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Ordinary differential operators in Hilbert spaces and Fredholm pairs

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Abstract. Let $A(t)$ be a path of bounded operators on a real Hilbert space, hyperbolic at $\pm\infty$. We study the Fredholm theory of the operator $F_A = d/dt - A(t)$. We relate the Fredholm property of F_A to the stable and unstable linear spaces of the associated system $X' = A(t)X$. Several examples are included to point out the differences with respect to the finite dimensional case, in particular concerning the role of the spectral flow. We define a general class of paths A for which many properties typical of the finite dimensional framework still hold. Our motivation is to develop the linear theory which is necessary for the set-up of Morse homology on Hilbert manifolds.

Introduction

Consider a smooth vector field ξ on the Euclidean space $\mathbb{R}^n$ and the corresponding system of differential equations

$$u'(t) = \xi(u(t)). \tag{1}$$

Let $x, y \in \mathbb{R}^n$ be equilibrium points for the above system, $\xi(x) = \xi(y) = 0$, which we assume to be hyperbolic, meaning that the Jacobian matrices $\nabla\xi(x)$ and $\nabla\xi(y)$ do not have purely imaginary eigenvalues. Assume that (1) has a solution u which connects x to y:

$$\lim_{t\to-\infty} u(t) = x, \quad \lim_{t\to+\infty} u(t) = y.$$

If one wants to examine the structure of the solutions of (1) connecting x to y and close to u, the object to be studied is the operator obtained by linearizing (1) along u:

$$v \mapsto v' - \nabla\xi(u)v,$$

defined on some space of curves $v : \mathbb{R} \to \mathbb{R}^n$ vanishing at $-\infty$ and $+\infty$. A natural domain for such an operator is $C_0^1(\mathbb{R};\mathbb{R}^n)$, the space of continuously differentiable curves vanishing at infinity together with their first derivatives. Another useful domain is $H^1(\mathbb{R};\mathbb{R}^n)$, the Hilbert space of square integrable curves whose weak derivatives are also square integrable[1]. Clearly, the domain can be chosen in a large class of function spaces, but this choice turns out to be not very relevant, see Remark 5.1. So one is lead to study a bounded operator of the form

$$F_A v(t) = v'(t) - A(t)v(t),$$

from C_0^1 to C_0^0 (or from H^1 to L^2, etc.) where A is a path of matrices admitting limits at $-\infty$ and $+\infty$ and such that $A(-\infty)$ and $A(+\infty)$ have no purely imaginary eigenvalues. Matrices without purely imaginary eigenvalues are said hyperbolic, so the paths with the above property will be called *asymptotically hyperbolic*. The following result is well known (see [Sch93], Propositions 2.12 and 2.16, or [RS95], Theorem 2.1).

Theorem A *Let A be an asymptotically hyperbolic path of n by n matrices. Then F_A is a Fredholm operator of index*

$$\operatorname{ind} F_A = \dim V^-(A(+\infty)) - \dim V^-(A(-\infty)).$$

Here $V^-(T)$ denotes the T-invariant subspace of $\mathbb{R}^n$ corresponding to the eigenvalues with negative real part in the spectral decomposition of T. When $F_{\nabla\xi(u)}$ is onto, the above theorem implies that its kernel has dimension $\dim V^-(\nabla\xi(y)) - \dim V^-(\nabla\xi(x))$: so, by the implicit function theorem, the set of solutions of (1) connecting x to y and close to u is a manifold of dimension $\dim V^-(\nabla\xi(y)) - \dim V^-(\nabla\xi(x))$.[2] If vector field ξ is the negative gradient of a Morse function f, the above result can be used as the starting point to develop a Morse homology for f, an alternative approach to Morse theory, based on the study of the gradient flow lines connecting critical points (notice that in this case $\nabla\xi(x) = \nabla^2 f(x)$, the Hessian of f in x, so the dimension of $V^-(\nabla\xi(x))$ is the Morse index of x). See [Sal90] or [Sch93].

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[1] This is a suitable space of perturbations of the curve u. Indeed, since the equilibrium points x and y are non-degenerate, all the connecting solutions converge to them exponentially fast, and in particular, they are square integrable together with their first derivatives.

[2] Such a result could be obtained also by looking at the intersection between the unstable manifold of x and the stable manifold of y. In fact, it can be proved that these manifolds have transversal intersection at $u(t)$ if and only if $F_{\nabla\xi(u)}$ is onto, see Sect. 8, or [Sal90] Theorem 3.3.

In this paper we present a detailed study of the properties of the operator F_A when A is an asymptotically hyperbolic path of bounded operators on a possibly infinite dimensional Hilbert space E. The aim is to provide a useful machinery which could be employed to develop Morse homology theories for functionals defined on infinite dimensional Hilbert manifolds. In [AM01] we constructed a Morse homology for functionals on a Hilbert space, consisting of the sum of a non-degenerate quadratic part and of a term with compact gradient. The generalization of Theorem A which was proved there is the following (see also [AvdV99])[3].

Theorem B *Assume that the asymptotically hyperbolic path A has the form $A(t) = A_0 + K(t)$, where A_0 is a hyperbolic operator and $K(t)$ is compact. Then F_A is Fredholm and*

$$\operatorname{ind} F_A = \dim(V^-(A(+\infty)), V^-(A(-\infty))).$$

Here $\dim(V, W)$ denotes the relative dimension of the (possibly infinite dimensional) subspace V with respect to W:

$$\dim(V, W) = \dim V \cap W^\perp - \dim V^\perp \cap W.$$

See Sect. 3 for more details on the relative dimension. Therefore, in the class of compact perturbations of some fixed hyperbolic operator, things go essentially as in the finite dimensional case. We will see that outside this class new phenomena occur.

Let X_A be the path of operators solving the Cauchy problem

$$\begin{cases} X_A'(t) = A(t) X_A(t), \\ X_A(0) = I. \end{cases}$$

Two important objects related to such a system are the stable and the unstable spaces:

$$W_A^s := \left\{ x \in E \mid \lim_{t \to +\infty} X_A(t) x = 0 \right\},$$
$$W_A^u := \left\{ x \in E \mid \lim_{t \to -\infty} X_A(t) x = 0 \right\}.$$

The fact that these are linear subspaces of E follows directly from the definition. Proving that they are closed and establishing further properties requires a fixed point argument: the following theorem is proved in Sect. 1 and 2.

[3] Actually, in [AvdV99] and [AM01] only paths of self-adjoint operators are considered, but the generalization stated here presents no difficulties. See also Sect. 6, where Theorem B is deduced as a consequence of more general facts.

Theorem C *Let A be an asymptotically hyperbolic path. Then W_A^s and W_A^u are closed subspaces of E. Moreover, $X_A(t)W_A^s$ converges to $V^-(A(+\infty))$ for $t \to +\infty$, while $X_A(t)W_A^u$ converges to $V^+(A(-\infty))$ for $t \to -\infty$.*

See Sect. 2 and Theorem 2.1 for the definition of convergence of subspaces and for a richer statement. We point out that this is also an existence result for the stable and the unstable space. More properties, such as the behavior of the stable and the stable spaces when the path A is perturbed by a path which either is small or consists of compact operators, will be established in Sect. 3. The importance of the stable and the unstable spaces can be seen from the following characterization, proved in Sect. 5, with a more detailed statement.

Theorem D *Let A be an asymptotically hyperbolic path. Then F_A has closed image if and only if $W_A^s + W_A^u$ is closed, F_A is Fredholm if and only if (W_A^s, W_A^u) is a Fredholm pair, in which case*

$$\operatorname{ind} F_A = \operatorname{ind}(W_A^s, W_A^u).$$

We recall that a Fredholm pair is a pair of closed subspaces (V, W) such that $V + W$ is closed and finite codimensional, $V \cap W$ is finite dimensional. The index of a Fredholm pair (V, W) is the integer

$$\operatorname{ind}(V, W) = \dim V \cap W - \operatorname{codim}(V + W).$$

The characterization given by Theorem D has many interesting consequences, on which more will be said in Sect. 5 and 7.

(i) The operator F_A is Fredholm whenever $V^-(A(+\infty))$ and $V^-(A(-\infty))$ are finite dimensional, regardless the behavior of $A(t)$ in between. In this case the formula of Theorem A still holds.
(ii) Since any two closed subspaces can be the stable and the unstable space of an asymptotically hyperbolic system, in general F_A may have very bad properties: it may not have a closed image, it may not have finite dimensional kernel and/or co-kernel.
(iii) Also when F_A is a Fredholm operator, it is not true anymore that, as in Theorems A, B and in (i), its index depends only on the end-points of the path, $A(+\infty)$ and $A(-\infty)$.
(iv) In general, the spectral flow of the path A, an algebraic count of the eigenvalues of $A(t)$ which cross the imaginary axis, has no connection whatsoever with the index of F_A. This is in sharp contrast with what happens with paths of self-adjoint unbounded operators with compact resolvent (see [RS95]).

Besides these negative results, it is possible to find classes of asymptotically hyperbolic paths much more general than the one of Theorem B, for which F_A is Fredholm and its index depends only on the end-points. A result which seems interesting for its implications in the calculus of variations is the following.

Theorem E *Assume that E has a splitting $E = E^- \oplus E^+$ and that the asymptotically hyperbolic path A has the form $A(t) = A_0(t) + K(t)$, where the operators $K(t)$ are compact, E^- and E^+ are $A_0(t)$-invariant for every t, $A_0(\pm\infty)$ are hyperbolic, and*

$$V^-(A_0(\pm\infty)) = E^-, \quad V^+(A_0(\pm\infty)) = E^+.$$

Then F_A is Fredholm and its index is

$$\operatorname{ind} F_A = \dim(V^-(A(+\infty)), V^-(A(-\infty))).$$

The proof of this result is contained in Sect. 6, together with more general statements. An example will show that in some sense, this is the most general situation in which a result of this kind is to be expected. In the last section we present no new results, but we explain how to use the theory developed so far in a nonlinear setting.

We wish to remark that all the statements proved in this paper could be generalized to paths of bounded operators on a Banach space.

1 The stable and unstable spaces

Let E be a real Hilbert space, with inner product $u \cdot v$ and related norm $|u|$. Denote by $\mathcal{L}(E)$ the Banach algebra of bounded linear operators on E, by $\mathcal{L}_c(E)$ the closed ideal of compact operators, by $\|T\|$ the norm of an operator $T \in \mathcal{L}(E)$, and by $\sigma(T)$ the spectrum of T.

Definition 1.1 *A bounded operator $T \in \mathcal{L}(E)$ is said hyperbolic if its spectrum does not meet the imaginary axis.*[4]

Every hyperbolic operator is invertible. An invertible self-adjoint operator is hyperbolic. The set of hyperbolic operators is open, by the semicontinuity of the spectrum (see [Kat80] IV.3). By definition, the spectrum of a hyperbolic operator T consists of two isolated closed components (one of which may be empty)

$$\sigma(T) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\} \quad \text{and} \quad \sigma(T) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

Let

$$E = V^-(T) \oplus V^+(T) \tag{2}$$

[4] In the framework of discrete dynamical systems, a hyperbolic operator is a bounded operator whose spectrum does not meet the unit circle. In that context, an operator satisfying Definition 1.1 should be called *infinitesimally hyperbolic*.

be the corresponding T-invariant splitting of E into closed subspaces, given by the spectral decomposition, with projections $P^-(T)$ and $P^+(T)$. So

$$\sigma(T|_{V^-(T)}) = \sigma(T) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\} \quad \text{and}$$
$$\sigma(T|_{V^+(T)}) = \sigma(T) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}.$$

Furthermore

$$V^-(-T^*) = V^-(T)^\perp, \quad V^+(-T^*) = V^+(T)^\perp.$$

If, moreover, T is normal, meaning that T and T^* commute, the splitting (2) is orthogonal and

$$Tx \cdot x \leq -\alpha |x|^2 \quad \forall x \in V^-(T), \quad Tx \cdot x \geq \alpha |x|^2 \quad \forall x \in V^+(T),$$

where $\alpha := \inf |\operatorname{Re} \sigma(T)|$.

If A is a piecewise continuous $\mathcal{L}(E)$-valued path on an interval J containing 0, let X_A be the associated linear flow, i.e. the solution of the linear problem[5]

$$\begin{cases} X'(t) = A(t)X(t), \\ X(0) = I. \end{cases}$$

The path of operators A will appear as a subscript of many objects we are going to introduce. We will omit such subscript whenever no ambiguity is possible. From the uniqueness of linear Cauchy problems, it is readily seen that

$$X_{-A^*} = (X_A^{-1})^*, \tag{3}$$
$$X_{A+B} = X_A \cdot X_{X_A^{-1} B X_A},$$
$$X_{A(\cdot+s)}(t) = X_A(t+s) X_A(s)^{-1}.$$

Moreover,

$$X_B(t) = X_A(t) + \int_0^t X_A(t) X_A(\tau)^{-1} (B-A)(\tau) X_B(\tau)\, d\tau. \tag{4}$$

Notice that if A is bounded, X_A satisfies an estimate of the kind

$$\|X_A(t) X_A(s)^{-1}\| \leq c e^{\lambda(t-s)}, \quad \text{for } t \geq s, \tag{5}$$

for some $c \geq 1$, $\lambda \in \mathbb{R}$. When the path A is constant, $A(t) = A_0$, the linear flow is $X_A(t) = e^{tA_0}$; for any number $\lambda > \sup \operatorname{Re} \sigma(A_0)$, the constant

$$c := \sup_{t \geq 0} \|e^{t(A_0 - \lambda I)}\|$$

[5] Working with piecewise continuous paths instead of continuous ones gives us some freedom which is sometimes useful to build examples. We have no good reason here for considering a further level of generality, such as $A \in L^\infty(J; \mathcal{L}(E))$.

is finite and (5) holds with the pair (c, λ). When A_0 is normal, we are allowed to choose $\lambda = \sup \operatorname{Re} \sigma(A_0)$ and $c = 1$.

