \sum_{p \geq 0} b_p Y^p.$$

It is essential also to assume that $b_0 = 0$, in other words that $\omega(T) \geq 1$. To each monomial $a_n X^n$ associate the formal series $a_n(T(Y))^n$, which has a meaning because the formal series in Y form an algebra. Since $b_0 = 0$, the order of $a_n(T(Y))^n$ is $\geq n$; thus the family of the $a_n(T(Y))^n$ (as n takes the values 0, 1, ...) is *summable*, and we can consider the formal series

$$\sum_{n \geq 0} a_n (T(Y))^n, \tag{4.1}$$

in which we regroup the powers of Y. This formal series in Y is said to be obtained *by substitution* of T(Y) for X in S(X); we denote it by S(T(Y)), or $S \circ T$ without specifying the indeterminate Y. The reader will verify the relations:

$$\begin{cases} (S_1 + S_2) \circ T = S_1 \circ T + S_2 \circ T, \\ (S_1 S_2) \circ T = (S_1 \circ T)(S_2 \circ T), \qquad 1 \circ T = 1. \end{cases} \tag{4.2}$$

But, note carefully that $S \circ (T_1 + T_2)$ is not, in general, equal to

$$S \circ T_1 + S \circ T_2.$$

The relations (4.2) express that, for given T (of order ≥ 1), the mapping $S \to S \circ T$ is a *homomorphism* of the ring K[[X]] in the ring K[[Y]] which transforms the unit element 1 into 1.

Note. If we substitute 0 in $S(X) = \sum_{n \geq 0} a_n X^n$, we find that the formal series reduces to its 'constant term' a_0.

If we have a summable family of formal series S_i and if $\omega(T) \geq 1$, then the family $S_i \circ T$ is summable and

$$\Big(\sum_i S_i\Big) \circ T = \sum (S_i \circ T), \tag{4.3}$$

which generalizes the first of the relations (4.2). For, let

$$S_i(X) = \sum_{n \geq 0} a_{n,i} X^n;$$

we have

$$\sum_i S_i(X) = \sum_{n \geq 0} \Big(\sum_i a_{n,i}\Big) X^n,$$

whence

$$\Big(\sum_i S_i\Big) \circ T = \sum_{n \geq 0} \Big(\sum_i a_{n,i}\Big) (T(Y))^n, \tag{4.4}$$

while

$$\sum S_i \circ T = \sum_i \Big(\sum_{n \geq 0} a_{n,i} (T(Y))^n\Big). \tag{4.5}$$

To prove the equality of the right hand sides of (4.4) and (4.5), we observe that the coefficient of a given power Y^p in each of them involves only a finite number of the coefficients $a_{n,i}$ and we apply the associativity law of (finite) addition in the field K.

Proposition 4.1. *The relation*

$$(S \circ T) \circ U = S \circ (T \circ U) \tag{4.6}$$

holds whenever $\omega(T) \geq 1$, $\omega(U) \geq 1$ (associativity of substitution).

Proof. Both sides of (4.6) are defined. In the case when S is a monomial, they are equal because

$$T^n \circ U = (T \circ U)^n \tag{4.7}$$

which follows by induction on n from the second relation in (4.2).

The general case of (4.6) follows by considering the series S as the (infinite) sum of its monomials $a_n X^n$; by definition,

$$S \circ T = \sum_{n \geq 0} a_n T^n,$$

and, from (4.3),

$$(S \circ T) \circ U = \sum_{n \geq 0} a_n (T^n \circ U),$$

which, by (4. 7), is equal to

$$\sum_{n\geqslant 0} a_n(T \circ U)^n = S \circ (T \circ U).$$

This completes the proof.

5. Algebraic inverse of a formal series

In the ring K[[Y]], the identity

$$(1 - Y)(1 + Y + \cdots + Y^n + \cdots) = 1 \qquad (5.1)$$

can easily be verified. Hence the series $1 - Y$ has an inverse in K[[Y]]

Proposition 5. 1. *For* $S(X) = \sum_n a_n X^n$ *to have an inverse element for the multiplication of* K[[X]], *it is necessary and sufficient that* $a_0 \neq 0$, *i.e. that* $S(0) \neq 0$.

Proof. The condition is necessary because, if

$$T(X) = \sum_n b_n X^n \quad \text{and if} \quad S(X)T(X) = 1,$$

then $a_0 b_0 = 1$ and so $a_0 \neq 0$. Conversely, suppose that $a_0 \neq 0$; we shall show that $(a_0)^{-1}S(X) = S_1(X)$ has an inverse $T_1(X)$, whence it follows that $(a_0)^{-1}T_1(X)$ is the inverse of S(X). Now

$$S_1(X) = 1 - U(X) \quad \text{with} \quad \omega(U) \geqslant 1,$$

and we can substitute U(X) for Y in the relation (5. 1), from which it follows that $1 - U(X)$ has an inverse. The proposition is proved.

Note. By considering the algebra of polynomials K[X] imbedded in the algebra of formal series K[[X]], it will be seen that any polynomial Q(X) such that $Q(0) \neq 0$ has an inverse in the ring K[[X]]; this ring then contains all the quotients P(X)/Q(X), where P and Q are polynomials and where $Q(0) \neq 0$.

6. Formal derivative of a series

Let $S(X) = \sum_n a_n X^n$; by definition, the *derived* series S′(X) is given by the formula

$$S'(X) = \sum_{n\geqslant 0} n a_n X^{n-1}. \qquad (6.1)$$

It can also be written $\frac{dS}{dX}$, or $\frac{d}{dX}S$. The derivative of a (finite or infinite) sum is equal to the sum of its derivatives. The mapping $S \to S'$ is a linear mapping of K[[X]] into itself. Moreover, the derivative of the product of two formal series is given by the formula

$$\frac{d}{dX}(ST) = \frac{dS}{dX}T + S\frac{dT}{dX}. \qquad (6.2)$$

For, it is sufficient to verify this formula in the particular case when S and T are monomials, and it is clearly true then.

If $S(0) \neq 0$, let T be the inverse of S (c.f. n°. 5). The formula (6. 2) gives

$$\frac{d}{dX}\left(\frac{1}{S}\right) = -\frac{1}{S^2}\frac{dS}{dX}. \qquad (6.3)$$

Higher derivatives of a formal series are defined by induction. If $S(X) = \sum a_n X^n$, its derivative of order n is

$$S^{(n)}(X) = n!\,a_n + \text{terms of order} \geqslant 1.$$

Hence,

$$S^{(n)}(0) = n!\,a_n, \qquad (6.4)$$

where $S^{(n)}(0)$ means the result of substituting the series 0 for the indeterminate X in $S^{(n)}(X)$.

7. Compositional inverse series

The series I(X) defined by $I(X) = X$ is a *neutral element* for the composition of formal series :

$$S \circ I = S = I \circ S.$$

Proposition 7. 1. *Given a formal series* S, *a necessary and sufficient condition for there to exist a formal series* T *such that*

$$T(0) = 0, \quad S \circ T = I \qquad (7.1)$$

is that

$$S(0) = 0, \quad S'(0) \neq 0. \qquad (7.2)$$

In this case, T *is unique, and* $T \circ S = I$: *in other words* T *is the inverse of* S *for the law of composition* $\circ$.

Proof. Let $S(X) = \sum_{n\geqslant 0} a_n X^n$, $T(Y) = \sum_{n\geqslant 1} b_n Y^n$. If

$$S(T(Y)) = Y, \qquad (7.3)$$

then equating the first two terms gives

$$(7.4) \qquad a_0 = 0, \qquad a_1 b_1 = 1.$$

Hence the conditions (7.2) are necessary.

Suppose that they are satisfied; we write down the condition that the coefficient of Y^n is zero in the left hand side of (7.3). This coefficient is the same as the coefficient of Y^n in

$$a_1 T(Y) + a_2 (T(Y))^2 + \cdots + a_n (T(Y))^n,$$

which gives the relation

$$(7.5) \qquad a_1 b_n + P_n(a_2, \ldots, a_n, b_1, \ldots, b_{n-1}) = 0,$$

where P_n is a known polynomial with non-negative integral coefficients and is linear in $a_2, \ldots, a_n$. Since $a_1 \neq 0$, the second equation (7.4) determines b_1; then, for $n \geqslant 2$, b_n can be calculated by induction on n from (7.5). Thus we have the existence and uniqueness of the formal series $T(Y)$. The series thus obtained satisfies $T(0) = 0$ and $T'(0) \neq 0$, and so the result that we have just proved for S can be applied to T, giving a formal series S_1 such that

$$S_1(0) = 0, \qquad T \circ S_1 = I.$$

This implies that

$$S_1 = I \circ S_1 = (S \circ T) \circ S_1 = S \circ (T \circ S_1) = S \circ I = S.$$

Hence S_1 is none other than S and, indeed, $T \circ S = I$, which completes the proof.

Remark. Since $S(T(Y)) = Y$ and $T(S(X)) = X$, we can say that the 'formal transformations'

$$Y = S(X), \qquad X = T(Y)$$

are inverse to one another; thus we call T the 'inverse formal series' of the series S.

Proposition 7.1 is an 'implicit function theorem' for formal functions.

2. Convergent power series

1. The complex field

From now on, the field K will be either **R** or **C**, where **R** denotes the field of real numbers and **C** the field of complex numbers.

Recall that a complex number $z = x + iy$ (x and y real) is represented by a point on the plane $\mathbf{R}^2$ whose coordinates are x and y. If we associate

with each complex number $z = x + iy$ its 'conjugate' $\bar{z} = x - iy$, we define an automorphism $z \to \bar{z}$ of the field **C**, since

$$\overline{z + z'} = \bar{z} + \bar{z}', \qquad \overline{zz'} = \bar{z}\bar{z}'.$$

The conjugate of $\bar{z}$ is z; in other words, the transformation $z \to \bar{z}$ is *involutive*, i.e. is equal to its inverse transformation.

The *norm, absolute value*, or *modulus* $|z|$ of a complex number z is defined by

$$|z| = (z \cdot \bar{z})^{1/2}.$$

It has the following properties:

$$|z + z'| \leqslant |z| + |z'|, \qquad |zz'| = |z| . |z'|, \qquad |1| = 1.$$

The norm $|z|$ is always $\geqslant 0$ and is zero only when $z = 0$. This norm enables us to define a *distance* in the field **C**: the distance between z and z' is $|z - z'|$, which is precisely the euclidean distance in the plane $\mathbf{R}^2$. The space **C** is a *complete* space for this distance function, which means that the *Cauchy criterion* is valid: for a sequence of points $z_n \in \mathbf{C}$ to have a limit, it is necessary and sufficient that

$$\lim_{\substack{m \to \infty \\ n \to \infty}} |z_m - z_n| = 0.$$

The Cauchy criterion gives the following well-known theorem: if a series $\sum_n u_n$ of complex numbers is such that $\sum_n |u_n| < +\infty$, then the series converges (we say that the series is absolutely convergent). Moreover,

$$\left|\sum_n u_n\right| \leqslant \sum_n |u_n|.$$

We shall always identify **R** with a sub-field of **C**, i. e. the sub-field formed by the z such that $\bar{z} = z$. The norm induces a norm on **R**, which is merely the absolute value of the real number. **R** is complete. The norm of the field **C** (or **R**) plays an essential role in what follows.

