Soluzioni 8.5.2011

1. a) $\int \arccos x \, dx = \int 1 \cdot \arccos x \, dx =$ (per parti)

$= x \arccos x - \int x \left(-\frac{1}{\sqrt{1-x^2}} \right) dx =$

$= x \arccos x - \sqrt{1-x^2} + c$

b) $\int \operatorname{arctg} x \, dx = \int 1 \cdot \operatorname{arctg} x \, dx =$ (per parti)

$= x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx = x \operatorname{arctg} x - \frac{1}{2} \log(1+x^2) + c$

c) $\int \operatorname{tg} x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\log|\cos x| + c$

d) $\int \frac{1}{\operatorname{tg} x} \, dx = \int \frac{\cos x}{\sin x} \, dx = \log|\sin x| + c$

2. a) $\int x(x+2-e)e^x \, dx = (x^2+2x-ex)e^x -$

$- \int (2x+2-e)e^x \, dx = (x^2+2x-ex)e^x - (2x+2-e)e^x$

$+ \int 2e^x \, dx = e^x(x^2+2x-ex-2x-2+e+2)$

$= e^x(x^2-ex+e)$

$\Rightarrow \int_0^1 x(x+2-e)e^x \, dx = \left[e^x(x^2-ex+e) \right]_0^1 =$

$= e(1-e+e) - e = 0$

b) $\int_1^e \frac{\log x}{\sqrt{x}} \, dx = \left[2\sqrt{x} \log x \right]_1^e - \int_1^e \frac{2\sqrt{x}}{x} \, dx =$

$= 2\sqrt{e} - \int_1^e \frac{2}{\sqrt{x}} \, dx = 2\sqrt{e} - \left[4\sqrt{x} \right]_1^e = 4 - 2\sqrt{e}$

c) $\int_1^e \log^3 x\,dx = \int_1^e 1\cdot\log^3 x\,dx = \left[x\log^3 x\right]_1^e -$

$- \int_1^e x\cdot 3\frac{\log^2 x}{x}\,dx = e - 3\int_1^e \log^2 x\,dx =$

$= e - \left[3x\log^2 x\right]_1^e + 3\int_1^e x\,\frac{2\log x}{x}\,dx =$

$= e - 3e + 6\int_1^e \log x\,dx = -2e + \left[6x\log x\right]_1^e -$

$- 6\int_1^e x\,\frac{1}{x}\,dx = -2e + 6e - 6(e-1) = 6 - 2e$

d) $\int_0^\pi x(\cos x + e^{x/\pi})\,dx = \left[x(\sin x + \pi e^{x/\pi})\right]_0^\pi -$

$- \int_0^\pi (\sin x + \pi e^{x/\pi})\,dx = \pi^2 e - \left[-\cos x + \pi^2 e^{x/\pi}\right]_0^\pi$

$= \pi^2 e - (1 + \pi^2 e + 1 - \pi^2) = \pi^2 - 2$

e) $\int_0^1 x^2 \arcsin x\,dx = \left[x^2(x\arcsin x + \sqrt{1-x^2})\right]_0^1 -$

$- \int_0^1 2x(x\arcsin x + \sqrt{1-x^2})\,dx = \arcsin 1 -$

$- 2\int_0^1 x^2\arcsin x\,dx - \int_0^1 2x\sqrt{1-x^2}\,dx =$

$= \arcsin 1 - 2\int_0^1 x^2\arcsin x\,dx - \left[-\frac{2}{3}(1-x^2)^{3/2}\right]_0^1$

$= \arcsin 1 - 2\int_0^1 x^2\arcsin x\,dx - \frac{2}{3}$

$\Longrightarrow\ 3\int_0^1 x^2\arcsin x\,dx = \arcsin 1 - \frac{2}{3}$

$\int_0^1 x^2\arcsin x\,dx = \frac{1}{3}\arcsin 1 - \frac{2}{9}$

f) $\int_0^{\pi/2} x\cos^2 x\,dx = \left[x\left(\frac{1}{2}\sin x\cos x + \frac{x}{2}\right)\right]_0^{\pi/2} -$

$-\int_0^{\pi/2}\left(\frac{1}{2}\sin x\cos x + \frac{x}{2}\right)dx = \frac{\pi^2}{8} - \frac{1}{2}\int_0^{\pi/2}\left(\frac{1}{2}\sin 2x + x\right)dx$