By the last identity in (3),

$$X_{A(\cdot+\tau)}(t) X_{A(\cdot+\tau)}(s)^{-1} = X_A(t+\tau) X_A(s+\tau)^{-1},$$

so inequality (5) holds with the same constant c, λ when we replace A by the translated path $A(\cdot + \tau)$. By the second identity in (3),

$$X_{A+\mu I}(t) X_{A+\mu I}(s)^{-1} = e^{\mu(t-s)} X_A(t) X_A(s)^{-1},$$

so inequality (5) holds with constants $c, \lambda+\mu$ when we replace A by $A+\mu I$.

The constant c in (5) plays a role when A is subject to more general perturbations. More precisely, it determines how much the constant λ is sensitive to perturbations:

Lemma 1.1 *Let A and H be piecewise continuous and bounded $\mathcal{L}(E)$-valued paths on $\mathbb{R}^+$, and let c, λ be such that X_A satisfies (5). Then*

$$\|X_{A+H}(t) X_{A+H}(s)^{-1}\| \leq c e^{\mu(t-s)}, \quad \textit{for } t \geq s \geq 0,$$

with $\mu := \lambda + c\|H\|_\infty$.

Proof. By our previous considerations, we may assume $s = 0$ and $\mu = 0$. Fix some $t > 0$. In this case, the curve $X_{A+H}|_{[0,t]}$ is a fixed point of the contraction

$$Y \mapsto X_A(\cdot) \left[I + \int_0^{\cdot} X_A(\tau)^{-1} H(\tau) Y(\tau)\, d\tau \right]$$

on $C([0,t]; \mathcal{L}(E))$ with the uniform norm. It is easy to check that the closed ball of radius c of $C([0,t]; \mathcal{L}(E))$ is invariant, so $\|X_{A+H}(t)\| \leq c$. □

Let A be a piecewise continuous $\mathcal{L}(E)$-valued path on $\mathbb{R}^+$, respectively on $\mathbb{R}^-$. The *stable space*, respectively *unstable space*, corresponding to the system $X' = A(t)X$ are the linear subspaces of E

$$W_A^s := \left\{ x \in E \mid \lim_{t \to +\infty} X_A(t) x = 0 \right\},$$

$$W_A^u := \left\{ x \in E \mid \lim_{t \to -\infty} X_A(t) x = 0 \right\}$$

The last identity of (3) implies that

$$X_A(t) W_A^s = W_{A(\cdot+t)}^s, \quad X_A(t) W_A^u = W_{A(\cdot+t)}^u. \tag{6}$$

When the path A is constant, $A(t) = A_0$ with A_0 a hyperbolic operator, the stable and unstable spaces are the invariant subspaces

$$W_A^s = V^-(A_0), \quad W_A^u = V^+(A_0).$$

If $A(t)$ is a small perturbation of a constant path, W_A^s and W_A^u are closed subspaces (in general not invariant) close to $V^-(A_0)$ and $V^+(A_0)$, as the next proposition shows.

Proposition 1.2 *Let A_0 be a hyperbolic operator, $E^- := V^-(A_0)$, $E^+ := V^+(A_0)$, and $c \geq 1$, $\lambda > 0$ be constants such that*

$$\|e^{tA_0}|_{E^-}\| \leq ce^{-\lambda t}, \quad \|e^{-tA_0}|_{E^+}\| \leq ce^{-\lambda t}, \quad \forall t \geq 0.$$

Let $A(t) := A_0 + H(t)$, where H is piecewise continuous on $\mathbb{R}^+$ and

$$\|H\|_\infty < \frac{1}{2}c^{-\frac{3}{2}}\lambda.$$

Set $\nu := \lambda - 2c^{\frac{3}{2}}\|H\|_\infty$ and $b := 2c^{\frac{3}{2}}$. Then the following facts hold ("evolution towards E^-"):

(i) for every $t \geq 0$, $X_A(t)W_A^s$ is the graph of an operator $S(t) \in \mathcal{L}(E^-, E^+)$;
(ii) $\|S(t)\| \leq c^2 \int_t^\infty e^{-\nu(\tau - t)}\|H(\tau)\|\, d\tau$;
(iii) the function S has as much differentiability as X_A;
(iv) for every $u_0 \in W_A^s$ and every $t \geq s \geq 0$ there holds

$$|X_A(t)u_0| \leq be^{-\nu(t-s)}|X_A(s)u_0|.$$

Moreover ("evolution from E^+"):

(v) for every $t \geq 0$, $X_A(t)E^+$ is the graph of an operator $T(t) \in \mathcal{L}(E^+, E^-)$;
(vi) $\|T(t)\| \leq c^2 \int_0^t e^{-\nu(t-\tau)}\|H(\tau)\|\, d\tau$;
(vii) the function T has as much differentiability as X_A;
(viii) for every $y_0 \in E^+$ and every $t \geq s \geq 0$ there holds

$$|X_A(t)y_0| \geq b^{-1}e^{\nu(t-s)}|X_A(s)y_0|.$$

Proof. Denote by P_- and P_+ the linear projections associated with the decomposition $E^- \oplus E^+$. Splitting u as $u = x + y$, $x = P_-u$, $y = P_+u$, the equation $u' = A(t)u$ becomes

$$\begin{cases} x' = A_-(t)x + A_\mp(t)y, \\ y' = A_\pm(t)x + A_+(t)y, \end{cases} \tag{7}$$

which is equivalent to the system of integral equations

$$x(t) = X_{A_-}(t)X_{A_-}(s)^{-1}x(s) + \int_s^t X_{A_-}(t)X_{A_-}(\tau)^{-1}A_\mp(\tau)y(\tau)\, d\tau, \tag{8}$$

$$y(t) = X_{A_+}(t)X_{A_+}(r)^{-1}y(r) - \int_t^r X_{A_+}(t)X_{A_+}(\tau)^{-1}A_\pm(\tau)x(\tau)\, d\tau. \tag{9}$$

Choosing $s \leq t \leq r$, Lemma 1.1 provides bounds for the above integrals, with any pair of continuous functions (x, y): setting $\mu := \lambda - c\|H\|_\infty$, since $\|H\|_\infty < c^{-\frac{3}{2}}\mu$, we obtain

$$\begin{aligned}
&\left| \int_s^t X_{A_-}(t) X_{A_-}(\tau)^{-1} A_\mp(\tau) y(\tau)\, d\tau \right| \\
&\leq c \left(\int_s^t e^{-\mu(t-\tau)} \|A_\mp(\tau)\|\, d\tau \right) \|y\|_{\infty,[s,t]} \\
&\leq c(1 - e^{-\mu(t-s)}) \frac{\|H\|_\infty}{\mu} \|y\|_{\infty,[s,t]} \\
&\leq c^{-\frac{1}{2}} (1 - e^{-\mu(t-s)}) \|y\|_{\infty,[s,t]},
\end{aligned} \tag{10}$$

and using the identity

$$\begin{aligned}
\|X_{A^+}(t) X_{A^+}(\tau)^{-1}\| &= \|(X_{A^+}(t) X_{A^+}(\tau)^{-1})^*\| \\
&= \|X_{A_+}(\tau)^{-1*} X_{A_+}(t)^*\| \\
&= \|X_{-A_+^*}(\tau) X_{-A_+^*}(t)^{-1}\|,
\end{aligned}$$

we obtain also

$$\begin{aligned}
&\left| \int_t^r X_{A_+}(t) X_{A_+}(\tau)^{-1} A_\pm(\tau) x(\tau)\, d\tau \right| \\
&\leq c \left(\int_t^r e^{-\mu(\tau-t)} \|A_\pm(\tau)\|\, d\tau \right) \|x\|_{\infty,[t,r]} \\
&\leq c(1 - e^{-\mu(r-t)}) \frac{\|H\|_\infty}{\mu} \|x\|_{\infty,[t,r]} \\
&\leq c^{-\frac{1}{2}} (1 - e^{-\mu(r-t)}) \|x\|_{\infty,[t,r]}.
\end{aligned} \tag{11}$$

Now, if $u = x + y$ solves $u' = A(t)u$ with $u(0) \in W^s$, Lemma 1.1 and estimate (11) allow to take the limit for $r \to +\infty$ in (9), so taking $s = 0$ in (8), (x, y) is a solution of the fixed point problem

$$\begin{pmatrix} x \\ y \end{pmatrix} = L_A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} X_{A_-}(\cdot) x_0 \\ 0 \end{pmatrix}, \tag{12}$$

where $x_0 = x(0)$ and L_A is the linear operator on $C_b(\mathbb{R}^+; E^- \oplus E^+)$ defined by

$$L_A \begin{pmatrix} x \\ y \end{pmatrix}(t) := \begin{pmatrix} \int_0^t X_{A_-}(t) X_{A_-}(\tau)^{-1} A_\mp(\tau) y(\tau)\, d\tau \\ -\int_t^\infty X_{A_+}(t) X_{A_+}(\tau)^{-1} A_\pm(\tau) x(\tau)\, d\tau \end{pmatrix}. \tag{13}$$

Conversely, taking $s = 0$ and $r = \infty$ in (10) and (11), we deduce that L_A is a linear contraction. Therefore problem (12) has a unique solution (x, y), which clearly satisfies (8) and (9); from Lemma 1.1 and (10), for $t \geq s$

$$\begin{aligned} |x(t)| &\leq ce^{-\mu(t-s)}|x(s)| + c^{-\frac{1}{2}}(1 - e^{-\mu(t-s)})\|y\|_{\infty,[s,\infty[} \\ &\leq \max\{c|x(s)|, c^{-\frac{1}{2}}\|y\|_{\infty,[s,\infty[}\}, \end{aligned} \tag{14}$$

hence

$$\|x\|_{\infty,[s,\infty[} \leq \max\{c|x(s)|, c^{-\frac{1}{2}}\|y\|_{\infty,[s,\infty[}\}, \tag{15}$$

while from (11),

$$|y(t)| \leq c^{-\frac{1}{2}}\|x\|_{\infty,[t,\infty[}, \tag{16}$$

hence

$$\|y\|_{\infty,[s,\infty[} \leq c^{-\frac{1}{2}}\|x\|_{\infty,[s,\infty[}. \tag{17}$$

Estimates (15) and (17) give

$$\|x\|_{\infty,[s,\infty[} \leq \max\{c|x(s)|, c^{-1}\|x\|_{\infty,[s,\infty[}\}.$$

Since $c^{-1} \leq 1$, the last inequality implies that

$$\|x\|_{\infty,[s,\infty[} \leq c|x(s)|, \tag{18}$$

and (16) becomes

$$|y(t)| \leq c^{\frac{1}{2}}|x(t)|. \tag{19}$$

So there exists a continuous path of rank-one operators $U(t) \in \mathcal{L}(E^-, E^+)$ such that

$$U(t)x(t) = y(t), \quad \|U(t)\| \leq c^{\frac{1}{2}}.$$

Then the first equation of (7) becomes

$$x' = [A_-(t) + A_\mp(t)U(t)]x,$$

and since $\|A_\mp(t)U(t)\| \leq c^{\frac{3}{2}}\|H\|_\infty$, a further application of Lemma 1.1 yields

$$|x(t)| \leq ce^{-\nu(t-s)}|x(s)|, \tag{20}$$

for $\nu := \mu - c^{\frac{3}{2}}\|H\|_\infty$. Again by (19),

$$|y(t)| \leq c^{\frac{3}{2}}e^{-\nu(t-s)}|x(s)|, \tag{21}$$

hence x and y vanish at infinity. The conclusion is that, for any $x_0 \in E^-$, there exists a unique $y_0 \in E^+$, namely $y_0 = y(0)$, such that $x_0 + y_0 \in W^s$:

that is, W^s is the graph of a (linear) operator $S_0 : E^- \to E^+$. From the first inequality in (11) with $t = 0, r = \infty$, and (18), we have

$$\begin{aligned} |S_0 x_0| = |y_0| &\leq c \left(\int_0^\infty e^{-\mu\tau} \|A_\pm(\tau)\| \, d\tau \right) \|x\|_\infty \\ &\leq c^2 \left(\int_0^\infty e^{-\nu\tau} \|H(\tau)\| \, d\tau \right) |x_0|, \end{aligned}$$

for $\nu < \mu$. This proves (i) and (ii) in the case $t = 0$; the general statements follow by considering the shifted path $A(\cdot + t)$ and using the identity $X_A(t) W^s_A = W^s_{A(\cdot+t)}$. Claim (iv) follows from (19) and (20): indeed

$$|u(t)| \leq |x(t)| + |y(t)| \leq (1 + c^{\frac{1}{2}}) c e^{-\nu(t-s)} |x(s)| \leq 2 c^{\frac{3}{2}} e^{-\nu(t-s)} |u(s)|.$$

Since graph $S(t) = X_A(t) W^s$, the representation

$$\begin{aligned} S(t) &= P_+ \left(P_-|_{X_A(t)W^s} \right)^{-1} \\ &= P_+ X_A(t)(I_{E^-} + S(0))[P_- X_A(t)(I_{E^-} + S(0))]^{-1} \end{aligned}$$

implies (iii).