We define

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad \mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

the 'real part' and the 'imaginary coefficient' of $z \in \mathbf{C}$.

2. Revision of the theory of convergence of series of functions

(For a more complete account of this theory, the reader is referred to Cours de Mathématiques I of J. Dixmier : Cours de l'A.C.E.S., Topologie, chapter VI, § 9.)

Consider functions defined on a set E taking real, or complex, values (or one could consider the more general case when the functions take values in a complete normed vector space; cf. *loc. cit.*). For each function u, we write

$$\|u\| = \sup_{x \in E} |u(x)|,$$

which is a number $\geqslant 0$, or may be infinite. Evidently,

$$\|u + v\| \leqslant \|u\| + \|v\|, \qquad \|\lambda u\| = |\lambda| . \|u\|$$

for any scalar λ, when $\|u\| < +\infty$: in other words, $\|u\|$ is a *norm* on the vector space of functions u such that $\|u\| < +\infty$.

We say that a series of functions u_n is *normally convergent* if the series of norms $\sum_n \|u_n\|$ is a convergent series of positive terms, in other words, if $\sum_n \|u_n\| < +\infty$. This implies that, for each $x \in E$, the series $\sum_n |u_n(x)|$ is convergent, and so the series $\sum_n u_n(x)$ is absolutely convergent; moreover, if $v(x)$ is the sum of this last series,

$$\|v\| \leqslant \sum_n \|u_n\|, \qquad \lim_{p \to \infty} \|v - \sum_{n=0}^{p} u_n\| = 0.$$

The latter relation expresses that the partial sums $\sum_{n=0}^{p} u_n$ converge *uniformly* to v as P tends to infinitiy. Thus, *a normally convergent series is uniformly convergent.* If A is a subset of E, the series whose general term is u_n is said to converge normally for $x \in A$ if the series of functions

$$u'_n = u_n | A \quad \text{(restriction of } u_n \text{ to A)}$$

is normally convergent. This is the same as saying that we can bound each $|u_n(x)|$ on A above by a constant $\varepsilon_n \geqslant 0$ in such a way that the series $\sum_n \varepsilon_n$ is convergent. Recall that the limit of a uniformly convergent sequence of *continuous* functions (on a topological space E) is continuous. In particular, *the sum of a normally convergent series of continuous functions is continuous.* An important consequence of this is :

Proposition 1.2. *Suppose that, for each* n, $\lim_{x \to x_0} u_n(x)$ *exists and takes the value* a_n. *Then, if the series* $\sum_n u_n$ *is normally convergent, the series* $\sum_n a_n$ *is convergent and*

$$\sum_n a_n = \lim_{x \to x_0} \Big(\sum_n u_n(x) \Big)$$

(changing the order cf the summation and the limiting process).

All these results extend to multiple series and, more generally, to summable families of functiors (cf. the above-mentioned course by Dixmier).

3. Radius of convergence of a power series

All the power series to be considered will have coefficients in either the field **R**, or the field **C**.

Note however that what follows remains valid in the more general case when coefficients are in any field with a complete, non-discrete, valuation, that is, a field K with a mapping $x \to |x|$ of K into the set of real numbers $\geqslant 0$ such that

$$\begin{cases} |x + y| \leqslant |x| + |y|, \qquad |xy| = |x| . |y|, \\ (|x| = 0) \Longleftrightarrow (x = 0), \end{cases}$$

and such that there exists some $x \neq 0$ with $|x| \neq 1$.

Let $S(X) = \sum_{n \geqslant 0} a_n X^n$ be a formal series with coefficients in **R** or **C**. We propose to substitute an element z of the field for the indeterminate X and thus to obtain a 'value' $S(z)$ of the series, which will be an element of the field; but this substitution is not possible unless the series $\sum_{n \geqslant 0} a_n z^n$ is convergent. In fact, we shall limit ourselves to the case when it is *absolutely convergent.*

To be precise, we introduce a real variable $r \geqslant 0$ and consider the series of positive (or zero) terms

$$\sum_{n \geqslant 0} |a_n| r^n,$$

called the *associated series* of S(X). Its sum is a well-defined number $\geqslant 0$, which may be infinity. The set of $r \geqslant 0$ for which

$$\sum_{n \geqslant 0} |a_n| r^n < +\infty$$

is clearly an interval of the half line $\mathbf{R}^+$, and this interval is non-empty since the series converges for $r = 0$. The interval can either be open or closed on the right, it can be finite or infinite, or it can reduce at the single point 0. In all cases, let ρ be the least upper bound of the interval, so ρ is a number $\geqslant 0$, finite, infinite, or zero; it is called the *radius of convergence* of the formal power series $\sum_{n \geqslant 0} a_n X^n$. The set of z such that $|z| < \rho$ is called the *disc of convergence* of the power series; it is an open set and it is empty if $\rho = 0$. It is an ordinary disc when the field of coefficients is the complex field **C**.

Proposition 3.1.

a) *For any* $r < \rho$, *the series* $\sum_{n \geqslant 0} a_n z^n$ *converges normally for* $|z| \leqslant r$. *In particular, the series converges absolutely for each* z *such that* $|z| < \rho$;

b) *the series* $\sum_{n\geqslant 0} a_n z^n$ *diverges for* $|z| > \rho$. (We say nothing about the case when $|z| = \rho$.)

Proof. Proposition 3.1 follows from

ABEL'S LEMMA. *Let* r *and* r_0 *be real numbers such that* $0 < r < r_0$. *If there exists a finite number* $M > 0$ *such that*

$$|a_n|(r_0)^n \leqslant M \qquad \textit{for any integer } n \geqslant 0,$$

then the series $\sum_{n\geqslant 0} a_n z^n$ *converges normally for* $|z| \leqslant r$.

For, $|a_n z^n| \leqslant |a_n| r^n \leqslant M(r/r_0)^n$, and $\varepsilon_n = M(r/r_0)^n$ is the general term of a convergent series — a geometric series with common ratio $r/r_0 < 1$. We now prove statement a) of proposition 3.1 : if $r < \rho$, choose r_0 such that $r < r_0 < \rho$; since $\sum_{n\geqslant 0} |a_n|(r_0)^n$ converges, its general term is bounded above by a fixed number M, and Abel's lemma ensures the normal convergence of $\sum_{n\geqslant 0} a_n z^n$ for $|z| \leqslant r$. Statement b) remains to be proved : if $|z| > \rho$, we can make $|a_n z^n|$ arbitrarily large by chosing the integer n suitably because, otherwise, Abels' lemma would give an r' with $\rho < r' < |z|$ such that the series $\sum_{n\geqslant 0} |a_n| r'^n$ were convergent and this would contradict the definition of ρ.

Formula for the radius of convergence (Hadamard) : we shall prove the formula

$$(3.\ 1) \qquad 1/\rho = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Recall, first of all, the definition of the upper limit of a sequence of real numbers u_n:

$$\limsup_{n\to\infty} u_n = \lim_{p\to\infty}\left(\sup_{n\geqslant p} u_n\right).$$

To prove (3. 1), we use a classical criterion of convergence : if v_n is a sequence of non-negative numbers such that $\limsup_{n\to\infty} (v_n)^{1/n} < 1$, then $\sum_n v_n < +\infty$; moreover, if they are such that $\limsup_{n\to\infty} (v_n)^{1/n} > 1$, then $\sum_n v_n = +\infty$ (this is " Cauchy's rule " and follows by comparing the series $\sum_n v_n$ with a geometric series).

Here we put $v_n = |a_n| r^n$ and find that

$$\limsup_{n\to\infty} (v_n)^{1/n} = r\left(\limsup_{n\to\infty} |a_n|^{1/n}\right),$$

and so the series $\sum_n |a_n| r^n$ converges for $1/r > \limsup_{n\to\infty} |a_n|^{1/n}$, and diverges for $1/r < \limsup_{n\to\infty} |a_n|^{1/n}$. This proves (3. 1).

Some examples. — The series $\sum_{n\geqslant 0} n!z^n$ has zero radius of convergence;
— the series $\sum_{n\geqslant 0} \frac{1}{n!} z^n$ has infinite radius of convergence;
— each of the series $\sum_{n\geqslant 0} z^n$, $\sum_{n>0} \frac{1}{n} z^n$, $\sum_{n>0} \frac{1}{n^2} z^n$ has radius of convergence equal to 1. It can be shown that they behave differently when $|z| = 1$.

4. ADDITION AND MULTIPLICATION OF CONVERGENT POWER SERIES.

PROPOSITION 4. 1. *Let* A(X) *and* B(X) *be two formal power series whose radii of convergence are* $\geqslant \rho$. *Let*

$$S(X) = A(X) + B(X) \quad \text{and} \quad P(X) = A(X).B(X)$$

be their sum and product. Then :

a) *the series* S(X) *and* P(X) *have radius of convergence* $\geqslant \rho$;

b) *for* $|z| < \rho$, we have

$$(4.1) \qquad S(z) = A(z) + B(z), \qquad P(z) = A(z)B(z).$$

Proof. Let

$$A(X) = \sum_{n\geqslant 0} a_n X^n, \quad B(X) = \sum_{n\geqslant 0} b_n X^n, \quad S(X) = \sum_{n\geqslant 0} c_n X^n, \quad P(X) = \sum_{n\geqslant 0} d_n X^n,$$

and let

$$\gamma_n = |a_n| + |b_n|, \qquad \delta_n = \sum_{0\leqslant p\leqslant n} |a_p| \cdot |b_{n-p}|.$$

We have $|c_n| \leqslant \gamma_n$, $|d_n| \leqslant \delta_n$. If $r < \rho$, the series $\sum_{n\geqslant 0} |a_n| r^n$ and $\sum_{n\geqslant 0} |b_n| r^n$ converge, thus

$$\sum_{n\geqslant 0} \gamma_n r^n = \left(\sum_{n\geqslant 0} |a_n| r^n\right) + \left(\sum_{n\geqslant 0} |b_n| r^n\right) < +\infty,$$
$$\sum_{n\geqslant 0} \delta_n r^n = \left(\sum_{p\geqslant 0} |a_p| r^p\right) \cdot \left(\sum_{q\geqslant 0} |b_q| r^q\right) < +\infty.$$

