On Product Structures in Floer Homology of Cotangent Bundles

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Abstract In an earlier paper we have shown that the pair-of-pants product on the Floer homology of the cotangent bundle of an oriented compact manifold Q corresponds to the Chas-Sullivan loop product on the singular homology of the free loop space of Q. We now give chain level constructions of further product structures in Floer homology, corresponding to the cup product on the homology of any path space, and to the Goresky-Hingston product on the relative cohomology of the free loop space modulo constant loops. Moreover, we give a explicit construction for the inverse isomorphism between Floer homology and loop space homology.

1 Introduction and Main Results

Let Q be a closed, smooth manifold, and let $H\colon \mathbb{T} \times T^*Q \to \mathbb{R}$ be a time-periodic smooth Hamiltonian on its cotangent bundle. The cotangent bundle is viewed as a symplectic manifold with the canonical symplectic structure $\omega = d\lambda$, where λ is the Liouville one-form, whose expression in local coordinates is $\lambda = \sum p_j\, dq_j$. The corresponding Liouville vector field Y, which is defined by $\omega(Y,\cdot) = \lambda$, has the local expression $Y = \sum p_j \frac{\partial}{\partial p_j}$.

We assume that H is of *quadratic type*, i.e., it satisfies the conditions

(H1) $dH(t,q,p)[Y] - H(t,q,p) \geq h_0|p|^2 - h_1$,
(H2) $|\nabla_q H(t,q,p)| \leq h_2(1+|p|^2)$, $|\nabla_p H(t,q,p)| \leq h_2(1+|p|)$,

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for every (t, q, p), for some positive constants h_0, h_1 and h_2. Here the norm $|\cdot|$ and the covariant derivative ∇ are induced by some fixed metric on Q, but the conditions are actually independent on the choice of this metric. Condition (H1) essentially says that H grows at least quadratically in p on each fiber of T^*Q, and that it is radially convex for $|p|$ large. Condition (H2) implies that H grows at most quadratically in p on each fiber. Such Hamiltonians include in particular physical Hamiltonians with magnetic fields,

$$H(t, q, p) = \frac{1}{2}|p - A(t, q)|^2 + V(t, q),$$

where $A(t, \cdot)$ is a one-form and $V(t, \cdot)$ is a smooth function on Q, both depending 1-periodically on $t \in \mathbb{R}$. Generically, the Hamiltonian system

$$\dot{x}(t) = X_H(t, x(t)), \tag{1}$$

for the Hamiltonian vector field X_H defined by $\omega(X_H, \cdot) = -dH$, has a discrete set $\mathcal{P}_1(H)$ of 1-periodic orbits. In fact, the following non-degeneracy condition holds for a generic set of H:

(H0) The time-1-map of the flow Φ_H^t generated by X_H has only non-degenerate fixed points, i.e. $D\Phi_H^1(x)$ has no eigenvalue 1 for any fixed point x of Φ_H^1.

The free abelian group $F_*(H)$ generated by the elements $x \in \mathcal{P}_1(H)$, which by $x \mapsto x(0)$ correspond exactly to the fixed points of Φ_H^1, graded by their Conley-Zehnder index $\mu_{cz}(x)$, supports a chain complex, the *Floer complex* $(F_*(H), \partial)$. The boundary operator ∂ is defined by an algebraic count of the maps u from the cylinder $\mathbb{R} \times \mathbb{T}$ to T^*Q, solving the Cauchy-Riemann type equation

$$\partial_s u(s,t) + J(t, u(s,t))\big(\partial_t u(s,t) - X_H(t, u(s,t))\big) = 0, \quad \text{for all } (s,t) \in \mathbb{R}\times\mathbb{T}, \tag{2}$$

in short $\overline{\partial}_{J,H} u = 0$, and converging to two 1-periodic orbits x, y for $s \to -\infty$ and $s \to \infty$. Here, J is an almost-complex structure on T^*Q calibrated by the symplectic structure in the sense that $\omega(J\cdot, \cdot)$ gives a positive definite and symmetric form.

The equation (2) can be seen as the negative L^2-gradient equation for the Hamiltonian action functional

$$\mathcal{A}_H : C^\infty(\mathbb{T}, T^*Q) \to \mathbb{R}, \quad \mathcal{A}(x) = \int_{\mathbb{T}} \big(x^*\lambda - H(t, x(t))\, dt\big). \tag{3}$$

The almost complex structure J is chosen in a generic way, so that for every pair (x, y) of 1-periodic orbits, the space of solutions of (2) with asymptotics x and y is the zero-set of a Fredholm section of a Banach bundle which is transverse to the zero-section, and in particular it is a finite dimensional manifold.

This construction is due to A. Floer (see e.g. [13–16]) in the case of a closed symplectic manifold (M,ω), in order to prove a conjecture of V. Arnold on the number of periodic Hamiltonian orbits. The extension to non-compact symplectic manifolds, such as the cotangent bundles we consider here, requires suitable conditions on the asymptotic behavior of both the Hamiltonian H and the almost complex structure J. A possibility is to assume that H satisfies the asymptotic quadratic growth conditions (H1) and (H2) and that J is C^0-close to the Levi-Civita almost complex structure on T^*Q which is induced by the Riemannian metric on Q (see [2]). Another possibility is to consider Hamiltonians which are superlinear functions of $|p|$ for $|p|$ large and almost complex structures which are of contact type with respect to λ (see e.g. [25]). Here we stick to the former set of conditions, although everything we say could also be adapted to the latter one.

The Floer complex obviously depends on the Hamiltonian H, but its homology often does not, so it makes sense to call this homology the *Floer homology* of the underlying symplectic manifold (M,ω), and to denote it by $HF_*(M)$. The Floer homology of a compact symplectic manifold M without boundary is isomorphic to the singular homology of M, as proved by A. Floer for special classes of symplectic manifolds, and later extended to larger and larger classes by several authors (the general case requiring special coefficient rings, see [18, 20, 21]).

Unlike the compact case, the Floer homology of a cotangent bundle T^*Q for Hamiltonians of quadratic type is a truly infinite-dimensional homology theory, being isomorphic to the singular homology of the free loop space ΛQ of Q. This fact was proved by C. Viterbo (see [26]) using a generating functions approach, later by D. Salamon and J. Weber using the heat flow for curves on a Riemannian manifold (see [23]) and then by the authors in [2].

In particular, our proof reduces the general case to the case of a Hamiltonian which is uniformly convex in the momenta, meaning that it satisfies the condition

(H3) $\nabla_{pp} H(t,q,p) \geq h_3 I$, for some $h_3 > 0$,

and for such a Hamiltonian it constructs an explicit isomorphism between the Floer complex $(F_*(H),\partial)$ and the *Morse complex* $(M_*(\mathbb{S}_L),\partial)$ of the action functional

$$\mathbb{S}_L(\gamma) = \int_{\mathbb{T}} L(t,\gamma(t),\dot\gamma(t))\,dt, \quad \gamma \in W^{1,2}(\mathbb{T},Q),$$

associated to the Lagrangian $L\colon \mathbb{T}\times TQ \to \mathbb{R}$ which is the Fenchel dual of H,

$$L(t,q,v) = \max_{p\in T_q^*Q} \big(\langle p,v\rangle - H(t,q,p)\big),$$

a Lagrangian of Tonelli type. The latter complex is the standard chain complex associated to the Lagrangian action functional $\mathbb{S}_L$. The domain of such a functional is the infinite dimensional Hilbert manifold $W^{1,2}(\mathbb{T},Q)$ consisting of closed loops of Sobolev class $W^{1,2}$ on Q. The functional $\mathbb{S}_L$ is bounded from below, it has non-degenerate critical points a with finite Morse index $i(a)$, it satisfies the Palais-Smale condition, and, although in general it is not of class C^2, it admits a smooth Morse-Smale pseudo-gradient flow (see [3]). The construction of the Morse complex in this infinite-dimensional setting and the proof that its homology is isomorphic to the singular homology of the ambient manifold are described in [1]. The isomorphism goes from the Morse to the Floer complex and is obtained by coupling the Cauchy-Riemann type equation on the half-cylinder $\mathbb{R}^+ \times \mathbb{T}$ with the pseudo-gradient flow equation for the Lagrangian action. We call this the *hybrid method*.

Since the space $W^{1,2}(\mathbb{T}, Q)$ is homotopy equivalent to ΛQ, we get the asserted isomorphism

$$\Phi^\Lambda : H_*(\Lambda Q) \xrightarrow{\cong} HF_*(T^*Q).$$

This isomorphism result was generalized in [6] for more general path spaces than the free loop space. In fact, given a closed submanifold $R \subset Q \times Q$, we can consider the path space

$$\Omega_R Q = \left\{ c \in W^{1,2}([0,1], Q) \mid (c(0), c(1)) \in R \right\}.$$

In particular, the choice $R = \triangle$, the diagonal in $Q \times Q$, produces the free loop space ΛQ, while the based loop space $\Omega_{q_o} Q$ is given by the choice $R = \{(q_o, q_o)\}$.

Given a submanifold $S \subset Q$ we have its associated conormal bundle

$$N^*S = \left\{ (q,p) \in T^*Q \mid q \in S,\ p_{|T_q S} \equiv 0 \right\},$$

which is a Lagrangian submanifold of $(T^*Q, d\lambda)$ on which the Liouville one-form λ vanishes identically. The non-degeneracy assumption for a Hamiltonian $H : [0,1] \times T^*Q \to \mathbb{R}$ is now that the Lagrangian submanifold

$$G_H = \left\{ \left(\alpha, \mathsf{C}\phi_H^1(\alpha)\right) \mid \alpha \in T^*Q \right\} \subset T^*Q \times T^*Q = T^*(Q \times Q)$$

should have a transverse intersection with N^*R in $T^*(Q \times Q)$, where $\mathsf{C} : (q,p) \mapsto (q, -p)$ is the anti-symplectic conjugation on T^*Q.

In [6] it was shown that we have an associated Floer homology HF_*^R, with the chain complex $F_*^R(H)$ generated by the Hamiltonian paths

$$\mathcal{P}_R(H) = \left\{ x : [0,1] \to T^*Q \mid \dot{x}(t) = X_H(t, x(t)),\ \left(x(0), \mathsf{C}x(1)\right) \in N^*R \right\}, \tag{4}$$

and the boundary operator $\partial : F_*^R \to F_{*-1}^R$ defined by counting the Floer trajectories

$$u : \mathbb{R} \times [0,1] \to T^*Q, \quad \overline{\partial}_{J,H} u = 0, \quad \left(u(s,0), \mathsf{C}u(s,1)\right) \in N^*R \quad \forall s \in \mathbb{R},$$

converging to x and $y \in \mathcal{P}_R(H)$ as $s \to -\infty$ and $s \to \infty$. Note that this is a well-posed Fredholm problem because N^*R is a Lagrangian submanifold of $T^*(Q \times Q)$. Compactness and energy estimates hold because $(\lambda \oplus \lambda)_{|N^*R} \equiv 0$.

Theorem 1.1. *[6] We have $HF_*^R(T^*Q) \cong H_*(\Omega_R Q)$ via an explicit chain complex isomorphism*

$$\Phi^R\colon M_*(\mathbb{S}_L|_{\Omega_R Q}) \xrightarrow{\cong} F_*^R(H)$$

where $L : [0,1]\times TQ \to \mathbb{R}$ is the Lagrangian which is Fenchel dual to the quadratic type Hamiltonian H.

The first aim of this paper is to give an explicit chain level construction of a chain complex homomorphism

$$\Psi^R\colon F_*^R(H) \to M_*(\mathbb{S}_L|_{\Omega_R Q})$$

which might not be a chain complex isomorphism, but which induces an isomorphism

$$\Psi_*^R\colon HF_*^R(H) \xrightarrow{\cong} HM_*(\mathbb{S}_L|_{\Omega_R Q})$$

such that $\Psi_*^R = \left(\Phi_*^R\right)^{-1}$. Such a chain map brings methodical advantages when comparing the ring structures on the Floer and on the topological side, as we are going to show.