As for the second part of the proposition, notice that for any $\bar{t} \geq 0$ and $\bar{u} \in E$, we have that $\bar{u}$ belongs to $X_A(\bar{t})E^+$ if and only if there exists a solution u of $u' = A(t)u$ such that $u(0) \in E^+$ and $u(\bar{t}) = \bar{u}$. In other terms, setting as before $x(t) = P_- u(t)$, $y(t) = P_+ u(t)$, $\bar{u}$ is in $X_A(\bar{t})E^+$ if and only if (x, y) is a solution of system (8) and (9) with conditions $x(0) = 0$, $x(\bar{t}) = \bar{x} := P_- \bar{u}$, $x(\bar{t}) = \bar{x} := P_- \bar{u}$. That is, (x, y) is a solution of the fixed point problem

$$\begin{pmatrix} x \\ y \end{pmatrix} = M_A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ X_{A_+}(\cdot) X_{A_+}(\bar{t})^{-1} \bar{y} \end{pmatrix}, \tag{22}$$

where M_A is the linear operator on $C([0, \bar{t}]; E^- \oplus E^+)$ defined by

$$M_A \begin{pmatrix} x \\ y \end{pmatrix}(t) := \begin{pmatrix} \int_0^t X_{A_-}(t) X_{A_-}(\tau)^{-1} A_\mp(\tau) y(\tau) \, d\tau \\ -\int_t^{\bar{t}} X_{A_+}(t) X_{A_+}(\tau)^{-1} A_\pm(\tau) x(\tau) \, d\tau \end{pmatrix}.$$

As before, from (10) and (11), M_A is a linear contraction. We conclude that, for any $\bar{y} \in E^+$, there exists a unique $\bar{x} \in E^-$, namely $\bar{x} = x(\bar{t})$, such that $\bar{x} + \bar{y} \in X_A(\bar{t})E^+$: that is, $X_A(\bar{t})E^+$ is the graph of a (linear) operator $T(\bar{t}) : E^+ \to E^-$. From (10), (22), for $0 \leq t \leq r \leq \bar{t}$,

$$|x(t)| \leq c^{-\frac{1}{2}} \|y\|_{\infty,[0,t]}, \tag{23}$$

hence

$$\|x\|_{\infty,[0,r]} \leq c^{-\frac{1}{2}} \|y\|_{\infty,[0,r]}. \tag{24}$$

From Lemma 1.1, (9), (11)) and (22),

$$\begin{aligned}|y(t)| &\leq ce^{-\mu(r-t)}|y(r)| + c^{-\frac{1}{2}}(1-e^{-\mu(r-t)})\|x\|_{\infty,[0,r]} \\ &\leq \max\{c|y(r)|, c^{-\frac{1}{2}}\|x\|_{\infty,[0,r]}\},\end{aligned}$$

hence

$$\|y\|_{\infty,[0,r]} \leq \max\{c|y(r)|, c^{-\frac{1}{2}}\|x\|_{\infty,[0,r]}\}. \tag{25}$$

Estimates (24) and (25) imply

$$\|y\|_{\infty,[0,r]} \leq \max\{c|y(r)|, c^{-1}\|y\|_{\infty,[0,r]}\}.$$

Since $c^{-1} \leq 1$, we have

$$\|y\|_{\infty,[0,r]} \leq c|y(r)|. \tag{26}$$

Then (23) becomes

$$|x(t)| \leq c^{\frac{1}{2}}|y(t)|. \tag{27}$$

From the first part of (10), with $s = 0$, and (26),

$$\begin{aligned}|T(\bar{t})\bar{y}| = |\bar{x}| = |x(\bar{t})| &\leq c\left(\int_0^{\bar{t}} e^{-\mu(\bar{t}-\tau)}\|A_\pm(\tau)\|\,d\tau\right)\|y\|_{\infty,[0,\bar{t}]} \\ &\leq c^2\left(\int_0^{\bar{t}} e^{-\nu(\bar{t}-\tau)}\|H(\tau)\|\,d\tau\right)|\bar{y}|,\end{aligned}$$

for $\nu < \mu$, proving (v) and (vi). Let $U(t)$ be a continuous path of rank-one operators in $\mathcal{L}(E^+, E^-)$ such that

$$U(t)y(t) = x(t), \quad \|U(t)\| \leq c^{\frac{1}{2}}.$$

Then the second equation of (7) becomes

$$y' = [A_\pm(t)U(t) + A_+(t)]y$$

and since $\|A_\pm(t)U(t)\| \leq c^{\frac{1}{2}}\|H\|_\infty$, we can apply directly Lemma 1.1 and obtain

$$|y(t)| \geq c^{-1}e^{\nu(t-s)}|y(s)|,$$

for $\nu := \mu - c^{\frac{3}{2}}\|H\|_\infty$. By (27),

$$|y(s)| \geq \frac{1}{2}c^{-\frac{1}{2}}|u(s)|,$$

so

$$|X_A(t)u_0| \geq |y(t)| \geq c^{-1}e^{\nu(t-s)}|y(s)| \geq \frac{1}{2}c^{-\frac{3}{2}}e^{\nu(t-s)}|X_A(s)u_0|,$$

proving (viii). Claim (vii) follows from the representation

$$T(t) = P_-\left(P_+|_{X_A(t)E^+}\right)^{-1} = P_-X_A(t)[P_+X_A(t)]^{-1}.$$

□

Remark 1.1 For further applications, we point out that the operator $S(0) = S_A(0)$ of the above proposition can be represented as

$$S(0)x_0 = P_+ ev_0(I - L_A)^{-1} X_{A_-}(\cdot)x_0,$$

where ev_0 denotes the evaluation map for $t = 0$.

2 Asymptotically hyperbolic systems

Although Proposition 1.2 would allow to handle a more general situation, we will be mainly interested in those paths of operators which have a limit for $t \to \pm\infty$.

Definition 2.1 *An asymptotically hyperbolic path of operators on* $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ *is a piecewise continuous map* $A : \overline{\mathbb{R}} \to \mathcal{L}(E)$ *such that* $A(+\infty)$ *and* $A(-\infty)$ *are hyperbolic.*

Similarly, we can define an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}^+ := [0, \infty]$, respectively $\overline{\mathbb{R}}^- := [-\infty, 0]$, by requiring that A is defined and piecewise continuous on $\overline{\mathbb{R}}^+$, respectively $\overline{\mathbb{R}}^-$, and that $A(+\infty)$, respectively $A(-\infty)$, is hyperbolic.

Let $\mathcal{G}(E)$ be the Grassmannian of E, i.e. the set of all closed subspaces of E. For $V \in \mathcal{G}(E)$, denote by P_V the orthogonal projection onto V. The distance

$$\mathrm{dist}\,(V, W) := \|P_V - P_W\|$$

makes $\mathcal{G}(E)$ a complete metric space, isometric to the subset of $\mathcal{L}(E)$ of all orthogonal projections. If $\mathrm{dist}\,(V, W) < 1$, then $P_V|_W : W \to V$ is an isomorphism, being the restriction to W of the isomorphism

$$I-(P_W-P_V)(P_W-P_{W^\perp}) = P_V P_W + P_{V^\perp} P_{W^\perp} : W \oplus W^\perp \to V \oplus V^\perp.$$

In particular, $\dim$ and codim are continuous functions from $\mathcal{G}(E)$ to $\mathbb{N} \cup \{\infty\}$. A useful equivalent distance is

$$\delta(V, W) := \max\{\rho(V, W), \rho(W, V)\},$$

where

$$\rho(V, W) := \sup_{\substack{v \in V \\ |v|=1}} \inf_{w \in W} |v - w| = \|P_{W^\perp} P_V\|.$$

Indeed, δ is a distance because of the inequality

$$\|P_{W^\perp} P_V\| \leq \|P_{W^\perp} P_Y P_V\| + \|P_{W^\perp} P_{Y^\perp} P_V\| \leq \|P_{W^\perp} P_Y\| + \|P_{Y^\perp} P_V\|,$$

and the equivalence between δ and dist follows from

$$\delta(V, W) = \max\{\|(P_V - P_W)P_V\|, \|(P_W - P_V)P_W\|\} \leq \|P_V - P_W\|,$$

and

$$\begin{aligned}\|P_V - P_W\| &\leq \|(P_V - P_W)P_V\| + \|(P_V - P_W)P_{V^\perp}\| \\ &= \|P_{W^\perp}P_V\| + \|P_W P_{V^\perp}\| \\ &= \rho(V,W) + \|P_{V^\perp}P_W\| \\ &= \rho(V,W) + \rho(W,V) \leq 2\delta(V,W).\end{aligned}$$

The image TV of a closed subspace V by an invertible operator T continuously depends on the pair $(T,V) \in GL(E) \times \mathcal{G}(E)$, as the identity

$$P_{TV} = TP_V[TP_V + T^{*-1}(I - P_V)]^{-1} \tag{28}$$

shows. In particular, if $E = E^- \oplus E^+$, a sequence of operators $(S_n) \subset \mathcal{L}(E^-, E^+)$ converges to S if and only if the graphs of S_n converge to the graph of S.

Theorem 2.1 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}^+$. Then W^s_A is a closed subspace and the following convergence results for $t \to +\infty$ hold:*

(i) W^s_A *is the only closed subspace W such that* $X_A(t)W \to V^-(A(+\infty))$*;*
(ii) $\|X_A(t)|_{W^s_A}\| \leq ce^{-\lambda(t-s)}\|X_A(s)|_{W^s_A}\|$ *for suitable $c, \lambda > 0$, and for every $t \geq s \geq 0$.*

If the closed linear subspace $V \subset E$ is topological complement of W^s_A,

(iii) $X_A(t)V \to V^+(A(+\infty))$*;*
(iv) $\inf\limits_{\substack{v\in V\\ |v|=1}} |X_A(t)v| \to \infty$ *exponentially fast.*

Furthermore

(v) $W^s_{-A^*} = (W^s_A)^\perp$.

Proof. If t is large enough, the shifted path $A(\cdot + t)$ satisfies the assumptions of Proposition 1.2, with $E^- := V^-(A(+\infty))$ and $E^+ := V^+(A(+\infty))$. So $X_A(t)W^s_A = W^s_{A(t+\cdot)}$ is the graph of an operator $S(t) : E^- \to E^+$. Since $X_A(t)$ is invertible, W^s_A is a closed subspace. By Proposition 1.2 (ii), $\|S(t)\|$ tends to zero for $t \to \infty$, hence $X_A(t)W^s_A \to E^-$, proving the first part of (i).

Conclusion (ii) readily follows from Proposition 1.2 (iv).

Up to a time-shift, we may assume that E^+ is in direct sum with W^s_A and that the conclusions (iv) and (viii) of Proposition 1.2 hold for some $b, \nu > 0$. Since V is also in direct sum with W^s_A, it is the graph of a bounded operator

L from E^+ to W^s_A. Therefore, if $v \in X_A(t)V$, writing $v = X_A(t)(y + Ly)$ with $y \in E^+$, we get that $u := X_A(t)y$ is an element of $X_A(t)E^+$ satisfying

$$|v - u| = |X_A(t)Ly| \leq be^{-\nu t}\|L\||y| \leq b^2 e^{-2\nu t}\|L\||X_A(t)y| \\ \leq b^2 e^{-2\nu t}\|L\|(|v| + |v - u|).$$

Hence $|v - u| = o(1)|v|$, so

$$\rho(X_A(t)V, X_A(t)E^+) \to 0. \tag{29}$$

On the other hand, for any $u = X_A(t)y \in X_A(t)E^+$, setting $v := u + X_A(t)Ly \in X_A(t)V$, we get that

$$|u - v| = |X_A(t)Ly| \leq be^{-\nu t}\|L\||y| \leq b^2 e^{-2\nu t}\|L\||u|,$$

so $|u - v| = o(1)|u|$ and

$$\rho(X_A(t)E^+, X_A(t)V) \to 0. \tag{30}$$

From (29) and (30), (iii) follows, because $X_A(t)E^+ \to E^+$ by Proposition 1.2 (v), (vi).

Now, as a consequence of (iii), we have that any closed subspace W such that $X_A(t)W \to V^-(A(+\infty))$, necessarily has null intersection $W \cap V = (0)$ with any topological complement V of W^s_A. This implies that $W \subset W^s_A$, so also $X_A(t)W \subset X_A(t)W^s_A$ for any t. As we have $\operatorname{dist}(X_A(t)W, X_A(t)W^s_A) < 1$ for large t, we conclude that $X_A(t)W = X_A(t)W^s_A$ and $W = W^s_A$. This shows the uniqueness of the subspace W and ends the proof of assertion (i).

For any $v \in V$, $v = y + Ly$, by Proposition 1.2 (iv) and (viii),

$$|X_A(t)v| \geq |X_A(t)y| - |X_A(t)Ly| \geq b^{-1}e^{\nu t}|y| - be^{-\nu t}|Ly| \\ \geq \left(b^{-1}e^{\nu t} - be^{-\nu t}\|L\|\right)|y| \geq \frac{1}{\|I + L\|}\left(b^{-1}e^{\nu t} - be^{-\nu t}\|L\|\right)|v|,$$

proving (iv).

