Soluzioni 1.12.2011

1) • $a'(x) = 100x^{99} - 50x^{49}$

• $b'(x) = \dfrac{(10x^9 - 25x^4)(3x^9 - 9x^3) - (x^{10} - 5x^5 + 1)(27x^8 - 27x^2)}{(3x^9 - 9x^3)^2}$

$= \dfrac{15x^7(2x^5 - 5)(x^6 - 3) - 27x^2(x^{10} - 5x^5 + 1)(x^6 - 1)}{9x^6(x^6 - 3)^2}$

$= \dfrac{5x^5(2x^{11} - 6x^5 - 5x^6 + 15) - 9(x^{16} - x^{10} - 5x^{11} + 5x^5 + x^6 - 1)}{3x^4(x^6 - 3)^2}$

$= \dfrac{x^{16} - 70x^{11} - 21x^{10} - 9x^6 + 30x^5 + 9}{3x^4(x^6 - 3)^2}$

• $c'(x) = \sin(x + \log x) + x(\cos(x + \log x))\left(1 + \frac{1}{x}\right)$

• $d'(x) = 7\,tg^6 x\,(1 + tg^2 x)$

• $e'(x) = -2\cos x \sin x\, e^{\cos^2 x}$

• $f'(x) = D\left(e^{x \log x}\right) = (\log x + 1)\, e^{x \log x} = (\log x + 1)\, x^x$

• $g'(x) = D\, e^{(\operatorname{arctg} e^x) \log \sin x} = \left(\frac{1}{1 + e^{2x}} e^x \log \sin x + \right.$

$\left. + \operatorname{arctg} e^x \frac{1}{\sin x} \cos x\right) e^{(\operatorname{arctg} e^x) \log \sin x} =$

$= \left(\frac{e^x}{1 + e^{2x}} \log \sin x + \frac{\operatorname{arctg} e^x}{tg\, x}\right) (\sin x)^{\operatorname{arctg} e^x}$

• $h'(x) = D\, e^{(\sin \sqrt{x}) \log(x \log x)} = \left(\frac{1}{2\sqrt{x}} \cos \sqrt{x} \log(x \log x) + \right.$

$\left. + \sin \sqrt{x} \left(\frac{1}{x \log x}(\log x + 1)\right)\right) e^{(\sin \sqrt{x}) \log(x \log x)}$

$= \left(\frac{\cos \sqrt{x}}{2\sqrt{x}} (\log x + \log \log x) + \sin \sqrt{x} \left(\frac{1}{x} + \frac{1}{x \log x}\right)\right) (x \log x)^{\sin \sqrt{x}}$

- $i'(x) = 100 (\log x)^{99} \frac{1}{x}$

- $j'(x) = \frac{1}{\sin \sqrt{x}} (\cos \sqrt{x}) \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x} \operatorname{tg} \sqrt{x}}$

- $k'(x) = \frac{1}{\sqrt{1 - \sin x}} \frac{1}{2} \frac{1}{\sqrt{\sin x}} \cos x = -\frac{1}{2} \frac{\sin x}{|\sin x|}$

- $l'(x) = \frac{1}{1 + \frac{1-\cos x}{1+\cos x}} \cdot \frac{1}{2} \sqrt{\frac{1+\cos x}{1-\cos x}} \frac{\sin x(1+\cos x) + \sin x(1-\cos x)}{(1+\cos x)^2}$

$= \frac{1+\cos x}{2} \cdot \frac{1}{2} \sqrt{\frac{1+\cos x}{1-\cos x}} \frac{2 \sin x}{(1+\cos x)^2}$

$= \frac{1}{2} (1 - \cos^2 x)^{-\frac{1}{2}} \sin x = \frac{1}{2} \sqrt{\sin x}$

- $m'(x) = \frac{-\sin^3 x - 2 \sin x \cos^2 x}{\sin^4 x} \frac{1}{\operatorname{tg} \frac{x}{2}} \left(1 + \operatorname{tg}^2 \frac{x}{2}\right) \cdot \frac{1}{2}$