An important structure in Floer homology is its canonical ring structure, the so-called *pair-of-pants product* in the case of the free loop space (see [24]), or triangle product in the case of the path space with endpoints on Lagrangian submanifolds. Already in the case of a closed symplectic $2n$-dimensional manifold (M,ω), the pair-of-pants product

$$m_\triangle\colon HF_*(M) \otimes HF_*(M) \to HF_{*-n}(M)$$

encodes a truly symplectic invariant. While $HF_*(M)$ as an abelian group is isomorphic to the ordinary singular homology of M, the pair-of-pants product in general deviates from the expected intersection product (note that the grading of $m_\triangle$ becomes consistent with that of the intersection product by the grading shift in the isomorphism $HF_*(M) \cong H_{*+n}(M)$). In fact, as shown in [22], Floer homology with the pair-of-pants product is ring isomorphic to the quantum homology of $QH_*(M,\omega)$ of (M,ω), a deformation of the intersection ring structure due to the presence of pseudoholomorphic spheres.

In the context of cotangent bundles, such a deformation by pseudoholomorphic spheres cannot occur, since they simply cannot exist for the exact symplectic structure $\omega = d\lambda$. But the question remains, what the pair-of-pants ring structure corresponds to in view of the isomorphism $HF_*(T^*Q) = HF_*^\Delta(H) \cong H_*(\Lambda Q)$. In [5], we finally give the proof that the same isomorphism Φ^Δ intertwines $m_\triangle$ with the Chas-Sullivan loop product (see [9]), provided that we consider closed and oriented smooth manifolds Q.

For the definition of the pair-of-pants product on chain level

$$m_\triangle \colon F_*^\triangle(H) \otimes F_*^\triangle(H) \to F_{*-n}^\triangle(H^{(2)}),$$

in [5] we use as a model for the domain surface the branched 2:1-covering of the standard cylinder, a smooth pair-of-pants surface with two cylindrical entrances and one cylindrical exit and a conformal structure globally given in the cylindrical coordinates as $s+it$. Note that, for precise energy estimates, we use the Hamiltonian $H^{(2)}(t,q,p) = 2H(2t,q,p)$ whose 1-periodic orbits equal the 2-periodic ones for H. Equivalently, we define $m_\triangle$ by counting the solutions of the following problem

$$\begin{aligned} &u = (u_1,u_2)\colon \mathbb{R}\times[0,1] \to T^*(Q\times Q),\ \bar\partial_{J,H} u_i = 0,\ i = 1,2, \\ &\big(u_1(s,0), \mathsf{C}u_1(s,1), u_2(s,0), \mathsf{C}u_2(s,1)\big) \in \begin{cases} N^*(\triangle_{12}\times\triangle_{34}),\ s\le 0, \\ N^*(\triangle_{14}\times\triangle_{23}),\ s \ge 0, \end{cases} \end{aligned} \tag{5}$$

with asymptotics $(x,y) \in \mathcal{P}_1(H)\times\mathcal{P}_1(H)$ for $s \to -\infty$ and $z \in \mathcal{P}_2(H)$ for $s \to \infty$ (see Fig. 1). Here

$$\begin{aligned} \triangle_{12}\times\triangle_{34} &= \big\{ (q,q,q',q') \,|\, q,q' \in Q \big\}, \\ \triangle_{14}\times\triangle_{23} &= \big\{ (q,q',q',q) \,|\, q,q' \in Q \big\}. \end{aligned} \tag{6}$$

Similarly, when $R = \{(q_0,q_0)\}$ we have the triangle product

$$m_{\{(q_0,q_0)\}} : HF_*^{\{(q_0,q_0)\}}(H) \otimes HF_*^{\{(q_0,q_0)\}}(H) \to HF_*^{\{(q_0,q_0)\}}(H^{(2)}),$$

and [5] contains the proof of the following:

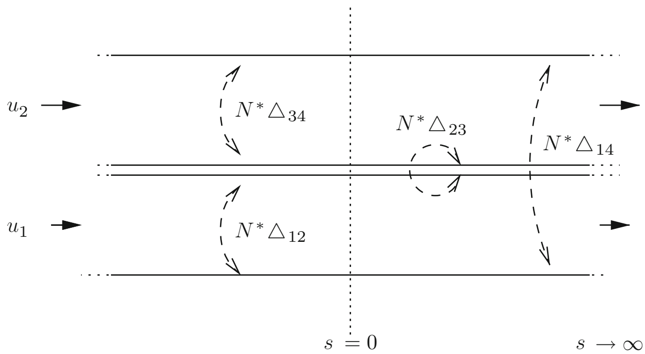

Fig. 1 The pair-of-pants product

Theorem 1.2. *The chain complex isomorphisms* $\Phi^R\colon M_*(\mathbb{S}_L|_{\Omega_R Q}) \to F_*^R(H)$, *for* $R = \Delta$ *or* $R = \{(q_0, q_0)\}$, *induces ring isomorphisms*

$$(H_*(\Lambda Q), \circ) \cong (HF_*^{\Delta}, m_{\Delta}), \quad (H_*(\Omega_{q_o} Q), \#) \cong (HF_*^{\{(q_0,q_0)\}}, m_{\{(q_0,q_0)\}}),$$

for the Chas-Sullivan product $\circ$ *on the singular homology of the free loop space and the Pontrjagin product* # *on the singular homology of the based loop space.*

If we view the submanifold $R \subset Q \times Q$ as a correspondence, these products have natural generalizations in terms of composition of correspondences. In fact, given two correspondences $R_1, R_2 \subset Q \times Q$, their composition is defined as $R_2 \circ R_1 = \pi_{13}\big((R_1 \times Q) \cap (Q \times R_2)\big)$, where $\pi_{13}\colon Q \times Q \times Q \to Q \times Q$ is the projection on the first and third coordinate. We actually have $R \circ R = R$ both for the free loop case $R = \Delta$ and for the based loop case $R = \{(q_o, q_o)\}$. When $R_1 \times Q$ and $Q \times R_2$ intersect cleanly in Q^3, and the restriction of π_{13} to such an intersection is regular, meaning that the kernel of its differential has constant dimension, then R_1 and R_2 are said to be smoothly composable. In this case, $R_2 \circ R_1$ is a closed submanifold of $Q \times Q$, so the Floer homology $HF_*^{R_2 \circ R_1}(H)$ is still defined.

One can show that the pair-of-pants product m_{Δ} on HF_*^{Δ} and the triangle product $m_{\{(q_0,q_0)\}}$ on $HF_*^{\{(q_0,q_0)\}}$ can be unified in terms of a binary operation

$$m_{R_1,R_2}\colon HF_*^{R_1} \otimes HF_*^{R_2} \to HF_{*-d(R_1,R_2)}^{R_2 \circ R_1}$$

for composable correspondences. In fact, in (5) we have to replace $\triangle_{12} \times \triangle_{34}$ for $s \le 0$ by $R_1 \times R_2$, and $\triangle_{14} \times \triangle_{23}$ for $s \ge 0$ by $(R_2 \circ R_1) \times \triangle_{23}$. Depending on the correspondences R_1 and R_2, there is a degree shift $d(R_1, R_2)$, which equals the codimension of the clean intersection $(R_1 \times R_2) \cap (Q \times \Delta \times Q)$ in $R_1 \times R_2$.

In general, m_{R_1,R_2} is isomorphic to a binary operator

$$H_*\big(\Omega_{R_1} Q\big) \otimes H_*\big(\Omega_{R_2} Q\big) \to H_{*-d(R_1,R_2)}\big(\Omega_{R_2 \circ R_1} Q\big),$$

generalizing the loop product. Such a binary operator is defined as the composition

$$\begin{aligned} H_j\big(\Omega_{R_1} Q\big) \otimes H_k\big(\Omega_{R_2} Q\big) &\xrightarrow{\times} H_{j+k}\big(\Omega_{R_1} Q \times \Omega_{R_2} Q\big) \\ &= H_{j+k}\big(\Omega_{R_1 \times R_2} Q \times Q\big) \to \\ &\xrightarrow{i_!} H_{j+k-d}\big(\Omega_{(R_1 \times R_2) \cap (Q \times \Delta \times Q)} Q \times Q\big) \longrightarrow H_{j+k-d}\big(\Omega_{R_2 \circ R_1} Q\big), \end{aligned}$$

where $\times$ is the exterior product, $i_!$ is the Umkehr morphism induced by the d-co-dimensional and co-oriented inclusion

$$i : \Omega_{(R_1 \times R_2) \cap (Q \times \Delta \times Q)} Q \times Q \hookrightarrow \Omega_{R_1 \times R_2} Q \times Q,$$

and the last homomorphism is induced by the concatenation map.

In this paper, we want to emphasize the general rule that Floer homology on cotangent bundles should be able to remodel any known algebro-topological structure in classical (co-)homology of loop spaces of closed, oriented manifolds. In fact, there should always be an independent chain level construction which, under the isomorphism Φ, is isomorphic to a corresponding structure on the classical side. This has been carried out successfully with the loop product and the Pontrjagin product in [5], where in fact, for the loop product, it was the pair-of-pants product which had been considered first, whereas the loop product had for whatever reason essentially eluded the topologists' attention until [9].

In the present paper we want to address in the same light two more product structures on the classical side. One is the cup-product on cohomology, which can be equivalently seen as a coproduct on the homology of $\Omega_R Q$,

$$\cup\colon H_*(\Omega_R Q) \to H_*(\Omega_R Q) \otimes H_*(\Omega_R Q)\,.$$

We give a Floer-theoretical construction of such a product, and we prove the following:

Theorem 1.3. *Given a generic triple of quadratic type Hamiltonians, we have a chain level operation* $u\colon F_*^R(H_1) \to F_*^R(H_2) \otimes F_*^R(H_3)$ *which induces a coproduct* $u_*\colon HF_*^R \to HF_*^R \otimes HF_*^R$ *isomorphic to the cup-coproduct on* $H_*(\Omega_R Q)$ *via the isomorphism* Φ_*^R.

An interesting question is whether the coalgebra structure u_* on HF_*^R can be seen to be an algebra homomorphism $(HF_*^R, m^R) \to (HF_*^R \otimes HF_*^R, m_R \otimes m_R)$, or equivalently, whether m_R is a coalgebra morphism for u_*. In other words, this is the question of whether (HF_*^R, m_R, u_*) carries a Hopf algebra structure, which for the based loop space homology $(H_*(\Omega Q), \#, \cup)$ is classically known to hold. Clearly, the fact that the isomorphism Φ^Ω intertwines # with $m_{\{(q_0,q_0)\}}$ and $\cup$ with u_* (Theorems 1.2 and 1.3) implies that the Hopf algebra structure also exists on the Floer side for $R = (q_o, q_o)$. In fact, this Hopf algebra property can be verified directly on chain level on the Floer side for the based loop space version. For general R with $R \circ R = R$, this Hopf algebra property cannot hold already for dimensional reasons, e.g. for the free loop space version $R = \triangle$.

The other structure we are interested in is a coproduct derived from the obvious pair-of-pants type coproduct with one entrance and two exits (see [11]). This coproduct, however, is essentially trivial, but it gives rise to a secondary coproduct on homology of loop space relative to the constant loops,

$$\square\colon H_*(\Lambda Q, Q) \to \big(H_*(\Lambda Q, Q) \otimes H_*(\Lambda Q, Q)\big)_{*-n+1}\,.$$

This coproduct has been constructed by M. Goresky and N. Hingston in [19], and computed for interesting examples such as spheres.

Given the special Hamiltonian $\frac{1}{2}|p|^2$ with a generic and small potential perturbations $V(t,q)$ we can consider Floer cohomology filtered by the action, $F^*_{\geq a}(H)$.

On the level of cohomology we can perform a limit for the perturbation $V \to 0$, and we have the following:

Theorem 1.4. *For every action values $a, b > 0$, Floer cohomology comes equipped with a product operation*

$$\tilde{w}\colon HF^*_{\geq a}(\tfrac{1}{2}|p|^2) \otimes HF^*_{\geq b}(\tfrac{1}{2}|p|^2) \;\to\; HF^{*+n-1}_{\geq a+b}(\tfrac{1}{2}|p|^2).$$

When the positive numbers a, b are small enough, the isomorphism Φ^* induces a ring isomorphism from $(HF^*_{>0}, \tilde{w})$ to $(H^*(\Lambda Q, Q), \Box)$.

In fact, it is possible to replace $\frac{1}{2}|p|^2$ by any superlinear $c|p|^{1+\delta}$, $\delta > 0$. This is not of quadratic type and requires a somewhat different argument for the C^0-estimates of the moduli spaces involved. In this paper, we give an explicit construction of $\tilde{w}$. The proof of the equivalence with $\Box$ will be given elsewhere.