By the first assertion, the curve of subspaces

$$X_{-A^*}(t)\left[(W^s_A)^\perp\right] = (X_A(t)^{-1})^*\left[(W^s_A)^\perp\right] = [X_A(t)W^s_A]^\perp$$

converges to $(E^-)^\perp = V^-(-A^*)$. By assertion (i), $(W^s_A)^\perp = W^s_{-A^*}$. □

We conclude this section with the characterization of the asymptotically hyperbolic paths for which the evolution of the stable space is constant.

Lemma 2.2 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}^+$. Then $X_A(t)W^s_A = V^-(A(+\infty))$ for every $t \geq 0$ if and only if $A(t)V^-(A(+\infty)) \subset V^-(A(+\infty))$ for every $t \geq 0$.*

Proof. The first condition implies the second one by differentiation. Conversely, the second condition implies $X_A(t)V^-(A(+\infty)) \subset V^-(A(+\infty))$, so $X_A(t)$ restricts to a continuous path of injective semi-Fredholm operators on $V^-(A(+\infty))$. Since $X_A(0) = I$, the continuity of the Fredholm index implies that $X_A(t)V^-(A(+\infty)) = V^-(A(+\infty))$ for any t. Then by Theorem 2.1 (i), $W_A^s = V^-(A(+\infty))$. □

3 Perturbations

It is interesting to see what happens to the stable space W_A^s when the path A is subject to a perturbation by a path which is either small in the L^∞ norm, or consists of compact operators. We begin with the small perturbations.

Theorem 3.1 *The stable space W_A^s depends continuously on the path A, with respect to the standard topology of $\mathcal{G}(E)$ and to the L^∞ topology on the set of asymptotically hyperbolic paths.*

Proof. Let $(A_n)_{n\geq 1}$ be a sequence of asymptotically hyperbolic paths converging uniformly to the asymptotically hyperbolic path A_∞. It is enough to prove that for some $t_0 \geq 0$, $(W^s_{A_n(\cdot+t_0)})$ converges to $W^s_{A_\infty(\cdot+t_0)}$ in $\mathcal{G}(E)$: the claim then follows from identity (6) and from the fact that $(X_{A_n}(t_0)^{-1})$ converges to $X_{A_\infty}(t_0)^{-1}$.

Therefore, up to a time shift and to a shift of the indices, we can assume that the hypotheses of Proposition 1.2 are satisfied, with $A_0 := A_\infty(\infty)$ and $H(t) := A_n(t) - A_0$, for any $1 \leq n \leq \infty$. As a consequence, keeping the notations of Proposition 1.2, we have that $W^s_{A_n}$ is the graph of an operator $S_{A_n} \in \mathcal{L}(E^-, E^+)$ and the estimates

$$\begin{aligned} \|X_{A_{n-}}(t)X_{A_{n-}}(\tau)^{-1}\| &\leq ce^{-\mu(t-\tau)}, \quad t \geq \tau, \\ \|X_{A_{n+}}(t)X_{A_{n+}}(\tau)^{-1}\| &\leq ce^{-\mu(\tau-t)}, \quad t \leq \tau, \end{aligned} \tag{31}$$

hold, where $A_{n-} := P_- A_n P_-$, $A_{n+} := P_+ A_n P_+$. By Remark 1.1,

$$S_{A_n} x_0 = P_+ ev_0 \circ (I - L_{A_n})^{-1} X_{A_{n-}}(\cdot) x_0, \quad \forall x_0 \in E^-,$$

where L_{A_n} is the bounded operator on $C_b(\mathbb{R}^+; E)$ defined as

$$L_{A_n}\begin{pmatrix} x \\ y \end{pmatrix}(t) := \begin{pmatrix} \int_0^t X_{A_{n-}}(t)X_{A_{n-}}(\tau)^{-1}A_{n\mp}(\tau)y(\tau)\,d\tau \\ -\int_t^\infty X_{A_{n+}}(t)X_{A_{n+}}(\tau)^{-1}A_{n\pm}(\tau)x(\tau)\,d\tau \end{pmatrix},$$

with operator norm $\|L_{A_n}\| \leq \alpha < 1$. Notice that

$$ev_0(I - L_{A_n})^{-1}X_{A_{n-}} = \sum_{k=0}^{\infty} L^k_{A_n} X_{A_{n-}}(0) \tag{32}$$

is absolutely convergent in $\mathcal{L}(E^-, E)$, uniformly in n:

$$\|L_{A_n}^k X_{A_{n-}}(0)\|_{\mathcal{L}(E^-,E)} \leq c\alpha^k, \quad 1 \leq n \leq \infty.$$

As we are going to prove, $(L_{A_n}^k X_{A_{n-}}(0))$ converges to $L_{A_\infty}^k X_{A_{\infty-}}(0)$ in $\mathcal{L}(E^-, E)$ for every k, so identity (32) passes to the limit. This shows that (S_{A_n}) converges to S_{A_∞}, concluding the proof.

To prove the claim, notice first that $(X_{A_{n-}}(t))$ converges to $X_{A_{\infty-}}(t)$ for every t; moreover, if $Z_n(t) := L_{A_n}^k X_{A_{n-}}(t)$ converges to $Z_\infty(t) := L_{A_\infty}^k X_{A_{\infty-}}(t)$, then $L_{A_{n-}}^{k+1} X_{A_{n-}}(t) = L_{A_n} Z_n(t)$ converges to $L_{A_\infty} Z_\infty(t) = L_{A_{\infty-}}^{k+1} X_{A_{\infty-}}(t)$, because the integrands in the expression for $L_{A_n} Z_n(t)$ pass to the limit pointwise, and by (31), are dominated by an integrable function. The claim follows by induction. □

In order to study the perturbations by paths of compact operators, it is necessary to recall some concepts from [AM01]. Two closed subspaces $V, W \subset E$ are said *commensurable* if their orthogonal projectors differ by a compact operator. The *relative dimension of W with respect to V* is the integer

$$\dim(W, V) := \dim W \cap V^\perp - \dim V \cap W^\perp = \operatorname{ind}(W, V^\perp).$$

If (V, Z) is a semi-Fredholm pair (see Sect. 5) and W is commensurable to V, then (W, Z) is also a semi-Fredholm pair, of index

$$\operatorname{ind}(W, Z) = \operatorname{ind}(V, Z) + \dim(W, V). \tag{33}$$

The following lemma is a simple variant of Proposition 2.2 in [AM01].

Lemma 3.2 *Let T, S be two hyperbolic operators such that $T - S$ is compact. Let $P^\pm(T)$ and $P^\pm(S)$ be the projectors given by the spectral decomposition of T and S. Then $P^-(T) - P^-(S)$ and $P^+(T) - P^+(S)$ are compact.*

The following lemma is proved in [AM01], Proposition 2.3.

Lemma 3.3 *Let E_1, E_2 be Hilbert spaces, and let $T, S \in \mathcal{L}(E_1, E_2)$ be operators with compact difference and closed image. Then $\ker T$ is commensurable to $\ker S$, $\operatorname{ran} T$, the range of T, is commensurable to $\operatorname{ran} S$ and*

$$\dim(\operatorname{ran} T, \operatorname{ran} S) = -\dim(\ker T, \ker S).$$

Lemmas 3.2 and 3.3 imply that, if T, S are hyperbolic operators on E with compact difference, then $V^-(T)$ is commensurable to $V^-(S)$, while $V^+(T)$ is commensurable to $V^+(S)$.

Identity (4) immediately implies:

Lemma 3.4 *Let $J \subset \mathbb{R}$ be an interval containing zero. If $A(t) - B(t)$ is compact for every $t \in J$, then $X_A(t) - X_B(t)$ is compact for every $t \in J$.*

Lemma 3.5 *Let $J \subset \overline{\mathbb{R}}$ be an interval and let $K : J \to \mathcal{L}(E)$ be an integrable path consisting of compact operators. Then*

$$C_b(J; E) \ni u \mapsto \int_J K(\tau)u(\tau)\, d\tau \in E$$

is a compact operator.

Proof. The claim is obvious when K is a characteristic function of a bounded interval. The general case follows by linearity and by a density argument. □

Theorem 3.6 *Let A and B be two asymptotically hyperbolic paths such that $A(t) - B(t)$ is compact for every t. Then W_A^s is commensurable to W_B^s and*

$$\dim(W_A^s, W_B^s) = \dim(V^-(A(+\infty)), V^-(B(+\infty))).$$

Proof. Assume at first that A and B satisfy the assumptions of Proposition 1.2 with $E^\pm$ equal to $V^\pm(A(+\infty))$, respectively $V^\pm(B(+\infty))$. In this case, W_A^s and W_B^s are graphs of two bounded operators, $S_A \in \mathcal{L}(V^-(A(+\infty)), V^+(A(+\infty)))$ and $S_B \in \mathcal{L}(V^-(B(+\infty)), V^+(B(+\infty)))$. Let P_A^- and P_A^+ denote the projections associated to the splitting $E = V^-(A(+\infty)) \oplus V^+(A(+\infty))$; then

$$W_A^s = \ker(P_A^+ - S_A P_A^-), \quad W_B^s = \ker(P_B^+ - S_A P_B^-). \tag{34}$$

Since $A(+\infty) - B(+\infty)$ is compact, $P_A^- - P_B^-$ and $P_A^+ - P_B^+$ are compact, by Lemma 3.2. We shall prove that $S_A P_A^-$ and $S_B P_B^-$ differ by a compact operator.

Recall, from the proof of Proposition 1.2, that $L_A \in \mathcal{L}(C_b(\mathbb{R}^+; E))$ writes as

$$\begin{aligned} L_A u(t) = &\int_0^t X_{P_A^- A P_A^-}(t) X_{P_A^- A P_A^-}(\tau)^{-1} P_A^- A(\tau) P_A^+ u(\tau)\, d\tau \\ &- \int_t^\infty X_{P_A^+ A P_A^+}(t) X_{P_A^+ A P_A^+}(\tau)^{-1} P_A^+ A(\tau) P_A^- u(\tau)\, d\tau. \end{aligned}$$

The operator L_A maps bounded subsets of $C_b(\mathbb{R}^+; E)$ into equicontinuous sets: indeed

$$\frac{d}{dt}(L_A u) = (P_A^+ A P_A^+ + P_A^- A P_A^-)(L_A - I)u + Au$$

is uniformly bounded, for u varying in a bounded subset of $C_b(\mathbb{R}^+;E)$. Since $P_A^- A(t) P_A^- - P_B^- B(t) P_B^-$ and $P_A^+ A(t) P_A^+ - P_B^+ B(t) P_B^+$ are compact for every $t \geq 0$, the operators

$$X_{P_A^- A P_A^-}(t) - X_{P_B^- B P_B^-}(t) \quad \text{and} \quad X_{P_A^+ A P_A^+}(t) - X_{P_B^+ B P_B^+}(t)$$

are compact for every $t \geq 0$ (Lemma 3.4). Hence, for every $t \geq 0$, the paths

$$\begin{aligned} K_-(\tau) := {} & X_{P_A^- A P_A^-}(t) X_{P_A^- A P_A^-}(\tau)^{-1} P_A^- A(\tau) P_A^+ \\ & - X_{P_B^- B P_B^-}(t) X_{P_B^- B P_B^-}(\tau)^{-1} P_B^- B(\tau) P_B^+, \\ K_+(\tau) := {} & X_{P_A^+ A P_A^+}(t) X_{P_A^+ A P_A^+}(\tau)^{-1} P_A^+ A(\tau) P_A^- \\ & - X_{P_B^+ B P_B^+}(t) X_{P_B^+ B P_B^+}(\tau)^{-1} P_B^+ B(\tau) P_B^-, \end{aligned}$$

consist of compact operators. Since K_- is integrable on $[0,t]$ and K_+ is integrable on $[t,\infty[$, by Lemma 3.5 the set

$$\{[(L_A - L_B)u](t) \mid u \in \mathcal{B}\}$$

is relatively compact, for any bounded set $\mathcal{B} \subset C_b(\mathbb{R}^+;E)$. Together with the equicontinuity stated before, by Ascoli-Arzelà theorem this implies that $L_A - L_B$ is a compact operator. Thus, $(I - L_A)^{-1}$ and $(I - L_B)^{-1}$ have a compact difference. By Remark 1.1,

$$S_A P_A^- = P_A^+ ev_0 (I - L_A)^{-1} X_{P_A^- A P_A^-}(\cdot) P_A^-,$$

and every operator in the above chain changes by a compact operator when we replace A by B. Therefore, $S_A P_A^- - S_B P_B^-$ is compact, so identities (34) and Lemma 3.3 imply that W_A^s is commensurable to W_B^s and

$$\begin{aligned} \dim(W_A^s, W_B^s) &= -\dim(\operatorname{ran}(P_A^+ - S_A P_A^-), \operatorname{ran}(P_B^+ - S_A P_B^-)) \\ &= -\dim(V^+(A(+\infty)), V^+(B(+\infty))) \\ &= \dim(V^-(A(+\infty)), V^-(B(+\infty))). \end{aligned}$$