$= \frac{\pi^2}{8} - \frac{1}{2}\left[-\frac{1}{4}\cos 2x + \frac{1}{2}x^2\right]_0^{\pi/2} = \frac{\pi^2}{8} - \frac{1}{8} - \frac{1}{4}\frac{\pi^2}{4} - \frac{1}{8}$

$= \frac{\pi^2}{16} - \frac{1}{4}$

3. a) $\int_0^{1/4}(4x+2)^3\,dx$; Sost. $4x+2=y$; $4\,dx = dy$

$= \int_2^3 y^3\cdot\frac{1}{4}\,dy = \left[\frac{y^4}{16}\right]_2^3 = 4 - 1 = 3$

b) $\int_0^1 \frac{x^3}{x^8+1}\,dx$; Sost. $x^4 = y$ $\quad 4x^3\,dx = dy$

$= \int_0^1 \frac{1}{y^2+1}\,\frac{1}{4}\,dy = \left[\frac{1}{4}\operatorname{arctg} y\right]_0^1 = \frac{\pi}{16}$

c) $\int_0^{\log 2}\sqrt{e^x-1}\,dx$ Sost. $t = \sqrt{e^x-1}$

$dt = \frac{e^x}{2\sqrt{e^x-1}}\,dx = \frac{e^x-1+1}{2t}\,dx = \frac{t^2+1}{2t}\,dx$

$\int_0^{\log 2}\sqrt{e^x-1}\,dx = \int_0^1 \frac{2t}{t^2+1}\,dt = \left[\log(t^2+1)\right]_0^1 = \log 2$

d) $\int_0^1 x^3 e^{-x^2}\,dx$; Sost. $x^2 = \log t$ ossia $t = e^{x^2}$

$2x\,dx = \frac{dt}{t} \qquad dx = \frac{1}{2x}\,\frac{1}{t}\,dt = \frac{1}{2\sqrt{\log t}\;t}\,dt$

$\int_0^1 x^3 e^{-x^2}\,dx = \int_1^e (\log t)^{3/2}\,\frac{1}{t}\,\frac{1}{2\sqrt{\log t}\;t}\,dt$

$= \frac{1}{2}\int_1^e \frac{\log t}{t^2}\,dt = \frac{1}{2}\left[-\frac{1}{t}\log t\right]_1^e - \frac{1}{2}\int_1^e\left(-\frac{1}{t}\right)\frac{1}{t}\,dt$

$$= -\frac{1}{2e} + \frac{1}{2}\int_1^e \frac{1}{t^2}\, dt = -\frac{1}{2e} + \frac{1}{2}\left[-\frac{1}{t}\right]_1^e =$$

$$= -\frac{1}{2e} - \frac{1}{2e} + \frac{1}{2} = \frac{1}{2} - \frac{1}{e}$$

e) $$\int_0^{\sqrt{\pi}} x \sin x^2\, dx = \int_0^{\sqrt{\pi}} \frac{d}{dx}\left(-\frac{1}{2}\cos x^2\right) dx =$$

$$= -\frac{1}{2}\cos\pi + \frac{1}{2}\cos 0 = 1$$

f) $$\int_2^3 \frac{1}{\sqrt{x(x-1)}}\, dx$$ Sostituzione $x - \frac{1}{2} = t$

$$= \int_{3/2}^{5/2} \frac{1}{\sqrt{(t+\frac{1}{2})(t-\frac{1}{2})}}\, dt = \int_{3/2}^{5/2} \frac{1}{\sqrt{t^2-\frac{1}{4}}}\, dt$$

$$= 2\int_{3/2}^{5/2} \frac{1}{\sqrt{4t^2-1}}\, dt$$ $2t = \cosh y$

$dt = \frac{1}{2}\sinh y\, dy$

$$= 2\int_{\operatorname{arccosh} 3}^{\operatorname{arccosh} 5} \frac{1}{\sqrt{\cosh^2 y - 1}}\,\frac{1}{2}\sinh y\, dy =$$

$$= 2\int_{\operatorname{arccosh} 3}^{\operatorname{arccosh} 5} \frac{1}{2}\, dy = \operatorname{arccosh} 5 - \operatorname{arccosh} 3$$