2 The Inverse Isomorphism

Let us recall the construction of the isomorphism from $H_*(\Omega_R Q)$ to $HF^R_*(T^*Q)$ from [2] and [6]. When the Hamiltonian $H \in C^\infty([0,1] \times T^*Q)$ satisfies (H1), (H2) and (H3), its Fenchel dual Lagrangian $L \in C^\infty([0,1] \times TQ)$ is well-defined and satisfies the analogous quadratic growth and strict convexity assumptions. We denote by $\mathbb{S}^R_L$ the restriction of the Lagrangian action functional

$$\mathbb{S}_L(\gamma) = \int_0^1 L(t, \gamma, \dot{\gamma})\, dt,$$

to the path space $\Omega_R Q$. Here $\Omega_R Q$ carries a $W^{1,2}$-Hilbert manifold structure, $\mathbb{S}^R_L$ is of class $C^{1,1}$ on $\Omega_R Q$ and it is twice Gateaux-differentiable. The fact that the Hamiltonian H is non-degenerate with respect to the correspondence R implies also the non-degeneracy of all critical points of $\mathbb{S}^R_L$. This fact allows to construct a smooth negative pseudo-gradient Morse vector field for $\mathbb{S}^R_L$, see [3]. We denote by $M_*(\mathbb{S}^R_L)$ the chain complex generated by the critical points $a \in \operatorname{Crit} \mathbb{S}^R_L$, graded by the non-negative Morse index $i(a)$, with boundary operator $\partial\colon M_*(\mathbb{S}^R_L) \to M_{*-1}(\mathbb{S}^R_L)$ defined by algebraically counting the unparametrized connecting trajectories for the generically chosen negative pseudo-gradient vector field for $\mathbb{S}^R_L$. A result from [1] shows that $H_*(M_*(\mathbb{S}^R_L), \partial) \cong H_*(\Omega_R Q)$ in a natural way, i.e. compatible with the continuation isomorphism $H_*(M_*(\mathbb{S}^R_L), \partial) \cong H_*(M_*(\mathbb{S}^R_{L'}), \partial)$ for homotopies of the Lagrangian.

In [2] and generalized for the path spaces $\Omega_R Q$ in [6], a chain complex isomorphism

$$\Phi^R\colon \big(M_*(\mathbb{S}^R_L), \partial\big) \xrightarrow{\cong} \big(F^R_*(H), \partial\big)$$

was constructed explicitly building on the Legendre-Fenchel duality of H and L. Given generators $x \in \mathcal{P}_R(H)$, $a \in \mathrm{Crit}(\mathbb{S}_L^R)$, we have the moduli space of hybrid type trajectories

$$\mathcal{M}_{a;x} = \big\{ u\colon [0,\infty) \times [0,1] \to T^*Q \,\big|\, \bar\partial_{J,H} u = 0,\ u(+\infty) = x,$$
$$\big(u(s,0), \mathsf{C}u(s,1)\big) \in N^*R, (\pi \circ u)(0,\cdot) \in W^u(\mathbb{S}_L^R; a)\big\}, \qquad (7)$$

where $W^u(\mathbb{S}_L^R; a)$ denotes the unstable manifold of a for the negative pseudo-gradient flow of $\mathbb{S}_L^R$. For generic choices of J and of the pseudo-gradient vector field, $\mathcal{M}_{a;x}$ is a manifold of dimension $i(a) - \mu^R(x)$, where $\mu^R(x)$ is the Maslov-type index of x as a solution of the non-local Lagrangian boundary value problem (4) (see [6] for the precise definition). Assuming arbitrary orientations for all unstable manifolds $W^u(\mathbb{S}_L^R; a)$ and using the concept of coherent orientation for Floer homology according to [17], we show in [2] that all $\mathcal{M}_{a;x}$ are orientable in a coherent way, that is, compatible with the splitting-off of boundary trajectories on either side. The compactness proof for this moduli space follows from the energy estimate for $u \in \mathcal{M}_{a;x}$

$$\mathbb{S}_L(a) \ge \mathbb{S}_L\big((\pi \circ u)(0)\big) \ge \mathcal{A}_H(u(0,\cdot)) \ge \mathcal{A}_H(x),$$

with equality if and only if $\pi \circ x = a$ and u is constant in s with $\pi(u(s,\cdot)) = a$, in particular $\#\mathcal{M}_{\pi(x);x} = 1$. The central estimate is an immediate consequence of the Fenchel-Legendre duality between L and H.

As a consequence from the identification of the generating sets, consistent even with index and critical value

$$\pi\colon \mathcal{P}_R(H) \xrightarrow{\cong} \mathrm{Crit}\, \mathbb{S}_L^R, \quad i(\pi(x)) = \mu^R(x),\ \mathbb{S}_L(\pi(x)) = \mathcal{A}_H(x),$$

the chain morphism

$$\Phi^R a = \sum_{\substack{x \in \mathcal{P}_R(H) \\ \mathcal{A}_H(x) \le \mathbb{S}_L(a)}} \big(\#_{\mathsf{alg}} \mathcal{M}_{a;x}\big) \cdot x,$$

gives a chain complex isomorphism, as it is representable by a semi-infinite triangular matrix with ± 1 on the diagonal.

We now give an equally explicit chain level construction of a chain morphism

$$\Psi^R\colon F_*^R(H) \to M_*(\mathbb{S}_L^R)$$

such that at the homology level $\Psi_*^R = (\Phi_*^R)^{-1}$. Here, we cannot give an argument why the given Ψ^R should already be a chain complex isomorphism, certainly not necessarily equal to $(\Phi^R)^{-1}$. However, the concrete form of Ψ^R allows for simpler proofs of ring isomorphism properties of Φ_*^R, compared with the construction from [5].

Let us consider the moduli space for $x \in \mathcal{P}_R(H)$,

$$\begin{aligned}\mathcal{M}_x^- = \big\{u\colon (-\infty,0]\times[0,1] \to T^*Q \,\big|\, \bar\partial_{J,H} u = 0,\ u(-\infty) = x,\\ u(0,\cdot) \in 0_Q,\ \big(u(s,0), \mathsf{C}u(s,1)\big) \in N^*R\big\},\end{aligned} \tag{8}$$

where 0_Q denotes the zero-section of T^*Q. For generic J, this is a smooth manifold of dimension $\mu^R(x)$, compact modulo splitting-off Floer trajectories at $-\infty$, in particular C^∞_{loc}-compact. Hence, we find an upper bound $c = c(x)$ depending on x for the Lagrangian action of the path $(\pi\circ u)(0,\cdot) \in \Omega_R Q$,

$$\mathbb{S}_L\big((\pi\circ u)(0,\cdot)\big) \le c(x) \quad \text{for all } u \in \mathcal{M}_x^- .$$

Given $x \in \mathcal{P}_R(H)$, $a \in \mathrm{Crit}\,\mathbb{S}_L^R$, we now set

$$\mathcal{M}_{x;a} = \big\{u \in \mathcal{M}_x^- \,\big|\, (\pi\circ u)(0) \in W^s(\mathbb{S}_L^R; a)\big\},$$

where $W^s(\mathbb{S}_L^R;a)$ denotes the stable manifold of a. Provided that $x \not\subset 0_Q$ or $\pi\circ x \ne a$ if $x \subset 0_Q$ (Fig. 2), we find for generic J and pseudo-gradient vector field for $\mathbb{S}_L^R$ that $\mathcal{M}_{x;a}$ is a smooth manifold of dimension $\mu^R(x) - i(a)$, compact up to splitting-off boundary trajectories, and oriented via coherent orientation. We set

$$\Psi^R\colon F_*^R(H) \to M_*(\mathbb{S}_L), \quad \Psi^R x = \sum_{\substack{a\in\mathrm{Crit}\,\mathbb{S}_L^R\\ \mathbb{S}_L(a)\le c(x)}} \big(\#_{\mathsf{alg}}\mathcal{M}_{x;a}\big)\cdot a,$$

and we obtain a chain complex morphism.

However, in general $c(x) > \mathcal{A}_H(x)$ is possible, in fact necessary if $\mathcal{M}_{x;\pi(x)} \ne \emptyset$, so that we cannot expect Ψ^R to be of triangular shape similarly to Φ^R. In fact, Ψ^R

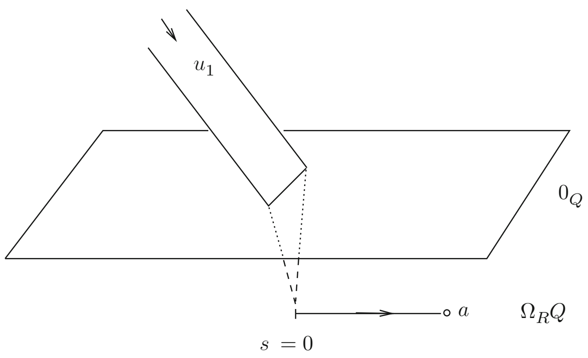

Fig. 2 The inverse construction, $\mathcal{M}_{x;a}$

can easily be defined for any pair (H, L) of a quadratic type Hamiltonian and a Lagrangian which does not need to be Fenchel dual to H.

The idea of using half-cylinders with boundary on the zero section of the cotangent bundle in order to provide cycles in the path space from cycles in the Floer chain complex via the evaluation at the zero section has been known for a while. In [10] this technique is used towards an isomorphism for linearized contact homology instead of Floer homology. The same idea is also used in [7].

Let us now give the proof that $\Psi^R \circ \Phi^R$ is chain homotopy equivalent to $\mathrm{id}_{M_*(\mathbb{S}_L)}$, which already implies that $\Psi^R_* = (\Phi^R_*)^{-1}$ since we know Φ^R_* to be an isomorphism.

Proposition 2.1. *Given H of quadratic type we have $\Psi^R \circ \Phi^R \simeq \mathrm{id}$ on $M_*(\mathbb{S}_L^R)$.*

Proof. Via the usual gluing result for Floer theory we clearly have that $\Psi^R \circ \Phi^R$ is chain homotopy equivalent to the chain morphism $M_*(\mathbb{S}_L^R) \to M_*(\mathbb{S}_L^R)$ defined by counting

$$\begin{aligned} \mathcal{M}^\sigma_{a,b} = \big\{ & w\colon [0,\sigma]\times[0,1] \to T^*Q \mid \bar\partial_{J,H} w = 0, \\ & \big(w(s,0), \mathsf{C}w(s,1)\big) \in N^*R,\ w(\sigma,\cdot) \subset 0_Q, \\ & (\pi\circ w)(0,\cdot) \in W^u(\mathbb{S}_L^R;a),\ (\pi\circ w)(\sigma,\cdot) \in W^s(\mathbb{S}_L^R;b) \big\} \end{aligned} \tag{9}$$

for $a, b \in \operatorname{Crit} \mathbb{S}_L^R$ with equal Morse index, and for $\sigma > 0$ fixed. The chain homotopy to $\mathrm{id}_{M_*(\mathbb{S}_L^R)}$ then follows from letting σ shrink to 0.