Let q be the above integer. In the general case, the preceding considerations show that for t large enough, $(X_A(t)W_A^s, X_B(t)W_B^s)$ is a pair of commensurable subspaces of relative dimension q. By Lemma 3.3, so is the pair $(W_A^s, X_A(t)^{-1}X_B(t)W_B^s)$. By Lemma 3.4, the invertible operator

$$X_B(t)^{-1}X_A(t) = I + X_B(t)^{-1}[X_A(t) - X_B(t)]$$

is a compact perturbation of the identity, so Lemma 3.3 implies that (W_A^s, W_B^s) is a pair of commensurable subspaces of relative dimension q, concluding the proof. $\square$

We conclude this section with the problem of when the evolution of the stable space remains in the same commensurable class. Let V be a closed subspace of E. Differentiating (28), it is easy to show that the orthogonal projector $P(t)$ onto $X_A(t)V$ verifies

$$P'(t) = (I - P(t))A(t)P(t) + P(t)A(t)^*(I - P(t)). \tag{35}$$

Denote by $a(t)$ and $p(t)$ the projections of $A(t)$ and $P(t)$ in the Calkin algebra $\mathcal{L}(E)/\mathcal{L}_c(E)$. By (35), p solves the following Riccati equation

$$q' = (1 - q)aq + qa^*(1 - q). \tag{36}$$

The subspaces $X_A(t)V$ are commensurable if and only if $p(t) = p(0)$ for any t. In this case, $p' = 0$, and from (36) we obtain

$$\begin{aligned}[][a(t), p(0)]p(0) &= (1 - p(0))a(t)p(0) = (1 - p(t))a(t)p(t) \\ &= p'(t)p(t) = 0.\end{aligned}$$

Conversely, if $[a(t), p(0)]p(0)] = 0$, then also

$$p(0)a(t)^*(1 - p(0)) = ([a(t), p(0)]p(0))^* = 0,$$

so the constant map $q(t) \equiv p(0)$ solves (36), and from the uniqueness of the solution of the Cauchy problem, we conclude that $p(t) = p(0)$. Thus we have proved:

Lemma 3.7 *Let $A : \mathbb{R}^+ \to \mathcal{L}(E)$ be a piecewise continuous path of operators, and let V be a closed subspace of E. Then $X_A(t)V$ is a path of commensurable subspaces if and only if $[A(t), P_V]P_V$ is compact for every $t \in \mathbb{R}^+$.*

Remark 3.1 Let P, Q be two projectors onto V. Since $[A, Q]Q = [[A, P]P, Q]Q$, if $[A, P]P$ is compact for a projector onto V, it is compact for any projector onto V, and more generally, onto any closed subspace commensurable to V.

Coming back to the evolution of the stable space, we have the following characterization (compare Lemma 2.2).

Proposition 3.8 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}^+$; let V be a closed subspace of E, and let P be a projector onto V. Then the following are equivalent:*

(i) *$X_A(t)W_A^s$ is commensurable to V for any $t \geq 0$;*
(ii) *$V^-(A(+\infty))$ is commensurable to V and $[A(t), P]P$ is compact for any $t \geq 0$.*

Proof. Assume (i). By Theorem 2.1 (i), $X_A(t)W_A^s$ converges to $V^-(A(+\infty))$ for $t \to +\infty$, proving the first part of (ii), because the commensurability classes are closed. By Lemma 3.7 and Remark 3.1, $[A(t), P]P$ is compact and (ii) holds.

Assume (ii): by Remark 3.1, the operator

$$B(t) := A(t) - [A(t), P_{V^-(A(+\infty))}]P_{V^-(A(+\infty))}$$

is a compact perturbation of $A(t)$. Moreover, $B(+\infty) = A(+\infty)$ and

$$B(t)V^-(A(+\infty)) \subset V^-(A(+\infty)).$$

By Lemma 2.2, $X_B(t)W_B^s = V^-(A(+\infty))$, and by Theorem 3.6 and identity (6), $X_A(t)W_A^s$ is commensurable to $X_B(t)W_B^s$, hence to V. $\square$

4 The operator F_A

Let $L^2(J; E)$ denote the space of square integrable E-valued functions on J, and $H^1(J; E)$ the space of square integrable E-valued functions on J whose weak derivatives are square integrable. For simplicity of notations, set

$$L^2 := L^2(\mathbb{R}; E), \quad L^2_+ := L^2(\mathbb{R}^+; E), \quad L^2_- := L^2(\mathbb{R}^-; E),$$
$$H^1 := H^1(\mathbb{R}; E), \quad H^1_+ := H^1(\mathbb{R}^+; E), \quad H^1_- := H^1(\mathbb{R}^-; E).$$

Let A be an asymptotically hyperbolic path on $\overline{\mathbb{R}}$ (see Definition 2.1). We can define the bounded linear operator F_A from H^1 to L^2 by setting

$$F_A u := u' - Au \quad \forall u \in H^1.$$

Similarly, if A is an asymptotically hyperbolic path on $\overline{\mathbb{R}}^+$, respectively on $\overline{\mathbb{R}}^-$, we have the operator F_A^+ from H^1_+ to L^2_+, respectively F_A^- from H^1_- to L^2_-, defined as

$$F_A^+ u = u' - Au \quad \forall u \in H^1_+,$$
$$F_A^- u = u' - Au \quad \forall u \in H^1_-.$$

Denote by P_A^s and P_A^u the orthogonal projections onto W_A^s and W_A^u. It is useful to introduce the following families of operators

$$\begin{aligned} G_A^+(t,\tau) &:= X_A(t)[1_{\mathbb{R}^+}(t-\tau)P_A^s - 1_{\mathbb{R}^-}(t-\tau)(I - P_A^s)]X_A(\tau)^{-1}, \\ &\qquad \forall (t,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ G_A^-(t,\tau) &:= X_A(t)[1_{\mathbb{R}^+}(t-\tau)(I - P_A^u) - 1_{\mathbb{R}^-}(t-\tau)P_A^u]X_A(\tau)^{-1}, \\ &\qquad \forall (t,\tau) \in \mathbb{R}^- \times \mathbb{R}^-. \end{aligned}$$

Lemma 4.1 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}^+$. Let $\lambda > 0$ be such that*

$$\sup \mathrm{Re}\,[\sigma(A(+\infty)) \cap \{z \in \mathbb{C} \mid \mathrm{Re}\, z < 0\}] < -\lambda.$$

Then there is a positive number c such that

$$\|G_A^+(t,\tau)\| \leq c e^{-\lambda|t-\tau|} \quad \forall (t,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+.$$

Proof. From the identity $G_A^+(\tau,t) = -G_{-A^*}^+(t,\tau)^*$, it is clearly sufficient to show the bound for all pairs (t,τ) with $t \geq \tau \geq 0$. In this case, using Theorem 2.1 (ii),

$$\begin{aligned} \|G^+(t,\tau)\| &= \|X(t)|_{W^s} P^s X(\tau)^{-1}\| \\ &\leq c e^{-\lambda(t-\tau)} \|X(\tau) P^s X(\tau)^{-1}\|. \end{aligned} \tag{37}$$

Now notice that $X(\tau)P^s X(\tau)^{-1}$ is the linear projection on the first factor of the splitting

$$E = X(\tau)W^s \oplus X(\tau)(W^s)^\perp.$$

From Theorem 2.1 (i) and (iii),

$$X(\tau)W^s \to V^-(A(+\infty)) \quad \text{and} \quad X(\tau)(W^s)^\perp \to V^+(A(+\infty)),$$

for $\tau \to \infty$. Therefore $X(\tau)P^s X(\tau)^{-1}$ is a continuous path of linear projectors which converges for $\tau \to \infty$. In particular, it is bounded and the conclusion follows from inequality (37) taking into account the initial observation. □

Proposition 4.2 *The operator F_A^+ is onto and a right-inverse $R_A^+ \in \mathcal{L}(L_+^2, L_+^2)$ is defined as*

$$(R_A^+ h)(t) := \int_0^\infty G_A^+(t,\tau) h(\tau)\, d\tau, \quad \forall h \in L_+^2,$$

Furthermore, the splitting $H_+^1 = \ker F_A^+ \oplus \mathrm{ran}\, R_A^+$ is given by the representation

$$u(t) = X_A(t) P_A^s u(0) + R_A^+ F_A^+ u(t), \quad \forall u \in H_+^1.$$

Analogous statements hold for F_A^- with

$$(R_A^- h)(t) := \int_{-\infty}^0 G_A^-(t,\tau) h(\tau)\, d\tau, \quad \forall h \in L_-^2.$$

Proof. First notice that R^+ is a well defined bounded operator: indeed, from Lemma 4.1, for any $h \in L^2_+$ and $t \geq 0$ we have

$$|(R^+h)(t)| = \left| \int_0^\infty G^+(t,\tau)h(\tau)\,d\tau \right| \leq c \int_0^\infty e^{-\lambda|t-\tau|}|h(\tau)|\,d\tau$$
$$= c[\exp(-\lambda|\cdot|) * |h(\cdot)|](t),$$

so, by Young's inequality, R^+h is in L^2_+ and

$$\|R^+h\|_{L^2} \leq \frac{2}{\lambda}\|h\|_{L^2}.$$

Furthermore, since

$$R^+h(t) = X(t)\Big[\int_0^t P^s X(\tau)^{-1}h(\tau)\,d\tau$$
$$- \int_t^\infty (I-P^s)X(\tau)^{-1}h(\tau)\,d\tau\Big],$$

by direct computation one gets that $R^+h \in H^1_+ = \operatorname{dom} F^+$ and $F^+R^+h = h$, as we wished to prove.

Since F^+ is continuous from H^1_+ to L^2_+ and R^+ is a right inverse for F^+, one has the corresponding splitting into closed subspaces

$$H^1_+ = \ker F^+ \oplus \operatorname{ran} R^+,$$

with projections $I - R^+F^+$ and R^+F^+. So, for any $u \in H^1_+$ there exists $v_0 \in W^s$ such that

$$u(t) = X(t)v_0 + R^+F^+u(t)$$

and

$$u(0) = v_0 + R^+F^+u(0).$$

By the definition of R^+, $R^+F^+u(0) \in (W^s)^\perp$, and we conclude that $v_0 = P_{W^s}u(0)$. □

Lemma 4.3 *The operator $h \mapsto R^+_A h(0)$ maps $C^\infty_c(]0,\infty[;E)$ onto $(W^s_A)^\perp$. The operator $h \mapsto R^-_A h(0)$ maps $C^\infty_c(]-\infty,0[;E)$ onto $(W^u_A)^\perp$.*

Proof. Let $\varphi \in C^\infty_c(]0,\infty[;\mathbb{R})$ be a function such that the operator $U := \int_0^\infty \varphi(\tau)X_A(\tau)^{-1}\,d\tau \in \mathcal{L}(E)$ is invertible. Given $v_0 \in (W^s_A)^\perp$, we can define $h \in C^\infty_c(]0,\infty[;E)$ as $h(t) := -\varphi(t)U^{-1}v_0$. Then

$$R^+_A h(0) = -\int_0^\infty (I-P^s)X_A(\tau)^{-1}h(\tau)\,d\tau$$
$$= (I-P^s)\int_0^\infty \varphi(\tau)X_A(\tau)^{-1}\,d\tau U^{-1}v_0 = (I-P^s)v_0 = v_0.$$

□

5 On the Fredholm property of F_A

We recall that a pair (V, W) of closed subspaces of E is said a *semi-Fredholm pair* if $V + W$ is closed and at least one of the numbers

$$\dim V \cap W, \quad \operatorname{codim}_E(V + W)$$

is finite. In this case the *Fredholm index* of the pair (V, W) is

$$\operatorname{ind}(V, W) = \dim V \cap W - \operatorname{codim}_E(V + W) \in \mathbb{Z} \cup \{-\infty, +\infty\}.$$

When $\operatorname{ind}(V, W) \in \mathbb{Z}$, (V, W) is called a *Fredholm pair*. If the closed subspaces W and Z are commensurable, the pair $(Z, W^{\perp})$ is Fredholm and

$$\operatorname{ind}(Z, W^{\perp}) = \dim(Z, W).$$

If (V, W) is a Fredholm pair and Z is commensurable to W, then also (V, Z) is a Fredholm pair and its index is

$$\operatorname{ind}(V, Z) = \operatorname{ind}(V, W) + \dim(Z, W).$$

Theorem 5.1 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}$. Then:*

(i) F_A has closed range if and only if the subspace $W_A^s + W_A^u$ is closed.
(ii) F_A is onto if and only if $W_A^s + W_A^u = E$;
(iii) F_A is injective if and only if $W_A^s \cap W_A^u = \{0\}$.
(iv) F_A is invertible if and only if $E = W_A^s \oplus W_A^u$.
(v) F_A is a semi-Fredholm operator if and only if (W_A^s, W_A^u) is a semi-Fredholm pair. In this case

$$\operatorname{ind} F_A = \operatorname{ind}(W_A^s, W_A^u).$$

The proof of the above theorem is based on the following useful characterizations.

Proposition 5.2 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}$. Then:*

(i) $\ker F_A = \{X_A(t)u_0 \mid u_0 \in W_A^s \cap W_A^u\}$;
(ii) $\operatorname{ran} F_A = \left\{h \in L^2 \mid (R_A^+ h)(0) - (R_A^- h)(0) \in W_A^s + W_A^u\right\}$;
(iii) $\overline{\operatorname{ran} F_A} = \left\{h \in L^2 \mid (R_A^+ h)(0) - (R_A^- h)(0) \in \overline{W_A^s + W_A^u}\right\}$;

Proof. (i). The first claim follows from the fact that the solutions of the linear system starting from $W^s \cap W^u$ have exponential decay at $\pm\infty$ (Theorem 2.1 (ii)).