$= \ldots = -\frac{2}{\sin^3 x}$

- $n'(x) = \frac{1}{\sqrt{1 - \frac{1-x^2}{x^2}}} \cdot \frac{1}{2} \sqrt{\frac{x^2}{1-x^2}} \left(-\frac{2}{x^3}\right)$

$= \ldots = \frac{1}{x \sqrt{(1-x^2)(2x^2-1)}}$

- $p'(x) = \frac{(\cos x - \cos x + x \sin x)(\cos x + x \sin x) - (\sin x - x \cos x)(-\sin x + \sin x + x \cos x)}{(\cos x + x \sin x)^2} = \ldots = \frac{x^2}{(\cos x + x \sin x)^2}$

- $q'(x) = \frac{1}{4} (\sin x^4)^{-\frac{3}{4}} \cos x^4 \cdot 4x^3 = \frac{x^3 \cos x^4}{\sqrt[4]{\sin^3 x^4}}$

- $r(x) = \sqrt[7]{8 \log x}$, $r'(x) = \frac{1}{7} (8 \log x)^{-6/7} \frac{8}{x} = \frac{8}{7x} (8 \log x)^{-6/7}$

2) a) Se $f > 0$ risulta

$$D \log |f(x)| = D \log f(x) = \frac{f'(x)}{f(x)} ,$$

Se $f < 0$ risulta

$$D \log |f(x)| = D \log (-f(x)) = \frac{-f'(x)}{-f(x)} = \frac{f'(x)}{f(x)}$$

b) $$f'(x) \overset{(a)}{=} \frac{D \log |x|}{\log |x|} \overset{(a)}{=} \frac{1}{x \log |x|} .$$

$$g'(x) \overset{(a)}{=} 1 + \frac{\sin\left(\frac{\pi}{4} - x\right)}{\cos\left(\frac{\pi}{4} - x\right)} = 1 + \operatorname{tg}\left(\frac{\pi}{4} - x\right)$$

3) • La funzione x è derivabile su tutto $\mathbb{R}$, mentre la funzione $|x-1|$ è derivabile su $\mathbb{R} \setminus \{1\}$. Quindi f è derivabile su $\mathbb{R} \setminus \{1\}$. Per $x = 1$ si ha

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0+} \frac{(1+h)|h|}{h} = \lim_{h \to 0+} (1+h) = 1$$

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0-} \frac{(1+h)|h|}{h} = \lim_{h \to 0-} (-1-h) = -1$$

dunque f non è derivabile in 1.

• g è senz'altro derivabile su $\mathbb{R} \setminus \{0\}$. Inoltre

$$g(x) = x^2 + 4|x|^2 - 4x|x| = 5x^2 - 4x|x|$$

La funzione $x|x|$ è derivabile in 0, con derivata nulla, dato che $x|x| = 0 + 0 \cdot x + o(x)$ per $x \to 0$. Quindi g è derivabile su tutto $\mathbb{R}$.

• Risulta $h(x) = e^{\alpha \log x}$, che è senz'altro derivabile in $]0, +\infty[$. Per $x = 0$:

$$\lim_{x \to 0+} \frac{h(x) - h(0)}{x} = \lim_{x \to 0+} x^{\alpha - 1} = \begin{cases} 0 & \text{se } \alpha > 1 \\ 1 & \text{se } \alpha = 1 \\ +\infty & \text{se } \alpha < 1 \end{cases}$$

Quindi h è derivabile in $0 \iff \alpha \geq 1$.

4) Dato che $f(1) = 2$ e $f'(x) = 2x$, si ha

$$Df^{-1}(2) = \frac{1}{f'(1)} = \frac{1}{2}$$

5) Dato che $f(0) = 0$ e $f'(x) = 1 + 3x^2$, si ha

$$Df^{-1}(0) = \frac{1}{f'(0)} = 1$$

6) a) Risulta $f'(x) = \sum_{h=0}^{n} h x^{h-1} = \sum_{k=0}^{n-1} (k+1) x^k$.