It follows that the series $\sum_{n\geqslant 0} |c_n| r^n$ and $\sum_{n\geqslant 0} |d_n| r^n$ converge and therefore that any $r < \rho$ is less than or equal to the radius of convergence of each of the series S(X) and P(X). Thus both radii of convergence are $\geqslant \rho$. The two relations (4. 1) remain to be proved. The first is obvious, and the second is obtained by multiplying convergent series; to be precise, we recall this classical result :

PROPOSITION 4. 2. *Let* $\sum_{n\geqslant 0} u_n$ *and* $\sum_{n\geqslant 0} v_n$ *be two absolutely convergent series. If*

$$w_n = \sum_{0\leqslant p\leqslant n} u_p v_{n-p},$$

then the series $\sum_{n\geqslant 0} w_n$ *is absolutely convergent and its sum is equal to the product*

$$\Big(\sum_{p\geqslant 0} u_p\Big)\cdot\Big(\sum_{q\geqslant 0} v_q\Big)$$

Write $\alpha_p = \sum_{n\geqslant p}|u_n|$, $\beta_q = \sum_{n\geqslant q}|v_n|$; we have

$$\sum_{n\geqslant 0}|w_n| \leqslant \sum_{p\geqslant 0}\sum_{q\geqslant 0}|u_p|.|v_q| = \alpha_0\beta_0;$$

moreover, if $m \geqslant 2n$,

$$\sum_{k\leqslant m} w_k - \Big(\sum_{k\leqslant n} u_k\Big)\cdot\Big(\sum_{k\leqslant n} v_k\Big)$$

is less than a sum of terms $|u_p|.|v_q|$, where for each term, at least one of the integers p and q is $> n$; thus, this sum is less than $\alpha_0\beta_{n+1}+\beta_0\alpha_{n+1}$, which tends to zero as n tends to infinity. It follows that $\sum_{k\leqslant m} w_k$ tends to the product of the infinite sums $\sum_{n\geqslant 0} u_n$ and $\sum_{n\geqslant 0} v_n$.

5. Substitution of a convergent power series in another

For two given formal power series S and T with T(0) = 0, we have defined the formal power series $S \circ T$ in paragraph 1, no. 4.

PROPOSITION 5.1. *Suppose* $T(X) = \sum_{n\geqslant 1} b_n X^n$. *If the radii of convergence* $\rho(S)$ *and* $\rho(T)$ *are* $\neq 0$, *then the radius of convergence of* $U = S\circ T$ *is also* $\neq 0$. *To be precise, there exists an* $r > 0$ *such that* $\sum_{n\geqslant 1}|b_n|r^n < \rho(S)$; *the radius of convergence of* U *is* $\geqslant r$, *and, for any* z *such that* $|z| \leqslant r$, *we have*

$$|T(z)| < \rho(S)$$

and

$$S(T(z)) = U(z). \tag{5. 1}$$

Proof. Put $S(X) = \sum_{n\geqslant 0} a_n X^n$. For sufficiently small $r > 0$, $\sum_{n\geqslant 1}|b_n|r^n$ is finite since the radius of convergence of T is $\neq 0$. Thus, $\sum_{n\geqslant 1}|b_n|r^{n-1}$ is finite for sufficiently small $r > 0$, and, consequently,

$$\sum_{n\geqslant 1}|b_n|r^n = r.\Big(\sum_{n\geqslant 1}|b_n|r^{n-1}\Big)$$

tends to 0 when r tends to 0. There exists, then, an $r > 0$ such that $\sum_{n\geqslant 1}|b_n|r^n < \rho(S)$ as required. It follows that

$$\sum_{p\geqslant 0}|a_p|\Big(\sum_{k\geqslant 1}|b_k|r^k\Big)^p$$

is finite. However, this is a series $\sum_{n\geqslant 0}\gamma_n r^n$, and, if we put $U(x) = \sum_{n\geqslant 0} c_n X^n$, we clearly obtain $|c_n| \leqslant \gamma_n$. Thus $\sum_{n\geqslant 0}|c_n|r^n$ is finite and the radius of convergence of U is $\geqslant r$.

Relation (5. 1) remains to be proved. Put $S_n(X) = \sum_{0\leqslant k\leqslant n} a_k X^k$ and let $S_n \circ T = U_n$. For $|z| \leqslant r$, we have

$$U_n(z) = S_n(T(z)),$$

since the mapping $T \to T(z)$ is a ring homomorphism and S_n is a polynomial. Since the series S converges at the point $T(z)$, we have

$$S(T(z)) = \lim_n S_n(T(z)).$$

On the other hand, the coefficients of $U - U_n = (S - S_n)\circ T$ are bounded by those of

$$\sum_{p>n}|a_p|\Big(\sum_{k\geqslant 1}|b_k|r^k\Big)^p,$$

a series whose sum tends to 0 as $n \to +\infty$. It follows that, for $|z| \leqslant r$, $U(z) - U_n(z)$ tends to 0 as $n \to +\infty$. Finally, we have

$$U(z) = \lim_{n\to\infty} U_n(z) = \lim_{n\to\infty} S_n(T(z)) = S(T(z)) \qquad \text{for} \qquad |z| \leqslant r,$$

which establishes relation (5. 1) and completes the proof.

Interpretation of relation (5. 1) : suppose r satisfies the conditions of proposition 5. 1. Denote the function $z \to T(z)$ by $\tilde{T}$, defined for $|z| \leqslant r$, and similarly denote the functions defined by the series S and U by $\tilde{S}$ and $\tilde{U}$ respectively. The relation (5. 1) expresses that, for $|z| \leqslant r$, the composite function $\tilde{S}\circ\tilde{T}$ is defined and is equal to $\tilde{U}$. Thus the relation $U = S\circ T$ between formal series implies the relation $\tilde{U} = \tilde{S}\circ\tilde{T}$ if the radii of convergence of S and T are $\neq 0$ and *if we restrict ourselves to sufficiently small values of the variable* z.

6. Algebraic inverse of a convergent power series

We know (§ 1, proposition 5. 1) that, if $S(X) = \sum_{n\geqslant 0} a_n X^n$ with $a_0 \neq 0$, there exists a unique formal series $T(X)$ such that $S(X)T(X)$ is equal to 1.

Proposition 6. 1. *If the radius of convergence of S is $\neq 0$, then the radius of convergence of the series T such that $ST = 1$ is also $\neq 0$.*

Proof. Multiplying $S(X)$ by a suitable constant reduces the proposition to the special case when $a_0 = 1$. Put $S(X) = 1 - U(X)$ so that $U(0) = 0$. The inverse series $T(X)$ is obtained by substituting $U(X)$ for Y in the series $1 + \sum_{n>0} Y^n$; moreover, the radius of convergence of the latter is equal to 1 and so $\neq 0$; proposition 6. 1 then follows from proposition 5. 1.

7. Differentiation of a convergent power series

Proposition 7. 1. *Let* $S(X) = \sum_{n\geqslant 0} a_n X^n$ *be a formal power series and let*

$$S'(X) = \sum_{n\geqslant 0} n a_n X^{n-1}$$

be its derived series (cf. § 1, no. 6). *Then the series S and S' have the same radius of convergence. Moreover, if this radius of convergence ρ is $\neq 0$, we have, for $|z| < \rho$,*

$$(7.1) \qquad S'(z) = \lim_h \frac{S(z+h) - S(z)}{h},$$

where h tends to 0 without taking the value 0.

Preliminary remark. If $|z| < \rho$, then $|z+h| < \rho$ for sufficiently small values of h (in fact, for $|h| < \rho - |z|$); thus $S(z+h)$ is defined. On the other hand, it is understood in relation (7. 1) that h tends to 0 through non-zero real values if the field of coefficients is the field $\mathbf{R}$, or by non-zero complex values if the field of coefficients is the field $\mathbf{C}$. In the case of the field $\mathbf{R}$, relation (7. 1) expresses that the function $z \to S(z)$ has *derivative* equal to $S'(z)$; in the case of the complex field $\mathbf{C}$, relation (7. 1) shows that we also have the notion of *derivative with respect to the complex variable z*. In both cases, the existence of a derived function $S'(z)$ obviously implies that the function $S(z)$ is continuous for $|z| < \rho$, which can also be proved directly.

Proof of proposition 7. 1. Let $\alpha_n = |a_n|$ and let ρ and ρ' be the radii of convergence of the series S and S' respectively. If $r < \rho'$, the series $\sum_{n\geqslant 0} n\alpha_n r^{n-1}$ converges, and so

$$\sum_{n\geqslant 1} \alpha_n r^n \leqslant r\left(\sum_{n\geqslant 0} n\alpha_n r^{n-1}\right) < +\infty,$$

and, consequently, $r \leqslant \rho$. Conversely, if $r < \rho$ choose an r' such that $r < r' < \rho$; then

$$n\alpha_n r^{n-1} = \frac{1}{r'}(\alpha_n r'^n) \,.\, n\left(\frac{r}{r'}\right)^{n-1};$$

since $r' < \rho$, there exists a finite $M > 0$ such that $\alpha_n r'^n \leqslant M$ for all n, whence

$$n\alpha_n r^{n-1} \leqslant \frac{M}{r} n \left(\frac{r}{r'}\right)^{n-1},$$

and, since the series $\sum_{n\geqslant 1} n\left(\frac{r}{r'}\right)^{n-1}$ converges, the series $\sum_{n\geqslant 1} n\alpha_n r^{n-1}$ also converges; thus $r \leqslant \rho'$. We have then that any number $< \rho'$ is $\leqslant \rho$ and any number $< \rho$ is $\leqslant \rho'$, from which it follows that $\rho = \rho'$.

Relation (7. 1) remains to be proved. Choose a fixed z with $|z| < \rho$ and an r such that $|z| < r < \rho$ and suppose that

$$(7.2) \qquad 0 \neq |h| \leqslant r - |z|$$

in what follows.

Then $S(z+h)$ is defined, and we have

$$(7.3) \qquad \frac{S(z+h) - S(z)}{h} - S'(z) = \sum_{n\geqslant 1} u_n(z, h),$$

where we have put

$$u_n(z, h) = a_n\{(z+h)^{n-1} + z(z+h)^{n-2} + \cdots + z^{n-1} - nz^{n-1}\}.$$

Since $|z|$ and $|z+h|$ are $\leqslant r$, we have $|u_n(z, h)| \leqslant 2n\alpha_n r^{n-1}$; and, since $r < \rho$, we have $\sum_{n\geqslant 1} n\alpha_n r^{n-1} < +\infty$; thus, given $\varepsilon > 0$, there exists an integer n_0 such that

$$\sum_{n>n_0} 2n\alpha_n r^{n-1} \leqslant \varepsilon/2.$$

With this choice of n_0, the finite sum $\sum_{n\leqslant n_0} u_n(z,h)$ is a polynomial in h which vanishes when $h = 0$; it follows that $\left|\sum_{n\leqslant n_0} u_n(z,h)\right| \leqslant \varepsilon/2$ when $|h|$ is smaller than a suitably chosen η. Finally, if h satisfies (7. 2) and $|h| \leqslant \eta$, we deduce from (7. 3) that

$$\left|\frac{S(z+h) - S(z)}{h} - S'(z)\right| \leqslant \left|\sum_{n\leqslant n_0} u_n(z,h)\right| + \sum_{n>n_0} 2n\alpha_n r^{n-1} \leqslant \varepsilon.$$

Thus we have proved the relation (7. 1).