(ii). The image of F consists of those $h \in L^2$ such that the equation

$$u' = A(t)u + h \tag{38}$$

has a solution u in L^2 (hence, from (38), in H^1). The solution of (38) such that $u(0) = u_0$ can be written as

$$u(t) = \begin{cases} X(t)[u_0 - R^+ h(0)] + R^+ h(t), \text{ for } t \geq 0, \\ X(t)[u_0 - R^- h(0)] + R^- h(t), \text{ for } t \leq 0. \end{cases}$$

So u belongs to L^2 if and only if

$$u_0 - R^+ h(0) \in W^s, \tag{39}$$

$$u_0 - R^- h(0) \in W^u. \tag{40}$$

If h is in the image of F, subtracting (40) from (39), we obtain

$$R^+ h(0) - R^- h(0) \in W^s + W^u. \tag{41}$$

Conversely, if h satisfies (41), $R^+ h(0) - R^- h(0) = v_s + v_u$, for some $v_s \in W^s$ and $v_u \in W^u$. Choosing

$$u_0 := R^+ h(0) - v_s = R^- h(0) + v_u,$$

(39) and (40) hold, so the corresponding u is in L^2.

(iii). By (ii) and by the fact that $H := \{h \in L^2 \mid R^+_A h(0) - R^-_A h(0) \in \overline{W^s_A + W^u_A}\}$ is closed, we get that $\overline{\operatorname{ran} F_A} \subset H$. By Lemma 4.3, according to the open mapping theorem, there exists $c > 0$ such that for every $w_0 \in (W^s_A)^\perp$ there exists $h_0 \in C^\infty_c(]0,\infty[; E)$ such that $R^+_A h_0(0) = w_0$ and $\|h_0\|_{L^2} \leq c|w_0|$.

Given $h \in H$ and $\epsilon > 0$, we may write $R^+_A h(0) - R^-_A h(0) = v_0 + w$, where $v_0 \in W^s_A + W^u_A$ and $|w| < \epsilon/c$. By the above claim, there exists $h_0 \in C^\infty_c(]0,\infty[; E)$, $\|h_0\|_{L^2} < \epsilon$, such that $R^+_A h_0(0) - R^-_A h_0(0) = R^+_A h_0(0) = (I - P^s)w$, and setting $h_\epsilon := h - h_0$, we have that $\|h_\epsilon - h\|_{L^2} < \epsilon$ and

$$R^+_A h_\epsilon(0) - R^-_A h_\epsilon(0) = v_0 + w - (I - P^s)w = v_0 + P^s w.$$

Since such a vector is in $W^s_A + W^u_A$, claim (ii) implies that $h_\epsilon \in \operatorname{ran} F_A$. Since ϵ is arbitrary, $h \in \overline{\operatorname{ran} F_A}$. □

Proof of Theorem 5.1. (i) Proposition 5.2 (ii)-(iii) implies that if $W^s + W^u$ is closed then F has closed range.

Conversely, assume that F has closed range. Let $v_0 \in \overline{W^s + W^u}$. By Lemma 4.3, there exists $h \in C^\infty_c(]0,\infty[; E)$ such that $R^+_A h(0) - R^-_A h(0) = (I - P^s)v_0$. Since $(I - P^s)v_0 = v_0 - P^s v_0 \in \overline{W^s + W^u}$, Proposition 5.2 (iii) implies that $h \in \overline{\operatorname{ran} F}$. Since $\overline{\operatorname{ran} F} = \operatorname{ran} F$, assertion (ii) of Proposition 5.2 implies that $(I - P^s)v_0 \in W^s + W^u$. Hence $v_0 = (I - P^s)v_0 + P^s v_0 \in W^s + W^u$. This proves that $\overline{W^s + W^u} \subset W^s + W^u$, hence $W^s + W^u$ is closed.

All the other statements follow immediately from Proposition 5.2 (i) and (ii), taking into account Lemma 4.3. □

Remark 5.1 All the results of sections 4 and 5 hold for different choices of the spaces between which F_A is defined. As a consequence, the fact of being Fredholm and the value of the index do not depend on these spaces. In particular, assuming the path A to be continuous, the proofs of all these results hold with no change for F_A as a bounded operator between the Banach spaces

$$C_0^1 := \left\{ u \in C^1(\mathbb{R}; E) \mid \lim_{t\to\pm\infty} u(t) = 0, \lim_{t\to\pm\infty} u'(t) = 0 \right\} \quad \text{and}$$
$$C_0^0 := \left\{ u \in C^0(\mathbb{R}; E) \mid \lim_{t\to\pm\infty} u(t) = 0 \right\}.$$

Remark 5.2 Seen as an unbounded operator on L^2 with domain H^1, F_A is closed. The domain of its adjoint operator is again H^1 and

$$F_A^* = -F_{-A^*}.$$

Compare this formula with the identities $X_{-A^*} = (X_A^{-1})^*$ and $W^s_{-A^*} = (W_A^s)^\perp$.

Example 1. When E is finite dimensional, F_A is always a Fredholm operator of index

$$\operatorname{ind} F_A = \dim V^-(A(+\infty)) - \dim V^-(A(-\infty)).$$

Indeed in a finite dimensional space (W_A^s, W_A^u) is always a Fredholm pair and since the evolution of W_A^s under the flow X_A converges to $V^-(A(+\infty))$ for $t \to +\infty$, while the evolution of W_A^u converges to $V^+(A(+\infty))$ for $t \to -\infty$ (Theorem 2.1 (i)),

$$\dim W_A^s = \dim V^-(A(+\infty)), \quad \dim W_A^u = \dim V^+(A(-\infty)).$$

Therefore F_A is a Fredholm operator of index

$$\begin{aligned} \operatorname{ind} F_A &= \operatorname{ind}(W_A^s, W_A^u) \\ &= \dim W_A^s + \dim W_A^u - \dim E \\ &= \dim V^-(A(+\infty)) - \dim V^-(A(-\infty)). \end{aligned} \qquad \square$$

Example 2. If $A(-\infty)$ and $A(+\infty)$ have negative eigenspaces of finite dimension, the same conclusions hold

$$\begin{aligned} \operatorname{ind} F_A &= \operatorname{ind}(V^-(A(+\infty)), V^+(A(-\infty))) \\ &= \dim V^-(A(+\infty)) - \dim V^-(A(-\infty)). \end{aligned}$$

no matter what is the form of $A(t)$ for $t \in \mathbb{R}$.

Indeed, by Theorem 2.1 (i), $X_A(t)W_A^s$ is a continuous path of closed subspace which converges to $V^-(A(+\infty))$ for $t \to +\infty$. The dimension is a continuous function on $\mathcal{G}(E)$, so $\dim W_A^s = \dim V^-(A(+\infty))$. On the other hand, $X_A(t)W_A^u$ is a continuous path of closed subspace which converges to $V^+(A(-\infty))$ for $t \to -\infty$, and the codimension is a continuous function on $\mathcal{G}(E)$, so $\operatorname{codim}_E W_A^u = \operatorname{codim}_E V^+(A(-\infty)) = \dim V^-(A(-\infty))$. A finite dimensional and a finite codimensional subspace always make a Fredholm pair, of index

$$\begin{aligned}\operatorname{ind}(W_A^s, W_A^u) &= \dim W_A^s - \operatorname{codim}_E W_A^u \\ &= V^-(A(+\infty)) - \dim V^-(A(-\infty)).\end{aligned}$$

The conclusion follows again by Theorem 5.1. □

But in infinite dimensional Hilbert spaces, more possibilities arise.

Example 3. Let V_- and V_+ be two arbitrary closed subspaces of E. Then there exists an asymptotically hyperbolic path of operators A on $\overline{\mathbb{R}}$ such that (i) A is smooth, (ii) $A(t)$ is self-adjoint for every t and (iii)

$$W_A^s = V_-, \quad W_A^u = V_+.$$

Indeed, let φ be a smooth function such that $\varphi = 1$ on $[-1/2, 1/2]$ and $\varphi = -1$ on $[-\infty, -1] \cup [1, \infty]$. A path satisfying the above conditions is the following

$$A(t) := \begin{cases} P_{V_+} + \varphi(t) P_{V_+^\perp}, \text{ for } t < 0, \\ P_{V_-^\perp} + \varphi(t) P_{V_-}, \text{ for } t \geq 0. \end{cases}$$

□

This example, together with the characterization given by Theorem 5.1, provides us with an easy way to build asymptotically hyperbolic paths A such that F_A does not have closed range, or such that F_A has closed range but an infinite dimensional kernel and/or co-kernel.

As we have seen in Examples 1 and 2, there are classes of asymptotically hyperbolic paths with the property that the corresponding F is Fredholm and its index depends only on the end-points of the path. In general, also when F happens to be Fredholm, this fact may not hold anymore.

Example 4. There exists a family $(A_k), k \in \mathbb{Z}$, of asymptotically hyperbolic paths of operators having identical end-points at $\pm\infty$, $A_k(\pm\infty) = A_0(\pm\infty)$ for any k, such that F_{A_k} is Fredholm of index k.

For this purpose, set

$$E := \ell^2(\mathbb{Z}) = \left\{ u = (u_n) \in \mathbb{R}^{\mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} |u_n|^2 < \infty \right\},$$

with its usual inner product

$$u \cdot v = \sum_{n \in \mathbb{Z}} u_n v_n.$$

Set

$$E^+ := \ell^2(\mathbb{Z}^+ \cup \{0\}) = \left\{ u \in \ell^2(\mathbb{Z}) \mid u_n = 0 \text{ for } n < 0 \right\},$$
$$E^- := \ell^2(\mathbb{Z}^-) = \left\{ u \in \ell^2(\mathbb{Z}) \mid u_n = 0 \text{ for } n \geq 0 \right\},$$

so that $E = E^- \oplus E^+$. If $k \in \mathbb{Z}$, let $S^k \in \mathcal{L}(E)$ be the k-shift

$$(S^k u)_n := u_{n+k}.$$

Since the set of invertible operators on the infinite dimensional Hilbert space E is connected[6] (a classical result due to Putnam and Wintner [PW51], see also [Kui65] for more information on the general group of a Hilbert space), there exists a smooth path $Y : \mathbb{R} \to GL(E)$ such that $Y(t) = I$ for $t \leq 0$ and $Y(t) = S^k$ for $t \geq 1$. Let φ be a smooth function such that $\varphi = 0$ on $[0, 1]$, $\varphi = 1$ on $]-\infty, -1] \cup [2, \infty[$ and set

$$A(t) = \begin{cases} \varphi(t)(P_{E^+} - P_{E^-}), & \text{for } t \leq 0, \\ Y'(t)Y(t)^{-1}, & \text{for } 0 \leq t \leq 1, \\ \varphi(t)(P_{E^+} - P_{E^-}), & \text{for } t \geq 1. \end{cases}$$

Then $A(+\infty) = A(-\infty) = P_{E^+} - P_{E^-}$ and $X_A(t) = Y(t)$ for $t \in [0, 1]$. It is readily seen that

$$W_A^u = E^+, \quad X_A(1) W_A^s = E^-,$$

and since $X_A(1) = Y(1) = S^k$, $W_A^s = S^{-k}E^-$. Therefore (W_A^s, W_A^u) is a Fredholm pair of index

$$\begin{aligned} \operatorname{ind}(W_A^s, W_A^u) &= \operatorname{ind}(S^{-k}E^-, E^+) \\ &= \dim S^{-k}E^- \cap E^+ - \operatorname{codim}(S^{-k}E^- + E^+) = k. \end{aligned}$$

Therefore F_A is a Fredholm operator of index k. $\square$

Let $\{e_n\}$ be the standard orthonormal system in $\ell^2(\mathbb{Z})$ and consider the invertible operator defined by

$$Te_n := \begin{cases} e_{2n}, & \text{if } n \geq 0, \\ e_{n/2}, & \text{if } n < 0 \text{ is even}, \\ e_{-n}, & \text{if } n < 0 \text{ is odd}. \end{cases}$$

[6] For example, I is connected to S^1 in $O(E)$ by the path $[0, \pi/2] \ni t \mapsto e^{-tJ} e^{tK}$, where $Je_i = \operatorname{sgn}(i + 1/2) e_{-i-1}$ and $Ke_i = (\operatorname{sgn} i) e_{-i}$.

Replacing the path Y by a path connecting I to either T or T^{-1} in $GL(E)$, the same construction shown above allows to build asymptotically hyperbolic path still having the same end-points, but index $-\infty$ or $+\infty$. If we connect I to the operator R such that $Re_n = e_{-n}$, we get an operator F_A with closed range but infinite dimensional kernel and co-kernel. We just mention that a suitable choice of Y would make A a path of self-adjoint operators in the examples above.

6 Essentially splitting paths

In this section we shall consider an asymptotically hyperbolic path A and the related operator F_A (either from H^1 to L^2 or from C_0^1 to C_0^0, in which case A is assumed to be continuous). It is interesting to see what happens when the path A is subject to a perturbation by a path of compact operators.