Dato che $f(x) = \frac{x^{n+1} - 1}{x - 1}$ per $x \neq 1$ e

$f(1) = n+1$, si ha anche

$$f'(x) = \frac{(n+1)x^n (x-1) - (x^{n+1} - 1)}{(x-1)^2} = \frac{n x^{n+1} - (n+1) x^n + 1}{(x-1)^2}$$

per $x \neq 1$.

b) $\sum_{h=1}^{n} h\,2^h = 2\sum_{h=1}^{n} h\,2^{h-1} = 2\sum_{k=0}^{n-1} (k+1)2^k \overset{(a)}{=} 2f'(2)$

$= 2\,\dfrac{n\,2^{n+1} - (n+1)2^n + 1}{(2-1)^2} = 2(n-1)2^n + 2$

c) Derivando l'identità $f'(x) = \sum_{k=0}^{n-1} (k+1)x^k$ si trova $f''(x) = \sum_{k=0}^{n-1} k(k+1)x^{k-1} = \sum_{k=1}^{n-1} k(k+1)x^{k-1}$

e per la seconda espressione per $f'(x)$,

$$f''(x) = \frac{(n(n+1)x^n - n(n+1)x^{n-1})(x-1)^2 - 2(x-1)(nx^{n+1} - (n+1)x^n + 1)}{(x-1)^4}$$

$$= \frac{(x-1)^2 n(n+1)x^{n-1} - 2(nx^{n+1} - (n+1)x^n + 1)}{(x-1)^3}$$

$$= \frac{n(n+1)x^{n+1} - 2n(n+1)x^n + n(n+1)x^{n-1} - 2nx^{n+1} + 2(n+1)x^n - 2}{(x-1)^3}$$

$$= \frac{n(n-1)x^{n+1} - 2(n^2-1)x^n + n(n+1)x^{n-1} - 2}{(x-1)^3}$$

Allora

$$\sum_{h=1}^{n} h^2 3^h = \sum_{h=1}^{n} h(h-1)3^h + \sum_{h=1}^{n} h\,3^h$$

La seconda somma è analoga a quella in (b). Calcoliamo la prima:

$$\sum_{h=1}^{n} h(h-1)3^h = \sum_{k=0}^{n-1} (k+1)k\,3^{k+1} = 9\sum_{k=1}^{n-1} k(k+1)3^{k-1} = 9f''(3)$$

$$= 9\,\frac{n(n-1)3^{n+1} - 2(n^2-1)3^n + n(n+1)3^{n-1} - 2}{2^3} =$$

$$= \frac{9}{8}\left(3n^2 - 3n - 2n^2 + 2 + \frac{n^2}{3} + \frac{n}{3}\right)3^n - \frac{9}{4} =$$

$$= \frac{9}{8}\left(\frac{4}{3}n^2 - \frac{8}{3}n + 2\right)3^n - \frac{9}{4} = \left(\frac{3}{2}n^2 - 3n + \frac{9}{4}\right)3^n - \frac{9}{4}$$

Calcoliamo adesso

$$\sum_{h=1}^{n} h\,3^h = 3\sum_{h=1}^{n} h\,3^{h-1} = 3\sum_{k=0}^{n-1}(k+1)3^k \overset{(a)}{=} 3\,f'(3)$$

$$= 3\,\frac{n3^{n+1} - (n+1)3^n + 1}{4} = \left(\frac{3}{2}n - \frac{3}{4}\right)3^n + \frac{3}{4}$$

Infine:

$$\sum_{h=1}^{n} h^2 3^h = \sum_{h=1}^{n} h(h-1)3^h + \sum_{h=1}^{n} h\,3^h =$$

$$= \left(\frac{3}{2}n^2 - 3n + \frac{9}{4}\right)3^n - \frac{9}{4} + \left(\frac{3}{2}n - \frac{3}{4}\right)3^n + \frac{3}{4}$$

$$= \frac{3}{2}\left(n^2 - n + 1\right)3^n - \frac{3}{2}$$

Quanche

$$\sum_{h=1}^{n} h^2 3^h = \frac{3}{2}\left(n^2 - n + 1\right)3^n - \frac{3}{2}$$

7) $D f^\alpha = D\,e^{\alpha \log f} = \frac{\alpha}{f} f' e^{\alpha \log f} = \frac{\alpha f'}{f} f^\alpha = \alpha f^{\alpha-1} f'$