Note. It can be shown that the convergence of $\frac{S(z+h)-S(z)}{h}$ towards $S'(z)$ is *uniform* with respect to z for $|z| \leqslant r$ (r being a fixed number strictly less than the radius of convergence ρ).

8. Calculation of the coefficients of a power series

Let $S(x)$ be a formal power series whose radius of convergence $\rho \neq 0$, so that $S(z)$ is the sum of the series $\sum_{n\geqslant 0} a_n z^n$ for $|z| < \rho$. The function $S(z)$ has for derivative the function $S'(z) = \sum_{n\geqslant 0} n a_n z^{n-1}$. We can again apply proposition 7. 1 to the series S' to obtain its derived function $S''(z)$, the sum of the power series $\sum_{n\geqslant 0} n(n-1)a_n z^{n-2}$, whose radius of convergence is also ρ. This process can be carried on indefinitely, and by induction we see that the function $S(z)$ is infinitely differentiable for $|z| < \rho$; its derivative of order n is

$$S^{(n)}(z) = n!a_n + T_n(z),$$

where T_n is a series of order $\geqslant 1$, in other words $T_n(0) = 0$. From this, we have

$$a_n = \frac{1}{n!} S^{(n)}(0). \tag{8. 1}$$

This fundamental formula shows, in particular, that, if the function $S(z)$ is known in some neighbourhood of 0 (however small), the coefficients a_n of the power series S are completely determined. Consequently, given a function $f(z)$ defined for all sufficiently small $|z|$, *there cannot exist more than one formal power series* $S(X) = \sum_{n\geqslant 0} a_n X^n$ whose radius of convergence is $\neq 0$, and such that $f(z) = \sum_{n\geqslant 0} a_n z^n$ for $|z|$ sufficiently small.

9. Compositional inverse series of a convergent power series.

Refer to § 1, proposition 7. 1.

Proposition 9.1. *Let* S *be a power series such that* $S(0) = 0$ *and* $S'(0) \neq 0$, *and let* T *be its inverse series, that is the series such that*

$$T(0) = 0, \qquad S \circ T = I.$$

If the radius of convergence of S *is* $\neq 0$, *then the radius of convergence of* T *is* $\neq 0$.

The reader can accept this proposition without proof because a proof (which does not use power series theory) will be given later (chap. IV, § 5, proposition 6. 1).

Here, however, a direct proof using power series theory is given to satisfy the reader with an inquisitive mind. It uses the idea of 'majorant series' (cf. chap. VII). Let us keep to the notations of the proof of proposition 7.1 in § 1 and let us consider relations (7. 5) of § 1 which enable us to calculate the unknown coefficients b_n of the required series $T(X)$. Along with the series $S(X)$, we consider a 'majorant' series, that is a series

$$\overline{S}(X) = A_1 X - \sum_{n\geqslant 2} A_n X^n$$

with coefficients $A_n > 0$ such that $|a_n| \leqslant A_n$ for all n; moreover we assume that $A_1 = |a_1|$. Applying § 1 proposition 7. 1 to the series $\overline{S}$, gives a series

$$\overline{T}(Y) = \sum_{n\geqslant 1} B_n Y^n$$

such that $\overline{S}(\overline{T}(Y)) = Y$; its coefficients B_n are given by the relations

$$A_1 B_n - P_n(A_2, \ldots, A_n, B_1, \ldots, B_{n-1}) = 0 \tag{9. 1}$$

which are analogs of (7. 5) of § 1. We obtain from them by induction on n

$$|b_n| \leqslant B_n. \tag{9. 2}$$

It follows that the radius of convergence of the series T is not less than that of the series $\overline{T}$. We shall prove proposition 9. 1 by showing that the radius of convergence of $\overline{T}$ is > 0.

To this end, we choose the series $\overline{S}$ as follows: let $r > 0$ be a number strictly less than the radius of convergence of the series S (by hypothesis, this radius of convergence is $\neq 0$); the general term of the series $\sum_{n\geqslant 1} |a_n| r^n$ is then bounded above by by a finite number $M > 0$ and, if we put

$$A_1 = |a_1|, \qquad A_n = M/r^n \quad \text{for } n \geqslant 2, \tag{9. 3}$$

we obtain the coefficients of a majorant series of S; its sum $\overline{S}(x)$ is equal to

$$\overline{S}(x) = A_1 x - M\frac{x^2/r^2}{1 - x/r} \qquad \text{for} \quad |x| < r.$$

We seek, then, a function $\overline{T}(y)$ defined for sufficiently small values of y which is zero for $y = 0$ and which satisfies the equation $\overline{S}(\overline{T}(y)) = y$ identically; $\overline{T}(y)$ must satisfy the quadratic equation

$$(A_1/r + M/r^2)\,\overline{T}^2 - (A_1 + y/r)\overline{T} + y = 0, \tag{9. 4}$$

which has for solution (which vanishes when $y = 0$)

$$\overline{T}(y) = \frac{A_1 + y/r - \sqrt{(A_1)^2 - 2A_1 y/r - 4My/r^2 + y^2/r^2}}{2(A_1/r + M/r^2)}.$$

When $|y|$ is sufficiently small, the surd is of the form $A_1\sqrt{1+u}$, with $|u| < 1$, and so $\overline{T}(y)$ can be expanded as a power series in y, which converges for sufficiently small $|y|$. Thus the radius of convergence of this series is $\neq 0$, as required.

3. Logarithmic and Exponential Functions

1. Exponential function

We have already remarked (§ 2, no. 3) that the formal series $\sum_{n\geq 0} \frac{1}{n!} X^n$ has infinite radius of convergence. For z complex, we define

$$e^z = \sum_{n\geq 0} \frac{1}{n!} z^n,$$

that is, the sum of an absolutely convergent series. This function has derivative

(1. 1) $$\frac{d}{dz}(e^z) = e^z$$

by proposition 7. 1 of § 2.

On the other hand, applying proposition 4. 2 of § 2 to two series with general terms

$$u_n = \frac{1}{n!} z^n, \qquad v_n = \frac{1}{n!} z'^n,$$

gives

$$w_n = \sum_{0\leq p\leq n} \frac{1}{p!(n-p)!} z^p z'^{n-p} = \frac{1}{n!}(z+z')^n.$$

Consequently

(1. 2) $$e^{z+z'} = e^z . e^{z'}$$

(the fundamental functional property of the exponential function). In particular,

(1. 3) $$e^z . e^{-z} = 1, \quad \text{so} \quad e^z \neq 0 \quad \text{for all } z.$$

Putting $z = x + iy$ (with x and y real) gives

$$e^{x+iy} = e^x . e^{iy},$$

so we need only study the two functions e^x and e^{iy}, where x and y are real variables. We have

(1. 4) $$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dy}(e^{iy}) = ie^{iy}.$$

2. Real exponential function e^x

We have seen that $e^x \neq 0$: what is more, $e^x = (e^{x/2})^2 > 0$. Moreover, the expansion $e^x = 1 + x + \frac{x^2}{2} + \cdots$ shows that $e^x > 1 + x$ when $x > 0$.

Thus

$$\lim_{x\to+\infty} e^x = +\infty;$$

substituting $-x$ for x leads to

$$\lim_{x\to-\infty} e^x = 0.$$

We deduce that the function e^x of the real variable x increases strictly from 0 to $+\infty$. The transformation $t = e^x$ has therefore a inverse transformation defined for $t > 0$; it is denoted by

$$x = \log t.$$

This function is also strictly monotonic increasing and increases from $-\infty$ to $+\infty$. The functional relation of e^x is written

(2. 1) $$\log(tt') = \log t + \log t',$$

and, in particular, $\log 1 = 0$.

On the other hand, the theorem about the derivative of an inverse function gives

(2. 2) $$\frac{d}{dt}(\log t) = 1/t.$$

Let us replace t by $1 + u$ $(u > -1)$; $\log(1+u)$ is the primitive of $\frac{1}{1+u}$ which vanishes for $u = 0$; moreover we have the following power series expansion

$$\frac{1}{1+u} = 1 - u + u^2 + \cdots + (-1)^{n-1}u^{n-1} + \cdots$$

whose radius of convergence is equal to 1. From proposition 7. 1 of § 2, it follows that the series of the primitive has the same radius of convergence and that its sum has derivative $\frac{1}{1+u}$; whence, for $|u| < 1$,

(2. 3) $$\log(1+u) = u - \frac{u^2}{2} + \cdots + (-1)^{n-1}\frac{u^n}{n} + \cdots$$

(in fact this expansion is also correct when $u = 1$).

Now put

(2. 4) $$S(X) = \sum_{n\geq 1} \frac{1}{n!} X^n, \qquad T(Y) = \sum_{n\geq 1} (-1)^{n-1} \frac{Y^n}{n},$$

and examine the composed series $U = S \circ T$. We have from proposition 5. 1 of § 2, for $-1 < u < +1$,

$$U(u) = S(T(u));$$

however, $T(u) = \log(1+u)$, $S(x) = e^x - 1$, so

$$U(u) = e^{\log(1+u)} - 1 = (1+u) - 1 = u.$$

This shows that the formal series U is merely I because of the uniqueness of the power series expansion of a function (cf. § 2, no. 8). Thus the series S and T are inverse.

3. The imaginary exponential function e^{iy} (y real)

The series expansion of e^{iy} shows that e^{-iy} is the complex conjugate of e^{iy}; thus $e^{iy}.e^{-iy}$ is the square of the modulus of e^{iy}; but this product is equal to 1 by relation (1. 3). Thus

$$|e^{iy}| = 1.$$

We note that, in the Argand plane representation of the complex field **C**, the point e^{iy} is on the *unit circle*, that is the locus of points whose distance from the origin 0 is equal to 1. The complex numbers u such that $|u| = 1$ form a *group* **U** under multiplication and the functional property

$$e^{i(y+y')} = e^{iy}.e^{iy'}$$

expresses the following : *the mapping $y \to e^{iy}$ is a homomorphism of the additive group* **R** *in the multiplicative group* **U**. This homomorphism will be studied more closely.