Theorem 6.1 *Let A and B be two asymptotically hyperbolic paths on $\overline{\mathbb{R}}$ such that $B(t) - A(t)$ is compact for every $t \in \overline{\mathbb{R}}$. Assume that F_A is semi-Fredholm. Then also F_B is semi-Fredholm and*

$$\begin{aligned}\operatorname{ind} F_B = \operatorname{ind} F_A + \dim(&V^-(B(+\infty)),\\ &V^-(A(+\infty))) - \dim(V^-(B(-\infty)), V^-(A(-\infty))).\end{aligned}$$

Notice that, if $B(t) - A(t)$ vanishes at $-\infty$ and at $+\infty$, this result is a simple consequence of the fact that the multiplication by a path of compact operators, infinitesimal at infinity, gives a compact operator from H^1 to L^2 and from C_0^1 to C_0^0, and that a perturbation of a semi-Fredholm operator by a compact one is semi-Fredholm with the same index. In general, everything follows from Theorem 3.6.

Proof. Since F_A is semi-Fredholm, (W_A^s, W_A^u) is a semi-Fredholm pair of index $\operatorname{ind} F_A$. By Theorem 3.6, W_B^s and W_B^u are commensurable to W_A^s and W_A^u, respectively, and

$$\begin{aligned}\dim(W_B^s, W_A^s) &= \dim(V^-(B(+\infty)), V^-(A(+\infty))), \\ \dim(W_B^u, W_A^u) &= \dim(V^+(B(-\infty)), V^+(A(-\infty))).\end{aligned} \tag{42}$$

Identity (33) shows that when we replace the elements of the semi-Fredholm pair (W_A^s, W_A^u) by their commensurable spaces W_B^s and W_B^u, we obtain another semi-Fredholm pair (W_B^s, W_B^u) of index

$$\operatorname{ind}(W_A^s, W_A^u) + \dim(W_B^s, W_A^s) - \dim(W_B^u, W_A^u).$$

The thesis follows from (42). □

Theorem E is a corollary of the above result:

Corollary 6.2 *Assume that E has a splitting $E = E^- \oplus E^+$ and that the asymptotically hyperbolic path A has the form $A(t) = A_0(t) + K(t)$, where the operators $K(t)$ are compact, A_0 is piecewise continuous, E^- and E^+ are $A_0(t)$-invariant for every t, $A_0(\pm\infty)$ are hyperbolic, and*

$$V^-(A_0(\pm\infty)) = E^-, \quad V^+(A_0(\pm\infty)) = E^+.$$

Then F_A is Fredholm and its index is

$$\operatorname{ind} F_A = \dim(V^-(A(+\infty)), V^-(A(-\infty))).$$

Proof. Under these assumptions, A_0 is an asymptotically hyperbolic path, and Theorem 2.1 (i) implies that

$$W^s_{A_0} = E^-, \quad W^u_{A_0} = E^+.$$

Hence, by Theorem 5.1 (iv), F_{A_0} is an invertible operator. So Theorem 6.1 implies that F_A is a Fredholm operator of index

$$\begin{aligned}\operatorname{ind} F_A &= \dim(V^-(A(+\infty)), E^-) - \dim(V^+(A(-\infty)), E^+)\\ &= \dim(V^-(A(+\infty)), E^-) + \dim(V^-(A(-\infty)), E^-)\\ &= \dim(V^-(A(+\infty)), V^-(A(-\infty))).\end{aligned}$$ □

This corollary generalizes Example 2 to a case where both the subspaces in the spectral decomposition of $A(\pm\infty)$ are infinite dimensional. It also generalizes [AvdV99] Lemmas 15, 16 and [AM01] Theorem 3.4.

In order to state a more general result, a few preliminaries are needed. A splitting $E = E_1 \oplus E_2$ is said to be *essentially invariant* for an operator $T \in \mathcal{L}(E)$ if it is an invariant splitting for some compact perturbation of T. Equivalently, denoting by P one of the two projections associated to such a splitting, the splitting is essentially invariant for T if and only if $[T, P] := TP - PT$ is compact. The *essential commutator of* T,

$$e\mathcal{C}(T) := \{S \in \mathcal{L}(E) \mid [S, T] \in \mathcal{L}_c(E)\},$$

is the pre-image of the commutator of $[T]$ in the Calkin algebra $\mathcal{L}(E)/\mathcal{L}_c(E)$. Therefore, $e\mathcal{C}(T)$ is a closed subalgebra with identity of $\mathcal{L}(E)$, invariant with respect to compact perturbations, and $e\mathcal{C}(T) = e\mathcal{C}(S)$ whenever $S - T$ is compact.

Here is a generalization of Corollary 6.2, which says that in the case of essentially splitting paths, the stable and the unstable spaces can be replaced by $V^-(A(+\infty))$ and $V^+(A(-\infty))$ in the characterization given by Theorem 5.1.

Theorem 6.3 *Let A be an asymptotically hyperbolic path of operators on $\overline{\mathbb{R}}$. Denote by $P^-(\pm\infty)$ and $P^+(\pm\infty)$ the projectors associated to the splittings $E = V^-(A(\pm\infty)) \oplus V^+(A(\pm\infty))$. Assume that $A(t) \in e\mathcal{C}(P^-(+\infty))$ for every $t \geq 0$, and $A(t) \in e\mathcal{C}(P^-(-\infty))$ for every $t \leq 0$. Then*

(i) *F has closed range if and only if the subspace $V^-(A(+\infty)) + V^+(A(-\infty))$ is closed;*

(ii) *F is a semi-Fredholm operator if and only if $(V^-(A(+\infty)), V^+(A(-\infty)))$ is a semi-Fredholm pair. In this case*

$$\operatorname{ind} F_A = \operatorname{ind}(V^-(A(+\infty)), V^+(A(-\infty))).$$

Proof. Set

$$A_0(t) := \begin{cases} P^-(+\infty)A(t)P^-(+\infty) + P^+(+\infty)A(t)P^+(+\infty), \text{ if } t \geq 0, \\ P^-(-\infty)A(t)P^-(-\infty) + P^+(-\infty)A(t)P^+(-\infty), \text{ if } t < 0. \end{cases}$$

The asymptotically hyperbolic path A_0 is such that $A(t) - A_0(t)$ is compact for every t. Theorem 2.1 (i) implies that

$$W^s_{A_0} = V^-(A(+\infty)), \quad W^u_{A_0} = V^+(A(-\infty)),$$

so Theorem 3.6 implies that W^s_A and W^u_A are commensurable to $V^-(A(+\infty))$ and $V^+(A(-\infty))$, respectively, and

$$\dim(W^s_A, V^-(A(+\infty)) = 0, \quad \dim(W^u_A, V^+(A(-\infty)) = 0.$$

Notice that if we replace two subspaces having a closed sum by commensurable ones, we obtain two subspaces still having a close sum. Then (i) follows from Theorem 5.1 (i). Claim (ii) follows from Theorem 5.1 (v) and identity (33). □

We would like to remark that the assumptions of Corollary 6.2 or of Theorem 6.3 (ii), are far from being a necessary condition in order to have F Fredholm. Indeed, we have seen that the Fredholm property of F can be expressed by looking at the relative position of the stable and unstable spaces. These spaces drastically change when A is changed by a time reparameterization, so the Fredholm property in general is not preserved under such a transformation. The assumption of Corollary 6.2, instead, is invariant for time reparameterizations.

However, the assumption of Corollary 6.2 is somehow sharp, if one wants to look at conditions involving only the form of $A(t)$ and not its dependence on t. To see this, notice that, when $A(t) = A_0(t)$, F is invertible (the stable and unstable spaces in this case are E^- and E^+). One could think that, if the invariant spaces of $A(t)$ are not fixed, but are allowed to vary within a small angle, F should still be invertible. In this way, one could hope to replace the compact perturbations by small perturbations in Corollary 6.2. In some sense this is true, simply because the set of invertible operators is open in the operators' norm and adding a small perturbation in L^∞ to A changes F_A by a small operator. However how small the norm of the perturbation must be highly depends on the whole path A. To show this, we begin with a finite dimensional example.

Example 5. For any $\theta \in]0, \pi[$ there exists a path of invertible symmetric 3 by 3 matrices $\{A(t)\}_{t\in\overline{\mathbb{R}}}$, such that all the positive eigenspaces $\{V^+(A(t))\}_{t\in\mathbb{R}}$ are one-dimensional and make angles not larger than θ with each other, and such that $W^s_A \cap W^u_A \neq (0)$.

To exhibit an example in $\mathbb{R}^3$, fix some $\alpha \in]0, \frac{\pi}{2}[$ such that

$$\frac{1-\cos(2\alpha)}{\sin(2\alpha)} < \tan(\frac{\theta}{2}) . \tag{43}$$

Set $\mu := \frac{1}{4}\sin(2\alpha)\tan(\frac{\theta}{2})$; set

$$a = (a_1, a_2, a_3) := \left(-\frac{1+\cos\alpha}{2}, -\frac{1-\cos\alpha}{2}, \frac{1-\cos(2\alpha)}{4}\right),$$

and

$$q := (q_1, q_2, q_3) = \left(\mu\cos(\frac{\alpha}{2}),\ \mu\sin(\frac{\alpha}{2}),\ \frac{1}{4}\sin(2\alpha)\right),$$

then define the matrices

$$A_0 := \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix} \qquad Q := \begin{pmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{pmatrix} \qquad L := A_0 - Q.$$

The eigenvalues of the matrix L are

$$\lambda_1 = a_3 - 1, \quad \lambda_2 = i\sqrt{\mu^2 - a_3^2}, \quad \lambda_3 = -i\sqrt{\mu^2 - a_3^2},$$

and $\mu^2 - a_3^2 > 0$ because of (43). The vector $e_3 := (0, 0, 1)$ belongs to the the eigenspace $V(L; \{\lambda_2, \lambda_3\})$. Indeed $V(L; \{\lambda_2, \lambda_3\}) = V(L^*; \{\lambda_1\})^\perp$; but $V(L^*; \{\lambda_1\})$ is generated by the eigenvector

$$\xi^* = \left(\begin{vmatrix} -q_3 & q_2 \\ \lambda_1 - a_2 & -q_1 \end{vmatrix}, - \begin{vmatrix} \lambda_1 - a_1 & q_2 \\ q_3 & -q_1 \end{vmatrix}, \begin{vmatrix} \lambda_1 - a_1 & -q_3 \\ q_3 & \lambda_1 - a_2 \end{vmatrix} \right) \neq 0,$$

and in fact $\xi^* \cdot e_3 = \begin{vmatrix} \lambda_1 - a_1 & -q_3 \\ q_3 & \lambda_1 - a_2 \end{vmatrix} = 0$. As a consequence, $\exp(tL)e_3 \cdot e_3$ has the form $c\sin(\sqrt{\mu^2 - a_3^2}(t - t_0))$ for some c and $t_0 > 0$, whence in particular $\exp(t_0 L)e_3 \cdot e_3 = 0$ We now define a continuous path of symmetric matrices, unitary conjugate to A_0, setting

$$A(t) := \begin{cases} A_0 & \text{for } t \leq 0, \\ \exp(tQ)A_0\exp(-tQ) & \text{for } 0 \leq t \leq t_0, \\ \exp(t_0 Q)A_0\exp(-t_0 Q) & \text{for } t_0 \leq t. \end{cases}$$

Hence $V^+(A(t)) = \exp(tQ)[V^+(A_0)] = \exp(tQ)[\mathbb{R}e_3]$ for $0 \le t \le t_0$. Now, since q is the axis of the rotation $\exp(tQ)$,

$$\begin{aligned}
\operatorname{ang}(\exp(tQ)e_3,\, q) &= \operatorname{ang}(\exp(tQ)e_3, \\
\exp(tQ)q) &= \operatorname{ang}(e_3,\, q) \\
&= \arcsin\left(\frac{|e_3 \times q|}{|q|}\right) \\
&= \arcsin\left(\frac{\sqrt{q_1^2+q_2^2}}{|q|}\right) \\
&= \arctan\left(\frac{\sqrt{q_1^2+q_2^2}}{q_3}\right) \\
&= \arctan\left(\frac{4\mu}{\sin(2\alpha)}\right) = \frac{\theta}{2}.
\end{aligned}$$

Thus the angle between $V^+(A(t))$ and $V^+(A(t'))$ is never larger than θ. The solution of $u' = A(t)u$ such that $u(0) = e_3$ is

$$u(t) := \begin{cases} \exp(ta_3)e_3 & \text{for } t \le 0, \\ \exp(tQ)\exp(tL)e_3 & \text{for } 0 \le t \le t_0, \\ \exp(t_0 Q)\exp((t-t_0)A_0)\exp(t_0 L)e_3 & \text{for } t_0 \le t. \end{cases}$$

Clearly $u(t) \to 0$ as $t \to -\infty$, but also as $t \to +\infty$, because $\exp(t_0L)e_3 \in V^-(A_0)$. Therefore $e_3 = u(0) \in W^s_A \cap W^u_A$. □

Example 6. Given $\theta > 0$, there exists a continuous path A of self-adjoint invertible operators such that $V^-(A(t))$ and $V^+(A(t))$ vary within an angle of θ, and such that F_A is not semi-Fredholm.