In order to simplify this argument, let us insert a further cobordism step. Namely, we clearly obtain a chain homotopy equivalence to the chain morphism on $M_*(\mathbb{S}_L^R)$ defined by counting

$$\begin{aligned} \widetilde{\mathcal{M}}^{\sigma,\lambda}_{a,b} = \big\{ & w\colon [0,\sigma]\times[0,1] \to T^*Q \mid \bar\partial_{J,H} w = 0, \\ & \big(w(s,0), \mathsf{C}w(s,1)\big) \in N^*R,\ w(\sigma,\cdot) \subset 0_Q, \\ & (\pi\circ w)(0,\cdot) \in W^u(\mathbb{S}_L^R;a),\ (\pi\circ w)(\lambda,\cdot) \in W^s(\mathbb{S}_L^R;b) \big\} \end{aligned} \tag{10}$$

for $\sigma > 0$ fixed and $\lambda \in [0,\sigma]$ given. For $\lambda = \sigma$ we have exactly $\mathcal{M}^\sigma_{a,b}$, and for $\lambda = 0$ we obtain

$$\widetilde{\mathcal{M}}^\sigma_{a,b} = \big\{ (c,w) \,\big|\, c \in W^u(\mathbb{S}_L^R;a) \cap W^s(\mathbb{S}_L^R;b),\ w \in \mathcal{M}^\sigma_c \big\}$$

with

$$\begin{aligned} \mathcal{M}^\sigma_c = \big\{ & w\colon [0,\sigma]\times[0,1] \to T^*Q \mid \bar\partial_{J,H} w = 0, \\ & \big(w(s,0), \mathsf{C}w(s,1)\big) \in N^*R, \\ & (\pi\circ w)(0,t) = c(t),\ w(\sigma,t) \in 0_Q\ \forall\, t \in [0,1] \big\}. \end{aligned} \tag{11}$$

If $i(a) = i(b)$ we have for $(c,w) \in \widetilde{\mathcal{M}}^\sigma_{a,b}$ that $a = b = c$, $w \in \mathcal{M}^\sigma_a$. The proof of the Proposition then follows from the following:

Lemma 2.2. *Given $c \in \Omega_R Q$ there exists a $\sigma_o = \sigma_o(c) > 0$ such that for each $\sigma \in (0, \sigma_o]$ the solution space $\mathcal{M}_c^\sigma$ contains a unique solution, compatible with coherent orientation.*

In fact, for $\sigma_n \to 0$, the solution sequence w_n converges uniformly with all derivatives to the path $(c, 0) \in \Omega_{N^*R} T^*Q$. Compatibility with coherent orientation implies that

$$\#_{\mathsf{alg}}\mathcal{M}_a^\sigma = \#\mathcal{M}_a^\sigma = 1 \quad \text{for } \sigma \in (0, \sigma_o],\ a \in \mathrm{Crit}\mathbb{S}_L\,.$$

Hence, counting $\widetilde{\mathcal{M}}_{a,b}^\sigma$ for $\sigma \in (0, \sigma_o]$ defines exactly the identity operator on $M_*(\mathbb{S}_L^R)$. This concludes the proof of Proposition 2.1 □

For the proof of Lemma 2.2 we refer to Proposition 4.10 in [5]. It follows from a uniform convergence analysis of solutions $w_n \in \mathcal{M}_c^{\sigma_n}$ as $\sigma_n \to 0$ together with a Newton type method to prove the unique existence of solutions for σ small enough. Note that, for example for $H_o = \frac{1}{2}|p|^2$, a first order approximation of solutions $w \in \mathcal{M}_c^\sigma$ is given by $w_{\mathrm{approx}}^\sigma(s,t) = \big(c(t), (\sigma - s)\dot{c}(t)\big)$, where we identify $TQ \cong T^*Q$ via the Legendre transformation from H_o.

Moreover, there is also a parametric version of Lemma 2.2, where we allow c to vary in a relatively compact family $K \subset \Omega_R Q$, for example an unstable manifold $W^u(\mathbb{S}_L; a)$. This would be the version to use in order to show $\Psi^R \circ \Phi^R \simeq \mathrm{id}$ directly by considering $\mathcal{M}_{a,b}^\sigma$ above for σ running from ∞ to 0.

3 Cup Product

We now show that also the cup-coproduct structure on path space homology

$$\cup\colon H_*(\Omega_R Q) \to H_*(\Omega_R Q) \otimes H_*(\Omega_R Q)$$

has a Floer theoretic counterpart given by a chain level construction, isomorphic to $\cup$ via Φ^R.

Given three R-nondegenerate Hamiltonians H_i, $i = 0, 1, 2$, we define a chain operation

$$u\colon F_*^R(H_0) \to F_*^R(H_1) \otimes F_*^R(H_2)$$

as follows. Given generators $x_i \in \mathcal{P}_R(H_i)$, $i = 0, 1, 2$, we consider three-fold Floer half-strips coupled by a conormal boundary condition

$$\begin{aligned} \mathcal{M}_{x_0;x_1,x_2}^{\cup,R} = \big\{ & u = (u_0, \bar{u}_1, \bar{u}_2)\colon (-\infty, 0] \times [0,1] \to T^*Q^3 \;\big| \\ & \bar{\partial}_{J,H_i} u_i = 0,\ i = 0,1,2,\ u_i(\pm\infty, \cdot) = x_i, \\ & \big(u_i(s,0), \mathsf{C}u_i(s,1)\big) \in N^*R,\ 0 \le |s| < \infty, \\ & u(0,t) \in N^*\triangle^{(3)} \big\}, \end{aligned} \tag{12}$$

where $\bar{u}_i(s,t) = \mathsf{C}u_i(-s,t)$ and $\triangle^{(3)} = \{\,(q,q,q)\,|\,q \in Q\} \subset Q^3$. Note that the conormal condition $u(0,\cdot) \in N^*\triangle^{(3)}$ means that

$$\begin{aligned} \pi \circ u_0(0,\cdot) \;&=\; \pi \circ u_1(0,\cdot) \;=\; \pi \circ u_2(0,\cdot) \;=:\; q(\cdot) \quad \text{and} \\ u_0(0,\cdot) \;&=\; u_1(0,\cdot) + u_2(0,\cdot) \quad \text{in } \; T^*_{q(\cdot)}Q\,. \end{aligned} \tag{13}$$

Hence, we have a well-posed Fredholm problem for $\mathcal{M}^{\cup,R}_{x_0;x_1,x_2}$ with

$$\dim \mathcal{M}^{\cup,R}_{x_0;x_1,x_2} \;=\; \mu^R(x_0) - \mu^R(x_1) - \mu^R(x_2)\,.$$

For the index formula for half-strips with piecewise conormal boundary condition see [5], Theorems 5.24 and 5.25. It remains to provide an energy estimate in order to obtain the usual compactness result. We compute with $u_i(0,\cdot) = \big(q(\cdot), p_i(\cdot)\big)$ and (13)

$$\begin{aligned} \mathcal{A}_{H_0}(x_0) \;\geq\; \mathcal{A}_{H_0}(u_0(0,\cdot)) \;&=\; \int_0^1 \big(\langle p_0, \dot q\rangle - H_0(t,q,p_0)\big)dt \\ &\overset{(13)}{=} \int_0^1 \big(\langle p_1 + p_2, \dot q\rangle - H_0(t,q,p_0)\big)dt \\ &=\; \mathcal{A}_{H_1}(u_1(0,\cdot)) + \mathcal{A}_{H_2}(u_2(0,\cdot)) \\ &\qquad + \int_0^1 \big(H_1(t,q,p_1) + H_2(t,q,p_2) - H_0(t,q,p_0)\big)dt\,. \end{aligned} \tag{14}$$

Thus, we obtain the required action monotonicity provided that the Hamiltonians satisfy

$$\begin{gathered} H_0(t,q,p+p') \;\leq\; H_1(t,q,p) + H_2(t,q,p') \\ \text{for all } \; t \in [0,1],\; q \in Q,\; p,p' \in T^*_q Q\,. \end{gathered}$$

For example, this is satisfied for geodesic type Hamiltonians with time-dependent potential perturbation,

$$H_0(t,q,p) = \frac{1}{2}|p|^2 + V(t,q), \quad H_1(t,q,p) = H_2(t,q,p) = |p|^2 + \frac{1}{2}V(t,q)\,.$$

Note that we have canonical isomorphisms $HF^R_*(H_0) \cong HF^R_*(H_i)$, $i = 1,2$ from the standard continuation argument. We define u by counting $\mathcal{M}^{\cup,R}_{x_0;x_1,x_2}$ with the usual orientation procedure,

$$u\colon F_*(H_0) \;\to\; F_*(H_1) \otimes F_*(H_2), \quad u(x) \;=\; \sum_{\substack{(y,z)\,\in\,\mathcal{P}_R(H_1)\times\mathcal{P}_R(H_2)\\ \mu^R(y)+\mu^R(z)=\mu^R(x)}} \big(\#_{\mathsf{alg}}\mathcal{M}^{\cup,R}_{x;y,z}\big)\, y \otimes z\,. \tag{15}$$

We shall prove the following:

Theorem 3.1. *The chain level operation* $u\colon F_*^R(H_0) \to F_*^R(H_1) \otimes F_*^R(H_2)$ *induces a coproduct* $u_*\colon HF_*^R \to HF_*^R \otimes HF_*^R$ *which is isomorphic to the cup coproduct on* $H_*(\Omega_R Q)$ *via the isomorphism* Φ_*^R.

Before proving the ring isomorphism property, let us remark that we have a variety of homotopically equivalent definitions for the cup coproduct in Floer homology. In fact, given $x_i \in \mathcal{P}_R(H_i)$, $i = 0, 1, 2$, we can consider the problem for $\lambda \in [0, 1]$,

$$u_0\colon (-\infty, 0] \times [0, 1] \to T^*Q,\ u_i\colon [0, \infty) \times [0, 1] \to T^*Q,\ i = 1, 2,$$
$$\overline{\partial}_{J_i,H_i} u_i = 0;\ u_1(-\infty) = x_1,\ u_i(+\infty) = x_i,\ i = 1, 2,$$

$$\begin{aligned}&\big(u_i(s, 0), \mathsf{C}u_i(s, 1)\big) \in N^*R,\ \text{f.a. } 0 \le |s| < \infty,\ i = 0, 1, 2,\\ &(\pi \circ u_0)(0, \cdot) = (\pi \circ u_1)(0, \cdot) = (\pi \circ u_2)(0, \cdot) =: q,\\ &\text{i.e. } u_i(0, \cdot) = (q, p_i),\ i = 0, 1, 2,\ p_0 = \lambda p_1 + (1 - \lambda) p_2\,. \end{aligned} \tag{16}$$

This is a well-posed Fredholm problem for all $\lambda \in [0, 1]$, and for $\lambda = 1/2$ we obtain a problem which is essentially equivalent to (12) (Fig. 3). In order to get compactness for the above problem, it is convenient to assume that the Hamiltonians H_0, H_1 and H_2 are physical Hamiltonians with the same kinetic part,

$$H_j(t, q, p) = \frac{1}{2}|p|^2 + V_j(t, q), \qquad \forall j = 0, 1, 2,$$

and that J is C^0-close enough to the Levi-Civita almost complex structure J_0. Under these assumptions we have the following compactness result, where as usual on the space of maps we consider the C^∞_{loc} topology:

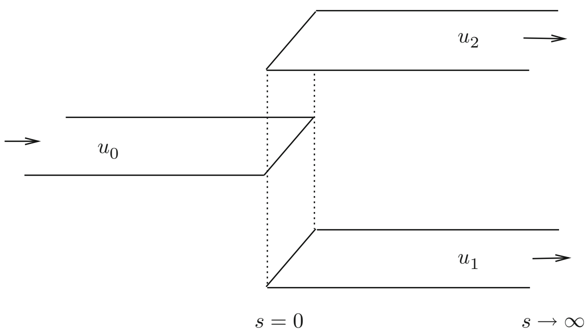

Fig. 3 The cup coproduct

Lemma 3.2. *For every triple $x_j \in \mathcal{P}_R(H_j)$, the space of solutions (λ, u_0, u_1, u_2) of (16) is pre-compact. Moreover, the existence of a solution (λ, u_0, u_1, u_2) of (16) gives rise to the estimate*

$$\lambda \mathcal{A}_{H_1}(x_1) \,+\, (1-\lambda)\mathcal{A}_{H_2}(x_2) \,\leq\, \mathcal{A}_{H_0}(x_0) + \|V_0\|_\infty + \max\big\{\|V_1\|_\infty, \|V_2\|_\infty\big\}. \tag{17}$$

Proof. By the special form of the Hamiltonians, we have

$$\begin{aligned} &H_0\big(t, q, \lambda p_1 + (1-\lambda)p_2\big) - \lambda H_1(t, q, p_1) - (1-\lambda)H_2(t, q, p_2) \\ &\quad \leq \|V_0\|_\infty + \max\big\{\|V_1\|_\infty, \|V_2\|_\infty\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{A}_{H_0}(u_0(0,\cdot)) \,&=\, \int p_0 dq - H_0(t,q,p_0)\, dt \\ &= \int \big(\lambda p_1 + (1-\lambda)p_2\big) dq - H_0(t,q,p_0)\, dt \\ &= \lambda \mathcal{A}_{H_1}(u_1(0,\cdot)) \,+\, (1-\lambda)\mathcal{A}_{H_2}(u_2(0,\cdot)) \\ &\quad - \int \big(H_0(t,q,p_0) - \lambda H_1(t,q,p_1) - (1-\lambda)H_2(t,q,p_2)\big)\, dt \\ &\geq \lambda \mathcal{A}_{H_1}(u_1(0,\cdot)) \,+\, (1-\lambda)\mathcal{A}_{H_2}(u_2(0,\cdot)) \\ &\qquad - \|V_0\|_\infty - \max\big\{\|V_1\|_\infty, \|V_2\|_\infty\big\}, \end{aligned} \tag{18}$$

and the estimate (17) follows from the bounds

$$\begin{aligned} \mathcal{A}_{H_0}(u_0(0,\cdot)) \leq \mathcal{A}_{H_0}(x_0), \quad \mathcal{A}_{H_1}(u_1(0,\cdot)) \geq \mathcal{A}_{H_1}(x_1), \\ \mathcal{A}_{H_2}(u_2(0,\cdot)) \geq \mathcal{A}_{H_2}(x_2). \end{aligned} \tag{19}$$