Theorem. *The homomorphism $y \to e^{iy}$ maps* **R** *onto* **U**, *and its 'kernel' (subgroup of the y such that $e^{iy} = 1$, the neutral element of* **U**) *is composed of all the integral multiples of a certain real number > 0. By definition, this number will be denoted by 2π.*

Proof. Let us introduce real and imaginary parts of e^{iy}; we put, by definition,

$$e^{iy} = \cos y + i \sin y,$$

which defines two real functions $\cos y$ and $\sin y$, such that

$$\cos^2 y + \sin^2 y = 1.$$

These functions can be expanded as power series whose radii of convergence are infinite :

$$(3.\,1)\qquad \begin{cases} \cos y = 1 - \dfrac{1}{2}y^2 + \cdots + \dfrac{(-1)^n}{(2n)!}y^{2n} + \cdots, \\ \sin y = y - \dfrac{1}{3!}y^3 + \cdots + \dfrac{(-1)^n}{(2n+1)!}y^{2n+1} + \cdots. \end{cases}$$

We shall study the way in which these two functions vary. Observe that separating the real and imaginary parts in the second equation (1. 4) gives

$$\frac{d}{dy}(\cos y) = -\sin y, \qquad \frac{d}{dy}(\sin y) = \cos y.$$

When $y = 0$, $\cos y$ is equal to 1; since $\cos y$ is a continuous function, there exists a $y_0 > 0$ such that $\cos y > 0$ for $0 \leqslant y \leqslant y_0$. Hence $\sin y$, whose derivative is $\cos y$, is a strictly increasing function in the interval $[0, y_0]$. Put $\sin y_0 = a > 0$. We shall show that $\cos y$ vanishes for a certain value of y which is > 0. Suppose in fact that $\cos y > 0$ for $y_0 \leqslant y \leqslant y_1$; we have

$$(3.\,2)\qquad \cos y_1 - \cos y_0 = -\int_{y_0}^{y_1} \sin y\, dy.$$

However, $\sin y \geqslant a$, because $\sin y$ is an increasing function in the interval $[y_0, y_1]$ where its derivative is > 0, thus

$$\int_{y_0}^{y_1} \sin y\, dy \geqslant a(y_1 - y_0).$$

By substituting this in (3. 2) and noting that $\cos y_1 > 0$, we find that

$$y_1 - y_0 < \frac{1}{a}\cos y_0.$$

This proves that $\cos y$ vanishes in the interval $\left[y_0, y_0 + \frac{1}{a}\cos y_0\right]$. Write $\frac{\pi}{2}$ for the smallest value of y which is > 0 and for which $\cos y = 0$ (this is a *definition* of the number π). In the interval $\left[0, \frac{\pi}{2}\right]$, $\cos y$ decreases strictly from 1 to 0, and $\sin y$ increases strictly from 0 to 1; thus the mapping $y \to e^{iy}$ is a bijective mapping of the compact interval $\left[0, \frac{\pi}{2}\right]$ onto the set of points (u, v) of the unit circle whose coordinates u and v are both $\geqslant 0$. By a theorem of topology about continuous, bijective, mappings of a *compact* space, we deduce :

Lemma. *The mapping $y \to e^{iy}$ is a homeomorphism of $\left[0, \frac{\pi}{2}\right]$ onto the sector of the unit circle $u^2 + v^2 = 1$ in the positive quadrant $u \geqslant 0$, $v \geqslant 0$.*

For $\frac{\pi}{2} \leqslant y \leqslant \pi$, we have $e^{iy} = ie^{i\left(y - \frac{\pi}{2}\right)}$, wnence we easily deduce that e^{iy} takes each complex value of modulus 1 whose abscissa is $\leqslant 0$ and whose ordinate is $\geqslant 0$, and takes each value precisely once. Analogous results can be deduced for the intervals $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$.

Thus, for $0 \leqslant y < 2\pi$, e^{iy} takes each complex value of modulus 1 precisely once, whereas $e^{2i\pi} = 1$. Therefore the function e^{iy} is periodic of period 2π, and the mapping $y \to e^{iy}$ maps **R** on **U**. This completes the proof of the theorem.

4. Measurement of angles. Argument of a complex number

Let $2\pi\mathbf{Z}$ denote the subgroup of the additive group **R** formed by the integral multiples of the number 2π. The mapping $y \to e^{iy}$ induces an *isomorphism* φ of the *quotient group* $\mathbf{R}/2\pi\mathbf{Z}$ on the *group* **U**. The inverse isomorphism φ^{-1} of **U** on $\mathbf{R}/2\pi\mathbf{Z}$ associates with any complex number u such that $|u| = 1$, a real number which is defined up to addition of an integral multiple of 2π; this class of numbers is called the *argument* of u and is denoted by $\arg u$. By an abuse of notation, $\arg u$ will also denote any one of the real numbers whose class modulo 2π is the argument of u; the function $\arg u$ is then an example of a many-valued function, that is, it can take many values for a given value of the variable u. This function resolves the problem of ' measure of angles ' (each angle is identified with the corresponding point of **U**) : the ' measure of an angle ' is a real number which is only defined modulo 2π.

We topologize the quotient group $\mathbf{R}/2\pi\mathbf{Z}$ by putting on it the *quotient* topology of the usual topology on the real line **R** : let p be the canonical mapping of **R** on its quotient $\mathbf{R}/2\pi\mathbf{Z}$, a subset A of $\mathbf{R}/2\pi\mathbf{Z}$ is said to be *open* if its inverse image $p^{-1}(A)$, which is a subset of **R** invariant under translation by 2π, is an open set of **R**. It is easily verified that the topological space $\mathbf{R}/2\pi\mathbf{Z}$ is Hausdorff (that is, that two distinct points have disjoint open neighbourhoods). Moreover, it is *compact*; for, if I is the closed interval $[0,2\pi]$, the natural mapping $I \to \mathbf{R}/2\pi\mathbf{Z}$ takes the compact space I onto the Hausdorff space $\mathbf{R}/2\pi\mathbf{Z}$ which is then compact by a classical theorem in topology. The homomorphism $\varphi : \mathbf{R}/2\pi\mathbf{Z} \to \mathbf{U}$ is continuous and is a bijective mapping of the compact space $\mathbf{R}/2\pi\mathbf{Z}$ onto the Hausdorff space **U**; hence φ is a *homeomorphism* of $\mathbf{R}/2\pi\mathbf{Z}$ on **U**.

General definition of argument : for any complex number $t \neq 0$, define the argument of t by the formula

$$\arg t = \arg\left(\frac{t}{|t|}\right).$$

The right hand side is defined already since $t/|t| \in \mathbf{U}$. (Note that the argument of 0 is not defined.) As above, $\arg t$ is only defined up to addition of integral multiples of 2π. We thus have

$$t = |t| e^{i \arg t}. \tag{4. 1}$$

Application. To solve the equation $t^n = a$ (where $a \neq 0$ is given) : the equation is equivalent to

$$|t| = |a|^{1/n}, \qquad \arg t = \frac{1}{n}\arg a,$$

and has n complex solutions t because one obtains for $\arg t$ a real number defined up to addition of an integral multiple of $2\pi/n$.

5. Complex Logarithms

Given a complex number t, we seek all the complex numbers z such that $e^z = t$. Such numbers exist only when $t \neq 0$. In this case, relation (4. 1) shows that the z that we seek are the complex numbers of the form

$$\log|t| + i \arg t. \tag{5. 1}$$

We define

$$\log t = \log|t| + i \arg t, \tag{5. 2}$$

which is a complex number defined only up to addition of an integral multiple of $2\pi i$. From this definition, we have $e^{\log t} = t$. When t is real and > 0, we again have the classical function $\log t$ if we allow only the value 0 for $\arg t$.
For any complex numbers t and t' both $\neq 0$ and for any values of $\log t$, $\log t'$ and $\log tt'$, we have

$$\log(tt') = \log t + \log t' \pmod{2\pi i}. \tag{5. 3}$$

Branches of the logarithm. So far we have not defined $\log t$ as a *function* in the proper sense of the word.

Definition. We say that a *continuous* function $f(t)$ of the complex variable t, defined in a *connected open set* D of the plane **C**, not containing the point $t = 0$, is *a branch* of $\log t$ if, for all $t \in D$, we have $e^{f(t)} = t$ (in other words, if $f(t)$ is one of the possible values of $\log t$).

We shall see later (chapter II, § 1, no. 7) what conditions must be satisfied by the open set D for branch of $\log t$ to exist in D. We shall now examine how it is possible to obtain all branches of $\log t$ if one exists.

Proposition 5. 1 *If there exists a branch $f(t)$ of $\log t$ in the connected open set* D, *then any other branch is of the form $f(t) + 2k\pi i$ (k an integer); conversely, $f(t) + 2k\pi i$ is a branch of $\log t$ for any integer k.*

Let us suppose the that $f(t)$ and $g(t)$ are two branches of $\log t$. The difference

$$h(t) = \frac{f(t) - g(t)}{2\pi i}$$

is a continuous function in D which takes only integral values; since D is assumed connected, such a function is necessarily *constant*. For, the set of points $t \in D$ such that $h(t)$ is equal to a given integer n is both open and closed. Thus the set is empty or is equal D. The constant must of course be an integer. That $f(t) + 2k\pi i$ is a branch of $\log t$ for any integer k is obvious.

One defines similarly what must be understood by a branch of $\arg t$ in a connected open set D which does not contain the origin. Moreover, any branch of $\arg t$ defines one of $\log t$ and vice-versa.

Example. Let D be the open half-plane $\mathrm{Re}(t) > 0$ (recall that $\mathrm{Re}(t)$ denotes the real part of t). For any t in this half-plane, there is a unique value of $\arg t$ which is $> -\frac{\pi}{2}$ and $< \frac{\pi}{2}$; denote this value by $\mathrm{Arg}\, t$. We shall show that $\mathrm{Arg}\, t$ is a *continuous function* and that consequently

$$\log |t| + i \,\mathrm{Arg}\, t$$

is a branch of $\log t$ in the half plane $\mathrm{Re}(t) > 0$. It will be called the *principal branch* of $\log t$. Since $\mathrm{Arg}\, t = \mathrm{Arg}\,(t/|t|)$ and since the mapping $t \to t/|t|$ is a continuous mapping of the half-plane $\mathrm{Re}(t) > 0$ on the set of u such that $|u| = 1$ and $\mathrm{Re}(u) > 0$, it is sufficient to show that the mapping $y = \mathrm{Arg}\, u$ is continuous. However, this is the inverse mapping of $u = e^{iy}$ as y ranges over the open interval $\left]-\frac{\pi}{2}, +\frac{\pi}{2}\right[$; the function $u = e^{iy}$ is a continuous bijective mapping of the compact interval $\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ on the set of u such that $|u| = 1$ and $\mathrm{Re}(u) \geqslant 0$; this then is a homeomorphism and the inverse mapping is indeed continuous, which completes the proof.