4 a) $$\int \frac{1}{a + b\cos x}\, dx =$$

$$= \frac{2}{\sqrt{a^2-b^2}}\, \operatorname{arctg}\left(\frac{a-b}{\sqrt{a^2-b^2}}\, \operatorname{tg}\left(\frac{1}{2}x\right)\right)$$

b) $\int e^{e^x + x}\, dx = \int e^x e^{e^x}\, dx = e^{e^x} + c$

c) $\int \frac{1}{e^{2x} - 2e^x}\, dx$ sostituzione $e^x = t$

$e^x\, dx = dt \qquad dx = \frac{dt}{e^x} = \frac{dt}{t}$

$$\int \frac{1}{e^{2x} - 2e^x}\, dx = \int \frac{1}{t^2 - 2t} \frac{1}{t}\, dt = \int \frac{1}{t^2(t-2)}\, dt$$

$\frac{1}{t^2(t-2)} = \frac{a}{t^2} + \frac{b}{t} + \frac{c}{t-2}$ Calcoliamo a, b, c:

$\frac{1}{t-2} = a + bt + c\frac{t^2}{t-2} \overset{t=0}{\Longrightarrow} a = -\frac{1}{2}$

$\frac{1}{t^2} = \frac{t-2}{t^2} a + \frac{t-2}{t} b + c \overset{t=2}{\Longrightarrow} c = \frac{1}{4}$

Per $t = 1$ $\quad \frac{1}{1 \cdot (-1)} = -\frac{1}{2} + b + \frac{1/4}{1-2}$

$-1 = -\frac{1}{2} + b - \frac{1}{4} \Longrightarrow b = -\frac{1}{4}$

$$\int \frac{1}{t^2(t-2)}\, dt = \int \left(-\frac{1}{2t^2} - \frac{1}{4t} + \frac{1}{4(t-2)} \right) dt$$

$$= \frac{1}{2t} - \frac{1}{4} \log|t| + \frac{1}{4} \log|t-2|$$

$$= \frac{1}{2e^x} - \frac{1}{4} x + \frac{1}{4} \log|e^x - 2|$$

d) $\int \sin\sqrt{x}\, dx \overset{t=\sqrt{x}}{=} \int \sin t \cdot 2t\, dt =$

$= 2t(-\cos t) - \int 2(-\cos t)\, dt = -2t\cos t + 2\int \cos t\, dt$

$= -2t\cos t + 2\sin t = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x}$

e) $\int x^3\sqrt{2-x^2}\,dx = \sqrt{2}\int x^3\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}\,dx =$

$\left(\frac{x}{\sqrt{2}} = \sin t\right) \qquad = \sqrt{2}\int (\sqrt{2})^3 \sin^3 t\,(\cos t)(-\sqrt{2}\cos t)\,dt$

$= +4\sqrt{2}\int \sin^3 t\cos^2 t\,dt = 4\sqrt{2}\int(\sin^3 t - \sin^5 t)\,dt$

$\int \sin^3 t\,dt = -\cos t\sin^2 t - \int(-\cos t)\,2\sin t\cos t\,dt$

$= -\cos t\sin^2 t + 2\int(\sin t - \sin^3 t)\,dt =$

$= -\cos t\sin^2 t - 2\cos t - 2\int \sin^3 t\,dt$

$\Rightarrow \int \sin^3 t\,dt = -\frac{2}{3}\cos t - \frac{1}{3}\cos t\sin^2 t$

$\int \sin^5 t\,dt = -\cos t\sin^4 t - \int(-\cos t)\,4\sin^3 t\cos t\,dt$

$= -\cos t\sin^4 t + 4\int \sin^3 t\,(1-\sin^2 t)\,dt =$

$= -\cos t\sin^4 t - \frac{8}{3}\cos t - \frac{4}{3}\cos t\sin^2 t - 4\int \sin^5 t\,dt$

$\Rightarrow \int \sin^5 t\,dt = -\frac{1}{5}\cos t\sin^4 t - \frac{8}{15}\cos t - \frac{4}{15}\cos t\sin^2 t$

(Sostituendo si trova)