By means of an isometric embedding of M into $\mathbb{R}^N$ and of the induced isometric embedding of T^*M into $\mathbb{R}^N \times \mathbb{R}^N \cong \mathbb{C}^N$, we can consider the map

$$v : [0, +\infty) \times [0,1] \to \mathbb{C}^N, \quad v = \lambda u_1 + (1-\lambda)u_2.$$

Then by (18), the quantity

$$\begin{aligned} \iint_{[0,+\infty)\times[0,1]} |\partial_s v|^2\, ds dt \leq \lambda \iint_{[0,+\infty)\times[0,1]} |\partial_s u_1|^2\, ds dt \\ + (1-\lambda) \iint_{[0,+\infty)\times[0,1]} |\partial_s u_2|^2\, ds dt \end{aligned}$$

$$= \lambda\big(\mathcal{A}_{H_1}(u_1(0,\cdot)) - \mathcal{A}_{H_1}(x_1)\big) + (1-\lambda)\big(\mathcal{A}_{H_2}(u_2(0,\cdot)) - \mathcal{A}_{H_2}(x_2)\big)$$
$$\le \mathcal{A}_{H_0}(x_0) + \|V_0\|_\infty + \max\big\{\|V_1\|_\infty, \|V_2\|_\infty\big\}$$
$$+ \big|\mathcal{A}_{H_1}(x_1)\big| + \big|\mathcal{A}_{H_2}(x_2)\big|$$

has a uniform bound. Since also $\|\partial_s u_0\|_2$ is uniformly bounded, because of (18) and (19), the L^2 norm of the s-derivative of the map

$$w : [0,+\infty) \times [0,1] \to \mathbb{C}^N \times \mathbb{C}^N, \quad w(s,t) = \big(\overline{u_0(-s,t)}, v(s,t)\big),$$

has a uniform bound. Since $\|J - J_0\|_\infty$ is small, w solves a Cauchy-Riemann type equation, and $w(0,t)$ belongs to the totally real space given by the conormal of the diagonal in $\mathbb{R}^N \times \mathbb{R}^N$, the argument of [2, Sect. 1.5] shows that w is uniformly bounded in C^∞. In particular, u_0 and

$$q(t) := \pi \circ u_0(0,t) = \pi \circ u_1(0,t) = \pi \circ u_2(0,t)$$

are uniformly bounded in C^∞, and we get uniform upper bounds for

$$\mathcal{A}_{H_1}(u_1(0,\cdot)) \le \mathbb{S}_{L_1}(q) \quad \text{and} \quad \mathcal{A}_{H_2}(u_2(0,\cdot)) \le \mathbb{S}_{L_2}(q).$$

Together with the lower bounds of (19), we conclude that $\|\partial_s u_1\|_2$ and $\|\partial_s u_2\|_2$ are both uniformly bounded. By [2, Theorem 1.14 (iii)] and the usual elliptic bootstrap argument, we conclude that also u_1 and u_2 have uniform C^∞ bounds. □

Let now, for given $\lambda \in [0,1]$, $W^\lambda_{x_0;x_1,x_2}$ denote the set of solutions of (16) with generically chosen J_i for each u_i, $i = 0,1,2$, as well as generically chosen triple (V_0, V_1, V_2) of perturbing potentials. Then, we can define a chain level operation

$$u_\lambda \colon F^R_*(H_0) \to \bigoplus_{i+j=*} F^R_i(H_1) \otimes F^R_j(H_2),$$

from counting $\#_{\mathsf{alg}} W^\lambda_{x_0;x_1,x_2}$. Using the full solution space $W_{x_0;x_1,x_2}$ of (16) with variable $\lambda \in [0,1]$ and accordingly generically chosen structures J_i and V_i and index relation $\mu^R(x_0) = \mu^R(x_1) + \mu^R(x_2) - 1$ we obtain easily the following:

Proposition 3.3. *The induced coproducts $(u_\lambda)_* \colon HF^R_*(H_0) \to HF^R_*(H_1) \otimes HF^R_*(H_2)$ do not depend on $\lambda \in [0,1]$, and they are equal to the cup-coproduct u.*

In fact, the cup-coproduct (15) is essentially given by $u_{\frac{1}{2}}$.

As a consequence, in dual cohomological formulation, we can apply the above action estimates to the notion of cohomologically critical values

$$c^*(\alpha, H) := \sup\big\{a \in \mathbb{R} \mid \alpha \in \operatorname{Im}\big(HF^*_{\ge a}(H) \to HF^*(H)\big)\big\}$$

for given $\alpha \in HF^*(H)$, where $HF^*_{\geq a}$ is the cohomology of the subcochain complex $F^*_{\geq a}(H) = \mathbb{Z}^{\{x\in\mathcal{P}_R(H)\,|\,\mathcal{A}_H(x)\geq a\}}$, and we are omitting the superscript R.

We have in this cohomological formulation, with $\cup$ dual to u:

Corollary 3.4 *Given $H_i(t,q,p) = \frac{1}{2}|p|^2 + V_i(t,q)$ as above, we have for $\alpha_i \in HF^*(H_i)$, $i = 1,2$ with $\alpha_1 \cup \alpha_2 \in HF^*(H_0)$*

$$c * (\alpha_1 \cup \alpha_2, H_0) \geq \max\left\{c^*(\alpha_1, H_1), c^*(\alpha_2, H_2)\right\} - \|V_0\|_\infty - \max\left\{\|V_1\|_\infty, \|V_2\|_\infty\right\}.$$

We now complete the proof of Theorem 3.1. At first, we give a Morse-homological definition of the cup-product.

Suppose we have three non-degenerate Lagrangians L_i, $i = 0,1,2$, such that $\mathbb{S}^R_{L_1}$ and $\mathbb{S}^R_{L_2}$ have no common critical points. Then we define

$$\cup\colon M_*(\mathbb{S}^R_{L_0}) \to M_*(\mathbb{S}^R_{L_1}) \otimes M_*(\mathbb{S}^R_{L_2}),$$

$$\cup\, a = \sum_{(b,c)\in \operatorname{Crit}\mathbb{S}^R_{L_1} \times \operatorname{Crit}\mathbb{S}^R_{L_2}} \langle a; b, c\rangle\, b \otimes c\,, \tag{20}$$

where $\langle a; b, c\rangle$ is the oriented count of

$$W^u(\mathbb{S}^R_{L_0}; a) \cap W^s(\mathbb{S}^R_{L_1}; b) \cap W^s(\mathbb{S}^R_{L_2}; c),$$

provided that we have chosen three generic pseudogradient fields so that the triple intersection is transverse. The dimensions of this intersection is $i(a) - i(b) - i(c)$, and the intersection is oriented if the unstable manifolds (which are all finite-dimensional) are oriented.

The usual splitting-off argument for boundary trajectories proves the Leibniz rule for $\cup$, and it is well-known see e.g. [8] that $\cup_*$ defines the cup-coproduct. One can also show Morse homologically that the cohomological product $\cup^*$ satisfies $\cup^* = \triangle^* \circ \times$, where $\times$ is the exterior product and $\triangle^*$ the pull-back by the diagonal embedding $\triangle\colon \Omega_R Q \hookrightarrow \Omega_R Q \times \Omega_R Q$, for which we also have Morse homological functoriality.

We now want to show that the isomorphism

$$\Psi^R_*\colon HF^R_*(H) \to HM_*(\mathbb{S}^R_L)$$

intertwines the coproducts u and $\cup_*$, i.e.

$$\cup \circ \Psi^R \simeq \left(\Psi^R \otimes \Psi^R\right) \circ u$$

are chain homotopic on F^R_*.

Clearly, $\cup \circ \Psi^R$ is chain homotopic to the operation

$$w_1 \colon F_*^R(H) \;\to\; M_*(\mathbb{S}^R_{L_1}) \otimes M_*(\mathbb{S}^R_{L_2}),$$
$$w_1(x) \;=\; \sum_{\substack{(b,c) \\ i(b)+i(c)=\mu^R(x)}} \big(\#_{\mathsf{alg}} \widetilde{\mathcal{M}}^{(1)}(x;b,c)\big) \cdot b \otimes c\,, \tag{21}$$

with

$$\widetilde{\mathcal{M}}^{(1)}(x;b,c) \;=\; \big\{ u \in \mathcal{M}_x^- \;\big|\; (\pi \circ u)(0,\cdot) \in W^s(\mathbb{S}^R_{L_1}; b) \cap W^s(\mathbb{S}^R_{L_2}; c) \big\}\,.$$

Then we find generic J for $\mathcal{M}_x^-$ and pseudo-gradient vector fields for $\mathbb{S}^R_{L_i}$, $i = 1,2$, such that $\widetilde{\mathcal{M}}^{(1)}(x;b,c)$ satisfies transversality for all x, b, c.

Next, we use Proposition 3.3, which allows us to replace u by u_λ for $\lambda = 1$. We obtain

$$\big(\Psi^R \otimes \Psi^R\big) \circ u_1 \simeq w_2,$$

with w_2 given by the oriented count of

$$\begin{aligned}\widetilde{\mathcal{M}}^{(2)}_\sigma(x;b,c) \;=\; \big\{ (u,v) \,\big|\, & u \in \mathcal{M}_x^-,\ (\pi \circ u)(0,\cdot) \in W^s(\mathbb{S}^R_{L_1}; c),\\ & v \colon [0,\sigma] \times [0,1] \to T^*Q,\ \big(v(s,0), \mathsf{C}v(s,1)\big) \in N^*R,\\ & (\pi \circ v)(0,\cdot) \;=\; (\pi \circ u)(-\sigma,\cdot),\\ & v(\sigma,\cdot) \subset 0_Q,\ (\pi \circ v)(\sigma,\cdot) \in W^s(\mathbb{S}^R_{L_2}; b) \big\}\end{aligned} \tag{22}$$

for any fixed $\sigma > 0$ (Figs. 4 and 5). Moreover, w_2 is clearly chain homotopic to $w_3 \colon F_*^R \to M_* \otimes M_*$ given by

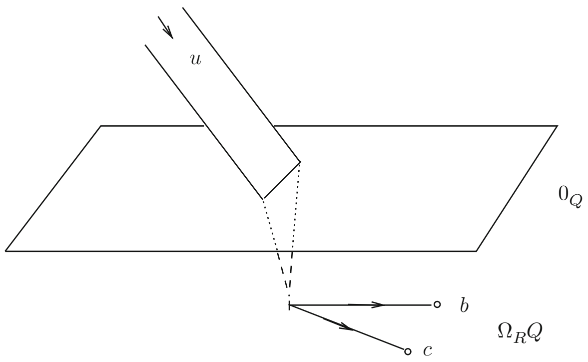

Fig. 4 $\widetilde{\mathcal{M}}^{(1)}_\sigma(x;b,c)$

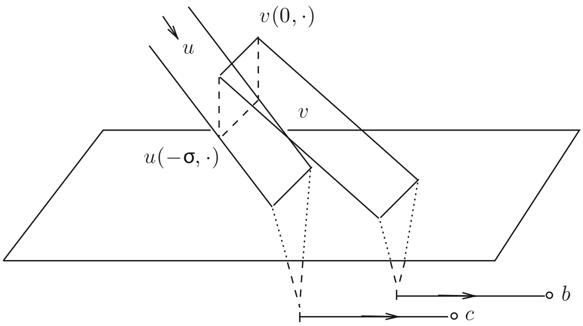

Fig. 5 $\widetilde{\mathcal{M}}_{\sigma}^{(2)}(x; b, c)$

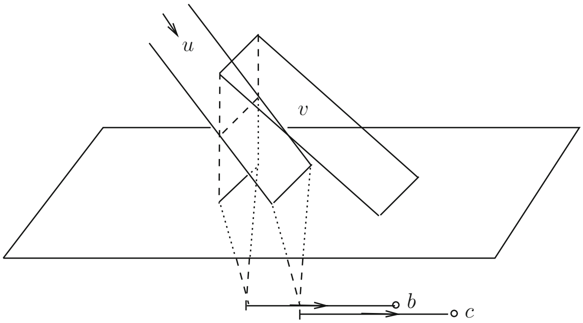

Fig. 6 $\widetilde{\mathcal{M}}_{\sigma}^{(3)}(x; b, c)$

$$\begin{aligned}\widetilde{\mathcal{M}}_{\sigma}^{(3)}(x; b, c) \;=\; \Big\{ (u, v) \,\Big|\, & u \in \mathcal{M}_x^-,\ (\pi \circ u)(0, \cdot) \in W^s(\mathbb{S}_{L_1}^R; c), \\ & v\colon [0, \sigma] \times [0, 1] \to T^*Q,\ \big(v(s, 0), \mathsf{C}v(s, 1)\big) \in N^*R, \\ & (\pi \circ v)(0, \cdot) \;=\; (\pi \circ u)(-\sigma, \cdot), \\ & v(\sigma, \cdot) \subset 0_Q,\ (\pi \circ v)(0, \cdot) \in W^s(\mathbb{S}_{L_2}^R; b) \Big\}, \end{aligned} \tag{23}$$

which differs from the previous space only for the value of s for which $\pi \circ v(s, \cdot)$ belongs to the stable manifold of b. Finally, w_3 is chain homotopic to w_4 given by (Fig. 6)

$$\begin{aligned}\widetilde{\mathcal{M}}_{\sigma}^{(4)}(x; b, c) \;=\; \Big\{ (u, v) \,\Big|\, & u \in \mathcal{M}_x^-,\ (\pi \circ u)(0, \cdot) \in W^s(\mathbb{S}_{L_1}^R; c), \\ & v\colon [0, \sigma] \times [0, 1] \to T^*Q,\ \big(v(s, 0), \mathsf{C}v(s, 1)\big) \in N^*R, \\ & (\pi \circ v)(0, \cdot) \;=\; (\pi \circ u)(0, \cdot) \in W^s(\mathbb{S}_{L_2}^R; b), \\ & v(\sigma, \cdot) \subset 0_Q \Big\} \end{aligned}$$