We can couple an infinite number of copies of the previous example: on the Hilbert sum

$$E := \bigoplus_{k=0}^{\infty} \mathbb{R}^3,$$

we consider the path $A(t) = \bigoplus_{k=0}^{\infty} A_3(t)$, where A_3 is the path constructed in Example 5. It is a continuous path of self-adjoint invertible operators such that $V^-(A(t))$ and $V^+(A(t))$ vary within an angle of θ. However the stable and unstable spaces have an infinite dimensional intersection and an infinite codimensional sum, so F_A is not semi-Fredholm. □

7 The spectral flow

If the Hilbert space E is finite dimensional and A is a continuous asymptotically hyperbolic path, its spectral flow $\operatorname{sf}(A)$ can be defined as the algebraic multiplicity of the eigenvalues of $A(t)$ whose real part changes from negative to positive, minus the multiplicity of those eigenvalues whose real part changes from positive to negative. Since the end-points of A are hyperbolic,

$$\mathrm{sf}(A) = \dim V^+(A(+\infty)) - \dim V^+(A(-\infty)),$$

so Theorem A can be restated by saying that the Fredholm index of F_A coincides with minus the spectral flow of A.

When E is infinite dimensional, there is still a large class of paths for which the spectral flow can be defined. An operator $T \in \mathcal{L}(E)$ is said *essentially hyperbolic* if its essential spectrum

$$\sigma_e(T) = \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not semi-Fredholm}\}$$

does not meet the imaginary axis. Equivalently, T is essentially hyperbolic if and only if it is a compact perturbation of a hyperbolic operator, or if and only if its purely imaginary spectrum consists of finitely many isolated eigenvalues having finite multiplicity. The set of essentially hyperbolic operators will be denoted by $e\mathcal{H}(E)$. The spectral flow $\mathrm{sf}(A)$ can be defined for continuous asymptotically hyperbolic paths in $e\mathcal{H}(E)$ (see [BBW93], [FPR99], or [Phi96] for rigorous definitions and for the properties stated in this section[7]). A natural question is whether $\mathrm{ind}\, F_A$ coincides with $-\mathrm{sf}(A)$ for such paths.

When E is finite dimensional, $e\mathcal{H}(E)$ coincides with $\mathcal{L}(E)$: in particular $e\mathcal{H}(E)$ is connected and contractible. When E is infinite dimensional, still separable, $e\mathcal{H}(E)$ consists of three connected components: the first one, $e\mathcal{H}_+(E)$, consisting of the operators whose essential spectrum has positive real part, the second one, $e\mathcal{H}_-(E)$, consisting of the operators whose essential spectrum has negative real part, the last one, $e\mathcal{H}_\pm(E)$, consisting of those operators whose essential spectrum contains numbers with positive real part and numbers with negative real part. The components $e\mathcal{H}_+(E)$ and $e\mathcal{H}_-(E)$ are star-shaped with respect to the operators I and $-I$, respectively: in particular, they are contractible. We already know that, if an asymptotically hyperbolic path takes value in $e\mathcal{H}_+(E)$ or in $e\mathcal{H}_-(E)$, F_A is Fredholm of index

$$\begin{aligned} \mathrm{ind}\, F_A &= \dim V^-(A(+\infty)) - \dim V^-(A(-\infty)), \quad \text{or} \\ \mathrm{ind}\, F_A &= \mathrm{codim} V^-(A(-\infty)) - \mathrm{codim} V^-(A(+\infty)), \end{aligned}$$

respectively: indeed, this is a particular case of Example 2. It would not be hard to show that, when such a path A is also continuous, the Fredholm index of F_A coincides with minus the spectral flow of A.

[7] Actually, in all these references, only self-adjoint Fredholm operators are considered, but the extension to essentially hyperbolic operators presents no difficulties.

The topology of $e\mathcal{H}_\pm(E)$ is more complicated. Since the spectral flow is invariant with respect to homotopies which fix the end-points, it defines a homomorphism

$$\mathrm{sf} : \pi_1(e\mathcal{H}_\pm(E)) \to \mathbb{Z}.$$

An interesting fact is that such a homomorphism is an isomorphism (see the main theorem in [Phi96], pag. 464). As a consequence, the spectral flow of a path in $e\mathcal{H}_\pm(E)$ does not depend only on the end-points.

Example 6 provides us with a continuous path A of self-adjoint invertible (hence hyperbolic) operators, such that F_A is not semi-Fredholm: this shows that there are asymptotically hyperbolic continuous paths in $e\mathcal{H}_\pm(E)$ such that the Fredholm index of F_A is not defined. However, such an A has spectral flow zero and F_A is almost a Fredholm operator of index zero, meaning that there are arbitrarily small perturbations of the path A which would make F_A Fredholm of index zero. In this sense, this is not a strong counterexample to the identity $\mathrm{ind}\, F_A = -\mathrm{sf}(A)$. A more striking example is the following.

Example 7. Let $k \in \mathbb{Z}$. There exists a smooth path $A \in C^\infty(\overline{\mathbb{R}}; \mathcal{L}(E))$ such that $A(t)$ is self-adjoint and invertible for every $t \in \overline{\mathbb{R}}$, and F_A is Fredholm of index k.

Consider two orthogonal splittings into infinite dimensional spaces

$$E = V^- \oplus V^+ = W^- \oplus W^+,$$

such that (V^-, W^+) is a Fredholm pair of index k. Let T be an orthogonal operator which maps W^+ onto V^+, and W^- onto V^-. Choose a smooth path of orthogonal operators $U : \mathbb{R} \to O(E)$ such that $U(t) = I$ for $t \leq 0$ and $U(t) = T$ for $t \geq 1$ (the orthogonal group of an infinite dimensional Hilbert spaces is connected, see [Kui65]). For every $\epsilon > 0$, consider the smooth path

$$A_\epsilon(t) := U(t/\epsilon)^{-1}[P_{V^+} - P_{V^-}]U(t/\epsilon), \quad t \in \mathbb{R},$$

The operators $A_\epsilon(t)$ are self-adjoint and invertible for every $t \in \overline{\mathbb{R}}$. When ϵ converges to zero, A_ϵ converges $L^1(\mathbb{R}; \mathcal{L}(E))$ to the piecewise continuous path

$$A_0(t) := \begin{cases} P_{V^+} - P_{V^-} & \text{if } t \leq 0, \\ P_{W^+} - P_{W^-} & \text{if } t > 0. \end{cases}$$

Since $W^s_{A_0} = V^-$ and $W^u_{A_0} = W^+$, F_{A_0} is a Fredholm operator of index k. The multiplication operator by $A_\epsilon - A_0$ converges to zero in $\mathcal{L}(H^1, L^2)$, so when ϵ is small enough, F_{A_ϵ} is a Fredholm operator of index k. □

The path $A = A_\epsilon$ constructed above is such that $\sigma(A(t)) = \sigma_e(A(t)) = \{-1, 1\}$, for every $t \in \overline{\mathbb{R}}$, so it is a path in $e\mathcal{H}_\pm(E)$ with spectral flow zero.

However F_A may have every index, so the identity $\operatorname{ind} F_A = -\mathrm{sf}(A)$ does not hold. It would hold in more restricted classes, such as the class of paths of operators which are both essentially hyperbolic and satisfy the hypothesis of Theorem E (a proof for paths of operators which satisfy the hypothesis of Theorem B can be found in [AvdV99]).

8 Nonlinear consequences

Let M be a C^2 Hilbert manifold modeled on the Hilbert space E. Let ξ be a C^1 tangent vector field on M and consider the system

$$u'(t) = \xi(u(t)), \quad u(t) \in M. \tag{44}$$

Let $\phi : \Omega \to M$, $\Omega \subset \mathbb{R} \times M$, be local flow defined by the above system, i.e. the maximally defined solution of

$$\begin{cases} \frac{\partial}{\partial t}\phi(t,p) = \xi(\phi(t,p)), \\ \phi(0,p) = p. \end{cases}$$

If x is an equilibrium point for ξ, that is $\xi(x) = 0$, a local representation of the Jacobian $\nabla\xi(x)$ is well defined up to conjugacy, meaning that, if $\varphi : (E,0) \to (U,x)$ is a local chart, the pull-back of the vector field

$$\xi_\varphi(y) := (d\varphi(y))^{-1}\xi(\varphi(y)), \quad y \in E,$$

is a C^1 map from E to E and its differential in 0 changes by a conjugacy when we choose a different chart. Therefore it makes sense to define a *hyperbolic equilibrium point* x as an equilibrium point such that $\nabla\xi(x)$ is a hyperbolic operator. Recall that a hyperbolic equilibrium point x has a stable and an unstable manifold

$$W^s(x) = \left\{ p \in M \mid \lim_{t \to +\infty} \phi(t,p) = x \right\},$$
$$W^u(x) = \left\{ p \in M \mid \lim_{t \to -\infty} \phi(t,p) = x \right\}.$$

These are immersed C^1 submanifolds of M, invariant for the local flow and such that $T_xW^s(x) \oplus T_xW^u(x) = E$. In the case of a gradient flow, they are actually embedded submanifolds.

Let $p \in W^s(x)$ and set $u(t) := \phi(t,p)$. We can choose a trivialization $U : \overline{\mathbb{R}}^+ \times E \to u^*(TM)$ and identify E with T_pM in such a way that $U(0) = I_E$. We may read the linearized flow on E as

$$X(t) := U(t)^{-1}d\phi(t,u(0)), \tag{45}$$

for $t \in \mathbb{R}^+$, where d denotes differentiation with respect to the variable on M. The function $X : \mathbb{R}^+ \to \mathcal{L}(E)$ is the solution of an asymptotically hyperbolic linear system on $\mathbb{R}^+$ in E

$$\begin{cases} X'(t) = A(t)X(t), \\ X(0) = I, \end{cases} \tag{46}$$

where $A(+\infty)$ is conjugated to a local representation of $\nabla\xi(x)$. Choosing a different trivialization, X is changed to $\tilde{X}(t) = G(t)X(t)$, for some $G \in C^1(\overline{\mathbb{R}}^+; GL(E))$. Correspondingly, A and F_A^+ are changed to

$$\tilde{A} = -G'G^{-1} + GAG^{-1}, \quad F_{\tilde{A}}^+ = G \circ F_A^+ \circ G^{-1}.$$

Thanks to the identification $E = T_pM$, we have that

$$W_A^s = T_pW^s(x).$$

Then Theorem 2.1 (iii) implies the following convergence result (where the definition of convergence of linear subspaces in TM is reduced via local charts to the convergence in $\mathcal{G}(E)$).

Corollary 8.1 *Let $p \in W^s(x)$ and let $V \subset T_pM$ be a closed subspace such that $T_pM = V \oplus T_pW^s(x)$. Then the path of subspaces $d\phi(t,p)V$ converges to $T_xW^u(x)$ for $t \to +\infty$.*

Now let x and y be two hyperbolic equilibrium points such that $W^u(x)$ and $W^s(y)$ have a non-empty intersection. In other words, there exists a solution u of (44) such that

$$\lim_{t\to-\infty} u(t) = x, \quad \lim_{t\to+\infty} u(t) = y.$$

As before, choosing a trivialization $U : \overline{\mathbb{R}} \times E \to u^*(TM)$ and identifying E with $T_{u(0)}M$ in such a way that $U(0) = I_E$, the linearized flow X is defined for all $t \in \mathbb{R}$ as in (45), and it solves (46), which is now an asymptotically hyperbolic linear system on $\mathbb{R}$. The operators $A(-\infty)$ and $A(+\infty)$ are conjugated to local representations of $\nabla\xi(x)$ and $\nabla\xi(y)$, and by the identification $E = T_{u(0)}M$,

$$W_A^u = T_{u(0)}W^u(x), \quad W_A^s = T_{u(0)}W^s(y). \tag{47}$$

Recall that two submanifolds $N, O \subset M$ are said transversal at $p \in N \cap O$ if $T_pN + T_pO = T_pM$. Similarly, two submanifolds $N, O \subset M$ are said to have a Fredholm intersection at $p \in N \cap O$ if (T_pN, T_pO) is a Fredholm pair of linear subspaces. Theorem 5.1 and (47) have the following consequence.

Corollary 8.2 *The following facts hold:*

(i) F_A *has closed range if and only if* $T_{u(t)}W^u(x) + T_{u(t)}W^s(y)$ *is closed for some* $t \in \mathbb{R}$ *(hence for every* $t \in \mathbb{R}$*);*

(ii) $\dim\ker F_A = \dim T_{u(t)}W^u(x) \cap T_{u(t)}W^s(y)$ *for some* $t \in \mathbb{R}$ *(hence for every* $t \in \mathbb{R}$*);*

(iii) F_A *is onto if and only if* $W^u(x)$ *and* $W^s(y)$ *are transversal at* $u(t)$*, for some* $t \in \mathbb{R}$ *(hence for every* $t \in \mathbb{R}$*);*

(iv) F_A *is Fredholm if and only if* $W^u(x)$ *and* $W^s(y)$ *have a Fredholm intersection at* $u(t)$*, for some* $t \in \mathbb{R}$ *(hence for every* $t \in \mathbb{R}$*).*

In particular, if Z is a connected component of $W^u(x) \cap W^s(y)$ such that $W^u(x)$ and $W^s(y)$ are transversal and have a Fredholm intersection at every $z \in Z$, then Z is an immersed manifold of dimension

$$\dim Z = \operatorname{ind} F_A = \dim\ker F_A,$$

where u is any solution of (44) belonging to Z.

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