$\int x^3\sqrt{2-x^2}\,dx = -\frac{1}{5}x^2(2-x^2)^{3/2} - \frac{4}{15}(2-x^2)^{3/2}$

f) $\int \frac{\sin 2x + \cos x}{1+\sin x}\,dx = 2\sin x - \log\cos x -$

$- \log\left(\frac{1}{\cos x} + \operatorname{tg} x\right)$

g) $\int \frac{1}{x(1-x)}\,dx = \int \left(\frac{1}{x} - \frac{1}{x-1}\right) dx = \log|x| - \log|x-1|$

h) $\int \frac{1}{x^2(1+x)}\,dx = \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{1+x}\right) dx =$

$= -\log|x| - \frac{1}{x} + \log|1+x|$

i) $\int \frac{x+3}{x^3+3x^2+2x}\,dx = \frac{3}{2}\log|x| + \frac{1}{2}\log|x+2|$

$- 2\log|1+x|$

j) $\int \frac{x^2-3}{(x+1)(x-1)^2}\,dx = \frac{1}{x-1} + \frac{3}{2}\log|x-1| - \frac{1}{2}\log|1+x|$

k) $\int \frac{1}{x^3(1+x)^2}\,dx = -\frac{1}{2x^2} + 3\log|x| + \frac{1}{1+x} +$

$+ \frac{2}{x} - 3\log|1+x|$

i) $\int \frac{1}{x^2-2x+2}\,dx = \operatorname{arctg}(x-1)$

5 - Integrando per parti ed usando il teo fondam

$$\int_a^x \left(\int_a^t f(s)\,ds\right) dt = \left[t \int_a^t f(s)\,ds\right]_a^x - \int_a^x t\,f(t)\,dt$$

$$= x\int_a^x f(s)\,ds - \int_a^x t\,f(t)\,dt = \int_a^x (x-t)\,f(t)\,dt$$

6 - Posto $F(y) = \int_a^y f(t)\,dt$ si ha

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}F(g(x)) = F'(g(x))\,g'(x)$$

$$= f(g(x))\,g'(x)$$

Da quanto appena dimostrato segue che

$$\frac{d}{dx}\int_{g(x)}^{b} f(t)\,dt = \frac{d}{dx}\left(-\int_{b}^{g(x)} f(t)\,dt\right) =$$

$$= -f(g(x))\,g'(x)$$

7 - Sia $x(t)$ la distanza in metri percorsa t secondi dopo l'inizio della frenata.

Quindi

$$x(0) = 0, \quad x'(0) = \frac{90\,000}{3600} = 25 \text{ m/s}$$

$$x''(t) = -K \quad \forall t \geq 0$$

Dunque

$$x'(t) = x'(0) + \int_0^t x''(s)\,ds = 25 + \int_0^t (-K)\,ds$$

$$= 25 - Kt$$

$$x(t) = x(0) + \int_0^t x'(s)\,ds = 0 + \int_0^t (25 - Ks)\,ds$$

$$= 25t - \frac{1}{2}Kt^2$$

All'istante finale t_0 risulta

$$x(t_0) = 25t_0 - \frac{1}{2}Kt_0^2 = 30$$

$$x'(t_0) = 25 - Kt_0 = 0$$

da cui $t_0 = \frac{25}{K}$ e $\frac{625}{K} - \frac{1}{2}K\frac{625}{K^2} = 30$

$$\Rightarrow \frac{625}{K} = 60 \Rightarrow K = \frac{625}{60} = \frac{125}{12} \simeq 10{,}42$$

Se la velocità iniziale è v, detto t_v il tempo di arresto, si ha

$$x'(t_v) = v - Kt_v = 0 \Rightarrow t_v = \frac{v}{K}$$

$$\Rightarrow x(t_v) = vt_v - \frac{1}{2}Kt_v^2 = \frac{v^2}{K} - \frac{1}{2}K\frac{v^2}{K^2} = \frac{1}{2}\frac{v^2}{K}$$

Per $v = 45 \text{ Km/h} = \frac{45\,000}{3600} = \frac{25}{2}$ m/s, l'auto si arresta dopo $x(t_v) = \frac{1}{2}\frac{625}{4}\frac{12}{125} = \frac{5 \cdot 3}{2} = \frac{15}{2} = 7,5$ metri

Per $v = 135\ Km/h = \frac{135000}{3600} = \frac{75}{2}\ m/s$ l'auto si

arresta dopo $x(t_v) = \frac{1}{2}\frac{5625}{4}\frac{12}{125} = \frac{135}{2} = 67,5$ metri