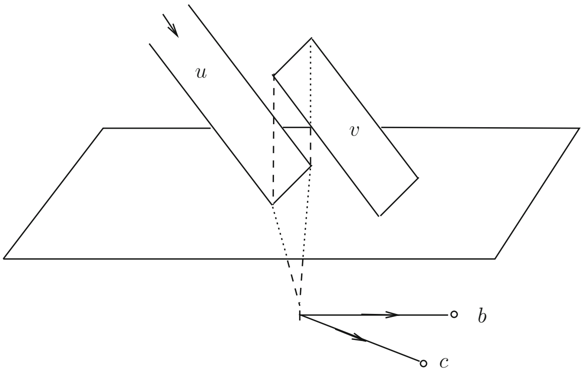

Fig. 7 $\widetilde{\mathcal{M}}^{(4)}_{\sigma}(x; b, c)$

$$
\begin{aligned}
&= \Big\{ (u,v) \,\Big|\, u \in \widetilde{\mathcal{M}}^{(1)}(x; b, c), \\
&\qquad v : [0,\sigma] \times [0,1] \to T^*Q,\ \big(v(s,0), \mathsf{C}v(s,1)\big) \in N^*R, \\
&\qquad (\pi \circ v)(0,\cdot) \;=\; (\pi \circ u)(0,\cdot)\,, \\
&\qquad v(\sigma,\cdot) \subset 0_Q \Big\}\,, \\
&= \Big\{ (u,v) \,\Big|\, u \in \widetilde{\mathcal{M}}^{(1)}(x; b, c),\ v \in \mathcal{M}^{\sigma}_{\pi\circ u(0)} \Big\}\,, \qquad (24)
\end{aligned}
$$

with $\mathcal{M}^{\sigma}_{\pi\circ u(0)}$ as in (11). The chain homotopy $w_4 \simeq w_1$ then follows from Lemma 2.2 if we choose $\sigma > 0$ small enough. This finishes the proof of Theorem 3.1 (Fig. 7). □

4 The Proof of the Hopf Algebra Property

Let us now analyze the compatibility of the coproduct $u_* : HF^R_*(H_0) \to HF^R_*(H_1) \otimes HF^R_*(H_2)$ with the product $m_R : HF^R_*(H) \otimes HF^R_*(H) \to HF^R_{*-d(R,R)}(H^{(2)})$ where $R \circ R = R$. In fact, we are interested in the relation

$$
u \circ m \;=\; (m \otimes m) \circ \tau \circ (u \otimes u), \qquad (25)
$$

where

$$
\begin{aligned}
\tau : & HF^R(H_1) \otimes HF^R(H_2) \otimes HF^R(H_1) \otimes HF^R(H_2) \\
&\to HF^R(H_1) \otimes HF^R(H_1) \otimes HF^R(H_2) \otimes HF^R(H_2)
\end{aligned}
$$

commutes the second and third factor. Obviously, a necessary condition for (8) to hold besides $R \circ R = R$ is that the degree $d(R, R)$ vanishes. Since this degree equals the codimension of $(R \times R) \cap (Q \times \Delta \times Q)$ in $R \times R$, $d(R, R)$ vanishes if and only if $R \times R \subset Q \times \Delta \times Q$, that is if and only if $R = \{(q_0, q_0)\}$ for some $q_0 \in Q$. Therefore, the only case to be considered is the classical case of based loop homology, that is we want to verify the Hopf-algebra property (8) for

$$\big(HF_*^R, m_R, u_*\big) \;\cong\; \big(H_*(\Omega Q), \#, \cup\big)$$

by Floer-theoretical arguments via chain level operations on $F_*^R(H_i)$. We replace the superscript $R = \{(q_o, q_o)\}$ by Ω and we prove the following:

Theorem 4.1. *The chain maps* $u^i: F_*^\Omega(H_0^{(i)}) \to F_*^\Omega(H_1^{(i)}) \otimes F_*^\Omega(H_2^{(i)})$*,* $i = 1, 2$*, and* $m_j: F_*^\Omega(H_j) \otimes F_*^\Omega(H_j) \to F_*^\Omega(H_j^{(2)})$*,* $j = 0, 1, 2$*, satisfy the chain homotopy property*

$$u^2 \circ m_0 \;\simeq\; (m_1 \otimes m_2) \circ \tau \circ (u^1 \otimes u^1)\,.$$

Proof. We recall from [5] that $m_j: F_*^\Omega(H_j) \otimes F_*^\Omega(H_j) \to F_*^\Omega(H_j^{(2)})$ is defined by counting

$$(u_1, u_2): \mathbb{R} \times [0,1] \to (T^*Q)^2,\; \bar\partial_{J,H_j} u_i = 0,\; i = 1, 2,$$
$$\big(u_1(s,0), \mathsf{C}u_1(s,1), u_2(s,0), \mathsf{C}u_2(s,1)\big) \in \begin{cases} (T^*_{q_o}Q)^4 \;=\; N^*(\{q_o\}^4), & s \le 0,\\ T^*_{q_o}Q \times N^*\triangle_{23} \times T^*_{q_o}Q, & s \ge 0\,. \end{cases}$$

Hence, using the definition of u^i via (12) and $H_j = \frac{1}{2}|p|^2 + V_j(t,q)$, with V_j 1-periodic in time, for $j = 0, 1, 2$, we obtain via the usual gluing argument that for every $\rho > 0$ the chain map $u^2 \circ m_0$ is chain homotopic to the operator

$$A_\rho: F_*^\Omega(H_0) \otimes F_*^\Omega(H_0) \to F_*^\Omega(H_1^{(2)}) \otimes F_*^\Omega(H_2^{(2)}),$$

which is defined by counting

$$\begin{aligned}
& w: (-\infty, 0] \times [0,1] \to (T^*Q)^6 \quad \text{with},\\
& w(s,t) \;=\; \big(u_{10}(s,t), \mathsf{C}u_{20}(s,1-t), \mathsf{C}u_{11}(-s,t), u_{21}(-s,1-t),\\
& \qquad\qquad \mathsf{C}u_{12}(-s,t), u_{22}(-s,1-t)\big)\\
& \bar\partial_{J,H_j} u_{ij} = 0,\; i = 1,2,\; j = 0,1,2,\\
& w(s,0) \in (T^*_{q_o}Q)^6,\; \text{for } -\infty < s \le 0,\\
& w(0,t) \in N^*\big(\triangle^{(3)} \times \triangle^{(3)}\big),\; 0 \le t \le 1,
\end{aligned} \tag{26}$$

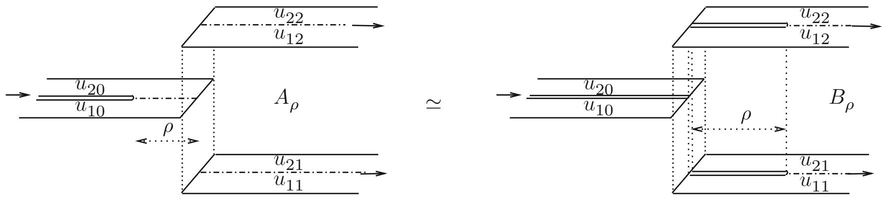

Fig. 8 The Hopf algebra argument

$$w(s,1) \in N^*\big(\{q_o\}^2 \times \triangle \times \triangle\big),\ -\infty < s \leq -\rho, \quad \text{and}$$

$$w(s,1) \in N^*\big(\triangle \times \triangle \times \triangle\big),\ -\rho \leq s \leq 0. \tag{27}$$

Likewise, we have that for every $\rho > 0$ the chain map $(m_1 \otimes m_2) \circ \tau \circ (u^1 \otimes u^1)$ is chain homotopic to the operator

$$B_\rho : F_*^\Omega(H_0) \otimes F_*^\Omega(H_0) \rightarrow F_*^\Omega(H_1^{(2)}) \otimes F_*^\Omega(H_2^{(2)}),$$

which is defined by counting $w: (-\infty, 0] \times [0,1] \rightarrow (T^*Q)^6$ as above satisfying instead of the last equation of (26) the condition

$$w(s,1) \in N^*(\{q_o\}^6) \quad \text{for } -\rho \leq s \leq 0\,. \tag{28}$$

We now need to show that A_ρ and B_ρ are chain homotopic. Instead of identifying the limit operators A_0 and B_0 as $\rho \to 0$, we apply a different argument from [5] (Fig. 8).

Note that it is not possible to homotope the last line of (26) into (28) through conormal boundary conditions N^*R for $s \in [-\rho, 0]$ since $\{q_o\}^6$ and $\triangle \times \triangle \times \triangle$ are not isotopic in Q^6 already by dimensional reasons. However, we have the following:

Lemma 4.2. *The chain maps $A_\rho \otimes B_\rho$ and $B_\rho \otimes A_\rho$ are chain homotopic.*

The proof is completely analogous to that of Proposition 4.7 in [5]. In order to deduce from the above lemma that A_ρ is chain homotopic to B_ρ, we can use the following algebraic fact, which is proven as Lemma 4.6 in [5]:

Lemma 4.3. *Let (C, ∂) and (C', ∂) be chain complexes, bounded from below. Let $\varphi, \psi : C \to C'$ be chain maps. Assume that there is an element $\epsilon \in C_0$ with $\partial\epsilon = 0$ and a chain map $\delta : C' \to (\mathbb{Z}, 0)$ such that*

$$\delta(\varphi(\epsilon)) = \delta(\psi(\epsilon)) = 1.$$

If $\varphi \otimes \psi$ is homotopic to $\psi \otimes \varphi$ then φ is homotopic to ψ.

Here $(\mathbb{Z}, 0)$ denotes the trivial chain complex all of whose groups vanish, except for the one in degree zero which is $\mathbb{Z}$.