6. Series expansion of the complex logarithm

Proposition 6. 1. *The sum of the power series*

$$T(u) = \sum_{n \geqslant 1} (-1)^{n-1} \frac{u^n}{n},$$

which converges for $|u| < 1$, is equal to the principal branch of $\log(1 + u)$.

Note first that if $|u| < 1$, $t = 1 + u$ remains inside an open disc contained in the half plane $\mathrm{Re}(t) > 0$. Again we use the notations of relation (2. 4) and remember that the series S and T are inverse to one another; proposition 5. 1 of § 2 shows that $S(T(u)) = u$ for any complex number u such that $|u| < 1$. In other words, $e^{T(u)} = 1 + u$; and consequently $T(u)$ is a branch of $\log(1 + u)$. To show that this is the principal branch, it is sufficient to verify that it takes the same value as the principal branch for a particular value of u, for instance, that it is zero when $u = 0$, which is obvious from the series expansion of $T(u)$.

Proposition 6. 2. *If $f(t)$ is a branch of $\log t$ in a connected open set D, the function $f(t)$ has derivative $f'(t)$ with respect to the complex variable t, and*

$$f'(t) = 1/t.$$

In fact, for h complex $\neq 0$ and sufficiently small, we have

$$\frac{f(t+h) - f(t)}{h} = \frac{f(t+h) - f(t)}{e^{f(t+h)} - e^{f(t)}};$$

and, when t tends to 0, this tends to the algebraic inverse of the limit of $\frac{e^{z'} - e^z}{z' - z}$ as z' tends to $z = f(t)$; the limit we seek is then the inverse of the value of the derivative of e^z for $z = f(t)$, which is equal to $e^{-f(t)} = 1/t$.

Note. This result checks with the fact that the derivative of the power series $T(u)$ is indeed equal to $\frac{1}{1+u}$.

Definition. For any pair of complex numbers $t \neq 0$ and α, we put

$$t^\alpha = e^{\alpha \log t}.$$

This is a many valued function of t for fixed α. A branch of t^α in a connected open set D is defined as above. Any branch of $\log t$ in D defines a branch of t^α in D.

Revision. Here the reader is asked to revise, if necessary, the power series expansions of the usual functions, arc tan x, arc sin x, etc. Moreover, for any complex exponent α and for x complex such that $|x| < 1$, we consider

$$(1 + x)^\alpha = e^{\alpha \log(1+x)},$$

where $\log(1 + x)$ denotes the principal branch (the function $(1 + x)^\alpha$ then takes the value 1 for $x = 0$); the reader should study its power series expansion.

4. Analytic Functions of a Real or Complex Variable

1. Definitions

Definition 1. 1. We say that a function $f(x)$, defined in some neighbourhood of x_0, has a *power series expansion* at the point x_0 if there exists a formal power series $S(X) = \sum_{n \geqslant 0} a_n X^n$ whose radius of convergence is $\neq 0$ and which satisfies

$$f(x) = \sum_{n \geqslant 0} a_n (x - x_0)^n \quad \text{for} \quad |x - x_0| \text{ sufficiently small.}$$

This definition applies equally well to the case when x is a real or a complex variable. The series $S(X)$, if it exists, is *unique* by no. 8 of § 2.

If $f(x)$ has a power series expansion at x_0, then the function f is infinitely differentiable in a neigbourhood of x_0 because the sum of a power series has this property. If the product fg of two functions f and g having power series expansions at x_0 is identically zero in some neighbourhood of x_0, then a least one of the functions f and g is identically zero in a neighbourhood of x_0; in fact, this is an immediate consequence of the fact that the ring of formal series is an integral domain (§ 1, proposition 3. 1). If f has a power series expansion at x, there exists a function g also having a power series expansion at x_0 and having derivative $g' = f$ in some neighbourhood of x_0; such a function is unique up to addition of a constant in some neighbourhood of x_0; to see why this is so, it is sufficient to examine the series of primitives of terms of a power series expansion of the function f.
We shall consider in what follows an *open set* D of the real line $\mathbf{R}$, or the complex plane $\mathbf{C}$. If D is open in $\mathbf{R}$, D is a union of open intervals and, if D is also connected, D is an open interval. We write x for a real or complex variable which varies over the open set D.

Definition 1. 2. A function $f(x)$ with real or complex values defined in the open set D, is said to be *analytic* in D if, for any point $x_0 \in D$, the function $f(x)$ has a power series expansion at the point x_0. In other words, there must exist a number $\rho(x_0) > 0$ and a formal power series $S(X) = \sum_{n \geqslant 0} a_n X^n$ with radius of convergence $\geqslant \rho(x_0)$ and such that

$$f(x) = \sum_{n \geqslant 0} a_n (x - x_0)^n \quad \text{for} \quad |x - x_0| < \rho(x_0).$$

The following properties are obvious: any analytic function in D is infinitely differentiable in D and all its derivatives are analytic in D. The sum and product of two analytic functions in D are analytic in D: that is to say, the analytic functions in D form a ring, and even an algebra. It follows from proposition 6. 1 of § 2 that, if $f(x)$ is analytic in D, then $1/f(x)$ is analytic in the open set D excluding the set of points x_0 such that $f(x_0) = 0$.

Finally, proposition 5. 1 of § 2 gives that, if f is analytic in D and takes its values in D′ and if g is analytic in D′, then the composed function $g \circ f$ is analytic in D.

Let f be an analytic function in a *connected* set D; if f has a primitive g, that is, if there exists a function g in D whose derivative g' is equal to f, then this primitive function is unique up to addition of a constant and it is an analytic function.

Examples of analytic functions. Polynomials in x are analytic functions on the whole of the real line (or in the complex plane). A rational function $P(x)/Q(x)$ is analytic in the complement of the set of points x_0 such that $Q(x_0) = 0$. It will follow from proposition 2. 1 that the function e^x is analytic. The function arc tan x is analytic for all real x since its derivative $\frac{1}{1 + x^2}$ is analytic.

2. Criteria of analyticity

Proposition 2. 1. *Let* $S(X) = \sum_{n \geqslant 0} a_n X^n$ *be a power series whose radius of convergence* ρ *is* $\neq 0$. *Let*

$$S(x) = \sum_{n \geqslant 0} a_n x^n$$

be its sum for $|x| < \rho$. *Then* $S(x)$ *is an analytic function in the disc* $|x| < \rho$.

This result is by no means trivial. It will be an immediate consequence of what follows, to be precise:

Proposition 2. 2 *With the conditions of proposition* 2. 1, *let* x_0 *be such that* $|x_0| < \rho$. *Then the power series*

$$(2.1) \qquad \sum_{n \geqslant 0} \frac{1}{n!} S^{(n)}(x_0) X^n$$

has radius of convergence $\geqslant \rho - |x_0|$ *and*

$$(2.2) \qquad S(x) = \sum_{n \geqslant 0} \frac{1}{n!} S^{(n)}(x_0)(x - x_0)^n \quad \textit{for} \quad |x - x_0| < \rho - |x_0|.$$

Proof of proposition 2. 2. Put $r_0 = |x_0|$, $\alpha_n = |a_n|$. We have

$$S^{(p)}(x_0) = \sum_{q \geq 0} \frac{(p+q)!}{q!} a_{p+q}(x_0)^q,$$

$$|S^{(p)}(x_0)| \leq \sum_{q \geq 0} \frac{(p+q)!}{q!} \alpha_{p+q}(x_0)^q.$$

For $r_0 \leq r < \rho$, we have

$$(2.\ 3) \quad \sum_{p \geq 0} \frac{1}{p!} |S^{(p)}(x_0)| (r - r_0)^p \leq \sum_{p,q} \frac{(p+q)!}{p!\,q!} \alpha_{p+q}(r_0)^q (r - r_0)^p,$$
$$\leq \sum_{n \geq 0} \alpha_n \left(\sum_{0 \leq p \leq n} \frac{n!}{p!(n-p)!} (r - r_0)^p (r_0)^{n-p} \right),$$
$$\leq \sum_{n \geq 0} \alpha_n r^n < +\infty.$$

Thus the radius of convergence of the series (2. 1) is $\geq r - r_0$. Since r can be chosen arbitrarily near to ρ, this radius of convergence is $\geq \rho - r_0$.

Now let x be such that $|x - x_0| < \rho - r_0$. The double series

$$\sum_{p,q} \frac{(p+q)!}{p!\,q!} a_{p+q}(x_0)^q (x - x_0)^p$$

is absolutely convergent by (2. 3). Its sum can therefore be calculated by regrouping the terms in an arbitrary manner. We shall calculate this sum in two different ways. A first grouping of terms gives

$$\sum_{n \geq 0} a_n \left(\sum_{0 \leq p \leq n} \frac{n!}{p!\,(n-p)!} (x - x_0)^p (x_0)^{n-p} \right) = \sum_{n \geq 0} a_n x^n = S(x);$$

another grouping gives

$$\sum_{p \geq 0} \frac{(x - x_0)^p}{p!} \left(\sum_{q \geq 0} \frac{(p+q)!}{q!} a_{p+q}(x_0)^q \right) = \sum_{p \geq 0} \frac{(x - x_0)^p}{p!} S^{(p)}(x_0).$$

Formula (2. 2) follows from a comparison of these two and this completes the proof.

Note 1. The radius of convergence of series (2. 1) may be strictly larger than $\rho - |x_0|$. Consider, for example, the series

$$S(X) = \sum_{n \geq 0} (iX)^n.$$

Then $S(x) = \frac{1}{1 - ix}$ for $|x| < 1$. Choose a real number for x_0, so we have

$$\frac{1}{1 - ix} = \frac{1}{1 - ix_0} \left(1 - i \frac{x - x_0}{1 - ix_0} \right)^{-1} = \sum_{n \geq 0} \frac{i^n}{(1 - ix_0)^{n+1}} (x - x_0)^n.$$

This series converges for $|x - x_0| < \sqrt{1 + (x_0)^2}$ and $\sqrt{1 + (x_0)^2}$ is strictly greater than $1 - |x_0|$.

Note 2. Let

$$A(r) = \sum_{n \geq 0} |a_n| r^n \quad \text{for} \quad r < \rho.$$

From inequality (2. 3), we have

$$(2.\ 4) \quad \left| \frac{1}{p!} S^{(p)}(x) \right| \leq \frac{A(r)}{(r - r_0)^p} \quad \text{for} \quad |x| \leq r_0 < r < \rho.$$

Note 3. If x is a *complex* variable, we shall see in chapter II that any function which is differentiable is analytic and is consequently infinitely differentiable. The situation is completely different in the case of a *real* variable : there exist functions which have a first derivative but no second derivative (one need only consider the primitive of a continuous function which is not differentiable). Moreover, there exist functions which are infinitely differentiable but which are not analytic; here is a simple example : the function $f(x)$, which is equal to zero for $x = 0$ and to e^{-1/x^2} for $x \neq 0$, is infinitely differentiable for all x; it vanishes with all its derivatives at $x = 0$ so, if it were analytic, it would be identically zero in some neighbourhood of $x = 0$, which is not the case.