It remains to find a cycle $\epsilon \in (F^\Omega(H_0) \otimes F^\Omega(H_0))_0$ and a chain map $\delta\colon F^\Omega(H_1^{(2)}) \otimes F^\Omega(H_2^{(2)}) \to (\mathbb{Z}, 0)$ such that

$$\delta(A_\rho(\epsilon)) = \delta(A_\rho(\epsilon)) = 1. \tag{29}$$

Without loss of generality, we can assume that the potentials V_j are time-independent Morse functions and that they have a common unique maximum at q_0. The constant orbit $x_0 = (q_0, 0)$ is an element of $F_0^\Omega(H_j^{(i)})$ for every $i = 1, 2$ and $j = 0, 1, 2$ and it defines a cycle

$$\epsilon = (x_0, x_0) \in (F^\Omega(H_0) \otimes F^\Omega(H_0))_0.$$

Since x_0 is the critical point with minimal action $\mathcal{A}_{H_j^{(i)}}$, for every $i = 1, 2$ and $j = 0, 1, 2$, we have

$$A_\rho(\epsilon) = B_\rho(\epsilon) = \epsilon. \tag{30}$$

Let $L_1^{(2)}$ and $L_2^{(2)}$ be the Lagrangians on TQ which are Fenchel dual to $H_1^{(2)}$ and $H_2^{(2)}$. Let

$$\tilde{\delta} : M\big(\mathbb{S}_{L_1^{(2)}}\big) \otimes M\big(\mathbb{S}_{L_2^{(2)}}\big) = M\big(\mathbb{S}_{L_1^{(2)}} \oplus \mathbb{S}_{L_2^{(2)}}\big) \to (\mathbb{Z}, 0)$$

be the standard augmentation on the Morse complex of the functional

$$\mathbb{S}_{L_1^{(2)}} \oplus \mathbb{S}_{L_2^{(2)}} : \Omega_{q_0} Q \times \Omega_{q_0} Q \to \mathbb{R}, \quad (\gamma_1, \gamma_2) \mapsto \mathbb{S}_{L_1^{(2)}}(\gamma_1) + \mathbb{S}_{L_2^{(2)}}(\gamma_2),$$

that is the homomorphism which maps each critical point of Morse index zero into 1. The homomorphism $\tilde{\delta}$ is a chain map because the boundary of every critical point γ of Morse index one has the form $\gamma_1 - \gamma_2$, where γ_1 and γ_2 are the critical points of Morse index zero to which the two sides of the one-dimensional unstable manifold of γ converge. We can now use the isomorphism Φ^Ω between the Morse complex and the Floer complex to read the chain map $\tilde{\delta}$ on the Floer complex, thus defining the chain map

$$\delta : F^\Omega(H_1^{(2)}) \otimes F^\Omega(H_2^{(2)}) \to (\mathbb{Z}, 0).$$

Since Φ^Ω is the identity mapping on global minimizers, we have

$$\delta(\epsilon) = 1.$$

Together with (30), this proves (29) and concludes the proof of the theorem. □

5 The Goresky-Hingston Coproduct

Throughout this section, we deal only with periodic boundary conditions, i.e. to the case $R = \triangle$. In order to simplify the notation, we omit the superscript $\triangle$ from all the objects which would require it (such as $F_*^\triangle$, $\mathbb{S}_L^\triangle$, $\mu^\triangle$, and so on).

Let us consider the coproduct of degree $-n$, $w\colon F_*(H_0) \to \big(F_*(H_1) \otimes F_*(H_2)\big)_{*-n}$ defined by counting

$$\begin{aligned}
&u = (u_1, u_2)\colon \mathbb{R} \times [0,1] \to T^*(Q \times Q), \quad \text{solving}\\
&\bar\partial_{J,H_0} u_i = 0 \text{ for } s \le 0,\ i = 1, 2,\\
&\bar\partial_{J,H_1} u_1 = \bar\partial_{J,H_2} u_2 = 0 \text{ for } s \ge 0\,,\\
&\big(u_1(s,0), \mathsf{C}u_1(s,1), u_2(s,0), \mathsf{C}u_2(s,1)\big) \in \begin{cases} N^*(\triangle_{14} \times \triangle_{23}), & s \le 0,\\ N^*(\triangle_{12} \times \triangle_{34}), & s \ge 0\,, \end{cases}
\end{aligned} \tag{30}$$

with asymptotics $x \in \mathcal{P}_2(H_0)$ for $s \to -\infty$ and $(y, z) \in \mathcal{P}_1(H_1) \times \mathcal{P}_1(H_2)$ for $s \to \infty$.

Then, completely analogous to the ring isomorphism $\Phi_*\colon \big(H_*(\Lambda Q), \circ\big) \xrightarrow{\cong} \big(HF_*(T^*Q), m\big)$, one can show that Φ_* identifies the coproduct w on HF_* with the comultiplication

$$\mu := \mu_{0,3}^{\text{top}}\colon H_*(\Lambda Q) \to \big(H_*(\Lambda Q) \otimes H_*(\Lambda Q)\big)_{*-n}\,,$$

of degree $-n$ from [11] (see Theorem 3).

We now give a short argument which explains why this coproduct is essentially trivial, i.e. 0 to large extents. Let us assume for simplicity that Q is simply connected and hence $H_0(\Lambda Q) \cong \mathbb{Z}$ generated by 1, where this class 1 is represented by any constant loop $q_o \in Q \subset \Lambda Q$ as a 0-cycle. Moreover, we denote by $e = [Q] \in H_n(\Lambda Q)$ the neutral element for the Chas-Sullivan loop product, which is given by the fundamental class of Q, as an n-cycle of constant loops. In Floer homology, $\Phi_*(e)$ is given by the Floer cycle

$$\sum_{\mu(x)=n} \big(\#_{\text{alg}}\mathcal{M}_x^+\big) \cdot x \ \in F_n(H),$$

$$\mathcal{M}_x^+ = \Big\{u\colon [0,\infty) \times \mathbb{T} \to T^*Q \;\Big|\; \bar\partial_{J,H} u = 0,\ u(+\infty) = x,\ \frac{\partial}{\partial t}(\pi \circ u)(0,\cdot) = 0\Big\} \tag{31}$$

for a generic J. We have $e \circ a = a \circ e = a$ for all $a \in H_*(\Lambda Q)$ and $\mu(e) = \alpha \cdot 1 \otimes 1$ for some $\alpha \in \mathbb{Z}$ by dimensional reasons. In fact, it is not hard to show that

$$\mu(e) = \chi(Q) \cdot 1 \otimes 1\,. \tag{32}$$

Lemma 5.1. *For any $a \in H_k(\Lambda Q)$, we have*

$$\mu(a) = \begin{cases} 0, & \text{if } k \neq n, \\ \beta \cdot 1 \otimes 1, & \text{if } k = n, \end{cases}$$

with $\beta \cdot 1 = \chi(Q) \cdot (a \circ 1) \in H_0(\Lambda Q)$.

Proof. From [11] or the property of HF_* to be a (noncompact) 2-dimensional topological field theory (see also [12]) it follows that

$$\begin{aligned}(\mathrm{id} \otimes m) \circ (\mu \otimes \mathrm{id}) &= (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mu) \\ &= \mu \circ m \colon H_*(\Lambda Q) \otimes H_*(\Lambda Q) \to \big(H_*(\Lambda Q) \otimes H_*(\Lambda Q)\big)_{*-2n} \qquad (33)\end{aligned}$$

where for notational clarity we write m for the loop product $\circ$. Applying this identity on $a \otimes e$ and $e \otimes a$ for the given $a \in H_k(\Lambda Q)$ gives

$$\begin{aligned}\mu(a) = (\mu \circ m)(a \otimes e) &= (m \times \mathrm{id}) \circ (\mathrm{id} \otimes \mu)(a \otimes e) \\ &= \chi(Q) \cdot (m \otimes \mathrm{id})(a \otimes 1 \otimes 1), \\ &= \chi(Q) \cdot m(a, 1) \otimes 1, \quad \text{as well as} \\ &= \chi(Q) \cdot 1 \otimes m(a, 1), \qquad (34)\end{aligned}$$

which leaves only the possibility $\chi(Q) \cdot m(a, 1) = 0$ in the case $k \neq n$ and $\mu(a) = \beta \cdot 1 \otimes 1, b \cdot 1 = \chi(Q) \cdot m(a, 1)$ if $k = n$. □

Hence, apart from degree n classes, the coproduct has to be trivial. This, however, can be seen as a possibility to define a secondary structure, namely a coproduct on relative homology $H_*(\Lambda Q, Q)$, or equivalently a cohomological product

$$\Box \colon H^*(\Lambda Q, Q) \otimes H^*(\Lambda Q, Q) \to H^{*+n-1}(\Lambda Q, Q)\,.$$

This cohomological product has been explicitly constructed and carefully analyzed in [19]. It gives an interesting nontrivial operation in particular for spheres $Q = S^n$.

Here, we now want to give an explicit chain-level construction for the Floer-homological counterpart of □. Let us consider a special Hamiltonian of physical type $H = \frac{1}{2}|p|^2 + V(t, q)$, where $V(t, q)$ is only a small potential perturbation in order to achieve Morse-nondegeneracy for the action $\mathcal{A}_H$. Let us pick $V(t, q)$ generically with $\|V\|_\infty$ small enough compared to the smallest length of a closed geodesic, so that the orbits $x \in \mathcal{P}_1(H)$ with $\mathcal{A}_H(x) > \epsilon$ for some $\epsilon > \|V\|_\infty$ can be seen as the generators of the quotient chain complex $F_*(H)/F_*^{\leq\epsilon}(H)$ which defines the homology $HF_*^{\epsilon>0}(H)$. Then, $HF_*(H)^{\epsilon>0}$ becomes isomorphic to $H_*(\Lambda Q, Q)$ under Φ_* for $\epsilon > 0$ small enough. Let us denote

$$HF_*^{>0}(T^*Q) := \lim_{\epsilon > \|V\|_\infty \to 0} HF_*^{\epsilon>0}(H)\,.$$

We will now construct a coproduct

$$\tilde{w}\colon HF_*^{>0}(T^*Q) \to \big(HF_*^{>0}(T^*Q) \otimes HF_*^{>0}(T^*Q)\big)_{*-n+1}\,. \tag{35}$$

Given $0 < \lambda < 1$ we consider the disjoint union of strips

$$\Sigma_\lambda = (-\infty, 0] \times [0, \lambda] \,\dot{\cup}\, (-\infty, 0] \times [\lambda, 1]\,.$$

Given 1-periodic solutions $x_i \in \mathcal{P}_1(H_i)$, $i = 0, 1, 2$ with $H_i = \frac{1}{2}|p|^2 + V_i(t, q)$ for a generic triple of small perturbations as above (V_0, V_1, V_2), we consider $\widetilde{\mathcal{M}}_{x_0;x_1,x_2}$ as the space of solutions (u, v, w, λ) of

$$\begin{aligned}
&\lambda \in (0,1),\ u\colon \Sigma_\lambda \to T^*Q,\ (v,w)\colon [0,\infty)\times\mathbb{T} \to T^*Q \times T^*Q,\\
&\big(v(+\infty), w(+\infty)\big) = (x_1, x_2),\ u(-\infty, t) = x_1(t) \text{ for } 0 \le t \le 1,\\
&\bar{\partial}_{J,H_1} v = \bar{\partial}_{J,H_2} w = 0,\\
&\bar{\partial}_{J,H_0} u(s,t) = 0 \text{ for all } 0 \le t \le 1,\ s \le -1,\\
&\bar{\partial}_{J,\frac{1}{2}|p|^2} u(s,t) = 0 \text{ for all } 0 \le t \le 1,\ -1 \le s \le 0,\\
&\big(u(s,0), u(s,\lambda+)\big) = \begin{cases} \big(u(s,\lambda-), u(s,1)\big), & -1 \le s \le 0,\\ \big(u(s,1), u(s,\lambda-)\big), & s \le -1,\end{cases}\\
&v(0,t) = u(0,\lambda t), \quad w(0,t) = u(0, \lambda + (1-\lambda)t) \quad \text{for all } 0 \le t \le 1\,.
\end{aligned} \tag{36}$$

Note that the variation of $\lambda \in (0, 1)$ can be equivalently regarded as a particular variation of the conformal structure on a pair-of-pants surface $\bar{\Sigma}$ with boundary (Fig. 9), given by $\Sigma_{\frac{1}{2}}$ sewed along $(s, 0) = (s, \frac{1}{2}-)$ and $(s, \frac{1}{2}+) = (s, 1)$ for $-1 \le s \le 0$ and $(s, 0) = (s, 1)$ and $(s, \frac{1}{2}-) = (s, \frac{1}{2}+)$ for $s \le -1$. In fact, $\bar{\Sigma}$ relative to $\partial\bar{\Sigma}$ has a topologically nontrivial Riemann moduli space and in order to define $\tilde{w}$ we are using a particular 1-cycle in its homology relative to its Deligne-Mumford compactification (Fig. 9).

Again, it is not hard to show that for generic choices of J and (V_0, V_1, V_2), $\widetilde{\mathcal{M}}_{x_0;x_1,x_2}$ is a smooth manifold of dimension

$$\dim \widetilde{\mathcal{M}}_{x_0;x_1,x_2} = \mu(x_0) - \mu(x_1) - \mu(x_2) - n + 1\,. \tag{37}$$

In order to obtain the important compactness modulo splitting-off of Floer trajectories, let us compute the energy estimate. We clearly have $\lambda|p|^2$, $(1-\lambda)|p|^2 \le |p|^2$

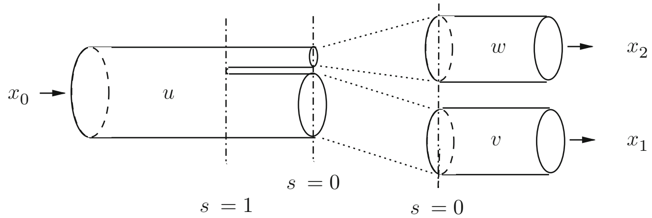

Fig. 9 The coproduct construction. $\widetilde{\mathcal{M}}_{x_1;x_2,x_3}$

for all $\lambda \in [0,1]$. Hence we have for any solution $(u,v,w,\lambda) \in \widetilde{\mathcal{M}}_{x_0;x_1,x_2}$

$$\mathcal{A}_{\frac{1}{2}|p|^2}(v(0)) \leq \mathcal{A}_{\lambda\frac{1}{2}|p|^2}(v(0)) = \mathcal{A}_{\frac{1}{2}|p|^2}\big(u(0,\cdot)_{|[0,\lambda]}\big) \quad \text{and}$$

$$\mathcal{A}_{\frac{1}{2}|p|^2}(w(0)) \leq \mathcal{A}_{(1-\lambda)\frac{1}{2}|p|^2}(w(0)) = \mathcal{A}_{\frac{1}{2}|p|^2}\big(u(0,\cdot)_{|[\lambda,1]}\big)\,.$$

Using $\epsilon \geq \|V_i\|_\infty$, we have

$$\mathcal{A}_{\frac{1}{2}|p|^2}(u(-1,\cdot)) = \mathcal{A}_{H_1}(u(-1,\cdot)) + \int_0^1 V_0(t,(\pi\circ u)(-1,t))dt$$

$$\leq \mathcal{A}_{H_1}(x_1) - \int_{-\infty}^{-1}\int_0^1 |\partial_s u|^2 ds dt + \epsilon$$

and

$$\mathcal{A}_{\frac{1}{2}|p|^2}(u(0,\cdot)) = \mathcal{A}_{\frac{1}{2}|p|^2}\big(u(0,\cdot)_{|[0,\lambda]}\big) + \mathcal{A}_{\frac{1}{2}|p|^2}\big(u(0,\cdot)_{|[\lambda,1]}\big)$$

$$\leq \mathcal{A}_{\frac{1}{2}|p|^2}(u(-1,\cdot)) - \int_{-1}^0\int_0^1 |\partial_s u|^2 ds dt\,.$$

Assembling all this gives

$$\mathcal{A}_{H_1}(x_1) \leq \mathcal{A}_{\frac{1}{2}|p|^2}(v(0,\cdot)) - \iint_0^\infty |\partial_s v|^2 ds dt + \epsilon$$

$$\mathcal{A}_{H_2}(x_2) \leq \mathcal{A}_{\frac{1}{2}|p|^2}(w(0,\cdot)) - \iint_0^\infty |\partial_s w|^2 ds dt + \epsilon$$

and

$$\begin{aligned}\mathcal{A}_{H_1}(x_1) + \mathcal{A}_{H_2}(x_2) &\le \mathcal{A}_{\frac{1}{2}|p|^2}(u(0,\cdot)) \\ &\qquad - \iint |\partial_s v|^2 ds dt - \iint |\partial_s w|^2 ds dt + 2\epsilon \\ &\le \mathcal{A}_{H_0}(x_0) - E(u,v,w) + 3\epsilon,\end{aligned}$$

that is

$$0 \le E(u,v,w) \le \mathcal{A}_{H_0}(x_0) - \mathcal{A}_{H_1}(x_1) - \mathcal{A}_{H_2}(x_2) + 3\epsilon, \tag{38}$$

with

$$E(u,v,w) = \int_{-\infty}^{0}\int_0^1 |\partial_s u|^2 ds dt + \int_0^\infty \int_0^1 \big(|\partial_s v|^2 + |\partial_s w|^2\big) ds dt .$$

With the usual arguments from the compactness theory for Floer trajectories in T^*Q for Hamiltonians of quadratic type, we see that $\widetilde{\mathcal{M}}_{x_0;x_1,x_2}$ is C^∞_{loc}-precompact.

The only new case here concerns sequences $(u_n, v_n, w_n, \lambda_n) \in \widetilde{\mathcal{M}}_{x_0;x_1,x_2}$ with $\lambda_n \to 0$ or $\lambda_n \to 1$. Assume without loss of generality $\lambda_n \to 0$. After choosing a C^∞_{loc}-convergent subsequence we view the restriction $u_n|[-1,0] \times [0,\lambda_n]$ as

$$u_n\colon [-1,0]\times\mathbb{R} \to T^*Q, \quad \overline{\partial}_{J,\frac{1}{2}|p|^2} u_n = 0 \quad \text{and}$$

$$u_n(s, t+\lambda_n) = u_n(s,t) \quad \text{for all } (s,t) \in [-1,0]\times\mathbb{R} .$$

We have $u_n \to u_\infty$ in C^∞_{loc},

$$u_\infty\colon [-1,0]\times\mathbb{R} \to T^*Q, \quad \partial_t u_\infty \equiv 0,$$

that is, $u_\infty(0) \in T^*Q$ is a point. On the other side

$$v_n(0,t) = u_n(0,\lambda_n t) \text{ f.a. } t\in\mathbb{R},\, n\in\mathbb{N}, \quad \text{and} \quad v_n \overset{C^\infty_{\mathrm{loc}}}{\to} v_\infty .$$

It follows that $v_\infty(0,t) = u_\infty(0)$ for all $t \in \mathbb{R}$. Hence

$$\mathcal{A}_{\frac{1}{2}|p|^2}\big(v_n(0)\big) \to \mathcal{A}_{\frac{1}{2}|p|^2}\big(u_\infty(0)\big) = -\frac{1}{2}|w_\infty(0)|^2 \le 0,$$

and thus

$$\mathcal{A}_{H_1}(x_1) \le \epsilon - \frac{1}{2}|u_\infty(0)|^2 \le \epsilon .$$

This proves

Proposition 5.2. *If* $\mathcal{A}_{H_1}(x_1)$, $\mathcal{A}_{H_2}(x_2) > \epsilon \geq \max\{\|V_1\|_\infty, \|V_2\|_\infty\}$, *then for all* $x_0 \in \mathcal{P}_1(H_0)$, *the solution space* $\widetilde{\mathcal{M}}_{x_0;x_1,x_2}$ *is compact modulo splitting of Floer trajectories.*

By counting the 0-dimensional solutions of $\widetilde{\mathcal{M}}_{x_0;x_1,x_2}$ we obtain a well-defined cochain operation on the Floer cochain complexes from the ascending $\mathcal{A}_H$-flow,

$$\tilde{w}^\bullet \colon F^k_{\geq a}(H_1) \otimes F^l_{\geq b}(H_2) \to F^{k+l+n-1}_{\geq a+b-3\epsilon}(H_0)$$
$$\tilde{w}^\bullet(x,y) = \sum_z \#_{\mathsf{alg}} \widetilde{\mathcal{M}}_{z;x,y}\, z\,,$$

for all $a, b > \epsilon$.

After using the usual continuation isomorphism of Floer theory in order to eliminate the perturbation V_i of $H = \frac{1}{2}|p|^2$, we obtain the product

$$\tilde{w} \colon HF^k_{\geq a}(H) \otimes HF^l_{\geq b}(H) \to HF^{k+l+n-1}_{\geq a+b}(H)$$

for all positive $a, b > 0$ and a ring $\big(HF^*_{>0}(H), \tilde{w}\big)$.

The proof that this product on cohomology is isomorphic to $\square$ from [19] will appear elsewhere.

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References

1. Abbondandolo, A., Majer, P.: Lectures on the Morse complex for infinite dimensional manifolds. In: Biran, P., Cornea, O., Lalonde, F. (eds.) Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology (Montreal), pp. 1–74. Springer, Berlin (2006)
2. Abbondandolo, A., Schwarz, M.: On the Floer homology of cotangent bundles. Comm. Pure Appl. Math. **59**, 254–316 (2006)
3. Abbondandolo, A., Schwarz, M.: A smooth pseudo-gradient for the Lagrangian action functional. Adv. Nonlinear Stud. **9**, 597–624 (2009)
4. Abbondandolo, A., Schwarz, M.: Estimates and computations in Rabinowitz-Floer homology. J. Topol. Anal. **1**(4), 307–405 (2009)
5. Abbondandolo, A., Schwarz, M.: Floer homology of cotangent bundles and the loop product. Geom. Topol. **14**(3), 1569–1722 (2010)
6. Abbondandolo, A., Portaluri, A., Schwarz, M.: The homology of path spaces and Floer homology with conormal boundary conditions. J. Fixed Point Theory Appl. **4**, 263–293 (2008)

7. Abouzaid, M.: A cotangent fibre generates the Fukaya category. Advances in Mathematics, **228**, 894–939 (2011)
8. Betz, M., Cohen, R.L.: Moduli spaces of graphs and cohomology operations. Turkish J. Math. **18**, 23–41 (1994)
9. Chas, M., Sullivan, D.: String topology. arXiv:math.GT/9911159 (1999)
10. Cieliebak, K., Latschev, J.: The role of string topology in symplectic field theory. In: Polterovich, L., Abreu, M., Lalonde, F. (eds.) New Perspectives and Challenges in Symplectic Field Theory CRM Proceedings and Lecture Notes, vol. 49, pp. 113–146. American Mathematical Society, Providence, RI (2009)
11. Cohen, R.L., Schwarz, M.: A Morse theoretic description of string topology. In: Polterovich, L., Abreu, M., Lalonde, F. (eds.) New Perspectives and Challenges in Symplectic Field Theory, CRM Proceedings and Lecture Notes, vol. 49, pp. 147–172. American Mathematical Society, Providence, RI (2009)
12. Cohen, R.L., Hess, K., Voronov, A.A.: String topology and cyclic homology. Lectures from the Summer School held in Almería, September 1620, 2003. Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2006
13. Floer, A.: A relative Morse index for the symplectic action. Comm. Pure Appl. Math. **41**, 393–407 (1988)
14. Floer, A.: The unregularized gradient flow of the symplectic action. Comm. Pure Appl. Math. **41**, 775–813 (1988)
15. Floer, A.: Symplectic fixed points and holomorphic spheres. Comm. Math. Phys. **120**, 575–611 (1989)
16. Floer, A.: Witten's complex and infinite dimensional Morse theory. J. Diff. Geom. **30**, 207–221 (1989)
17. Floer, A., Hofer, H.: Coherent orientations for periodic orbit problems in symplectic geometry. Math. Zeit. **212**, 13–38 (1993)
18. Fukaya, K., Ono, K.: Arnold conjecture and Gromov-Witten invariant. Topology **38**(5), 933–1048 (1999)
19. Goresky, M., Hingston, N.: Loop products and closed geodesics. Duke Math. J. **150**(1), 117–209 (2009)
20. Hofer, H. Salamon, D.: Floer homology and Novikov rings. In: Hofer, H., Taubes, C., Weinstein, A., Zehnder, E. (eds.) The Floer Memorial Volume, Progress in Mathematics, vol. 133, pp. 483–524. Birkhäuser, Berlin (1995)
21. Liu, G. Tian, G.: Floer homology and Arnold conjecture. J. Differential Geom. **49**(1), 1–74 (1998)
22. Piunikhin, S., Salamon, D. Schwarz, M.: Symplectic Floer-Donaldson theory and quantum cohomology. In: Thomas, C.B. (ed.) Contact and Symplectic Geometry (Cambridge 1994) Publ. Newton Inst., vol. 8, pp. 171–200. Cambridge University Press, London (1996)
23. Salamon, D., Weber, J.: Floer homology and the heat flow. Geom. Funct. Anal. **16**, 1050–1138 (2006)
24. Schwarz, M.: Cohomology operations from S^1-cobordisms in Floer homology, Ph.D. thesis, Swiss Federal Inst. of Techn. Zurich, Diss. ETH No. 11182 (1995)
25. Seidel, P.: A biased view of symplectic cohomology. in *Current Developments in Mathematics, 2006*. Edited by D. Jenison, B. Mazur, T. Mrowka, W. Schmid, R. Stanley and S.T. Yau. International Press, Somerville, MA, 2008, pp. 211–253
26. Viterbo, C.: Functors and computations in Floer homology with applications, II, preprint (revised in 2003) (1996)