THEOREM. *In order that an infinitely differentiable function of a real variable x in an open interval D should be analytic in D, it is necessary and sufficient that any point $x_0 \in$ D has a neighbourhood V with the following property : there exist numbers M and t, finite and > 0, such that*

$$(2.\ 5) \quad \left| \frac{1}{p!} f^{(p)}(x) \right| \leq M.t^p \quad \textit{for any } x \in V \textit{ and any integer } p \geq 0.$$

Indication of proof. The condition is shown to be necessary by using inequality (2. 4). It is shown to be sufficient by writing a finite Taylor expansion of the function $f(x)$ and using (2. 5) to find an upper bound for the Lagrange remainder.

3. Principle of analytic continuation

THEOREM. *Let f be an analytic function in a connected open set D and let $x_0 \in$ D. The following conditions are equivalent :*

a) $f^{(n)}(x_0) = 0$ *for all integers* $n \geq 0$;

b) f is identically zero in a neighbourhood of x_0;

c) f is identically zero in D.

Proof. It is obvious that *c*) implies *a*). We shall show that *a*) implies *b*) and *b*) implies *c*). Suppose *a*) is satisfied. We have then $f^{(n)}(x_0) = 0$ for all $n \geqslant 0$ with the convention that $f^{(0)} = f$. But $f(x)$ has a power series expansion in powers of $(x - x_0)$ in a neigbourhood of x_0 and the coefficients $\frac{1}{n!} f^n(x_0)$ are zero; thus $f(x)$ is identically zero in a neighbourhood of x_0 which proves *b*).
Suppose conditions *b*) is satisfied. To show that f is zero at all points of D, it is sufficient to show that the set D′ of points $x \in D$ *in a neighbourhood of which f is identically zero* is both open and closed (D′ is not empty because of *b*), thus, since D is connected, D′ will be equal to D). It follows from the definition of D′ that it is open. It remains to be proved that, if $x_0 \in D$ is in the closure of D′, then $x_0 \in D'$. However, $f^{(n)}(x) = 0$ for each $n \geqslant 0$ at points arbitrarily close to x_0 (in fact, at the points of D′); thus $f^{(n)}(x_0) = 0$ because of the continuity of $f^{(n)}$; this holding for all $n \geqslant 0$ implies as above that $f(x)$ is identically zero in a neighbourhood of x_0. Thus $x_0 \in D'$, which completes the proof.

COROLLARY 1. *The ring of analytic functions in a connected open set* D *is an integral domain.*

For, if the product fg of two analytic functions in D is identically zero and if $x_0 \in D$, then one of the functions f, g is identically zero in a neighbourhood of x_0 because the ring of formal power series is an integral domain. But, if f is identically zero in some neighbourhood of x_0, then f is zero in the whole of D by the above theorem.

COROLLARY 2. (Principle of analytic continuation) *If two analytic functions f and g in a connected open set* D *coincide in a neighbourhood of a point of* D, *then they are identical in* D.

The *problem of analytic continuation* is the following : given an analytic function h in a connected open set D′ and given a connected open set D containing D′, we ask if there exists an analytic function f in D which extends h. Corollary 2 shows that such a function f is unique if it exists.

4. Zeros of an analytic function

Let $f(x)$ be an analytic function in a neighbourhood of x_0 and let

$$f(x) = \sum_{n \geqslant 0} a_n (x - x_0)^n$$

be its power series expansion for sufficiently small $|x - x_0|$. Suppose that $f(x_0) = 0$ and that $f(x)$ *is not identically zero* in a neighbourhood of x_0.

Let k be the smallest integer such that $a_k \neq 0$. The series

$$\sum_{n \geqslant k} a_n (x - x_0)^{n-k}$$

converges for sufficiently small $|x - x_0|$ and its sum $g(x)$ is an analytic function such that $g(x_0) \neq 0$ in some neighbourhood of x. Thus, for x near enough to x_0, we have

$$f(x) = (x - x_0)^k g(x), \qquad g(x_0) \neq 0. \tag{4. 1}$$

The integer $k > 0$ thus defined is called the *order of multiplicity* of the zero x_0 for the function f. It is characterized by relation (4. 1), where $g(x)$ is analytic in a neighbourhood of x_0. The order of multiplicity k is also characterized by the condition

$$f^{(n)}(x_0) = 0 \quad \text{for} \quad 0 \leqslant n < k, \quad f^{(k)}(x_0) \neq 0.$$

If $k = 1$, we call x_0 a *simple* zero. If $k \geqslant 2$, we call x_0 a *multiple* zero.
Relation (4. 1) and continuity of $g(x)$ imply

$$f(x) \neq 0 \qquad \text{for} \qquad 0 < |x - x_0| < \varepsilon \qquad (\varepsilon > 0 \text{ sufficiently small}).$$

In other words the point x_0 has a neighbourhood in which it is the *unique* zero of the function $f(x)$.

PROPOSITION 4. 1. *If f is an analytic function in a connected open set* D *and if f is not identically zero, then the set of zeros of f is a discrete set* (in other words, all the points of this set are *isolated*).

For, corollary 2 of no. 3 gives that f is not identically zero in a neigbourhood of any point of D, so one can apply the above reasoning to each zero of f.

In particular, any *compact* subset of D contains only a *finite* number of zeros of the function g.

5. Meromorphic functions

Let f and g be two analytic functions in a connected open set D, and suppose that g is not identically zero. The function $f(x)/g(x)$ is defined and analytic in a neighbourhood of every point x_0 of D such that $g(x_0) \neq 0$, that is to say, in the whole of D except perhaps in certain isolated points.

Let us see how $f(x)/g(x)$ behaves in a neighbourhood of a point x_0 which is a zero of $g(x)$; if $f(x)$ is not identically zero, we have

$$f(x) = (x - x_0)^k f_1(x), \qquad g(x) = (x - x_0)^{k'} g_1(x),$$

where k and k' are integers with $k \geqslant 0$ and $k' > 0$, f_1 and g_1 are analytic in some neighbourhood of x_0 with $f_1(x_0) \neq 0$ and $g_1(x_0) \neq 0$; hence, for $x \neq x_0$ but near to x_0,

$$\frac{f(x)}{g(x)} = (x - x_0)^{k-k'} \frac{f_1(x)}{g_1(x)}.$$

The function $h_1(x) = f_1(x)/g_1(x)$ is analytic in a neighbourhood of x_0 and we have that $h_1(x_0) \neq 0$. Two cases arise:

1° $k \geqslant k'$; then the function

$$(x - x_0)^{k-k'} h_1(x)$$

is analytic in some neighbourhood of x_0 and coincides with $f(x)/g(x)$ for $x \neq x_0$. Hence the extension of f/g to the point x_0 is analytic in a neighbourhood of x_0 and admits x_0 as a zero if $k > k'$.

2° $k < k'$: then

$$\frac{f(x)}{g(x)} = \frac{1}{(x - x_0)^{k'-k}} h_1(x), \qquad h_1(x_0) \neq 0.$$

We say in this case that x_0 is a *pole* of the function f/g; the integer $k' - k$ is called the *order of multiplicity* of the pole. As x tends to x_0, $\left|\frac{f(x)}{g(x)}\right|$ tends to $+\infty$. We can agree to extend the function f/g by giving it the value "infinity" at x_0. We shall return later to the introduction of this unique number infinity, denoted ∞.
If $f(x)$ analytic and has x_0 as a zero of order $k > 0$, then x_0 is clearly a pole of order k of $1/f(x)$.

Definition. A *meromorphic* function in an open set D is defined to be a function $f(x)$ which is defined and analytic an the open set D' obtained from D by taking out a set of isolated points each of which is a *pole* of $f(x)$.

In a neighbourhood of each point of D (without exception), f can be expressed as a quotient $h(x)/g(x)$ of two analytic functions, the denominator being not identically zero. The sum and product of two meromorphic functions are defined in the obvious way: the meromorphic functions in D form a ring and even an algebra. In fact they form a field because, if $f(x)$ is not identically zero in D, it is not identically zero in any neighbourhood of any point of D by the theorem of no. 3; so $1/f(x)$ is then analytic, or has at most a pole at each point of D and is consequently meromorphic in D.

PROPOSITION 5. 1. *The derivative f' of a meromorphic function f in D is meromorphic in D; the functions f and f' have the same poles; if x_0 is a pole of order k of f, then it is a pole of order $k + 1$ of f'.*

For, f' is defined and analytic at each point of D which is not a pole of f. It remains to be proved that, if x_0 is a pole of f, x_0 is also a pole of f'. Moreover, for x near x_0,

$$f(x) = \frac{1}{(x - x_0)^k} g(x),$$

$g(x)$ being analytic with $g(x_0) \neq 0$, $k > 0$. Hence, for $x \neq x_0$,

$$f'(x) = \frac{1}{(x - x_0)^{k+1}}[(x - x_0)g'(x) - kg(x)] = \frac{1}{(x - x_0)^{k+1}} g_1(x),$$

and as $g_1(x_0) \neq 0$, x_0 is a pole of f' of order $k + 1$.

Exercises

1. Let K be a commutative field, X an indeterminate and E = K[[X]] the algebra of formal power series with coefficients in K. For S, T in E, define

$$d(\mathrm{S}, \mathrm{T}) = \begin{cases} 0 & \text{if } \mathrm{S} = \mathrm{T}, \\ e^{-k} & \text{if } \mathrm{S} \neq \mathrm{T}, \end{cases} \quad \text{and} \quad \omega(\mathrm{S} - \mathrm{T}) = k.$$

a) Show that d defines a distance function in the set E.

b) Show that the mappings $(\mathrm{S}, \mathrm{T}) \to \mathrm{S} + \mathrm{T}$ and $(\mathrm{S}, \mathrm{T}) \to \mathrm{ST}$ of $\mathrm{E} \times \mathrm{E}$ into E are continuous with respect to the metric topology defined by d.

c) Show that the algebra K[X] of polynomials is everywhere dense in E when considered as a subset of E.

d) Show that the metric space E is complete. (If (S_n) is a Cauchy sequence in E, note that for any integer $m > 0$, the first m terms of S_n do not depend on n for sufficiently large n.)

e) Is the mapping $\mathrm{S} \to \mathrm{S}'$ (the derivative of S) continuous?

2. Let p, q be integers $\geqslant 1$. Let $\mathrm{S}_1(\mathrm{X})$ be the formal series

$$1 + \mathrm{X} + \mathrm{X}^2 + \cdots + \mathrm{X}^n + \cdots,$$

and put

$$\mathrm{S}_p(\mathrm{X}) = (\mathrm{S}_1(\mathrm{X}))^p.$$

a) Show, by induction on n, that

$$(1) \quad 1 + p + \frac{p(p+1)}{2!} + \cdots + \frac{p(p+1)\ldots(p+n-1)}{n!} = \frac{(p+1)\ldots(p+n)}{n!},$$

and deduce (by induction on p), the expansion

$$(2) \qquad S_p(X) = \sum_{n\geqslant 0} \binom{p+n-1}{n} X^n,$$

where $\binom{k}{h}$ denotes the binomial coefficient $\dfrac{k!}{h!(k-h)!}$

b) Use $S_p(X).S_q(X) = S_{p+q}(X)$ to show that

$$(3) \qquad \sum_{0\leqslant l\leqslant n} \binom{p+l+1}{l}\binom{q+n+l-1}{n-l} = \binom{p+q+n+1}{n}$$

(which is a generalisation of (1), the case when $q = 1$).

3. Find the precise form of the polynomials P_n in the proof of proposition 7. 1, § 1, for $n \leqslant 5$ and calculate the terms of degree $\leqslant 5$ of the formal (compositional) inverse series of

$$S(X) = X - \frac{1}{3}X^3 + \frac{1}{5}X^5 + \cdots + (-1)^p \frac{1}{2p+1} X^{2p+1} + \cdots.$$

4. Find the radii of convergence of the following series :

a) $\displaystyle\sum_{n\geqslant 0} q^{n^2} z^n \qquad (|q| < 1),$

b) $\displaystyle\sum_{n\geqslant 0} n^p z^n \qquad (p \text{ integer} > 0),$

c) $\displaystyle\sum_{n\geqslant 0} a_n z^n$, with $a_{2n+1} = a^{2n+1}$, $a_{2n} = b^{2n}$ for $n \geqslant 0$,

where a and b are real and $0 < a$, $b < 1$.

5. Given two formal power series

$$S(X) = \sum_{n\geqslant 0} a_n X^n \quad \text{and} \quad T(X) = \sum_{n\geqslant 0} b_n X^n \quad (b_n \neq 0),$$

let

$$U(X) = \sum_{n\geqslant 0} (a_n)^p X^n, \quad V(X) = \sum_{n\geqslant 0} a_n b_n X^n, \quad W(X) = \sum_{n\geqslant 0} (a_n/b_n) X$$

(where p is an integer). Prove the following relations :

$$\rho(U) = (\rho(S))^p, \qquad \rho(V) \geqslant \rho(S).\rho(T),$$

and, if $\rho(T) \neq 0$,

$$\rho(W) \leqslant \rho(S)/\rho(T).$$

6. Let a, b and c be elements of $\mathbf{C}$, c not an integer $\leqslant 0$. What is the radius of convergence of the series

$$S(X) = 1 + \frac{ab}{c}X + \frac{a(a+1).(b+1)}{2!c(c+1)}X^2 + \cdots + \frac{a(a+1)\dots(a+n-1).b(b+1)\dots(b+n-1)}{n!c(c+1)\dots(c+n-1)}X^n + \cdots$$

Show that its sum $S(z)$, for $|z| < \rho(S)$, satisfies the differential equation

$$z(1-z)S'' + (c - (a+b+1)z)S' - abS = 0.$$

7. Let $S(X) = \sum_{n\geqslant 0} a_n X^n$ be a formal power series such that $\rho(S) = 1$. Put

$$s_n = a_0 + \cdots + a_n, \quad t_n = \frac{1}{n+1}(s_0 + s_1 + \cdots + s_n) \quad \text{for} \quad n \geqslant 0,$$

and put

$$U(X) = \sum_{n\geqslant 0} s_n X^n, \qquad V(X) = \sum_{n\geqslant 0} t_n X^n.$$

Show that : (i) $\rho(U) = \rho(V) = 1$, (ii) for all $|z| < 1$,

$$\frac{1}{1-z}\left(\sum_{n\geqslant 0} a_n z^n\right) = \sum_{n\geqslant 0} s_n z^n.$$

8. Let $S(X) = \sum_{n\geqslant 0} a_n X^n$ be a formal power series whose coefficients are defined by the following recurrence relations :

$$a_0 = 0, \; a_1 = 1, \; a_n = \alpha a_{n-1} + \beta a_{n-2} \quad \text{for} \quad n \geqslant 2,$$

where α, β are given real numbers.

a) Show that, for $n \geqslant 1$, we have $|a_n| \leqslant (2c)^{n-1}$ where $c = \max(|\alpha|, |\beta|, 1/2)$ and deduce that the radius of convergence $\rho(S) \neq 0$.

b) Show that

$$(1 - \alpha z - \beta z^2)S(z) = z, \quad \text{for} \quad |z| < \rho(S),$$

and deduce that, for $|z| < \rho(S)$,

$$(1) \qquad S(z) = \frac{z}{1 - \alpha z - \beta z^2}.$$

c) Let z_1, z_2 be the two roots of $\beta X^2 + \alpha X - 1 = 0$. By decomposing the right hand side of (1) into partial fractions, find an expression for the a_n in terms of z_1 and z_2 and deduce that

$$\rho(S) = \min(|z_1|, |z_2|).$$

(Note that, if $S(X) = S_1(X) . S_2(X)$, then $\rho(S) \geqslant \min(\rho(S_1), \rho(S_2))$.)

9. Show that, if x, y are real and n is an integer $\geqslant 0$, then

$$\sum_{0 \leqslant p \leqslant n} \sin(px + y) = \sin\left(\frac{n}{2}x + y\right) \sin\frac{n+1}{2}x / \sin\frac{x}{2}.$$
$$\sum_{0 \leqslant p \leqslant n} \cos(px + y) = \cos\left(\frac{n}{2}x + y\right) \sin\frac{n+1}{2}x / \sin\frac{x}{2},$$

(Use $\cos(px + y) + i\sin(px + y) = e^{i(px+y)} = e^{iy}(e^{ix})^p$.)

10. Prove the following inequalities for $z \in \mathbf{C}$:

$$|e^z - 1| \leqslant e^{|z|} - 1 \leqslant |z| e^{|z|}.$$

11. Show that, for any integer $n \geqslant 1$ and any complex number z,

$$\left(1 + \frac{z}{n}\right)^n = 1 + z + \sum_{2 \leqslant p \leqslant n} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{p-1}{n}\right) \frac{z^p}{p!},$$

and deduce that

$$e^z = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n.$$

12. Show that the function of a complex variable z defined by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \left(\text{resp. } \sin z = \frac{e^{iz} - e^{-iz}}{2i}\right)$$

is the analytic extension to the whole plane $\mathbf{C}$ of the function $\cos x$ (resp. $\sin x$) defined in § 3, no. 3, Prove that, for any $z, z' \in \mathbf{C}$,

$$\cos(z + z') = \cos z \cos z' - \sin z \sin z',$$
$$\sin(z + z') = \sin z \cos z' + \cos z \sin z';$$

$$\cos^2 z + \sin^2 z = 1.$$

13. Prove the relations

$$\frac{2}{\pi} x \leqslant \sin x \leqslant x \quad \text{for } x \text{ real and } 0 \leqslant x \leqslant \pi/2.$$

14. Let $z = x + iy$ with x, y real.

(i) Show that

$$|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y,$$
$$|\cos(x + iy)|^2 = \cos^2 x + \sinh^2 y;$$

(ii) determine the zeros of the functions $\sin az$, $\cos az$ (where a is a real number $\neq 0$);

(iii) Show that, if $-\pi < a < \pi$ and n is a positive integer,

$$\left|\frac{\sin az}{\sin \pi z}\right| \leqslant \frac{\cosh ay}{\cosh \pi y}, \quad \text{for} \quad z = n + \frac{1}{2} + iy,$$

and

$$\left|\frac{\sin az}{\sin \pi z}\right| \leqslant \frac{\cosh a\left(n + \frac{1}{2}\right)}{\sinh \pi\left(n + \frac{1}{2}\right)}, \quad \text{for} \quad z = x + i\left(n + \frac{1}{2}\right).$$

(N. B. By definition, $\cosh z = \cos(iz)$, $\sinh z = -i\sin(iz)$.)

15. Let I be an interval of the real line $\mathbf{R}$. Show that, if $f(x)$ is an analytic function (of a real variable but with complex values) in I, it can be extended to an analytic function in a connected open set D of the complex plane containing I.

16. (i) Let (α_n), (β_n) be two sequences of numbers with the following properties:

a) there is a constant $M > 0$ such that

$$|\alpha_1 + \alpha_2 + \cdots + \alpha_n| \leqslant M \quad \text{for all} \quad n \geqslant 1,$$

b) the β_n are real $\geqslant 0$ and $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_n \geqslant \cdots$.
Show that, for all $n \geqslant 1$,

$$|\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n| \leqslant M\beta_1.$$

(Introduce $s_n = \alpha_1 + \cdots + \alpha_n$ and write

$$\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = (\beta_1 - \beta_2)s_1 + \cdots + (\beta_{n-1} - \beta_n)s_{n-1} + \beta_n s_n.)$$

(ii) Let $S(X) = \sum_{n \geqslant 0} a_n X^n$ be a formal power series with complex coefficients such that $\rho(S) = 1$, and that $\sum_{n \geqslant 0} a_n$ is convergent. Use (i) to show that the series $\sum_{n \geqslant 0} a_n x^n$ is uniformly convergent in the closed interval $[0, 1]$ of $\mathbf{R}$, and deduce that

$$\lim_{\substack{x \to 1 \\ 0 < x < 1}} \sum_{n \geqslant 0} a_n x^n = \sum_{n \geqslant 0} a_n.$$

(iii) Let $S(X) = \sum_{n \geqslant 1} X^n/n^2$ now and let D be the intersection of the open disc $|z| < 1$ and of the open disc $|z - 1| < 1$. Show that there exists a constant a such that

$$S(z) + S(1 - z) = a - \log z \log (1 - z) \quad \text{for} \quad z \in D,$$

where log denotes the principal branch of the complex logarithm in the half-plane $\mathrm{Re}(z) > 0$ (which contains D).

(Note that, if $z \in D$, then $\log (1 - z) = - T(z)$ with

$$T(X) = X.S'(X),$$

because of proposition 6. 1 of § 3, and that proposition 6. 2 of § 3 gives

$$\frac{d}{dz}(\log z \log (1 - z)) = \frac{\log (1 - z)}{z} - \frac{\log z}{1 - z} \quad \text{for} \quad z \in D.)$$

Finally, use (ii) to show that

$$a = \sum_{n \geqslant 1} 1/n^2,$$

$$a - (\log 2)^2 = \sum_{n \geqslant 1} 1/n^2 2^{n-1}.$$

(Cf. chapter v, § 2, no. 2, the application of proposition 2. 1.)